Removing singularities from loop amplitudes

Babis Anastasiou ETH Zurich

in collaboration with George Sterman (arxiv:1812.03753) and G. Sterman, R. Haindl, Z. Yang, M. Zeng (in progress)

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Introduction

- Feynman diagrams and amplitudes: main tools for quantitative predictions for high energy processes.
- Difficult to compute, an active field of research over the span of many decades.
- Very satisfying progress for the purposes of LHC phenomenology: most processes at NLO, many processes at NNLO, few important processes at N3LO.
- Spectacular agreement of theory QCD predictions and experimental measurements.
- The LHC is a precision physics machine

Future precision mc(3 GeV) 0*.*986 GeV ↵S(*mZ*) 0*.*118 PDF PDF4LHC15_100 [49] All quark masses are treated in the MS scheme. To derive numerical predictions we use the program

outline in Ref. [47]. We use the following inputs: $\mathcal{A}^{\mathcal{A}}$

- A projection of Higgs crosssection measurements at the end of the high-luminosity LHC effects of electromagnetic order \mathbf{Q}_i of the collider energy. Each coloured band represents the size of one particular source of uncertainty as knowledge of the strong coupling constant and of parton distribution functions combined in quadrature. $\overline{}$ LO ((scale)).
	- experiment component will be an important component • Theoretical predictions for Standard Model cross-sections of the total uncertainty.

Fig. 28: (left) Summary plot showing the total expected *±*1 uncertainties in S2 (with YR18 systematic 1902.00134

A wish list…

Are we ready for such a leap?

Challenges

- One big challenge is the proliferation of Feynman diagrams.
- The integrands are simple rational functions of loop-momenta
- But established integration methods for loop amplitudes perform numerous costly operations on the integrands before final integrations.
- These operations are necessitated by the presence of divergences

$$
(In\; q\bar{q} \to Q\bar{Q})
$$

(Similar pattern for increasing the number of external legs)

NEED TO THINK OF ALTERNATIVES

Powerful schemes which have lead to impressive breaktroughs. But, I feel, that we have already achieved most of what is possible with them.

Alternative approach

- Generate amplitudes in momentum space.
- Integrate them directly after subtracting or deforming the integration contour away from singularities.
- The theoretical foundation for this program lies in the proofs of factorization for perturbative QCD *(Collins, Soper, Sterman)*
- For wide-angles and high energy, scattering amplitudes can be separated into short-distance (hard functions) and long-distance factors (jet and soft functions)

lines
Each by the momentum transfer. Each distribution of the momentum transfer. Each line connection the so-soft, i **Factorization in momentum-space**

$$
G = \int_C [dk] \left[\mathcal{F}(k) - \mathcal{F}_{approx}(k) \right]
$$
 Monte-Carlo Integration

$$
+\int_{-\infty}^{\infty} [dk] \mathcal{F}_{approx}(k)
$$

Factorization / Analytic Integration or combination with reak-radiation approximations

Subtraction of singularities

Outline

- Origin of singularities
- General method of nested subtractions
- Application to scalar integrals
- Application to two-loop QCD amplitudes
- Future prospects and possibilities.

Review of the origin of singularities The Service Constitution Containstant Constitution Constitution Constitution Constitution Constitution Constitu e estadounidense de la participa de la particip (propagate) from one

- Loop amplitudes contain the probability amplitude for propagation of particles in between vertices of Feynman graphs.
- These are singular when particles are on-shell. for the tiniest events cac are arrigarar writers
rtiolee are on-chell laws of physics and physics are the physics of physics and physics are $\frac{1}{2}$
- Do these singularities lead to divergent integrals?

$$
Ampl(A \to B) = \frac{\cdots}{E^2 - \omega^2}
$$

$$
\omega \equiv \sqrt{m^2 + \vec{p}^2}
$$

 ∞

$$
\left. \frac{E^2 - \omega^2}{E = \pm \omega} \right|_{E = \pm \omega} =
$$

"Infinities" from classical behaviour

$$
\int_{-\infty}^{\infty} dE \dots \frac{\cdots}{E^2 - \omega + i\delta} = \int_{-\infty}^{\infty} dE \dots \frac{\cdots}{\omega} \left(\frac{1}{E - \omega + i\delta} - \frac{1}{E + \omega - i\delta} \right)
$$

- The poles lie inside the domain of integration for the energy of virtual particles.
- If we can deform the path of integration away from the poles, then they lead to no singularities
- but the integral acquires both a real and imaginary part.

 $\overline{\omega}$ → $\overline{\omega}$ – *iδ* with *δ* → 0

"Infinities" from classical behaviour

$$
\int_{-\infty}^{\infty} dE \dots \frac{\cdots}{E^2 - \omega + i\delta} = \int_{-\infty}^{\infty} dE \dots \frac{\cdots}{\omega} \left(\frac{1}{E - \omega + i\delta} - \frac{1}{E + \omega - i\delta} \right)
$$

- The poles lie inside the domain of integration for the energy of virtual particles.
- If we can deform the path of integration away from the poles, then they lead to no singularities
- but the integral acquires both a real and imaginary part.

 $ω → ω − iδ$ with $δ → 0$

Soft massless particles

 $\mathbf{r} \infty$

- Poles due to soft massless particles.
- These singularities pinch the integration path from both sides.
- Condition for a TRUE INFINITY

$$
\int_{-\infty}^{\infty} dE \dots \frac{\dots}{(E + i\delta)(E - i\delta)}
$$

Collinear singularities

- A second source of infinities due to massless collinear particles.
- A singularity of one particle in the lower half-plane lines up with the singularity of a collinear particle in the higher half-pane.
- The singularities pinch the integration path from both sides.
- We cannot deform the path, a condition for a TRUE INFINITY!

Pinch singularities

- To know if a singularity develops, we need to study the behaviour of the integral in the vicinity of the pinch surface.
- We can calculate a degree of divergence.
- Scale variables which are perpendicular to the pinched surface with a small parameter and calculate the scaling of the integrand as the parameter is driven to zero.

Soft
$$
k^{\mu} \sim \delta Q
$$
, $d^4 k \sim \delta^4$

Collinear $k = xp + \alpha\eta + \beta p_\perp$, $x \sim \delta^0$, $\alpha \sim \delta$, $\beta \sim \delta^{\frac{1}{2}}$ $d^4k \sim \delta^2$

Integrand: d^4k *S*(*k*) ∼ δ^n **Convergent:** *n* > 0 Divergent: $n \leq 0$

Nested subtractions for an arbitrary number of loops in physical space The quantities *t*⇢ [Eq. (20)] can also be thought of as counterterms for ultraviolet divergences associated with the

- Singular regions are interconnected. How can we arbitrary $\frac{3}{2}$ channel as (*n*) $\frac{3}{2}$ channel as (*n*) $\frac{3}{2}$ channel as (*n*) $\frac{3}{2}$ contrary $\frac{3}{2}$ contrary $\frac{3}{2}$ contrary $\frac{3}{2}$ contrary $\frac{3}{2}$ contrary $\frac{3}{2}$ create systematically an **procedure a regulate a region of** \blacksquare approximation of the loop integrals in all singular regions? limits *x*²
- Order the singular regions by **and any then write for the full partonic partonic partonic partonic partonic partonic partonic part of P(***n***) = P(***n***) = P(***n***) = P(***n***) = P(***n***) = P(***n***) = P(n) = P(n) = P(n) = P(n) = P(n)** their "volume"
-
- Then, proceed to the next with the corresponding pinch surface. volume and repeat until there are no more singularities to remove.

III in the partonic matrix elements in the partonic matrix elements in the partonic matrix $\mathbf{P} = \mathbf{P} \mathbf{P} \mathbf{P}$

each PS separately. As a more general result, however, we will show that all divergent contributions to amplitudes

$$
R^{(n)}\,\gamma^{(n)}\;\;=\;\;\gamma^{(n)}\;\;+\;\sum_{N\in\mathcal{N}[\gamma^{(n)}]}\;\prod_{\rho\in N}\left(\,-\,t_\rho\right)\gamma^{(n)}\,,\quad \ \, \Bigg\vert\,
$$

where N is the set of all nonempty nestings for diagram . We will refer to R and subtraction operator at α

Nested subtractions d. C. André D. Ned

Ozan Erdogan, George Sterman

- Order the singular regions by **procedure a regeneral conducts** their "volume"
- Subtract an approximation of **We will refer to all nonempty nestings for all nonempty nestings for all references** *nthe integrand in the smallest nth-order <i>x*_I -integrand in the smallest *nth-order <i>x* volume
- remove. neighborhood of the corresponding pinch surface.

$$
R^{(n)}\,\gamma^{(n)}\;\;=\;\;\gamma^{(n)}\;+\;\sum_{N\in{\cal N}[\gamma^{(n)}]} \;\prod_{\rho\in N} \left(\,-\,t_{\rho}\right)\gamma^{(n)}\,,
$$

each PS separately. As a more general result, however, we will show that all divergent contributions to amplitudes

Nested subtractions d. C. André D. Ned The quantities **the quantities of associated of associated with the set of ultraviolet divergences associated with the set of associated with** Ozan Erdogan, George Sterman

- Order the singular regions by their "volume" limits *x*² $\overline{\mathbf{a}}$ in the partonic matrix \mathbf{b} and with $\overline{\mathbf{a}}$ and with $\overline{\mathbf{a}}$. We will denote an amplitude $\overline{\mathbf{a}}$ are is operatory and it one-particle in the *x*^{*I*} channel as (*n*) α ^{*I*} *n*) as (*n*) as (*n*) as (*n*) as (*n*) and (*n*) a procedure of Ref. [7], we define a regulated version of (*n*) by
- Subtract an approximation of the integrand in the smallest volume
- Then, proceed to the next volume *next in the full amplitude (6), and the full amplitude (6),* α α β γ and repeat until there are no more singularities to remove.
- perturbation theory.
- This structure gives rise to the corresponding pinch surface. factorisation into Jet, Soft and Hard functions for scattering amplitudes.

$$
R^{(n)} \gamma^{(n)} \;\; = \;\; \gamma^{(n)} \;\; + \;\; \sum_{N \in \mathcal{N}[\gamma^{(n)}]} \; \prod_{\rho \in N} \left(\,-\, t_{\rho} \right) \gamma^{(n)} \,,
$$

where N is the set of all nonempty nestings for diagram . We will refer to R and subtraction operator at α

each PS separately. As a more general result, however, we will show that all divergent contributions to amplitudes

An one-loop example which is one consistent with \mathbf{z} $\bf U \bf G \bf M$ dini $\bf U \bf G$ one-loop box fall into two categories, illustrated by the examples of Fig. 3a and b. First, the box is divergent in the four soft limits *kⁱ* ⇠ ! 0, for which the leading regions are four disjoint points in loop momentum space, illustrated by Fig. 3a. In terms of the power counting of Eq. (2.4) these regions all have *L^S* = 1, box diagram, to which we turn as a warm-up exercise in the following subsection. The following subsection in th **OIIE-IOOD EXQ** As a pedagogical example, we will apply the method of nested subtractions to the

p

2

p 3

out to be useful. This will also be appearent in our first example, the one-loop our first example, the one-loop α

Box (*s, t,*✏) = ¹

- One-loop massless box has both soft and collinear **A**
A²*k k*² *k*² *k* singularities internal momenta are related by
- A soft singularity occurs in a single point in momentum space (smallest volume). Needs to be subtracted first.
- A collinear singularity occurs in an one-dimensional space (larger volume). Needs to be subtracted after the soft.

assless box has	\n $\text{Box} = \int \frac{d^d k_1}{i \pi^{\frac{d}{2}} A_1 A_2 A_3 A_4}$,\n volume \n
arity occurs in a	\n softmax \n
asatisfy occurs in a	\n softmax \n
isingularities	\n $\frac{d^d k_2}{A_1 A_2 A_3 A_4} \rightarrow \frac{d^d k_2}{(-2k_2 \cdot p_1) k_2^2 (2k_2 \cdot p_2) t} \sim \mathcal{O}(\delta^{d-4})$ \n
est volume.	\n subtracted first. \n
isingularities	\n $\frac{d^d k_2}{A_1 A_2 A_3 A_4} \rightarrow \frac{d^d k_2}{A_1 A_2 s t x_1 (1 - x_1)} \sim \mathcal{O}(\delta^{\frac{d}{2} - 2})$ \n
isingularities	\n signal \n
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 $k_2 \rightarrow k_2$

 $\mathcal{L}(\mathcal{$

*A*² = *k*²

p

✏ + (*t*)

ventionally represented by "reduced diagrams", in which lines that are o 4σ

2

–8–

*A*¹ = (*k*² *p*1) *A*³ = (*k*² + *p*2)

² ⇠ *^O*(²

 $\boldsymbol{\mathcal{J}}$ ⇤

)*,*

 \mathcal{L}_{p}

p 3

 χ \sim 0, and the denominators scale as χ

² ⇠ *^t* ⁺ *^O*()*.* (2.11)

An one-loop example box diagram, to which we turn as a warm-up exercise in the following subsection. The following subsection in th **OIIE-IOOD EXQ** As a pedagogical example, we will apply the method of nested subtractions to the *x*¹ ⇠ *O*(1)*,* ¹ ⇠ *O*()*, k*¹? ⇠ *O*(p gions *kⁱ* = *xipi,* 0 *< xⁱ* 1 and, in the method of nested subtractions, they ought o anglong next. In the collinear limits, two propagators are only and two propagators are only and two propaga hard. We note as well that each soft region is an end-point of two collinear regions. We remove the divergence of the integral in the *k*² ! 0 limit by subtracting a let us label the subtraction operator for the *k*² ! 0 as *t^S*² . This operator acts as *t^S*² : *A*¹ ! 2*p*¹ *· k*² *,* = Z *d^dk*¹ *i*⇡ *d* ı **h** *A*1*A*2*A*3*A*⁴ *.* (2.21) This subtraction is certainly one of the possible choices that guarantees that the

mensions.

*k*¹ = *x*1*p*1+1⌘1+*k*¹?*, x*¹ ⌘

on-shell and one propagator is hard. The collinear singularities extend to larger re-

convenient, in which the *only approximation* is to neglect *k*² on the o s

*t^S*² : *Aⁱ* ! *Aⁱ , i* = 1*,* 2*,* 3 *,*

-
- Need to construct an in the soft limits.
- Options are not unique. Can have significant differences in

$$
\text{Box} \equiv \int \frac{d^d k_1}{i \pi^{\frac{d}{2}}} \frac{1}{A_1 A_2 A_3 A_4},
$$
\n
$$
\text{Subtractions which come first.}
$$
\n
$$
\text{Box}(s, t, \epsilon) = \frac{1}{st} \left\{ \frac{2c_\Gamma}{\epsilon^2} \left[(-s)^{-\epsilon} + (-t)^{-\epsilon} \right] - \pi^2 - \ln^2 \left(\frac{t}{s} \right) \right\} + \mathcal{O}(\epsilon)
$$
\n
$$
\text{Need to construct an}
$$

, ¹ ⌘

2*p*¹ *·* ⌘¹

1

, ⌘²

some purposes, in particular in particular in proofs of factorization, a choice in which we keep α

1

ⁱ = 0*,* ⌘*i·pⁱ* 6= 0*,* (2.14)

Z *d^dk*¹

out to be useful. This will also be appearent in our first example, the one-loop our first example, the one-loop α

2*p*¹ *·* ⌘¹

Z *d^dk*¹

approximation of the integrand

\n
$$
t_{S_2}: A_1 \rightarrow -2p_1 \cdot k_2
$$
\nin the soft limits.

\n1. $t_{S_2}: A_2 \rightarrow A_2$, $t_{S_2}: A_i \rightarrow A_i$, $i = 1, 2, 3$, $t_{S_2}: A_3 \rightarrow 2p_2 \cdot k_2$, $t_{S_2}: A_4 \rightarrow t$. (Nagy Soper)

\n1. $t_{S_2}: A_3 \rightarrow 2p_2 \cdot k_2$, $t_{S_3}: A_4 \rightarrow t$. (Nagy Soper)

\n2. $t_{S_2}: A_4 \rightarrow t$.

\n3. $t_{S_2}: A_3 \rightarrow 2p_2 \cdot k_2$, $t_{S_3}: A_4 \rightarrow t$. (Nagy Soper)

\n4. $t_{S_2}: A_4 \rightarrow t$.

\n5. $t_{S_2}: A_4 \rightarrow t$.

\n6. $t_{S_2}: A_4 \rightarrow t$.

\n7. $t_{S_2}: A_4 \rightarrow t$.

\n8. $t_{S_2}: A_4 \rightarrow t$.

\n9. $t_{S_2}: A_4 \rightarrow t$.

\n1. $t_{S_2}: A_4 \rightarrow t$.

\n2. $t_{S_2}: A_4 \rightarrow t$.

\n3. $t_{S_2}: A_4 \rightarrow t$.

\n4. $t_{S_2}: A_4 \rightarrow t$.

\n5. $t_{S_2}: A_4 \rightarrow t$.

\n6. $t_{S_2}: A_4 \rightarrow t$.

\n7. $t_{S_2}: A_4 \rightarrow t$.

\n8. $t_{S_2}: A_4 \rightarrow t$.

\n9. $t_{S_2}: A_4 \rightarrow t$.

\n1. $t_{S_2}: A_4 \rightarrow t$.

\n1. $t_{S_2}: A$

The subtraction in (2.21) associated with the *k*² = 0 singularity, for example,

It is easy to verify that this integral is not singular at any of the *k^µ*

*A*¹ = (*k*² *p*1) *A*³ = (*k*² + *p*2)

An one-loop example *i*=1 Box = ^Z *^d^dk*¹ *i*⇡ *d* 2 with *^N*Box = 1 *^A*²⁴ *^t ^A*¹³ *^s .* (2.23)

by comparing Eqs. (2.17) and (2.24).

Ξ

- The subtracted integral is now finite in all soft limits. The subtraction in (2.21) associated with the *k*² = 0 singularity, for example, is simply 1*/t* times a scalar triangle. When regulated dimensionally, the explicit expression for the subtraction is easily integrated, and the four subtraction terms of the four subtraction te
- Observation: The "soft" counterterms are easier to compute than the original integral (triangle integrals)
- The subtracted integral does not have quadratic poles in epsilon.
- In fact, it does not have single poles in epsilon either….

$$
t_{S_2} \text{ Box}(s, t, \epsilon) = t_{S_4} \text{ Box}(s, t, \epsilon) = \frac{c_{\Gamma}}{st\epsilon^2}(-s)^{-\epsilon}
$$

$$
t_{S_1} \text{ Box}(s, t, \epsilon) = t_{S_3} \text{ Box}(s, t, \epsilon) = \frac{c_{\Gamma}}{st\epsilon^2}(-t)^{-\epsilon}.
$$

ⁱ ! 0 soft limits.

!

It is easy to verify that this integral is not singular at any of the *k^µ*

$$
Box_R = -\frac{1}{st} \left[\pi^2 + \ln^2 \left(\frac{t}{s} \right) \right]
$$

In summary, for our introductory one-loop example, the method of nested sub-

tractions employed here yields the same separation of finite and divergent terms as

An one-loop example along with the leading behavior of each term in the numerator *N*Box, Eq. (2.22), that defines the sum of soft subtractions, evaluated at the pinch surface, *k*¹ = *xp*1, 0 *<x<* 1. Representing the action of the *p*1-collinear approximation by *t^C*¹ , we *t^C*¹ *A*¹ = *A*¹ *, t^C*¹ *A*² = *A*² *,*

have, in particular,

*N*Box*|*

- Let's consider a collinear limit
- Observation: The "soft" counterterms are easier to compute than the original integral (triangle integrals)
- The collinear limit approximation is potentially UV divergent.
- We introduce a UV counterterm to the Collinear counterterm as well *(Nagy, Soper)*.
- In this example, the numerator of the collinear counterterm vanishes.
- ..which explains why our softresult. (where *A*¹ = *A*² = 0),

Let's consider a collinear limit
\n
$$
t_{C_1} A_1 = A_1
$$
,
\nObservation: The "soft" counterterms are
\neasier to compute than the original
\nintegral (triangle integrals)
\nintegral (triangle integrals)
\n $t_{C_1} A_3 = (1 - x)s$,
\n $t_{C_1} A_4 = xt$.
\nThe collinear limit approximation is
\npotentially UV divergent.
\nWe introduce a UV counterterm to the
\nCollinear counterterm as well (*Nagy*,
\n s_{open}).
\nIn this example, the numerator of the
\ncollinear counterterm vanishes.
\n
$$
\int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \left[\frac{1}{A_1 A_2 s t x_1 (1 - x_1)} \right].
$$

\n
$$
\int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \left[\frac{N_{\text{Box}}}{A_1 A_2 A_3 A_4} - \frac{\frac{\mu^2}{\mu^2 - A_1} N_{\text{Box}}}{A_1 A_2 s t x_1 (1 - x_1)} \right].
$$

\n
$$
\text{which explains why our soft-}\n\begin{bmatrix}\n\frac{d^d k_1}{i\pi^{\frac{d}{2}}} \left[\frac{N_{\text{Box}}}{A_1 A_2 A_3 A_4} - \frac{\frac{\mu^2}{\mu^2 - A_1} N_{\text{Box}}}{A_1 A_2 s t x_1 (1 - x_1)} \right].\n\end{bmatrix}
$$

0 *<x<* 1. Representing the action of the *p*1-collinear approximation by *t^C*¹ , we

$$
N_{\text{Box}}|_{k_1 = -x_1 p_1} = \left[1 - \frac{A_{13}}{s} - \frac{A_{24}}{t}\right]|_{k_1 = -x_1 p_1}
$$

= 1 - (1 - x_1) - x_1
= 0.

Does the method work at two-loops?

A complicated web of interconnected divergences….

Nested subtractions at 2- \log *t*

- Order of subtractions: arbitrary *n*-loop diagram that is one-particle irreducible in the *x^I* channel as (*n*)
	- double-soft
	- soft-collinear
	- double-collinear
	- single-soft
	- single-collinear
- Approximations in singular regions do not need to be strict limits!
- introduce ultraviolet divergences
- Good approximations should be easy to integrate exactly.

each PS separately. As a more general result, however, we will show that all divergent contributions to amplitudes

Example: two-loop cross-box *A*3 *s ^t* ⁺ *u ^A*¹ *s u t*₂₂ ⁺ (*^t ^u*)² is free of all singularities associated with two independent loop momenta pinched independent loop momenta pinched in \mathbb{R} $\frac{p_1}{s_2}$ *^t* ⁺ *u s u t tu* ¹ *^A*³ **EXEL** + **1**ple: two *t s*2 **tu** (3.86) is free of all singularities associated with two independent locations and two independent locations and independent locations are in the standard independent locations and the standard induced induced induced in the stand *N*⁵ = two ◆² + *A*² *tu* (*A*² ⁺ *^s ^A*13) ✓ ¹ *^A*¹ *s* **free of all soft-collinear singularities** ✓*A*⁷ ◆ variable *s*. We can therefore modify our counterterm as follows: *N*⁴ = *N*³ + *A*2(*A*² + *s A*13) *tu* (3.85) The integral Xbox[*N*4] is now free of all double-soft singularities. We also find that is We therefore proceed with the subtraction of *two-collinear pairs/two-loop-collinear* types of singularities. These singular limits do not pose any special challenges and t are subtracted along the lines of our planet of our planet double-box example. We find that α

✓*A*⁶

◆

*A*⁶

◆

*A*2*A*⁴⁵⁶⁷

◆

◆ ✓*A*⁴

✓*A*⁷

 \mathcal{L}

*A*⁷

◆

and

 $_z$ </sub>

¹ *^A*¹

◆ ✓*A*⁵

F(1*c*)

*B*1*B*²

¹

i

1

*A*2*A*³

1

(*^N*⁵

*B*2*B*³

1

s(1 *x*3)

 $k_5 = -x_4p_4$

 $\frac{1}{2}$

¹

plication, due to the presence of power-like (rather than logarithmic) double-soft

*k*1=*x*1*p*¹

*k*2=0)

*N*⁵

*N*⁵

*A*4*A*5*A*6*A*⁷

*k*2=0)

For convenience below, and as for the planar box, we introduce the integral with an i $F_{Xbox} = F_{Xbox}^{(2)} + F_{Xbox}^{(1s)} + F_{Xbox}^{(1c)}$ d *ouble i*⇡ *^d* 2 *i*⇡ *^d* 2 *A*1*A*2*A*3*A*4*A*5*A*6*A*⁷ *-collineard^dk*⁵ *double N*(*k*2*, k*5) *i* $\sqrt{1}$ *i k*¹ = *k, k*² = *k* + *p*1*, k*³ = *k* + *p*12*, k*⁴ = *l p*12*,* $\kappa(A_4 \cup A_6)$ $\left| A_2 A_{4567} \right|$ $\overline{A_2}$ are $\overline{A_3}$ and $\overline{A_1}$ in removing the infrared singularities of $\overline{A_1}$ $p_1 + p_2 + p_3 + p_4 = 0$. We follow the same procedure as follows: $\begin{array}{ccc} u & v \end{array}$ and previous examples. Namely, we remove the singularities of \mathcal{U} iteratively, following the order: double-soft, soft-collinear, two-collinear pairs/two-Of the sixteen distinguishable double-soft regions of the crossed box, two have the property three lines are forced to \mathbf{v}_1 \mathbf{u} and \mathbf{v}_0 and \mathbf{v}_1 $\left[\frac{A_{4}A_{5}A_{6}A_{7}}{A_{4}A_{5}A_{6}A_{7}} \right]_{k_{2}=0} \quad\quad\quad F_{Xbox}=-\frac{1}{A_{1}A_{2}A_{3}}\left[\frac{A_{4}A_{5}A_{6}A_{7}}{A_{4}A_{5}A_{6}A_{7}} \right]_{k_{2}=0}$ $\binom{11}{2}$ 13 $\binom{11}{4}$ ¹ $\binom{3}{1}$ and $\binom{3}{1}$ k₂=0 configurations imply as well that *S*⁵ and *S*⁴ carry vanishing momentum, respectively. $A_{\rm eff}$ at configurations of the cross-box like this, we encounter an additional com-**Example 100 Single Single** all singularities: where, following the notation of the planar double box, the planar double box, the planar double box, the plan *N*⁵ $N_5 = \left(1 - \frac{14}{s}\right) + \frac{12}{tu}(A_2 + A_3)$ $\frac{1}{2}$ (3.888) $\frac{A_3}{s} \left(\frac{A_7}{t} + \frac{A_5}{u} \right) - \frac{A_1}{s} \left(\frac{A_3}{u} \right)$ *A*1*A*2*A*³ *A*4*A*5*A*6*A*⁷ *k*2=0 (3.89) *Xbox* = B_2B_3] $s(1-x_3)$ $\left[\frac{A_4A_5A_6A_7}{A_4A_5}\right]_{k_3=-x_2p_2}$
 $-\left[\frac{1}{A_4A_5}-\frac{1}{B_4B_5}\right]\left[\frac{N_5}{A_1A_2A_3A_6A_7}\right]_{k_5=-x_3p_3}$ $\left[\frac{1}{A_6A_7} - \frac{1}{B_6B_7}\right] \left[\frac{N_5}{A_1A_2A_3A_4A_5}\right],$ *k*2=0) $\overline{}$ **soft collinear limits** + *A*2*A*⁴⁵⁶⁷ a special kinematic configuration (soft or collinear). Γ integral subtractive of all double-soft singular integral is Γ (1*x*) in Γ (1*x*) is not all double-soft singular integral is not all double-soft singular integral in Γ (1*x*) is not all double-soft singular where, following the notation of the planar double box, the planar double box, the planar double box, the planar $F_{Xbox}^{(2)} = \frac{N_5}{4.4 \cdot 4.4 \cdot 4.4}$ $A_1A_2A_3A_4A_5A_6A_7$ $\overline{A_1} \left(\overline{A_1} \right) \left(\overline{A_2} \right) \left(\overline{A_3} \right)$ Γ ^(1c) $\begin{bmatrix} 1 & 1 \end{bmatrix}$ $\left[\frac{1}{B_1B_2}\right] \frac{1}{s(1-x_1)} \left\{ \left[\frac{N_5}{A_4A_5A_6A_7}\right]_{k=-\infty} - \left[\frac{N_5}{A_4A_5R_6A_7}\right]_{k=-\infty} \right\}$ (3.89) 1 $\left[\frac{1}{A_6A_7} - \frac{1}{B_6B_7}\right] \left[\frac{1}{A_1A_2A_3A_4A_5}\right]_{k_5=-k_5}$ *k*2=0) variable *s*. We can therefore modify our counterterm as follows: *tu* (3.85) $f\cdot T_{Xbox} + T_{Xbox} + T_{Xbox}$ We therefore proceed with the subtraction of *two-collinear pairs/two-loop-collinear* types of singularities. These singularities and pose and pose and pose and pose and pose any special challenges and pose an \sim subtracted along the lines of our planet of our planet of our planet of our planet of \sim $N_5 =$ $\bigg(1-\frac{A_{13}}{s}\bigg)$ \setminus^2 $+\frac{A_2}{tu}(A_2+s-A_{13})$ $\overline{}$ $\bigg(1-\frac{A_1}{s}$ ◆ ✓*A*⁵ $\frac{A_5}{t} + \frac{A_7}{u}$ *u* ◆ \equiv $\bigg(1-\frac{A_3}{s}$ ◆ ✓*A*⁴ $\frac{A_4}{u} + \frac{A_6}{t}$ *t* ◆ $+\frac{A_2A_{4567}}{4}$ *tu* $-\frac{A_3}{s}$ ✓*A*⁷ $\frac{A_7}{t} + \frac{A_5}{u}$ *u* $\bigg) - \frac{A_1}{s}$ ✓*A*⁶ $\frac{4}{u} + \frac{A_4}{t}$ *t* $+\frac{(t-u)^2}{s^2}$ $\frac{A_1 A_3}{t u}$ is free of all singularities associated with two independent loop momenta pinched in $\begin{bmatrix} N_5 & 1 \end{bmatrix}$ $\left.A_5A_6A_7\right]_{k_2=0}$ N_5 in $\Gamma(1s)$ integrations, we find that the following integrations in the following integration is free of 1 integrations, we find the following integration is free of 1 integrations, we find the following integr $A_5A_6A_7\rfloor_{k_2=0}$ $F_{1}F_{2}F_{3}$ $F_{4}F_{5}$ $k_2=0$ *i* **o**-loop single single *A*⁵ ✓*A*⁶ ⁺ (*^t ^u*)² *tu* (3.86) $\overline{\mathbf{r}}$ as $\overline{\mathbf{r}}$ as $\overline{\mathbf{r}}$ (1s) as $\overline{\mathbf{r}}$ (1c) $\frac{d}{dx} x + \frac{d}{dx} x + \frac{d$ $\textbf{double} \quad \textbf{double} \quad \textbf{double}$ Λ $\frac{F}{t}$ + $\frac{A_2}{t}$ (A₂ + *s* - A₁₃) $A_4A_5A_6A_7$ $\left[\frac{1-\frac{1}{s}}{\frac{1}{t}+\frac{1}{u}}-\frac{1-\frac{1}{s}}{\frac{1}{s}}\frac{\frac{1}{u}+\frac{1}{t}}{\frac{1}{u}+\frac{1}{t}}\right]$ $\left(\frac{2}{u}\right) - \frac{2}{s}\left(\frac{2}{u} + \frac{2}{t}\right) + \frac{1}{u}$ *A*1*A*2*A*3*A*4*A*5*A*6*A*⁷ *,* (3.88) $F_{Xbox}^{(1s)} = -\frac{1}{A_1A_2A_3}$ N_5 $A_4A_5A_6A_7$ $\overline{}$ $k_2=0$ $-\left[\frac{1}{A_2A_7}-\frac{1}{B_2B_7}\right]\left[\frac{N_5}{A_1A_2A_3A_4A_7}\right]$ *s* $\frac{1}{2}$ *tu* F_{Xbox} $= F (X_{\text{2}}^{\prime\prime})^{2} + F(X_{\text{2}}^{\prime})^{2} + F(X_{\text{2}}$ $\frac{1}{\lambda}$ $\Box(2)$ and N_5 $\overline{A_2A_3A_4A_5A_6}$ A_{π} ['] $\overline{}$ $F_{Xbox}^{(1c)} = -\left[\frac{1}{A_{1c}}\right]$ $\frac{1}{A_1 A_2} - \frac{1}{B_1 B_2}$ 1 $s(1 - x_1)$ $\int \left[\right]$ *N₅* $A_4A_5A_6A_7$ 1 $k_1 = -x_1p_1$ N_5 $A_4A_5A_6A_7$ $\bigg\}$ _{$k_2=0$} $\bigg\}$ Ξ $\lceil 1$ $\frac{1}{A_2A_3} - \frac{1}{B_2B_3}$ 1 $s(1 - x_3)$ $\int \left[\right]$ N_5 $A_4A_5A_6A_7$ 1 *k*3=*x*2*p*² N_5 $A_4A_5A_6A_7$ $\bigg\}$ _{$k_2=0$} $\bigg\}$ $\lceil 1$ $\frac{1}{A_4A_5} - \frac{1}{B_4B_5}$ $\bigcap_{i=1}^n$ $A_1A_2A_3A_6A_7$ 1 *k*5=*x*3*p*³ $\lceil 1$ $\frac{1}{A_6A_7} - \frac{1}{B_6B_7}$ \bigcap_{s} $A_1A_2A_3A_4A_5$ 1 *.* (3.90) *-soft*

*A*4*A*5*A*6*A*⁷

*B*1*B*²

*N*⁵

*N*⁵

¹

*A*1*A*²

and

i

(*^N*⁵

*N*⁵

Ξ

*k*2=0)

Xbox ⁼ ¹

s(1 *x*1)

F(1*s*)

*A*4*A*5*A*6*A*⁷

*A*4*A*5*A*6*A*⁷

Example: two-loop cross-box In the above, *Bⁱ* = *Aⁱ µ*². Upon direct analytic integration, using the integration techniques des des counters in the countertext of the analytic section for the analytic section f $r_{\rm F}$ for the crossed double-box integral, we verify that $r_{\rm F}$ result of the cross integral, we verify the cross ckoc Z *d^dk*² *d^dk*⁵ *d* **00** 2 *FXbox* = *O*(✏ In the above, *Bⁱ* = *Aⁱ µ*². Upon direct analytic integration, using the integration

result of \overline{S} for the crossed double-box integral, we verify that \overline{S}

Specifically, for *s >* 0 and *y* ⌘ *t/s* 2 [0*,* 1], we find

+44 Li3(*y*) log(1 *^y*) + 96 Li3(*y*) log(*y*) 188 Li4(*y*) + ¹⁷

+

⇡² log(1 *^y*)

⁴ log(*y*)

³ log(1 *^y*)

2

² log(*y*)

³ log(*y*)

² ⇡² + log(*y*) log(1 *^y*)

^GR(*y*) = 12 Li2(*y*) log(*y*) + 12 Li3(*y*) + ²

2

² log(*y*)

6

^ER(*y*) = ⁸ ⇡² Li2(*y*) + 8 Li2(*y*) log(1 *^y*)

4 log(*y*) ⇣³ ¹

+7 log(*y*) log(1 *^y*) ⇡² ²⁵

where

*A*1*A*2*A*3*A*4*A*5*A*6*A*⁷

¹² log(1 *^y*)

⇡² log(1 *^y*)

3

³ log(*y*) ⇡²

(3.94)

 $A_{\rm eff}$ at configurations of the cross-box like this, we encounter an additional complication, due to the presence of power-like (rather than logarithmic) double-soft

, (3.95)

⇡² log(1 *^y*) ⁸

² log(1 *^y*) + ²

$$
X_{\rm box}{}^{\rm fin}\equiv\int\frac{d^dk_2}{i\pi^\frac{d}{2}}\frac{d^dk_5}{i\pi^\frac{d}{2}}F_{Xbox}=\mathcal O(\epsilon^0).\qquad \text{ s3X$_{\rm box}$^{fin}=\frac{f_{X_{\rm box}}(y)}{y}+\frac{f_{X_{\rm box}}(1-y)}{1-y}},
$$

 $\overline{}$

fin ⌘

$$
f_{X_{\text{box}}}(y) = [G_R(y) + i\pi G_I(y)] \log \left(\frac{\mu^2}{s}\right) + E_R(y) + i\pi E_I(y)
$$

$$
E_R(y) = -8\pi^2 \operatorname{Li}_2(y) + 8 \operatorname{Li}_2(y) \log(1 - y)^2 - 28 \log(y) \operatorname{Li}_2(y) \log(1 - y) - 18 \operatorname{Li}_2(y) \log(y)^2
$$

+44 \operatorname{Li}_3(y) \log(1 - y) + 96 \operatorname{Li}_3(y) \log(y) - 188 \operatorname{Li}_4(y) + \frac{17}{36} \pi^4 + \frac{1}{12} \log(1 - y)^4
+7 \log(y) \log(1 - y) \pi^2 - \frac{25}{6} \pi^2 \log(1 - y)^2 - \frac{3}{2} \log(y)^2 \pi^2 + \log(y) \log(1 - y)^3
+44 \operatorname{S}_{12}(y) \log(1 - y) - 52 \operatorname{S}_{12}(y) \log(y) + 84 \operatorname{S}_{13}(y) + 88 \operatorname{S}_{22}(y) - 44 \zeta_3 \log(1 - y)
-4 \log(y) \zeta_3 - \frac{1}{4} \log(y)^4 + \log(y)^3 \log(1 - y) - \frac{9}{2} \log(y)^2 \log(1 - y)^2,

+44 *S*12(*y*) log(1 *y*) 52 *S*12(*y*) log(*y*) + 84 *S*13(*y*) + 88 *S*22(*y*) 44 ⇣³ log(1 *y*)

, (3.95)

Complexity of counterterms at two-loops V U Introduction stu↵... loops IE VV $\mathbf{1} \cdot \mathbf{1}$

Figure 1. The two-loop planar box *^FP box* ⁼ *^F*(2) In double-soft approximations:

P box = 1 *^P*²⁵⁷

1 **(all known). Double-soft counterterms are integrals with at most six massless propagators** $\mathbb R$ representative reduced diagrams for double-soft pinches. In (a) and (b), the formula $\mathbb R$ denominator of single line is fixed at *t* and *s*, respectively, while in (c) two lines are o↵-shell

Complexity of counterterms at two-loops

- Collinear counterterms for a Feynman diagrams or a Feynman integral require the convolution of a subgraph
- At two-loops, we have to integrate over one-loop infrared-subtracted subgraphs
- It can be done analytically, in principle… it requires a good calculator of one-loop integrals and a good dictionary for the integration of polylogarithms
- it can also be done numerically, with little effort
- Collinear counterterms are much simpler commoditions for physical amplitudes (exploiting QCD factorization) char

CCS
\n
$$
\int_0^1 \frac{dx}{x} \left[S_{12} \left(\frac{(x-y)(xy-1)}{y(x-1)^2} \right) - 2 \text{Li}_2 \left(\frac{(x-y)(xy-1)}{y(x-1)^2} \right) \log(1-x) - \zeta_3 \right]
$$
\n
$$
= -\frac{1}{24} \log(y)^4 - 2 \text{Li}_2(y)^2 + \frac{13}{45} \pi^4 - \text{Li}_2(y) \log(y)^2 + 4 \text{Li}_3(y) \log(y)
$$
\n
$$
-4 \zeta_3 \log(y) - \frac{4}{3} \pi^2 \text{Li}_2(y) - 8 \text{Li}_4(y) + 8 S_{22}(y). \tag{3.98}
$$

Of course, by using dimensional regularization we abandon the use of point-by-point cancellation in Eq. (3.16). Nevertheless, it will enable us to confirm the finiteness of

Subtractions for QCD amplitudes

with Rayan Haindl, George Sterman, Zhou Yang, Mao Zeng

- This is work in its infancy…
- From first principles, we expect that nested subtractions can separate the short distance (finite part) of physical amplitudes from the long distance (singularities) part.
- Significant simplifications occur in comps
- Singularities are at most logarithmic
- Factorisation of all singular limits when physical sets of Feynman diagrams are combined together
- Hope Generic subtraction terms for all processes.

- Consider the process for the production of a heavy colourless final-state from the scattering of a massless quarkantiquark pair.
- This encompasses a large set of processes (multi Z,W, photon production and combinations)
- Easy to verify at one-loop that a simple set of local counterterms exists for all these processes.

- Per tree-diagram, there is one 1-loop diagram with a soft singularity.
- The soft limit is (up to trivial factors), an one-loop scalar integral times a tree-diagram.

- Per tree-diagram, there is one 1-loop diagram with a soft singularity.
- The soft limit is (up to trivial factors), an one-loop scalar integral times a tree-diagram.

- Many graphs yield collinear divergences.
- Summing over all such graphs, cancellations take place ("Ward"-identity)
- The net-result is factorization of the amplitude in the collinear limit in terms of a splitting-functions and a treediagram.

- The same mechanisms factorise the singular limits of two-loop amplitudes as well
- We have derived the factorisation of the singular limits explicitly for the abelian part of two-loop amplitudes of colourless final-states. nicitly fo nto explicitly for the abo
- All limits work in a straightforward manner —except the single collinear limit for lines with self-energy

 $\boldsymbol{\lambda}$

Pi ki Pl ^K Da 4

<u>x1</u>¹ · 001 ⌘¹ *· p*¹ , *^y*² ⁼ ⌘² *· ^l* ⌘² *· p*² (A.2) **FACTORIZATION OF TWO-LOOP AMPLITUDE IN ITS SINGULAR LIMITS**

⌘¹ *· p*¹

where $\mathcal{L}_{\mathcal{A}}$ and $\mathcal{L}_{\mathcal{A}}$ are fixed reference light-like vectors.

planar double soft

⌘² *· p*²

non-planar double soft

planar soft-collinear

FACTORIZATION OF TWO-LOOP AMPLITUDE IN ITS SINGULAR LIMITS

non-planar soft-collinear

single soft

Numerical validation $q + \bar{q} \rightarrow \gamma + \gamma$

Numerical validation $q + \bar{q} \rightarrow \gamma + \gamma$

 $q + \bar{q} \rightarrow \gamma + \gamma$

Numerical integration

- Can such subtractions be used for evaluating loop amplitudes numerically?
- They are an important ingredient! They remove "pinch" singularities. 1 $\frac{1}{2}$ *irnperlant*
e "pinch"
- Other singularities which can be avoided with appropriate contour-deformations are equally important.
- A very challenging problem! Very encouraging progress by Z. Capatti, V. Hirschi, D. Kermanschah, A. Pelloni, B. Ruijl at ETH and other groups.

integrand with large variance

د،ا-

QIME

 $+\omega$

DEFORMED PATH

One-loop direct momentumspace integration *Numerical evaluation of NLO multiparton processes* C. Reuschle

- Foundational work by Nagy and Soper
- and by Becker and Weinzierl
- Good results in computing challenging one-loop amplitudes. 1 4*N*3 $\overline{}$
- Tough competition at one-loop with OPP/unitarity/semianalytic methods. calculation and an analytic calculation. The error bars from the MC integration are shown are shown and are almost α

Figure 6: Comparison of the NLO corrections to the two-, three- and four-jet rate between the numerical **(Becker, Goetz, Reuschle, Schwan, Weinzierl)4-jet production at NLO**

Loop-Tree Duality Analytically integrate over the loop energies using Cauchy's theorem.

$k^2 = 0$

- The energy component of the loop-momenta can be integrated out simply, using Cauchy's theorem.
- Leading to a nice mathematical structure at any loop order^{ded on}
- It appears to be advantageous numerically as well. $(k - p_1)^2 = 0$

Manufation For Multiple-learning for Multiple-learning for Multiple-learning ACAT 11.01.2019 Catani, Gleisberg, Krauss, Bierrenbaum, Catani, Draggiotis, Rodrigo; Buchta, Chachamis, Draggiotis, Rodrigo; Runkel, Szor, Vesga, Weinzierl; Capatti, Hirschi, Kermanschah, Pelloni, Ruijl

Numerical integration of one-,two- and three-loop off-shell planar box after LTD **- Euclidean region) Numerical implementation of Loop-Tree Duality beyond one loop** Formal derivation: @N-loops: Rodrigo & al. [1007.0194, 1211.5048] Weinzierl & al. [1902.02135] @one-loop: Catani & al. [0807.0531] Z. Capatti, V. Hirschi, D. Kermanschah, A. Pelloni, B. Ruijl **PRELIMINARY PRELIMINARY**- Deformation: Minkowski = complicated

Capatti, Hirschi, Kermanschah, Ruijl

A spin-off

Small mass expansions

Physical regulators

- The subtraction counterterms are local.
- They can be invented with dimensional regularisation in mind, but they can also be adapted to other regularisation schemes for the IR divergences.
- Small quark masses act as physical regulators.
- In such case, the infrared counterterms integrate to yield the logarithmically enhanced terms of the integral.

Large logs from small masses easily determined. **1** Ω *st* 1 *A*3*A*⁴ ii *^x*³ + (1 *^x*3)*^M*² *s* $\overline{}$ (5.16) $\overline{2}$ 2*p*¹ *·* ⌘¹ *, x*³ = *±* 2*k*³ *·* ⌘³ 2*p*³ *·* ⌘³ *.* (5.17) 5.3 Two-loop massive diagonal box with two o↵-shell legs

¹ *^A*¹

*^s ^A*⁴

J^R =

i⇡ *^d*

$$
D_{\text{box}} = \left[\frac{1}{(A_1 - m^2)(A_2 - m^2)} - \frac{1}{(A_1 - \mu^2)(A_2 - \mu^2)} \right] \left[\frac{1}{A_3 A_4 A_5} \right]_{k_1 = -x_2 p_2}
$$

+
$$
\left[\frac{1}{(A_3 - m^2)(A_4 - m^2)} - \frac{1}{(A_3 - \mu^2)(A_4 - \mu^2)} \right] \left[\frac{1}{A_1 A_2 A_5} \right]_{k_4 = x_4 p_4}
$$

-
$$
\left[\frac{1}{(A_1 - m^2)(A_2 - m^2)(A_3 - m^2)(A_4 - m^2)} \right]
$$

×
$$
\left[-\frac{1}{(A_1 - \mu^2)(A_2 - \mu^2)(A_3 - \mu^2)(A_4 - \mu^2)} \right] \left[\frac{1}{A_5} \right]_{k_4 = x_4 p_4, k_1 = -x_2 p_2}
$$

+
$$
D_{\text{box}}|_{\text{fin}} + \mathcal{O}(m^2)
$$

fin is given by Eq. 3.57. We have checked our result against numerical

where Dbox*|*

Large logs from small masses easily determined. **1** Ω *st* 1 *A*3*A*⁴ ii *^x*³ + (1 *^x*3)*^M*² *s* $\overline{}$ (5.16) $\overline{2}$ 2*p*¹ *·* ⌘¹ *, x*³ = *±* 2*k*³ *·* ⌘³ 2*p*³ *·* ⌘³ *.* (5.17) 5.3 Two-loop massive diagonal box with two o↵-shell legs that achieves this purpose for generic multi-loop integrals has been presented in Ref [46]. For the full finite part, including the original diagram, we have from the above, Dbox*|* fin ⁼ (1 + ✏)² ✏2 *dxdy A*(*x, y*) $\sqrt{25}$ (1 + *e*)²(1 3✏) *A*(*x, y*) 2✏ **1** $\frac{1}{2}$ ⁼ (1 + ✏)² ✏² (*µ*²) 2✏ Z ¹ where *A*(*x, y*) is given by Eq. (3.13). The remaining integrals can then be done in a straightforward manner analytically in terms of rank-two polylogarithmic functions.

are applied. We emphasize again that the development of an ecient numerical

¹ *^A*¹

*^s ^A*⁴

J^R =

i⇡ *^d*

$$
u = m_1^2 + m_3^2 - s - t, \quad K = m_1^2 m_3^2 - st,
$$

$$
v_1 = \frac{um_1^2}{K}, \quad v_3 = \frac{um_3^2}{K}, \quad v_s = \frac{us}{K}, \quad v_t = \frac{ut}{K}
$$

. (3.24)

*x*⁴ ⌘ *y* integrations, respectively. For this term, we find in this way,

$$
u D_{\text{box}}|_{\text{fin}}(\mu) = 2 \text{Li}_4(v_1) + 2 \text{Li}_4(v_3) - 2 \text{Li}_4(v_s) - 2 \text{Li}_4(v_t)
$$

\n
$$
-2 \text{Li}_3(v_1) L_{\mu}(m_1^2) - 2 \text{Li}_3(v_3) L_{\mu}(m_3^2) + 2 \text{Li}_3(v_s) L_{\mu}(s) + 2 \text{Li}_3(v_t) L_{\mu}(t)
$$

\n
$$
+ \text{Li}_2(v_1) L_{\mu}^2(m_1^2) + \text{Li}_2(v_3) L_{\mu}^2(m_3^2) - \text{Li}_2(v_s) L_{\mu}^2(s) - \text{Li}_2(v_t) L_{\mu}^2(t)
$$

\n
$$
+ \frac{1}{3} \ln(1 - v_1) L_{\mu}^3(m_1^2) + \frac{1}{3} \ln(1 - v_3) L_{\mu}^3(m_3^2) - \frac{1}{3} \ln(1 - v_s) L_{\mu}^3(s)
$$

\n
$$
- \frac{1}{3} \ln(1 - v_t) L_{\mu}^3(t).
$$

Eq. (3.25) and log(*µ*) in Eq. (3.27) all cancel. This is in accordance with expectations, since the strongest singularity is due to two-collinear pairs capable of producing at

Concluding remarks

- Nested subtractions can separate at the integrand, the pinch-singularities of Feynman diagrams and Feynman amplitudes.
- We aim to formulate a subtraction method for two-loop amplitudes of generic processes.
- This can be the basis for a purely numerical evaluation of two-loop amplitudes with an affordable computational cost.
- Substantial amount of work is needed in achieving that…it requires an excellent understanding of both pinched and integrable singularities (contour deformations)
- Spin-off: Nested subtractions are potentially useful for small mass expansions of loop amplitudes (e.g. bottom/charm-quark loop-induced processes, very high energy collider processes).