Removing singularities from loop amplitudes

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in collaboration with
George Sterman (arxiv:1812.03753)
and G. Sterman, R. Haindl, Z. Yang, M. Zeng (in progress)

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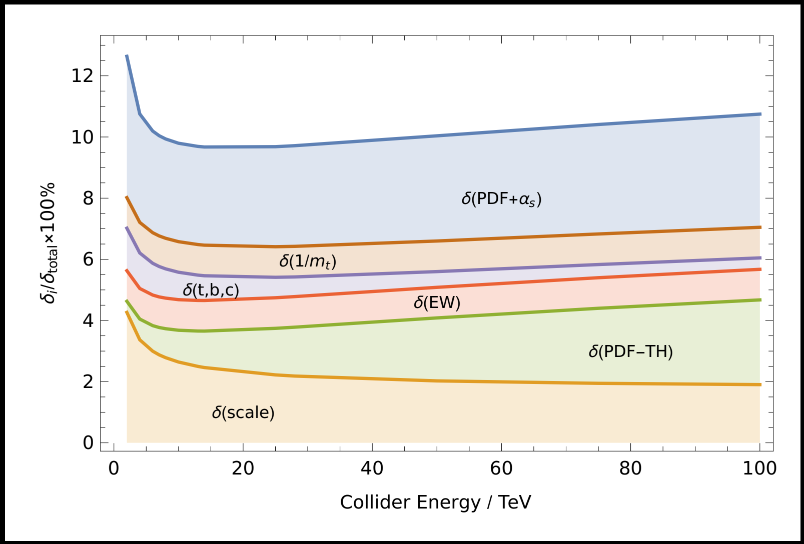
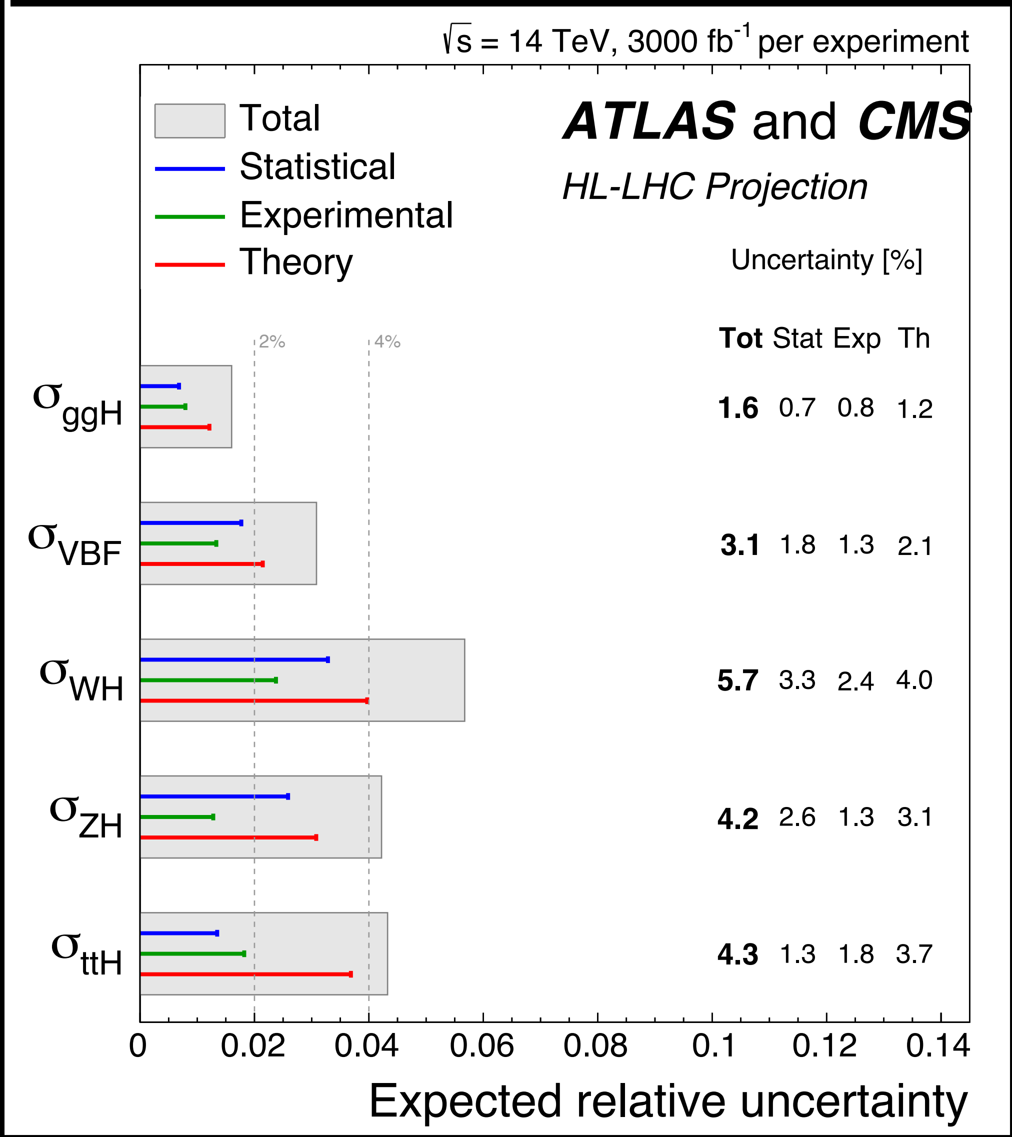
Introduction

- Feynman diagrams and amplitudes: main tools for quantitative predictions for high energy processes.
- Difficult to compute, an active field of research over the span of many decades.
- Very satisfying progress for the purposes of LHC phenomenology: most processes at NLO, many processes at NNLO, few important processes at N3LO.
- Spectacular agreement of theory QCD predictions and experimental measurements.
- The LHC is a precision physics machine

Future precision



- A projection of Higgs cross-section measurements at the end of the high-luminosity LHC programme.
- Theoretical predictions for Standard Model cross-sections will be an important component of the total uncertainty.



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A wish list…

PROCESSS CLASS	EXAMPLES	STATUS	POSSIBLE Phenomenolog y motivated GOAL
$2 \to 1$	H,W,Z,WH,ZH	N3LO	N3LO
$2 \to 2$	jet inclusive, diboson, top-pair, photon-jet, …	NNLO	N3LO
$2 \to 3$	ttH,diphton+jet, WW/ZZ/ZW+jet, top-pair+jet,…	NLO	NNLO

Are we ready for such a leap?

Challenges

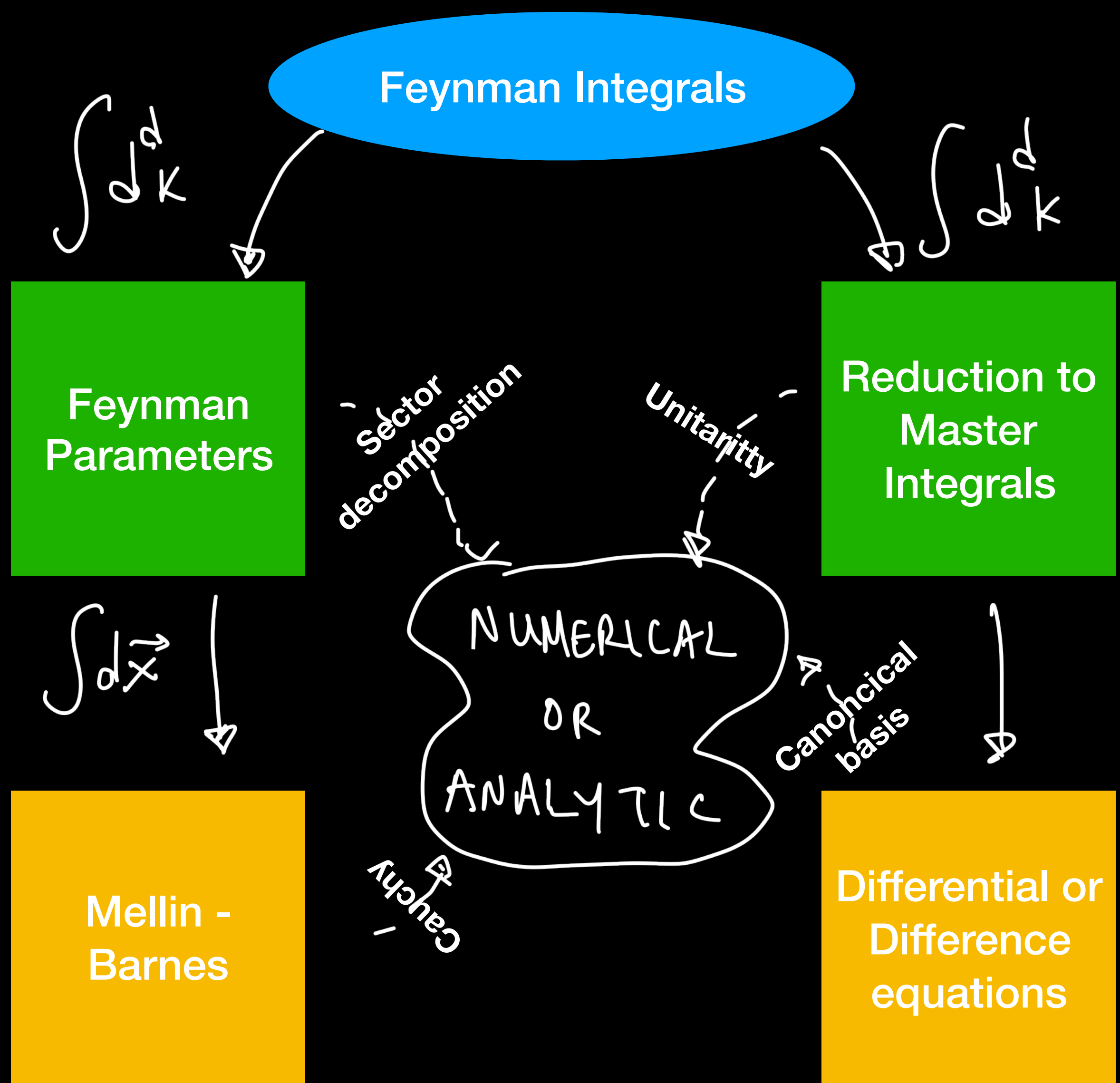
- One big challenge is the proliferation of Feynman diagrams.
- The integrands are simple rational functions of loop-momenta
- But established integration methods for loop amplitudes perform numerous costly operations on the integrands before final integrations.
- These operations are necessitated by the presence of divergences

$(In\ q\bar{q} \to Q\bar{Q})$

Order	Diagrams
tree	1
1-loop	10
2-loop	189
3-loop	134225

(Similar pattern for increasing the number of external legs)

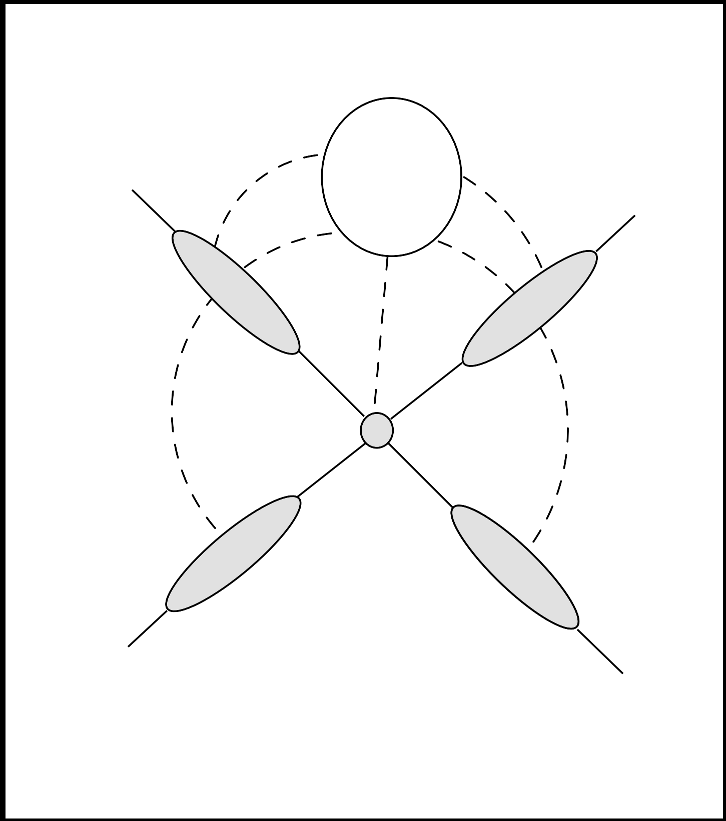
NEED TO THINK OF ALTERNATIVES



Powerful schemes which have lead to impressive breaktroughs.
But, I feel, that we have already achieved most of what is possible with them.

Alternative approach

- Generate amplitudes in momentum space.
- Integrate them directly after subtracting or deforming the integration contour away from singularities.
- The theoretical foundation for this program lies in the proofs of factorization for perturbative QCD *(Collins, Soper, Sterman)*
- For wide-angles and high energy, scattering amplitudes can be separated into short-distance (hard functions) and long-distance factors (jet and soft functions)



Factorization in momentum-space

Basic idea

Amplitude

$$G = \int_{-\infty}^{\infty} [dk] \mathcal{I}(k)$$

$$G = \int_{C} [dk] \Big[\mathcal{I}(k) - \mathcal{I}_{approx}(k) \Big]$$

Monte-Carlo Integration

$$+ \int_{-\infty}^{\infty} [dk] \mathcal{I}_{approx}(k)$$

Factorization / Analytic Integration or combination with reak-radiation approximations

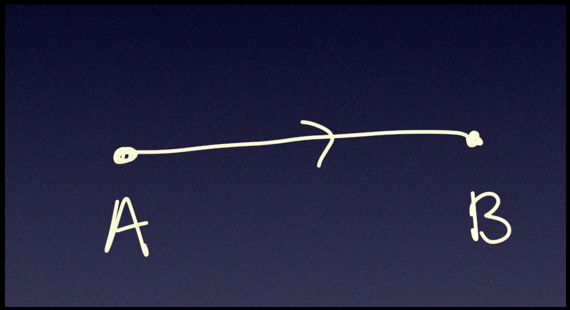
Subtraction of singularities

Feynman parameter space	Momentum space
• IR/UV counterterms can be found algorithmically for arbitrary loops • A sector-decomposition algorithm can disentangle overlapping singularities (*Binoth, Heinrich)* • Contour deformations can be produced algorithmically for arbitrary loops *(Nagy, Soper)*	• IR/UV counterterms have been found only at one-loop *(Nagy, Soper)* • Contour deformations are known at one-loop and beyond for processes. *(Nagy, Soper; Becker, Weinzierl), But not efficient!* • A promising field of research with space for new ideas (e.g. loop-tree-duality by Catani, Rodrigo et al.)

Outline

- Origin of singularities
- General method of nested subtractions
- Application to scalar integrals
- Application to two-loop QCD amplitudes
- Future prospects and possibilities.

Review of the origin of singularities



- Loop amplitudes contain the probability amplitude for propagation of particles in between vertices of Feynman graphs.

$$\text{Ampl}(A \to B) = \frac{\cdots}{E^2 - \omega^2}$$

- These are singular when particles are on-shell.

$$\omega \equiv \sqrt{m^2 + \vec{p}^{\,2}}$$

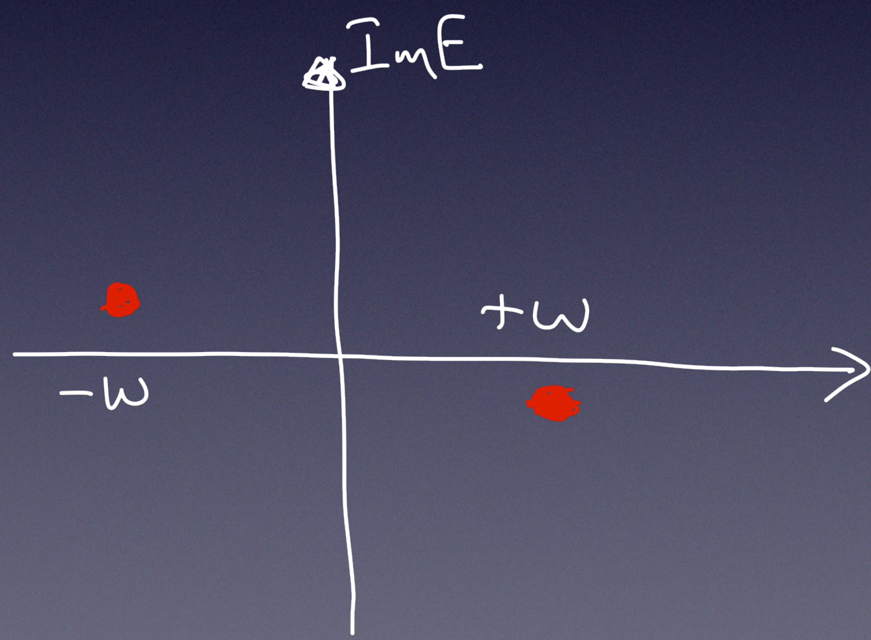
- Do these singularities lead to divergent integrals?

$$\left.\frac{\cdots}{E^2 - \omega^2}\right|_{E=\pm\omega} = \infty$$

"Infinities" from classical behaviour

$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{E^2 - \omega + i\delta} = \int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{\omega} \left(\frac{1}{E - \omega + i\delta} - \frac{1}{E + \omega - i\delta} \right)$$

- The poles lie inside the domain of integration for the energy of virtual particles.
- If we can deform the path of integration away from the poles, then they lead to no singularities
- but the integral acquires both a real and imaginary part.



$\omega \to \omega - i\delta$ with $\delta \to 0$

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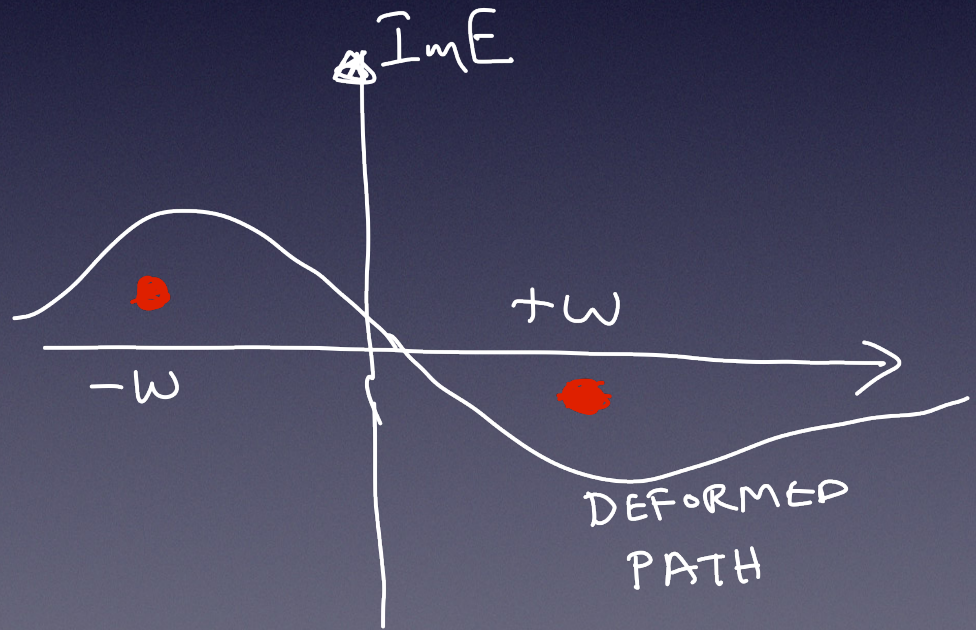


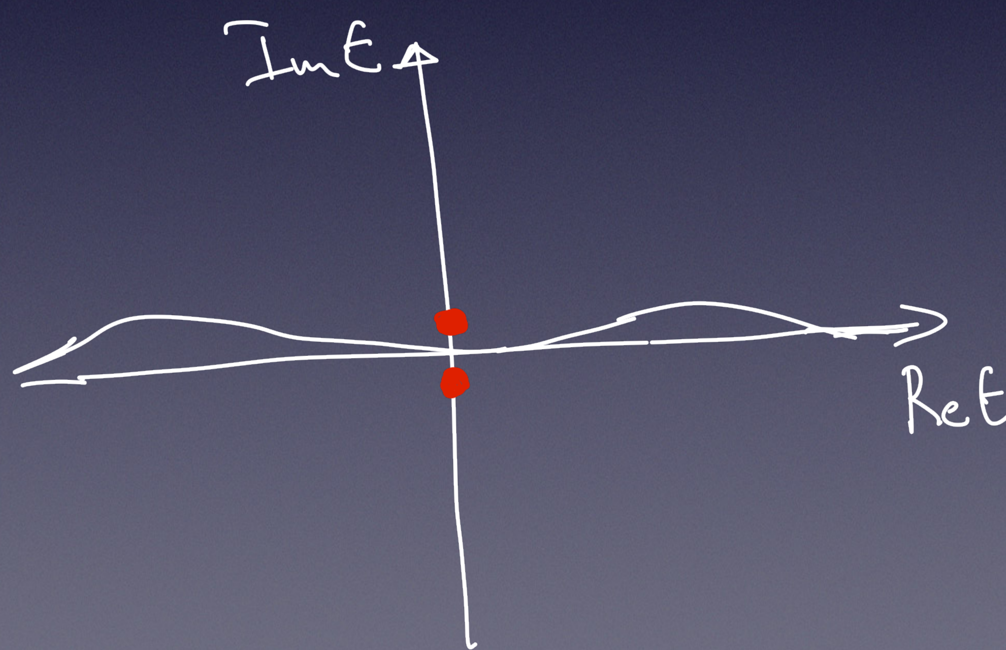
$\omega \to \omega - i\delta$ with $\delta \to 0$

Soft massless particles

- Poles due to soft massless particles.

$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{(E + i\delta)\,(E - i\delta)}$$

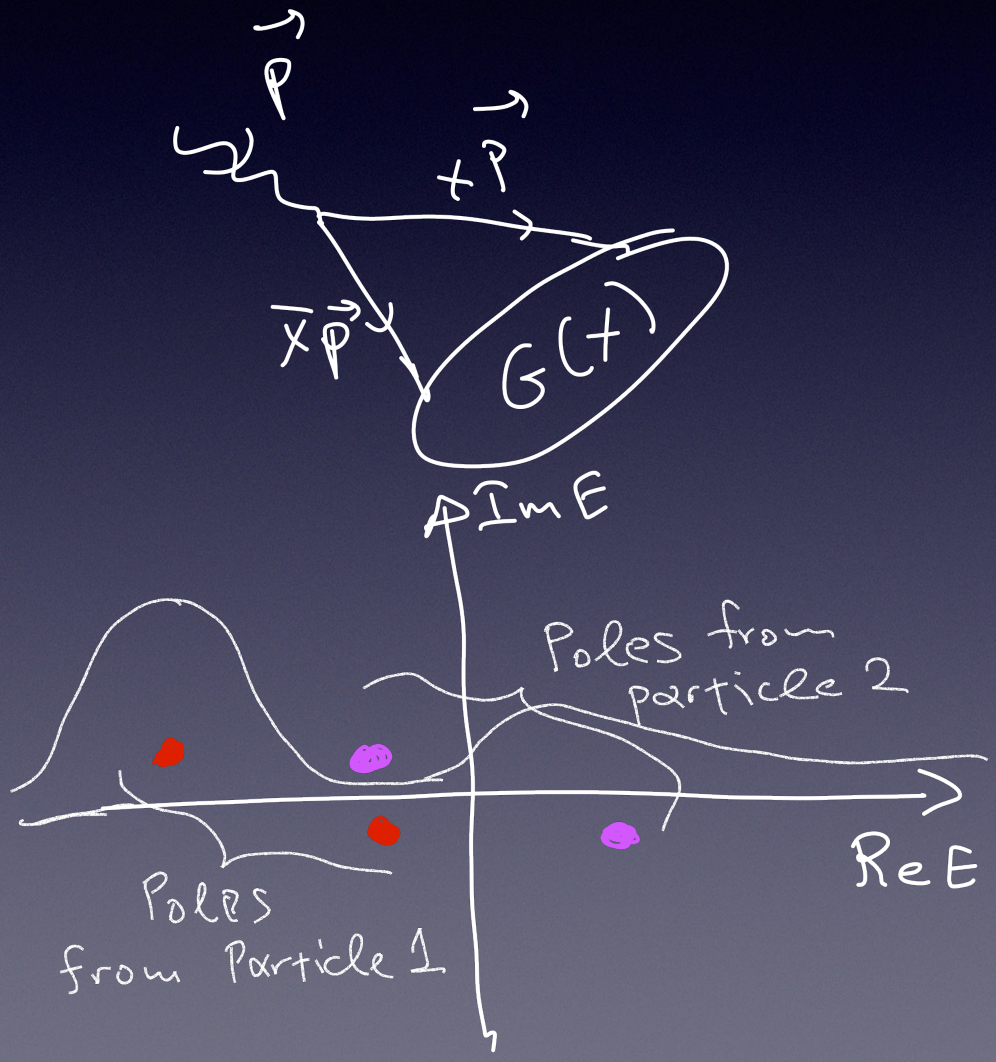
- These singularities pinch the integration path from both sides.



- Condition for a TRUE INFINITY

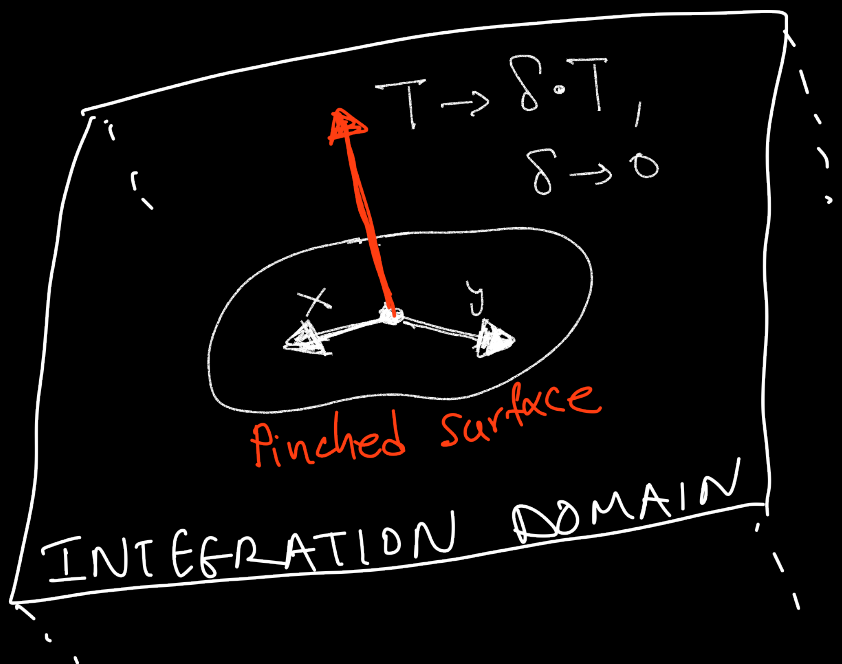
Collinear singularities

- A second source of infinities due to massless collinear particles.
- A singularity of one particle in the lower half-plane lines up with the singularity of a collinear particle in the higher half-pane.
- The singularities pinch the integration path from both sides.
- We cannot deform the path, a condition for a TRUE INFINITY!



Pinch singularities

- To know if a singularity develops, we need to study the behaviour of the integral in the vicinity of the pinch surface.
- We can calculate a degree of divergence.
- Scale variables which are perpendicular to the pinched surface with a small parameter and calculate the scaling of the integrand as the parameter is driven to zero.



Soft $k^\mu \sim \delta Q, \quad d^4k \sim \delta^4$

Collinear $k = xp + \alpha\eta + \beta p_\perp, \quad x \sim \delta^0, \alpha \sim \delta, \beta \sim \delta^{\frac{1}{2}} \quad d^4k \sim \delta^2$

Integrand: $d^4k\mathcal{I}(k) \sim \delta^n$

Divergent: $n \leq 0$

Convergent: $n > 0$

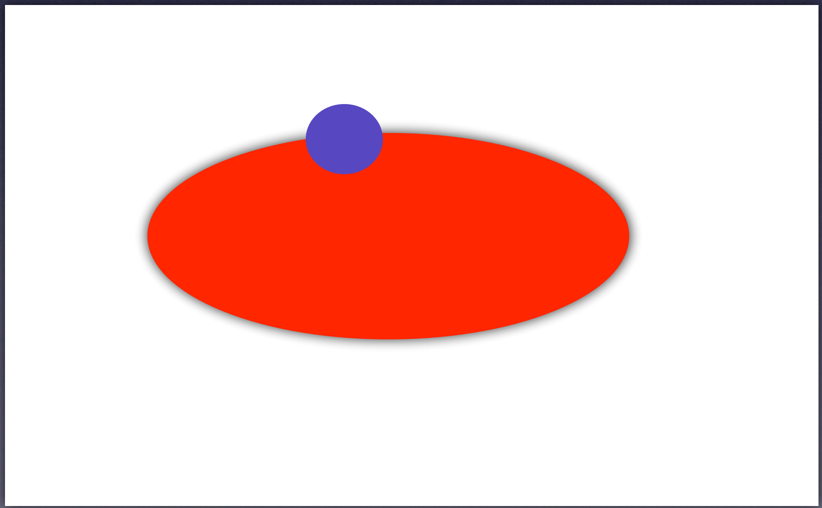
Nested subtractions for an arbitrary number of loops in physical space

Ozan Erdogan, George Sterman

- Singular regions are interconnected. How can we create systematically an approximation of the loop integrals in all singular regions?

$$R^{(n)}\,\gamma^{(n)} \;=\; \gamma^{(n)} + \sum_{N\in\mathcal{N}[\gamma^{(n)}]}\;\prod_{\rho\in N}\left(-t_\rho\right)\gamma^{(n)}\,,$$

- Order the singular regions by their "volume"
- Subtract an approximation of the integrand in the smallest volume
- Then, proceed to the next volume and repeat until there are no more singularities to remove.



Nested subtractions

Ozan Erdogan, George Sterman

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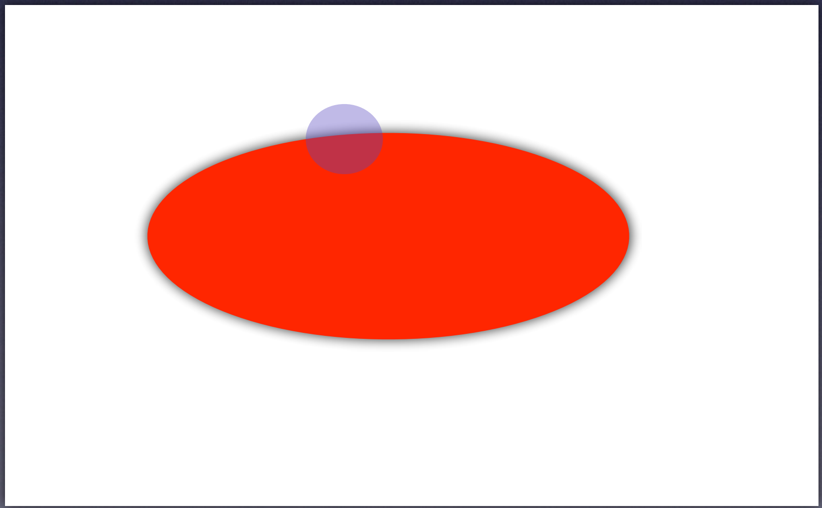
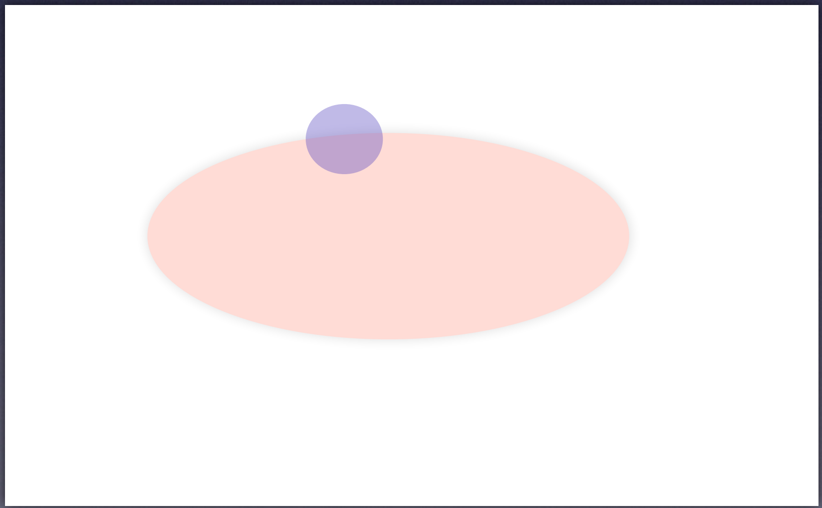


Nested subtractions

Ozan Erdogan, George Sterman

- Order the singular regions by their "volume"
- Subtract an approximation of the integrand in the smallest volume
- Then, proceed to the next volume and repeat until there are no more singularities to remove.
- Method should work at all orders in perturbation theory.
- This structure gives rise to factorisation into Jet, Soft and Hard functions for scattering amplitudes.

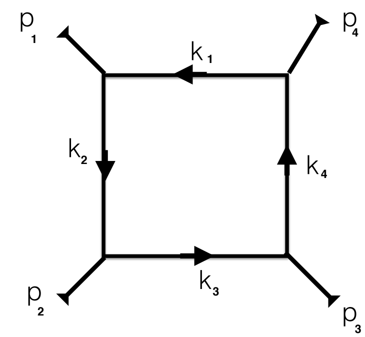
$$R^{(n)}\,\gamma^{(n)} \;=\; \gamma^{(n)} + \sum_{N \in \mathcal{N}[\gamma^{(n)}]} \prod_{\rho \in N} (-t_\rho)\,\gamma^{(n)}\,,$$



An one-loop example

- One-loop massless box has both soft and collinear singularities
- A soft singularity occurs in a single point in momentum space (smallest volume). Needs to be subtracted first.
- A collinear singularity occurs in an one-dimensional space (larger volume). Needs to be subtracted after the soft.

$$\mathrm{Box} \equiv \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{1}{A_1 A_2 A_3 A_4},$$

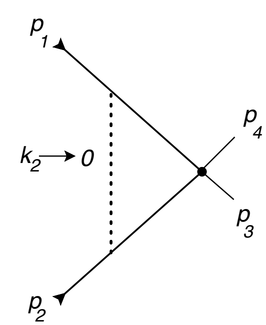


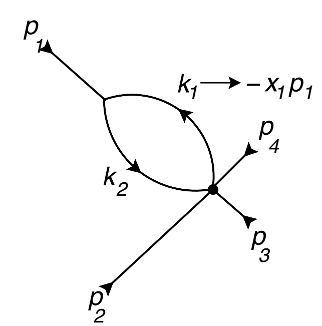
soft singularities

$$\frac{d^d k_2}{A_1 A_2 A_3 A_4} \to \frac{d^d k_2}{(-2k_2 \cdot p_1) k_2^2 (2k_2 \cdot p_2) t} \sim \mathcal{O}(\delta^{d-4})$$

collinear singularities

$$\frac{d^d k_2}{A_1 A_2 A_3 A_4} \to \frac{d^d k_2}{A_1 A_2 s t x_1 (1 - x_1)} \sim \mathcal{O}(\delta^{\frac{d}{2}-2}).$$

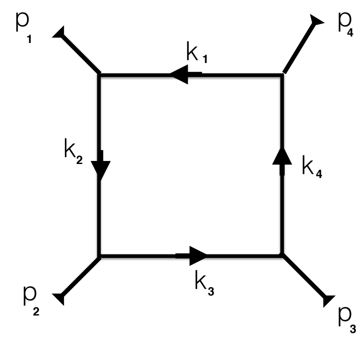




An one-loop example

- Let's focus on the soft-subtractions which come first.
- Need to construct an approximation of the integrand in the soft limits.
- Options are not unique. Can have significant differences in their UV behaviour.

$$\text{Box} \equiv \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{1}{A_1 A_2 A_3 A_4},$$



$$\text{Box}\,(s,t,\epsilon) = \frac{1}{st}\left\{\frac{2c_\Gamma}{\epsilon^2}\left[(-s)^{-\epsilon} + (-t)^{-\epsilon}\right] - \pi^2 - \ln^2\left(\frac{t}{s}\right)\right\} + \mathcal{O}(\epsilon)$$

$$t_{S_2} : A_1 \to -2p_1 \cdot k_2$$
$$t_{S_2} : A_2 \to A_2 \,,$$
$$t_{S_2} : A_3 \to 2p_2 \cdot k_2 \,,$$
$$t_{S_2} : A_4 \to t \,.$$

OR

$$t_{S_2} : A_i \to A_i \,, \quad i = 1, 2, 3 \,,$$
$$t_{S_2} : A_4 \to t \,.$$ *(Nagy Soper)*

$$\text{Box}_R \equiv \left(1 - \sum_{i=1}^{4} t_{S_i}\right) \text{Box} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{N_{\text{Box}}}{A_1 A_2 A_3 A_4},$$

$$N_{\text{Box}} = 1 - \frac{A_{24}}{t} - \frac{A_{13}}{s} \,.$$

An one-loop example

- The subtracted integral is now finite in all soft limits.
- Observation: The “soft” counterterms are easier to compute than the original integral (triangle integrals)
- The subtracted integral does not have quadratic poles in epsilon.
- In fact, it does not have single poles in epsilon either….

$$
\begin{aligned}
t_{S_2}\,\mathrm{Box}(s,t,\epsilon) &= t_{S_4}\,\mathrm{Box}(s,t,\epsilon) &= \frac{c_\Gamma}{st\epsilon^2}(-s)^{-\epsilon} \\
t_{S_1}\,\mathrm{Box}(s,t,\epsilon) &= t_{S_3}\,\mathrm{Box}(s,t,\epsilon) &= \frac{c_\Gamma}{st\epsilon^2}(-t)^{-\epsilon}\,.
\end{aligned}
$$

$$
\mathrm{Box}_R = -\frac{1}{st}\left[\pi^2 + \ln^2\left(\frac{t}{s}\right)\right]
$$

An one-loop example

- Let's consider a collinear limit
- Observation: The "soft" counterterms are easier to compute than the original integral (triangle integrals)
- The collinear limit approximation is potentially UV divergent.
- We introduce a UV counterterm to the Collinear counterterm as well *(Nagy, Soper)*.
- In this example, the numerator of the collinear counterterm vanishes.
- ..which explains why our soft-subtractions sufficed to yield a finite result.

$$
\begin{aligned}
t_{C_1} A_1 &= A_1 \,, \\
t_{C_1} A_2 &= A_2 \,, \\
t_{C_1} A_3 &= (1-x)s \,, \\
t_{C_1} A_4 &= xt \,.
\end{aligned}
$$

$$
t_{C_1}\,\mathrm{Box} \equiv \int \frac{d^dk_1}{i\pi^{\frac{d}{2}}} \left(\frac{1}{A_1} - \frac{1}{A_1-\mu^2}\right) \frac{1}{A_2} \left[\frac{1}{stx_1(1-x_1)}\right]
$$

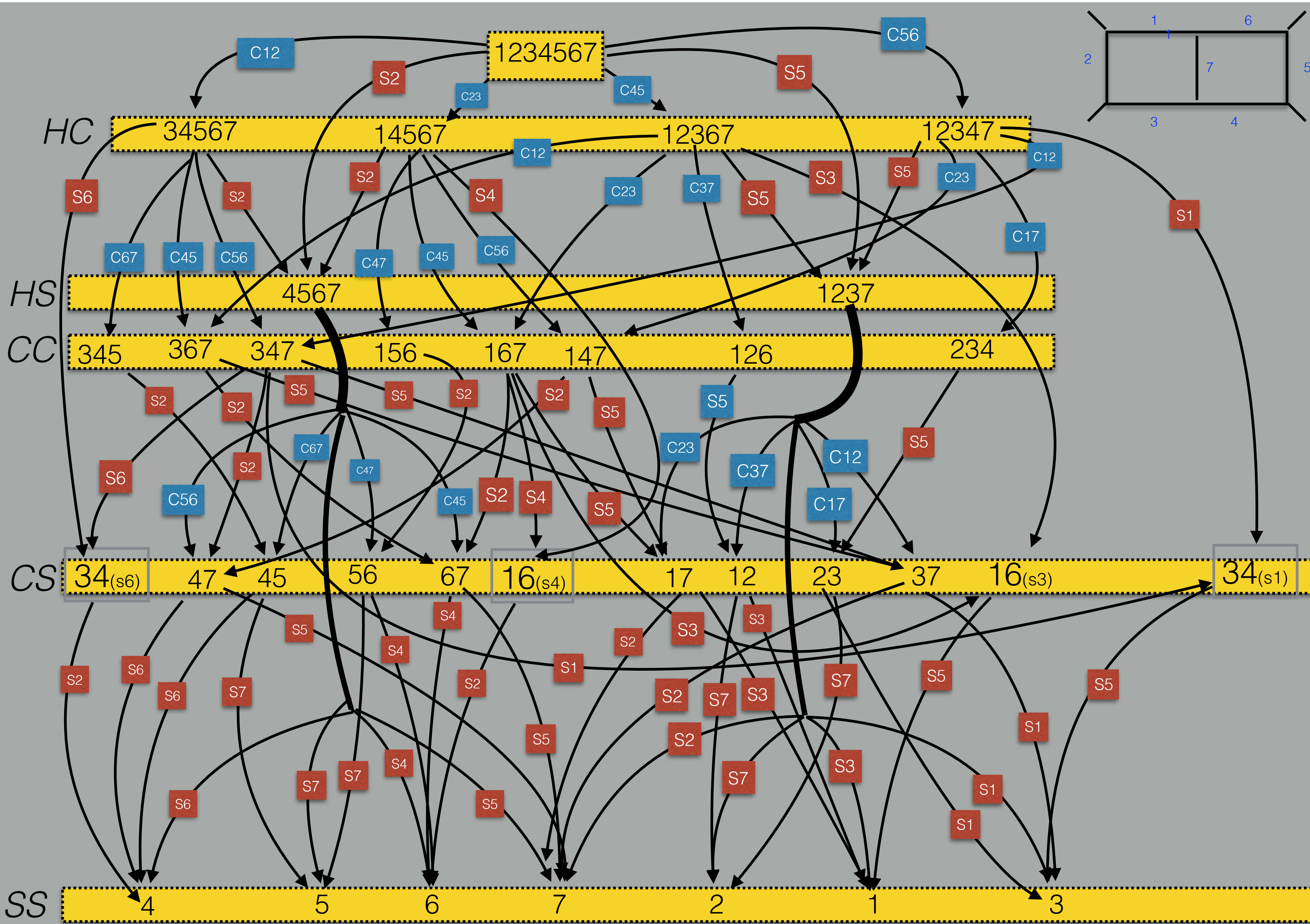
$$
= \int \frac{d^dk_1}{i\pi^{\frac{d}{2}}} \left[\frac{\frac{\mu^2}{\mu^2-A_1}}{A_1A_2stx_1(1-x_1)}\right] .
$$

$$
\int \frac{d^dk_1}{i\pi^{\frac{d}{2}}} \left[\frac{N_{\mathrm{Box}}}{A_1A_2A_3A_4} - \frac{\frac{\mu^2}{\mu^2-A_1} N_{\mathrm{Box}}\big|_{k_1=-x_1p_1}}{A_1A_2stx_1(1-x_1)}\right] .
$$

$$
\begin{aligned}
N_{\mathrm{Box}}\big|_{k_1=-x_1p_1} &= \left[1 - \frac{A_{13}}{s} - \frac{A_{24}}{t}\right]\Big|_{k_1=-x_1p_1} \\
&= 1-(1-x_1)-x_1 \\
&= 0 \,.
\end{aligned}
$$

Does the method work at two-loops?

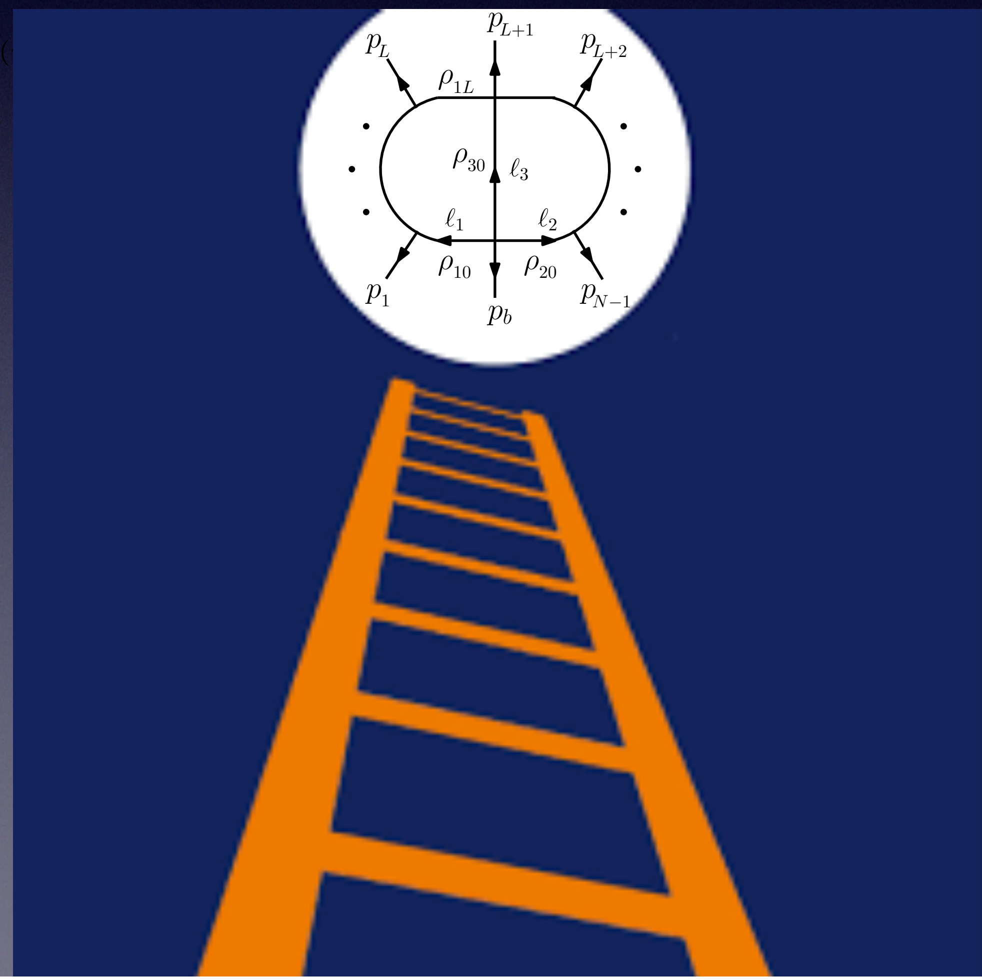
A complicated web of interconnected divergences….



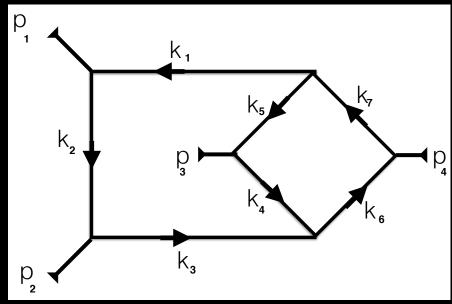
HC
HS
CC
CS
SS
1234567
34567
14567
12367
12347
4567
1237
345
367
347
156
167
147
126
234
34(s6)
47
45
56
67
16(s4)
17
12
23
37
16(s3)
34(s1)
4
5
6
7
2
1
3
C12
C23
C45
C56
C67
C47
C37
C17
S1
S2
S3
S4
S5
S6
S7

Nested subtractions at 2-loops

- Order of subtractions:
  - double-soft
  - soft-collinear
  - double-collinear
  - single-soft
  - single-collinear
- Approximations in singular regions do not need to be strict limits!
- Good approximations should not introduce ultraviolet divergences
- Good approximations should be easy to integrate exactly.



Example: two-loop cross-box



two-loop limits — **single soft** — **single collinear**

$$F_{Xbox} = F^{(2)}_{Xbox} + F^{(1s)}_{Xbox} + F^{(1c)}_{Xbox},$$

double -soft — *double -collinear*

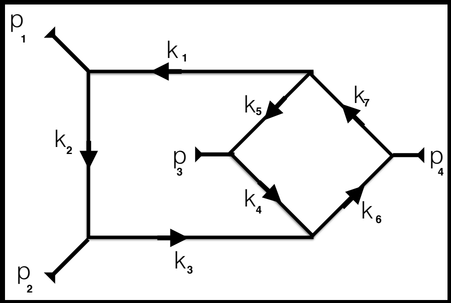
$$F^{(2)}_{Xbox} = \frac{N_5}{A_1A_2A_3A_4A_5A_6A_7},$$

$$N_5 = \left(1-\frac{A_{13}}{s}\right)^2 + \frac{A_2}{tu}\left(A_2+s-A_{13}\right) - \left(1-\frac{A_1}{s}\right)\left(\frac{A_5}{t}+\frac{A_7}{u}\right) - \left(1-\frac{A_3}{s}\right)\left(\frac{A_4}{u}+\frac{A_6}{t}\right) + \frac{A_2A_{4567}}{tu} - \frac{A_3}{s}\left(\frac{A_7}{t}+\frac{A_5}{u}\right) - \frac{A_1}{s}\left(\frac{A_6}{u}+\frac{A_4}{t}\right) + \frac{(t-u)^2}{s^2}\frac{A_1A_3}{tu}$$

$$F^{(1c)}_{Xbox} = -\left[\frac{1}{A_1A_2} - \frac{1}{B_1B_2}\right]\frac{1}{s(1-x_1)}\left\{\left[\frac{N_5}{A_4A_5A_6A_7}\right]_{k_1=-x_1p_1} - \left[\frac{N_5}{A_4A_5A_6A_7}\right]_{k_2=0}\right\}$$
$$-\left[\frac{1}{A_2A_3} - \frac{1}{B_2B_3}\right]\frac{1}{s(1-x_3)}\left\{\left[\frac{N_5}{A_4A_5A_6A_7}\right]_{k_3=-x_2p_2} - \left[\frac{N_5}{A_4A_5A_6A_7}\right]_{k_2=0}\right\}$$
$$-\left[\frac{1}{A_4A_5} - \frac{1}{B_4B_5}\right]\left[\frac{N_5}{A_1A_2A_3A_6A_7}\right]_{k_5=-x_3p_3}$$
$$-\left[\frac{1}{A_6A_7} - \frac{1}{B_6B_7}\right]\left[\frac{N_5}{A_1A_2A_3A_4A_5}\right]_{k_5=-x_4p_4}.$$

$$F^{(1s)}_{Xbox} = -\frac{1}{A_1A_2A_3}\left[\frac{N_5}{A_4A_5A_6A_7}\right]_{k_2=0}$$

Example: two-loop cross-box



$$\mathrm{X_{box}}^{\mathrm{fin}} \equiv \int \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{d^d k_5}{i\pi^{\frac{d}{2}}} F_{Xbox} = \mathcal{O}(\epsilon^0).$$

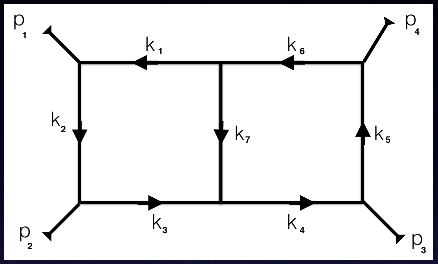
$$s^3 \mathrm{X_{box}}^{\mathrm{fin}} = \frac{f_{\mathrm{X_{box}}}(y)}{y} + \frac{f_{\mathrm{X_{box}}}(1-y)}{1-y},$$

$$f_{\mathrm{X_{box}}}(y) = [G_R(y) + i\pi G_I(y)] \log\left(\frac{\mu^2}{s}\right) + E_R(y) + i\pi E_I(y)$$

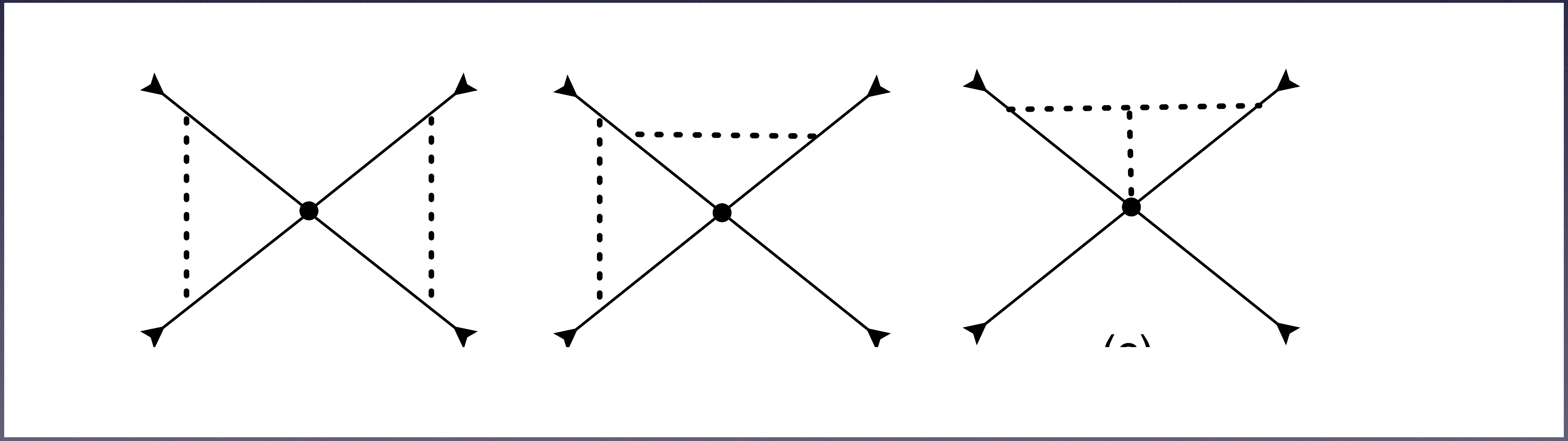
$$\begin{aligned} E_R(y) = & -8\,\pi^2\,\mathrm{Li}_2(y) + 8\,\mathrm{Li}_2(y)\,\log(1-y)^2 - 28\,\log(y)\,\mathrm{Li}_2(y)\,\log(1-y) - 18\,\mathrm{Li}_2(y)\,\log(y)^2 \\ & +44\,\mathrm{Li}_3(y)\,\log(1-y) + 96\,\mathrm{Li}_3(y)\,\log(y) - 188\,\mathrm{Li}_4(y) + \frac{17}{36}\,\pi^4 + \frac{1}{12}\,\log(1-y)^4 \\ & +7\,\log(y)\,\log(1-y)\,\pi^2 - \frac{25}{6}\,\pi^2\,\log(1-y)^2 - \frac{3}{2}\,\log(y)^2\,\pi^2 + \log(y)\,\log(1-y)^3 \\ & +44\,S_{12}(y)\,\log(1-y) - 52\,S_{12}(y)\,\log(y) + 84\,S_{13}(y) + 88\,S_{22}(y) - 44\,\zeta_3\,\log(1-y) \\ & -4\,\log(y)\,\zeta_3 - \frac{1}{4}\,\log(y)^4 + \log(y)^3\,\log(1-y) - \frac{9}{2}\,\log(y)^2\,\log(1-y)^2, \end{aligned}$$

...

Complexity of counterterms at two-loops



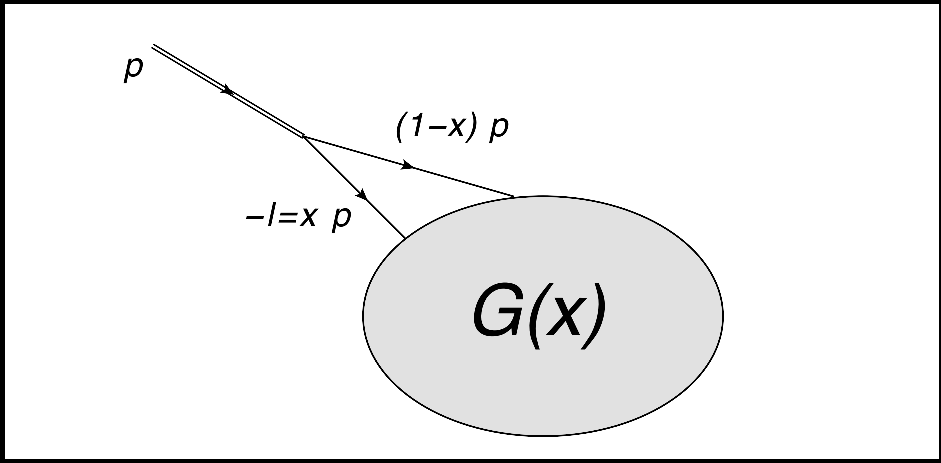
In double-soft approximations:



Double-soft counterterms are integrals with at most six massless propagators (all known).

Complexity of counterterms at two-loops

- Collinear counterterms for a Feynman diagrams or a Feynman integral require the convolution of a subgraph
- At two-loops, we have to integrate over one-loop infrared-subtracted subgraphs
- It can be done analytically, in principle… it requires a good calculator of one-loop integrals and a good dictionary for the integration of polylogarithms
- it can also be done numerically, with little effort
- Collinear counterterms are much simpler (no convolutions) for physical amplitudes (exploiting QCD factorization)



$$I_G = \frac{\Gamma(1+\epsilon)}{\epsilon}\left(\mu^2\right)^{-\epsilon}\int_0^1 dx G(x)$$

$$\begin{aligned} &\int_0^1 \frac{dx}{x}\left[S_{12}\left(\frac{(x-y)(xy-1)}{y(x-1)^2}\right) - 2\mathrm{Li}_2\left(\frac{(x-y)(xy-1)}{y(x-1)^2}\right)\log(1-x) - \zeta_3\right] \\ = & -\frac{1}{24}\log(y)^4 - 2\,\mathrm{Li}_2(y)^2 + \frac{13}{45}\pi^4 - \mathrm{Li}_2(y)\,\log(y)^2 + 4\,\mathrm{Li}_3(y)\,\log(y) \\ & -4\,\zeta_3\,\log(y) - \frac{4}{3}\pi^2\,\mathrm{Li}_2(y) - 8\,\mathrm{Li}_4(y) + 8\,S_{22}(y). \qquad (3.98) \end{aligned}$$

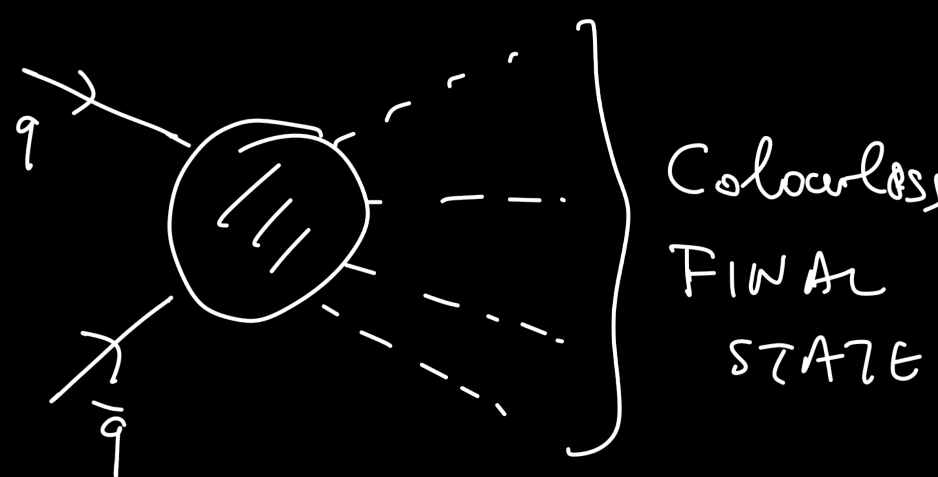
Subtractions for QCD amplitudes

with Rayan Haindl, George Sterman, Zhou Yang, Mao Zeng

- This is work in its infancy…
- From first principles, we expect that nested subtractions can separate the short distance (finite part) of physical amplitudes from the long distance (singularities) part.
- Significant simplifications occur in comps
- Singularities are at most logarithmic
- Factorisation of all singular limits when physical sets of Feynman diagrams are combined together
- Hope Generic subtraction terms for all processes.

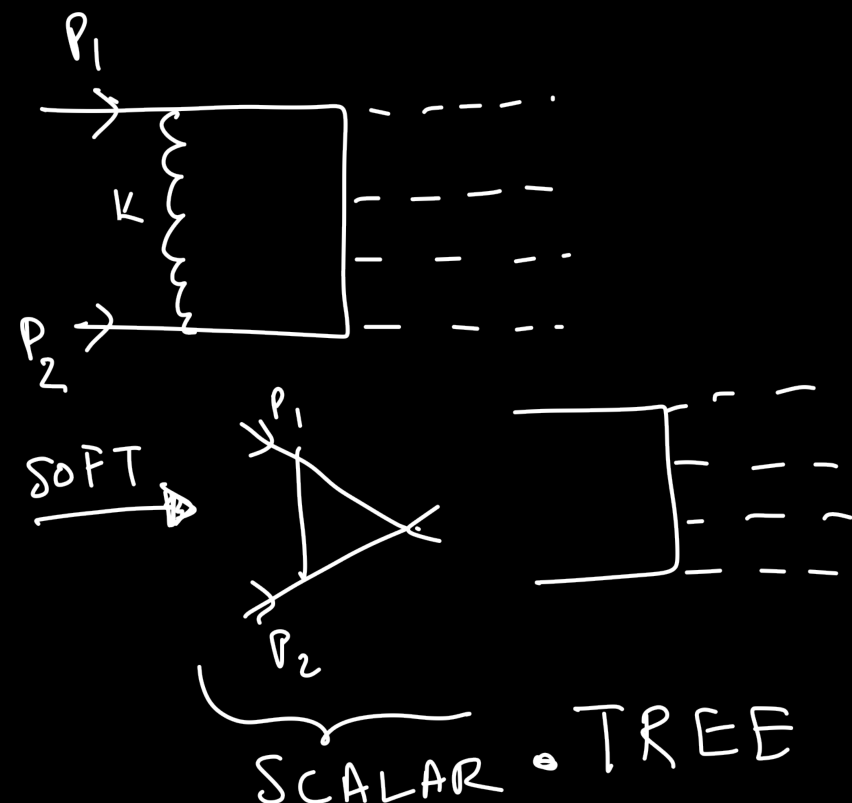
Application to amplitudes

- Consider the process for the production of a heavy colourless final-state from the scattering of a massless quark-antiquark pair.
- This encompasses a large set of processes (multi Z,W, photon production and combinations)
- Easy to verify at one-loop that a simple set of local counterterms exists for all these processes.



Application to amplitudes

- Per tree-diagram, there is one 1-loop diagram with a soft singularity.
- The soft limit is (up to trivial factors), an one-loop scalar integral times a tree-diagram.



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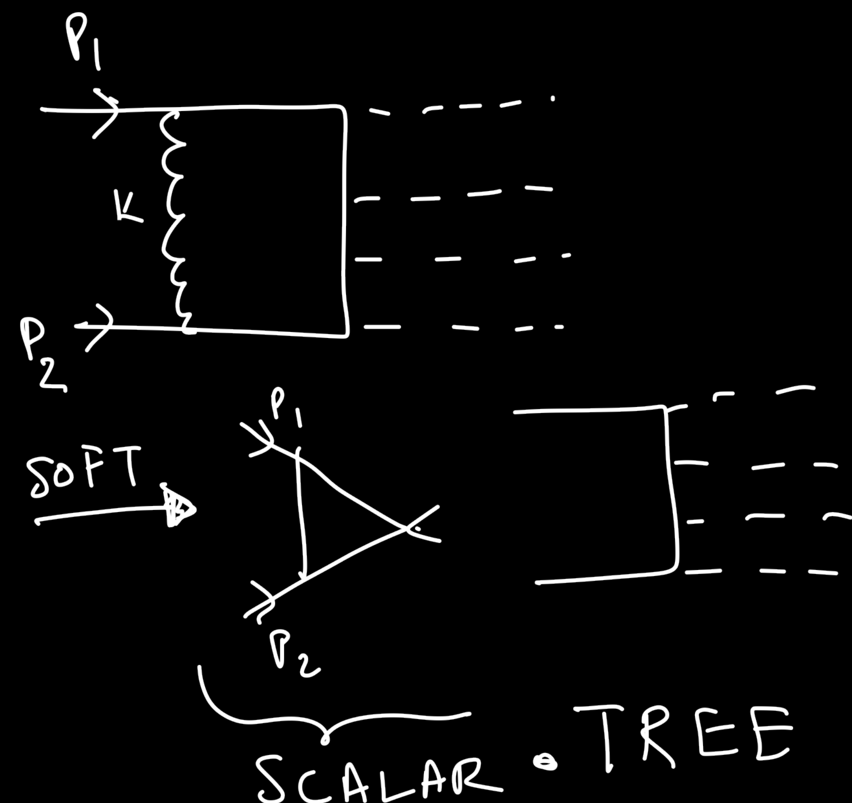
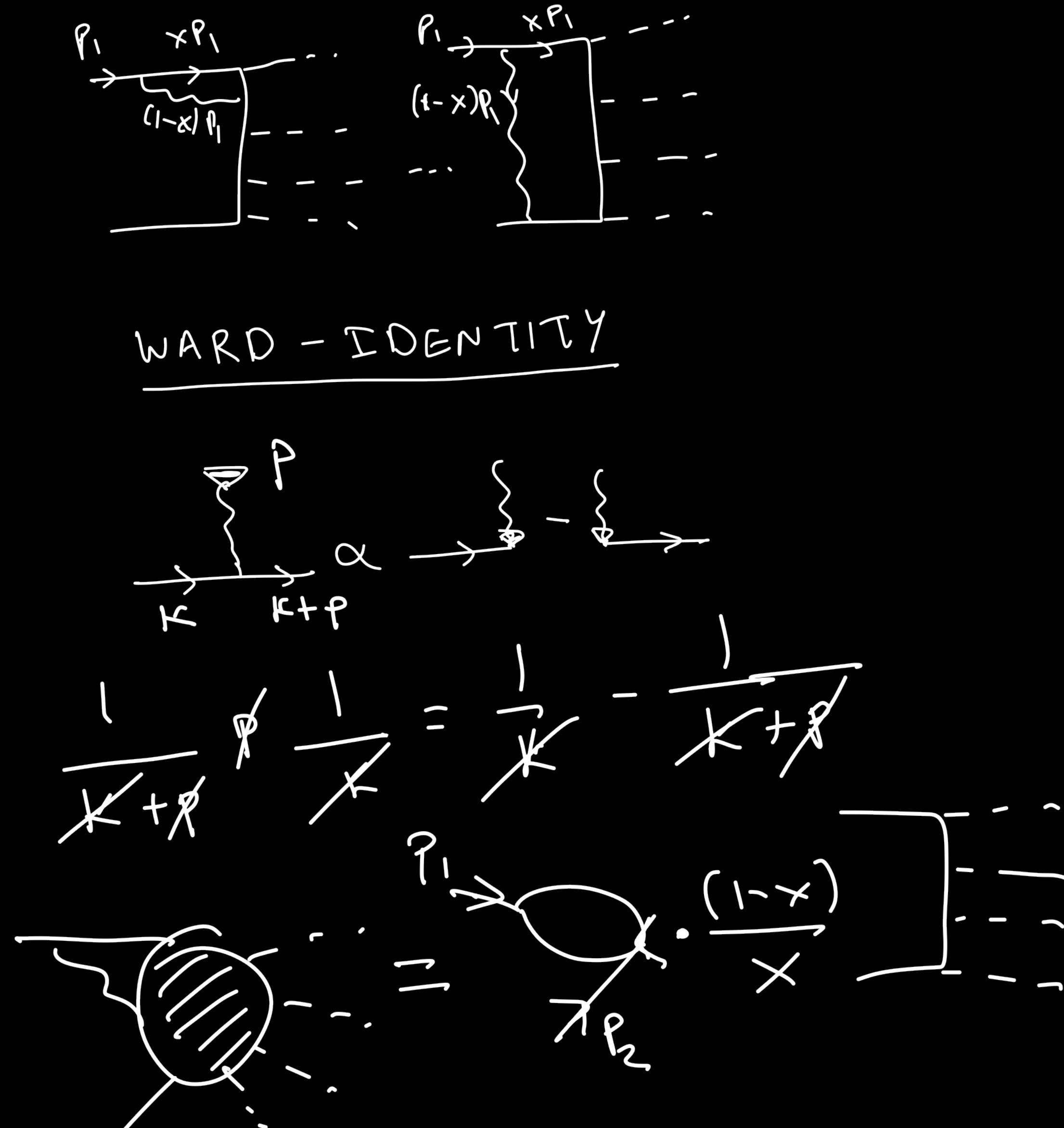


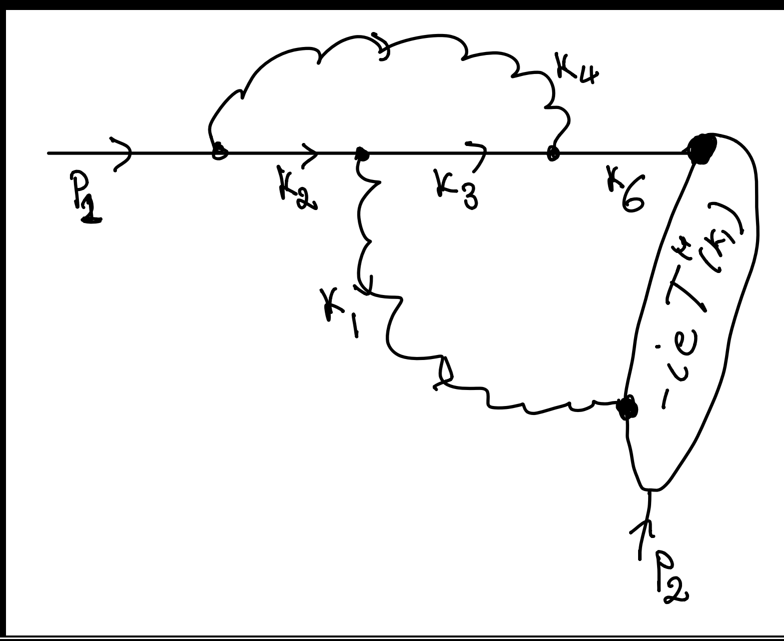
Application to amplitudes

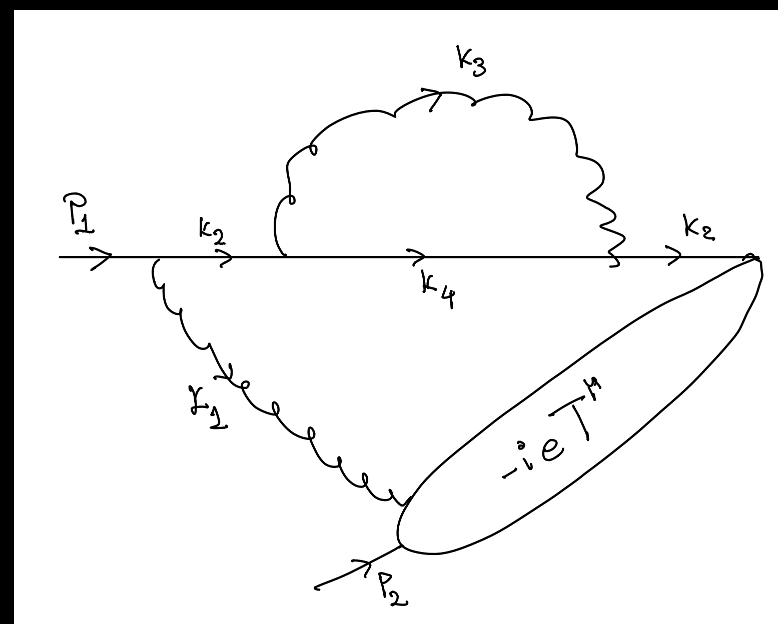
- Many graphs yield collinear divergences.
- Summing over all such graphs, cancellations take place ("Ward"-identity)
- The net-result is factorization of the amplitude in the collinear limit in terms of a splitting-functions and a tree-diagram.



Applications to amplitudes

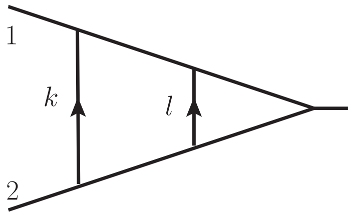
- The same mechanisms factorise the singular limits of two-loop amplitudes as well
- We have derived the factorisation of the singular limits explicitly for the abelian part of two-loop amplitudes of colourless final-states.
- All limits work in a straightforward manner…except the single collinear limit for lines with self-energy or vertex-corrections (collinear emissions from hard loops).

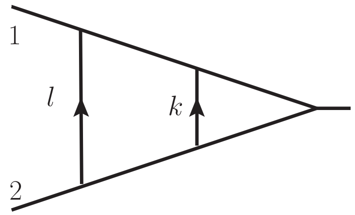




FACTORIZATION OF TWO-LOOP AMPLITUDE IN ITS SINGULAR LIMITS

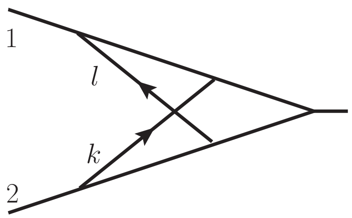
planar double soft

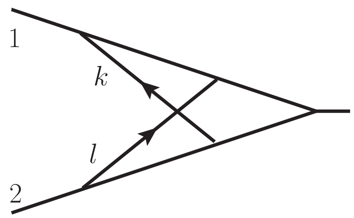
$-2e^4 s^2$ (


$+$


) $\cdot$ Tree (A.3)

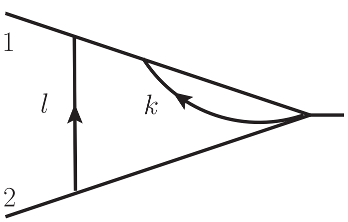
non-planar double soft

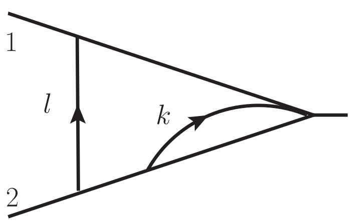
$-2e^4 s^2 \left(1 + \frac{2k \cdot l}{s}\right)$ (


$+$


) $\cdot$ Tree (A.4)

planar soft-collinear

$2e^4 s$ (


$+$


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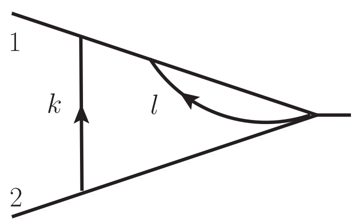
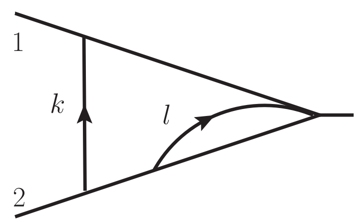


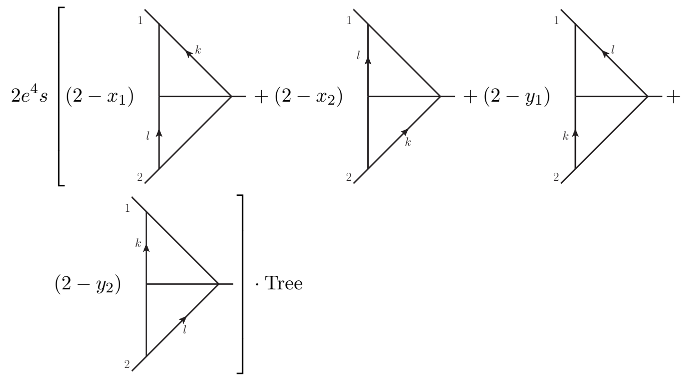
$+$


) $\cdot$Tree

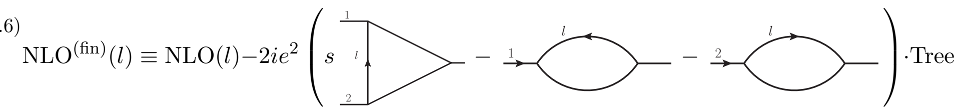
(A.5)

FACTORIZATION OF TWO-LOOP AMPLITUDE IN ITS SINGULAR LIMITS

non-planar soft-collinear

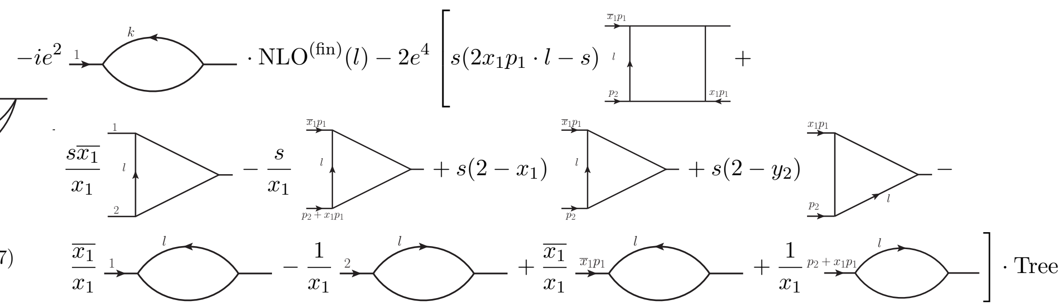


(A.6)



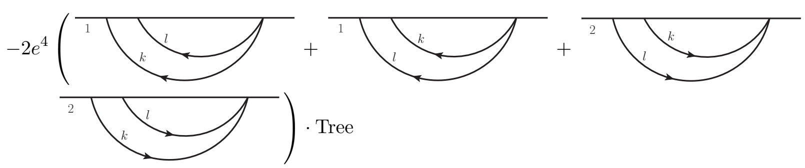
(A.10)

single collinear (at $k \to -x_1 p_1$)



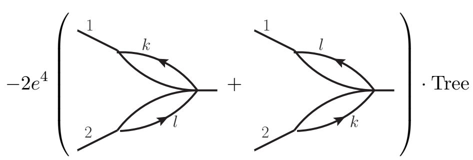
(A.11)

triple collinear



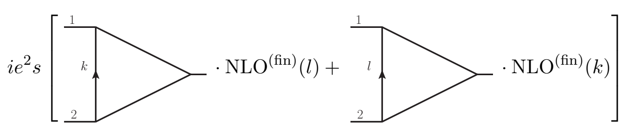
(A.7)

two collinear pairs



(A.8)

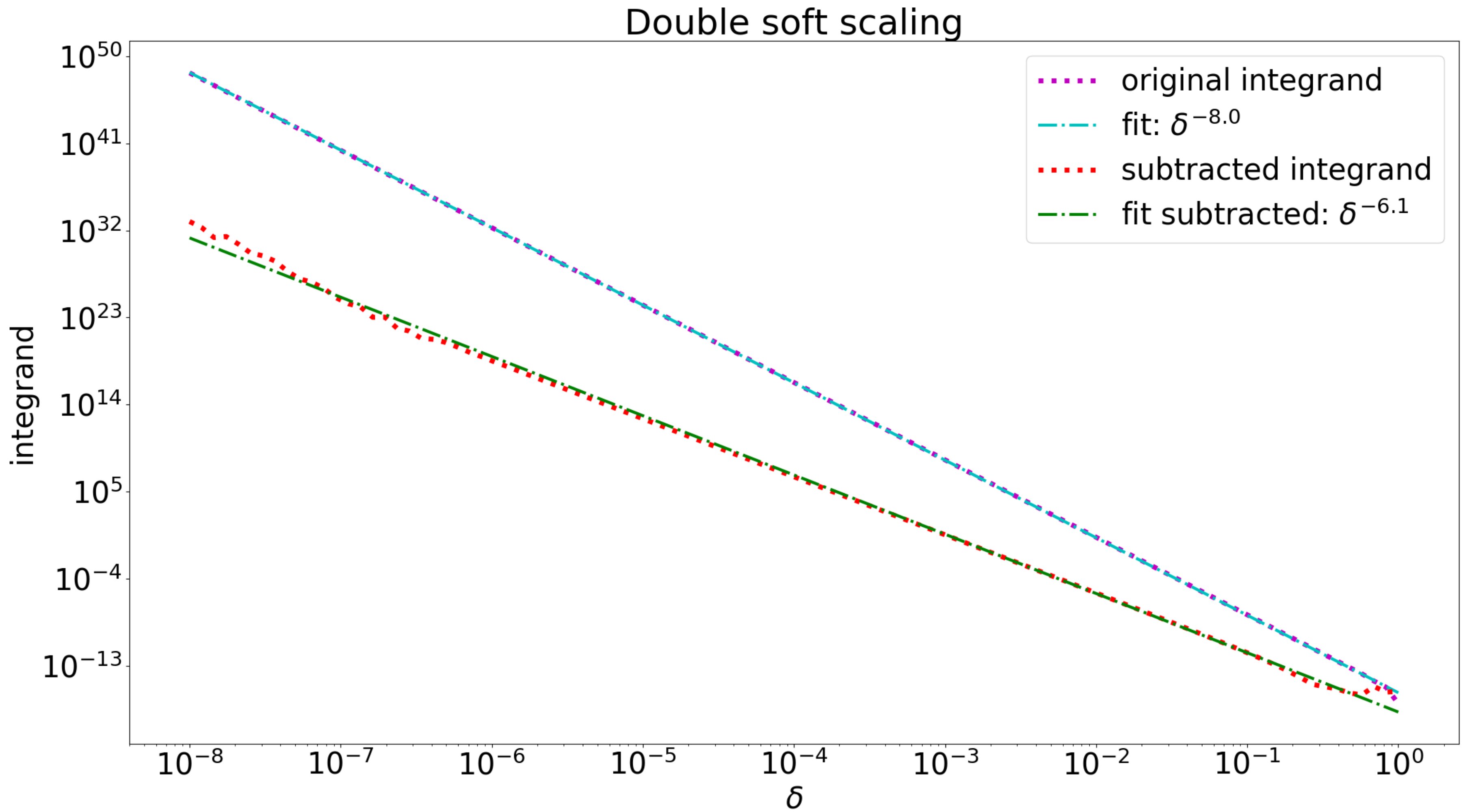
single soft



(A.9)

Numerical validation

$$q + \bar{q} \to \gamma + \gamma$$



Numerical validation

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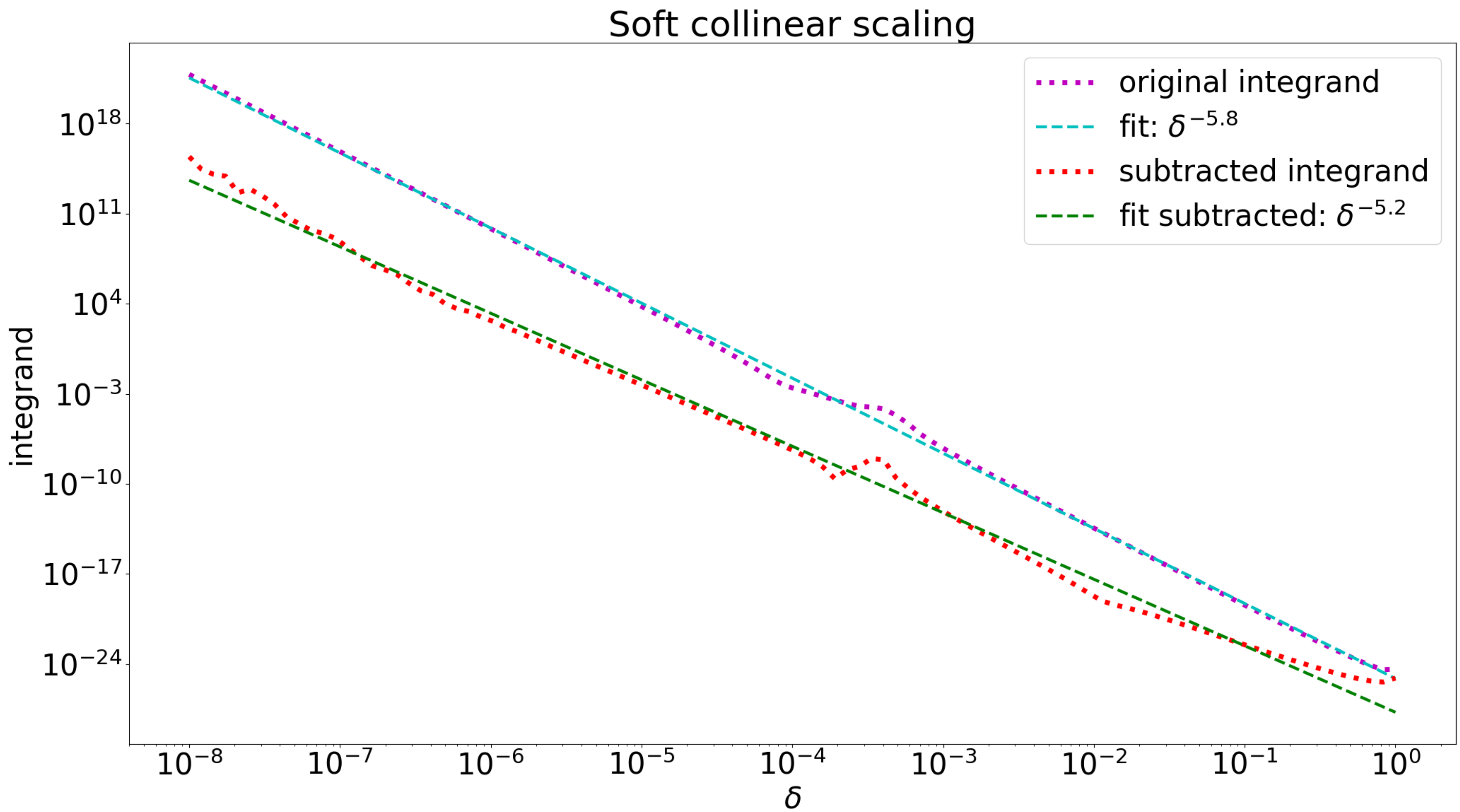

Numerical validation

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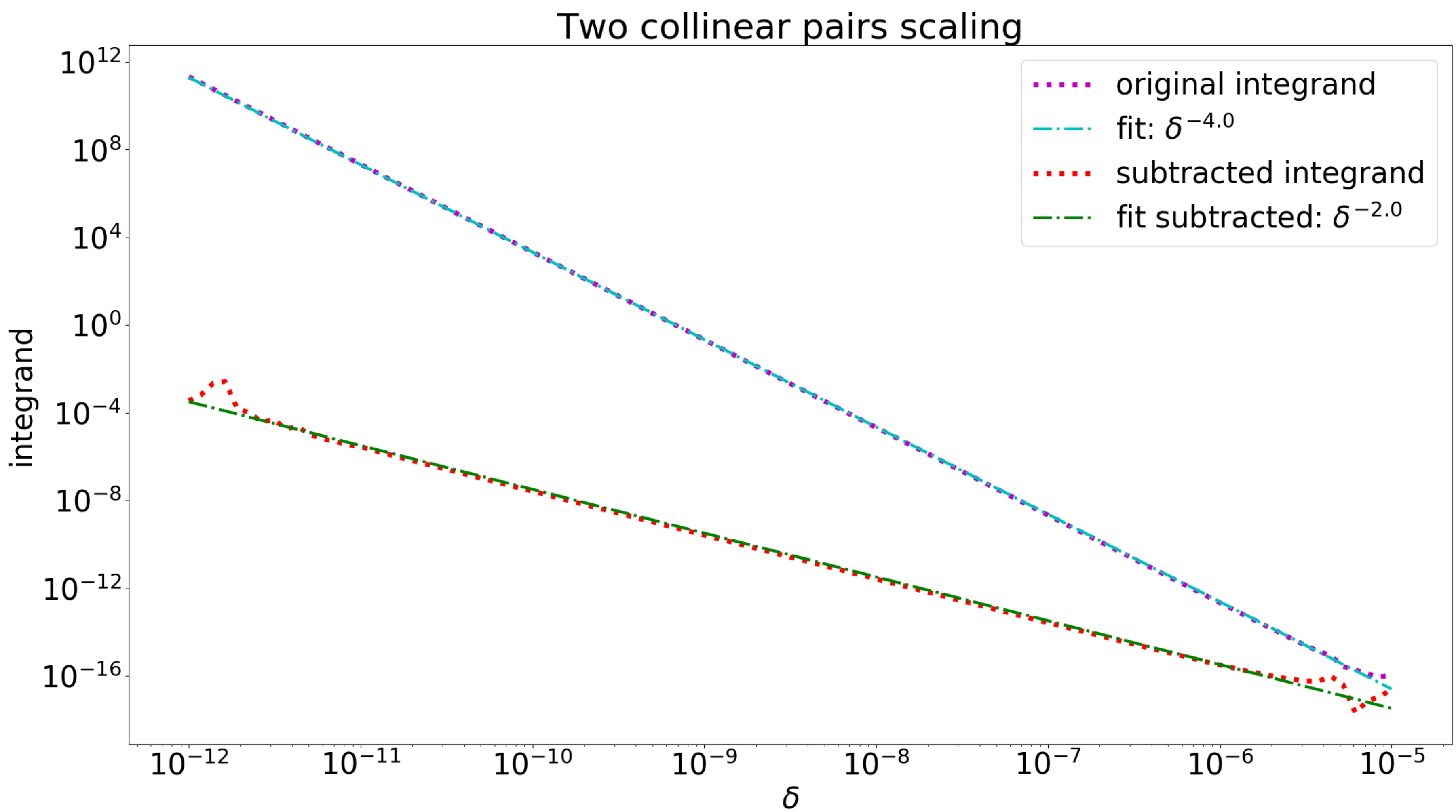
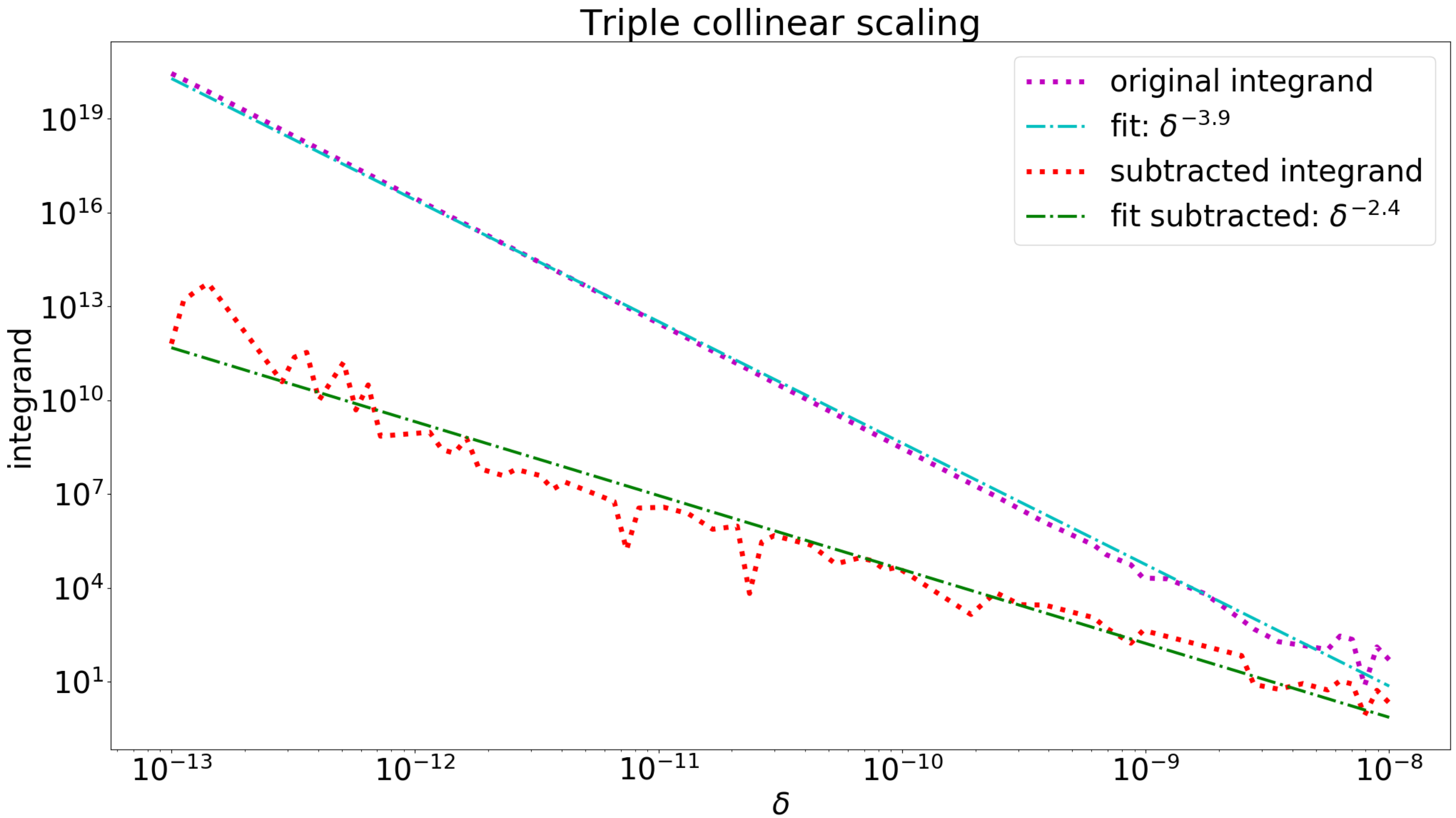
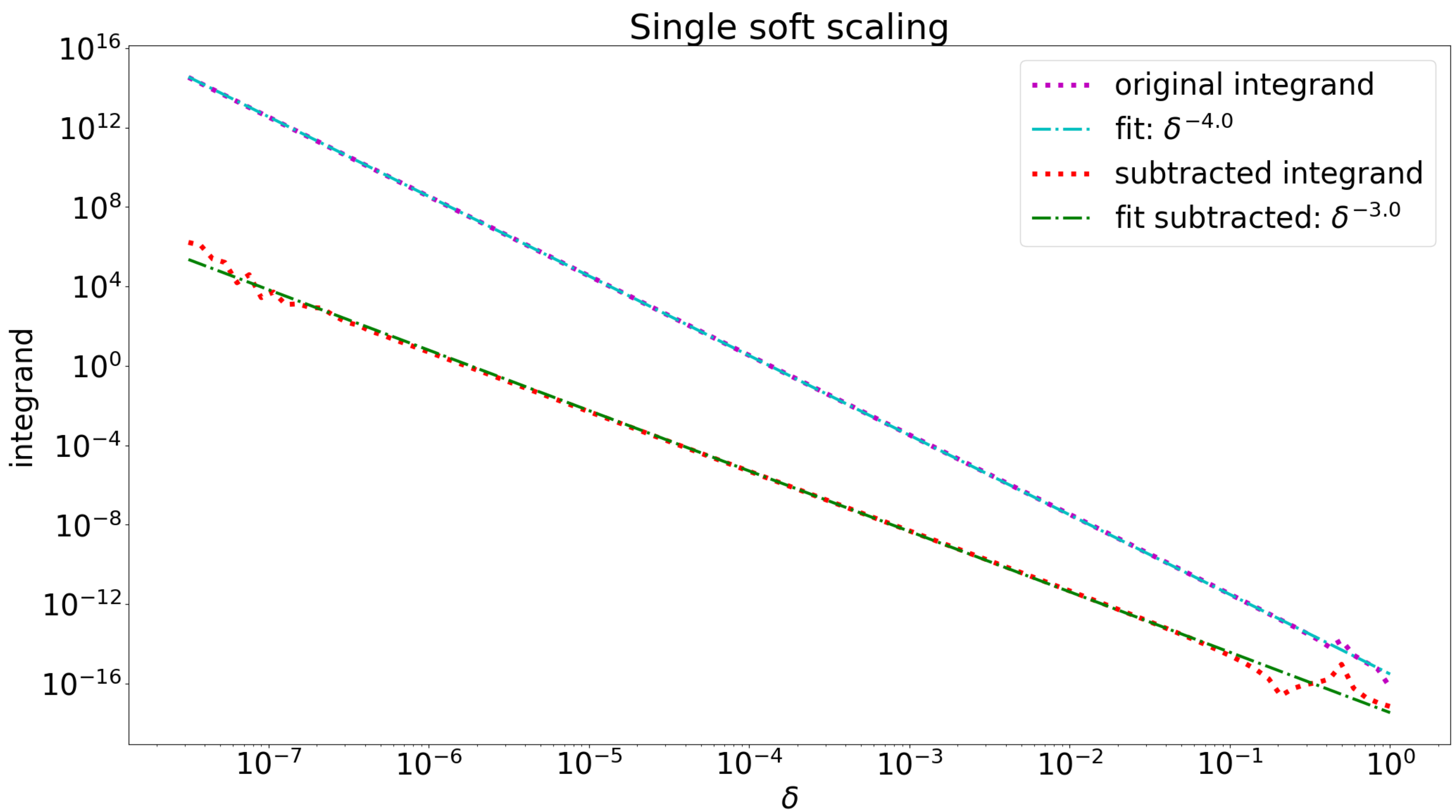


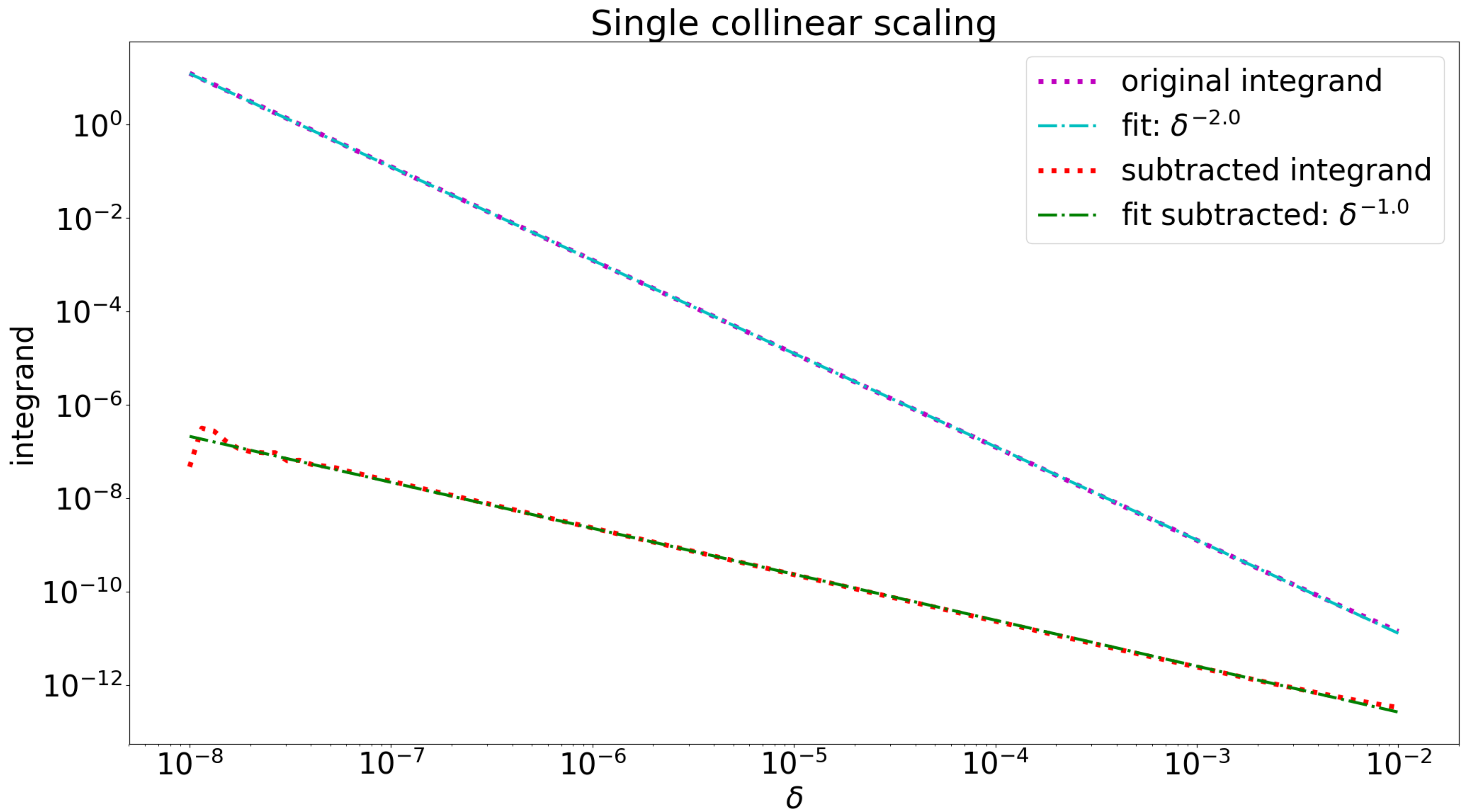
Numerical validation



Numerical validation

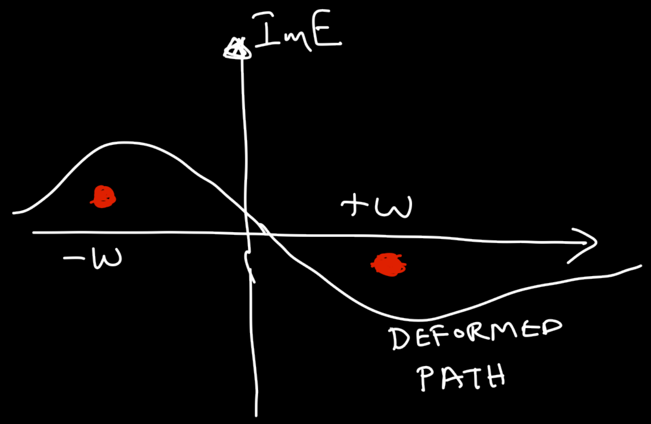


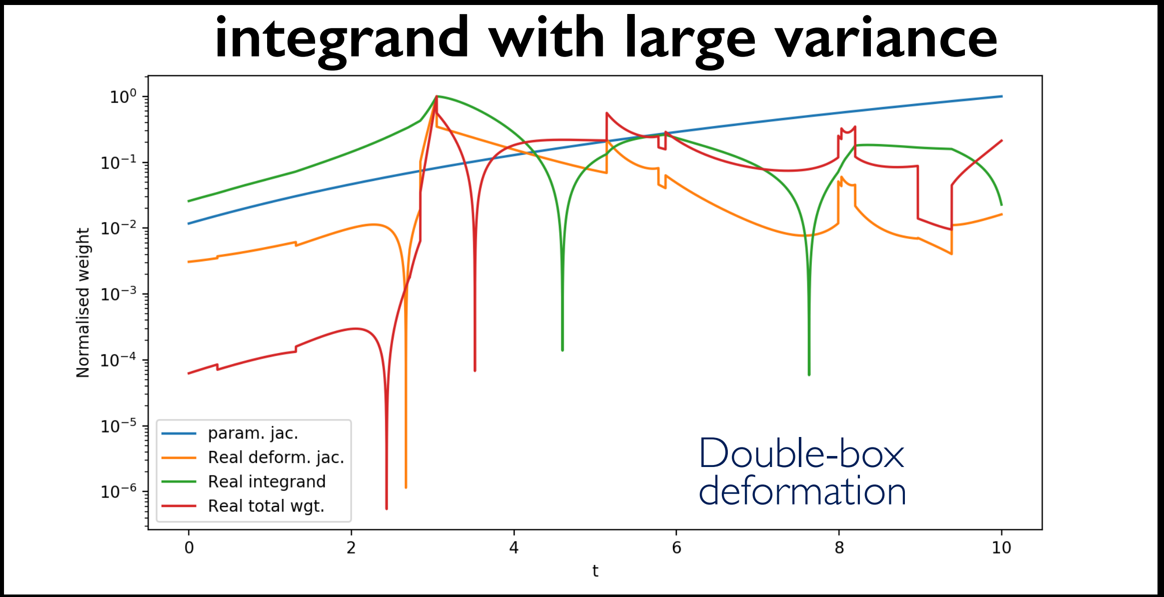
Numerical validation



Numerical integration

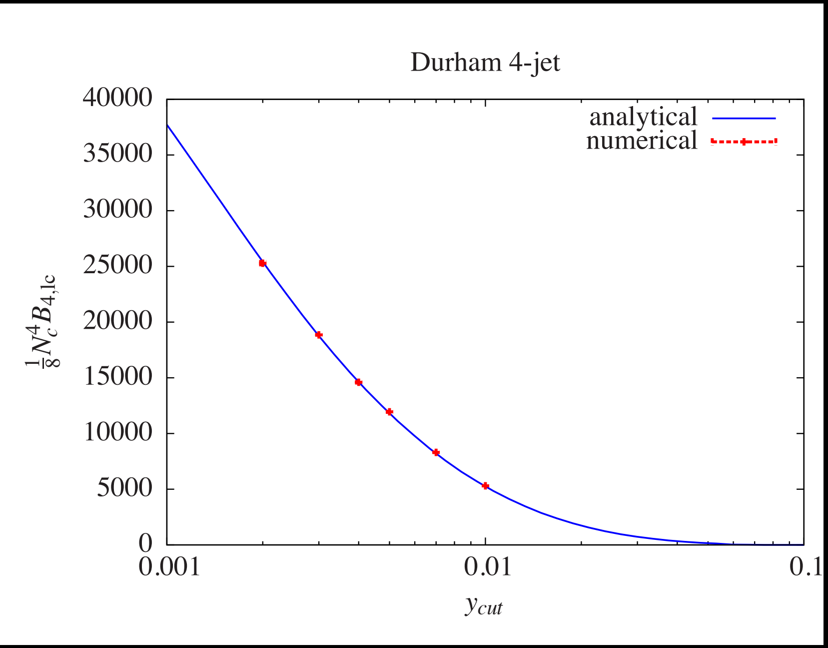
- Can such subtractions be used for evaluating loop amplitudes numerically?
- They are an important ingredient! They remove “pinch” singularities.
- Other singularities which can be avoided with appropriate contour-deformations are equally important.
- A very challenging problem! Very encouraging progress by Z. Capatti, V. Hirschi, D. Kermanschah, A. Pelloni, B. Ruijl at ETH and other groups.





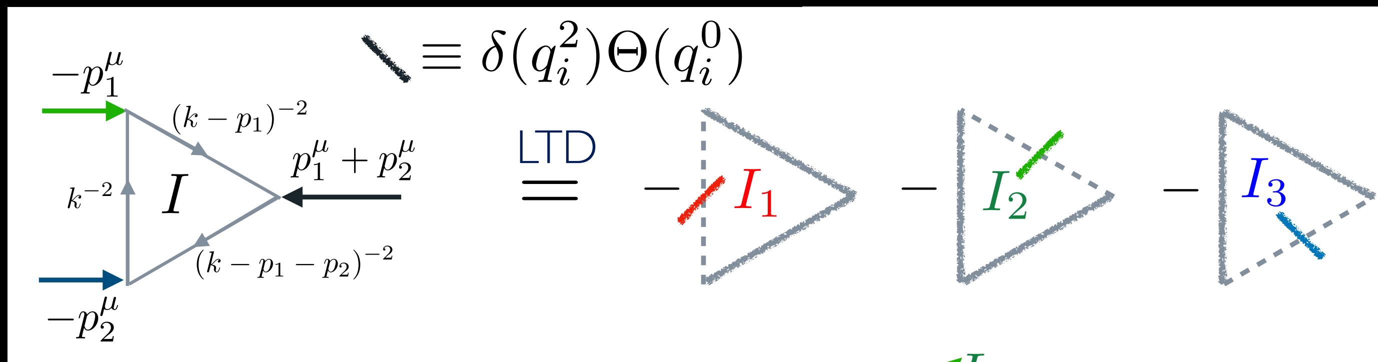
One-loop direct momentum-space integration

- Foundational work by Nagy and Soper
- and by Becker and Weinzierl
- Good results in computing challenging one-loop amplitudes.
- Tough competition at one-loop with OPP/unitarity/semi-analytic methods.



4-jet production at NLO (Becker, Goetz, Reuschle, Schwan, Weinzierl)

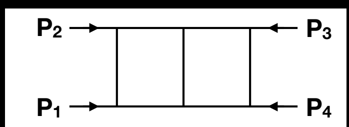
Loop-Tree Duality

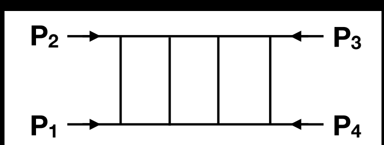


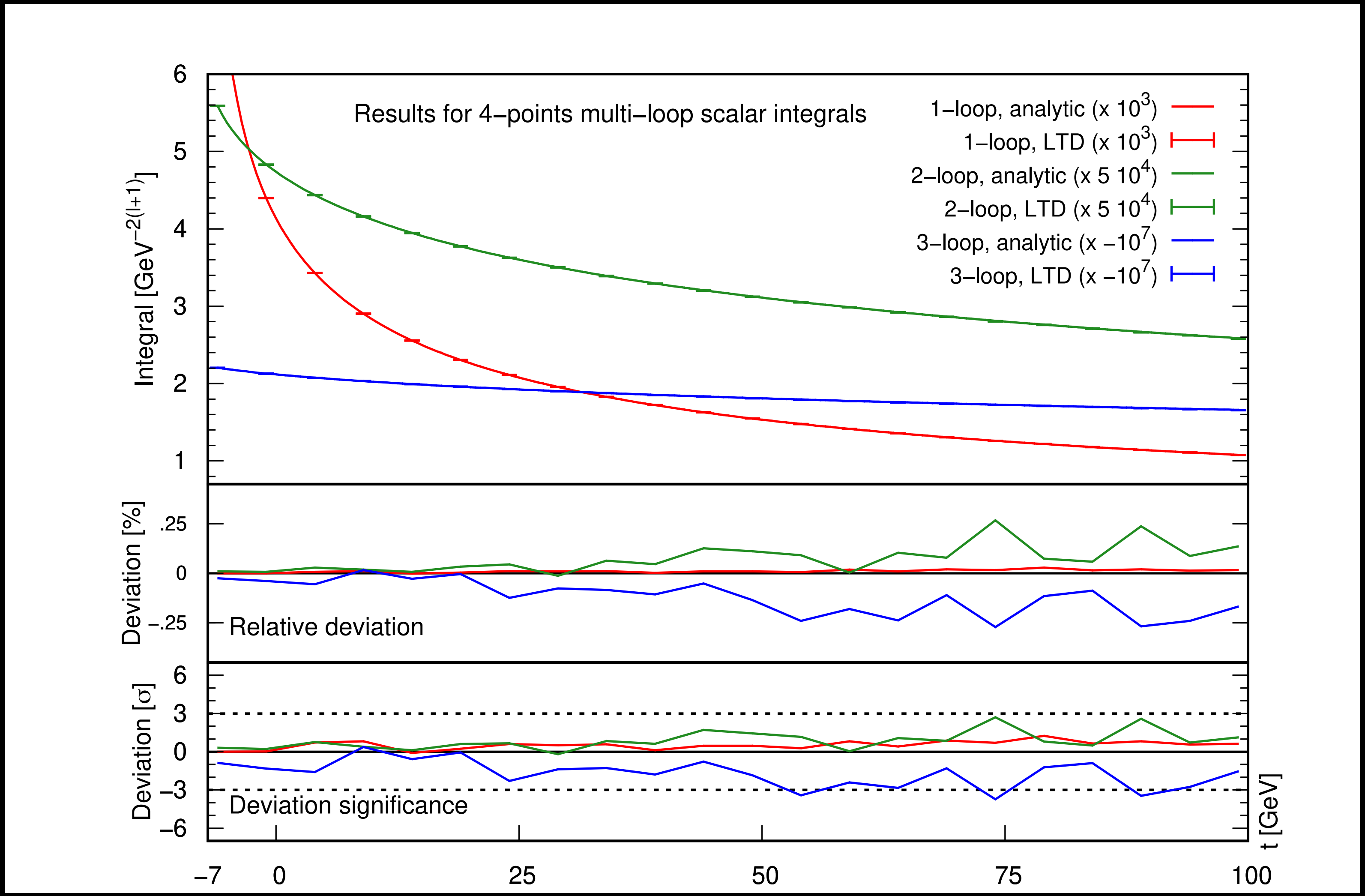
- The energy component of the loop-momenta can be integrated out simply, using Cauchy's theorem.
- Leading to a nice mathematical structure at any loop order.
- It appears to be advantageous numerically as well.

Catani, Gleisberg, Krauss, Rodrigo, Winter; Bierrenbaum, Catani, Draggiotis, Rodrigo; Buchta, Chachamis, Draggiotis, Rodrigo; Runkel, Szor, Vesga, Weinzierl; Capatti, Hirschi, Kermanschah, Pelloni, Ruijl

Numerical integration of one-,two- and three-loop off-shell planar box after LTD (Euclidean region)







Capatti, Hirschi, Kermanschah, Ruijl

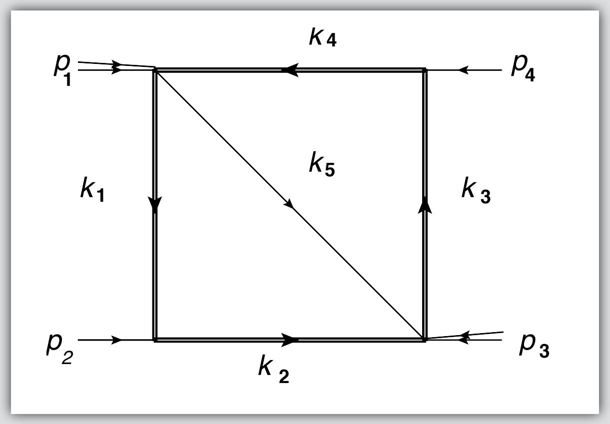
A spin-off

Small mass expansions

Physical regulators

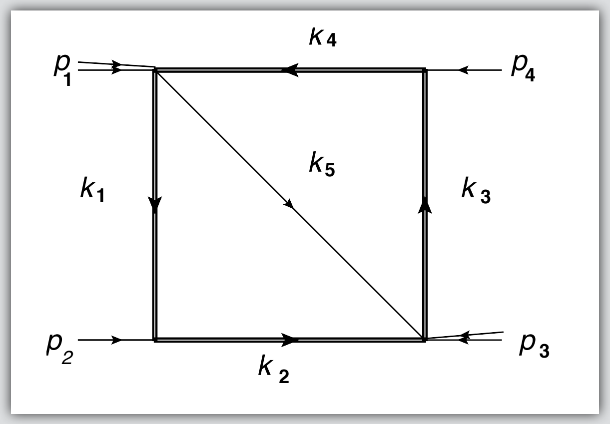
- The subtraction counterterms are local.
- They can be invented with dimensional regularisation in mind, but they can also be adapted to other regularisation schemes for the IR divergences.
- Small quark masses act as physical regulators.
- In such case, the infrared counterterms integrate to yield the logarithmically enhanced terms of the integral.

Large logs from small masses easily determined.



$$
\begin{aligned}
\mathrm{D_{box}} = & \left[\frac{1}{(A_1-m^2)\,(A_2-m^2)} - \frac{1}{(A_1-\mu^2)\,(A_2-\mu^2)}\right]\left[\frac{1}{A_3A_4A_5}\right]_{k_1=-x_2p_2} \\
& + \left[\frac{1}{(A_3-m^2)\,(A_4-m^2)} - \frac{1}{(A_3-\mu^2)\,(A_4-\mu^2)}\right]\left[\frac{1}{A_1A_2A_5}\right]_{k_4=x_4p_4} \\
& - \left[\frac{1}{(A_1-m^2)\,(A_2-m^2)\,(A_3-m^2)\,(A_4-m^2)}\right. \\
& \qquad \times \left[-\frac{1}{(A_1-\mu^2)\,(A_2-\mu^2)\,(A_3-\mu^2)\,(A_4-\mu^2)}\right]\left[\frac{1}{A_5}\right]_{\substack{k_4=x_4p_4,\\ k_1=-x_2p_2}} \\
& \qquad + \mathrm{D_{box}}|_{\mathrm{fin}} + \mathcal{O}(m^2)
\end{aligned}
$$

Large logs from small masses easily determined.



$$u = m_1^2 + m_3^2 - s - t, \quad K = m_1^2 m_3^2 - st\,,$$

$$v_1 = \frac{u m_1^2}{K}, \; v_3 = \frac{u m_3^2}{K}, \quad v_s = \frac{us}{K}, \; v_t = \frac{ut}{K}$$

$$\begin{aligned}
u\, \mathrm{D_{box}}|_{\mathrm{fin}}(\mu) &= 2\mathrm{Li}_4(v_1) + 2\mathrm{Li}_4(v_3) - 2\mathrm{Li}_4(v_s) - 2\mathrm{Li}_4(v_t) \\
&-2\mathrm{Li}_3(v_1)L_\mu(m_1^2) - 2\mathrm{Li}_3(v_3)L_\mu(m_3^2) + 2\mathrm{Li}_3(v_s)L_\mu(s) + 2\mathrm{Li}_3(v_t)L_\mu(t) \\
&+\mathrm{Li}_2(v_1)L_\mu^2(m_1^2) + \mathrm{Li}_2(v_3)L_\mu^2(m_3^2) - \mathrm{Li}_2(v_s)L_\mu^2(s) - \mathrm{Li}_2(v_t)L_\mu^2(t) \\
&+\frac{1}{3}\ln(1-v_1)L_\mu^3(m_1^2) + \frac{1}{3}\ln(1-v_3)L_\mu^3(m_3^2) - \frac{1}{3}\ln(1-v_s)L_\mu^3(s) \\
&-\frac{1}{3}\ln(1-v_t)L_\mu^3(t)\,.
\end{aligned}$$

Concluding remarks

- Nested subtractions can separate at the integrand, the pinch-singularities of Feynman diagrams and Feynman amplitudes.
- We aim to formulate a subtraction method for two-loop amplitudes of generic processes.
- This can be the basis for a purely numerical evaluation of two-loop amplitudes with an affordable computational cost.
- Substantial amount of work is needed in achieving that…it requires an excellent understanding of both pinched and integrable singularities (contour deformations)
- Spin-off: Nested subtractions are potentially useful for small mass expansions of loop amplitudes (e.g. bottom/charm-quark loop-induced processes, very high energy collider processes).