Three-body Interactions in Lattice QCD and Phenomenology

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MA 7156/1-1

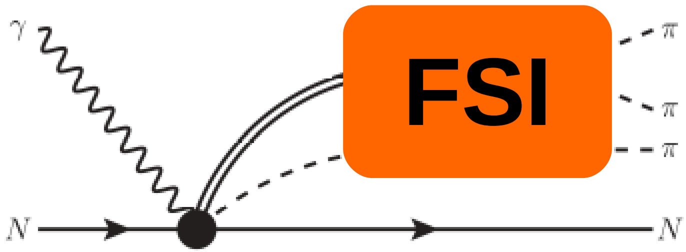
[Many slides from **Maxim Mai**]

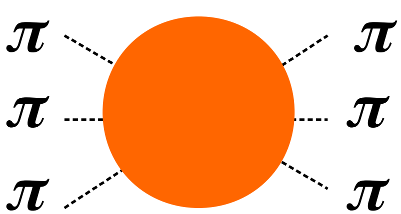
Outline

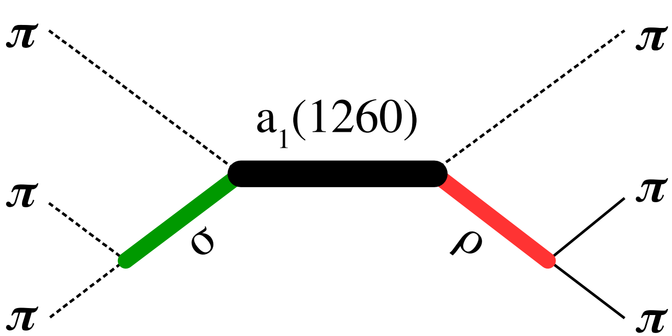
- Three-body dynamics in infinite volume
- The Finite-volume problems (application: 2-body)
- **Three-body dynamics in finite volume**
  - The 3-pion system at maximal isospin:
    Interpretation of recent lattice QCD data

3-body dynamics for mesons and baryons

Light mesons

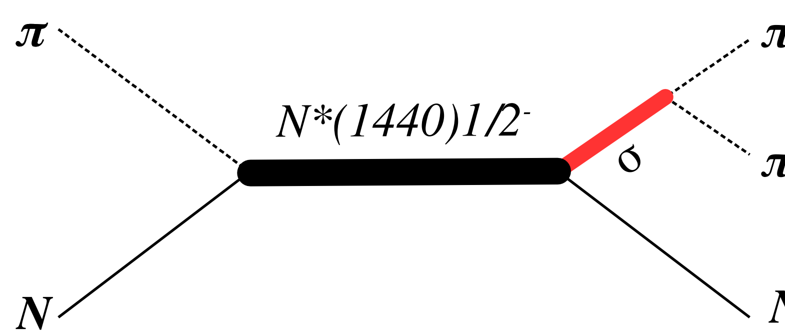






- Important channel in GlueX @ JLab
- Finite volume spectrum from lattice QCD:
  Lang, Leskovec, Mohler, Prelovsek (2014)
  Woss, Thomas et al. [HadronSpectrum] (2018)
  Hörz, Hanlon (2019), ...

Light baryons



- Roper resonance is debated for ~50 years in experiment. Can only be seen in PWA.
- 1st calculation w. meson-baryon operators on the lattice: **Lang et al. (2017)**

Three-body Interactions with Isobars

Mai, Hu, M. D., Pilloni, Szczepaniak

Eur. Phys. J. A53 (2017) 177

3-body Unitarity

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle = i \int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

$$\times \prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4 k_\ell}{(2\pi)^4}(2\pi)\boxed{\delta^+(k_\ell^2 - m^2)}\right](2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3} k_\ell\right)$$

delta function sets all intermediate particles on-shell

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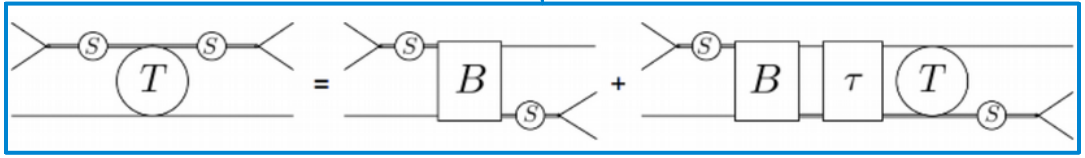

General Ansatz for the isobar-spectator interaction
→ **B & τ** are **new** unknown functions

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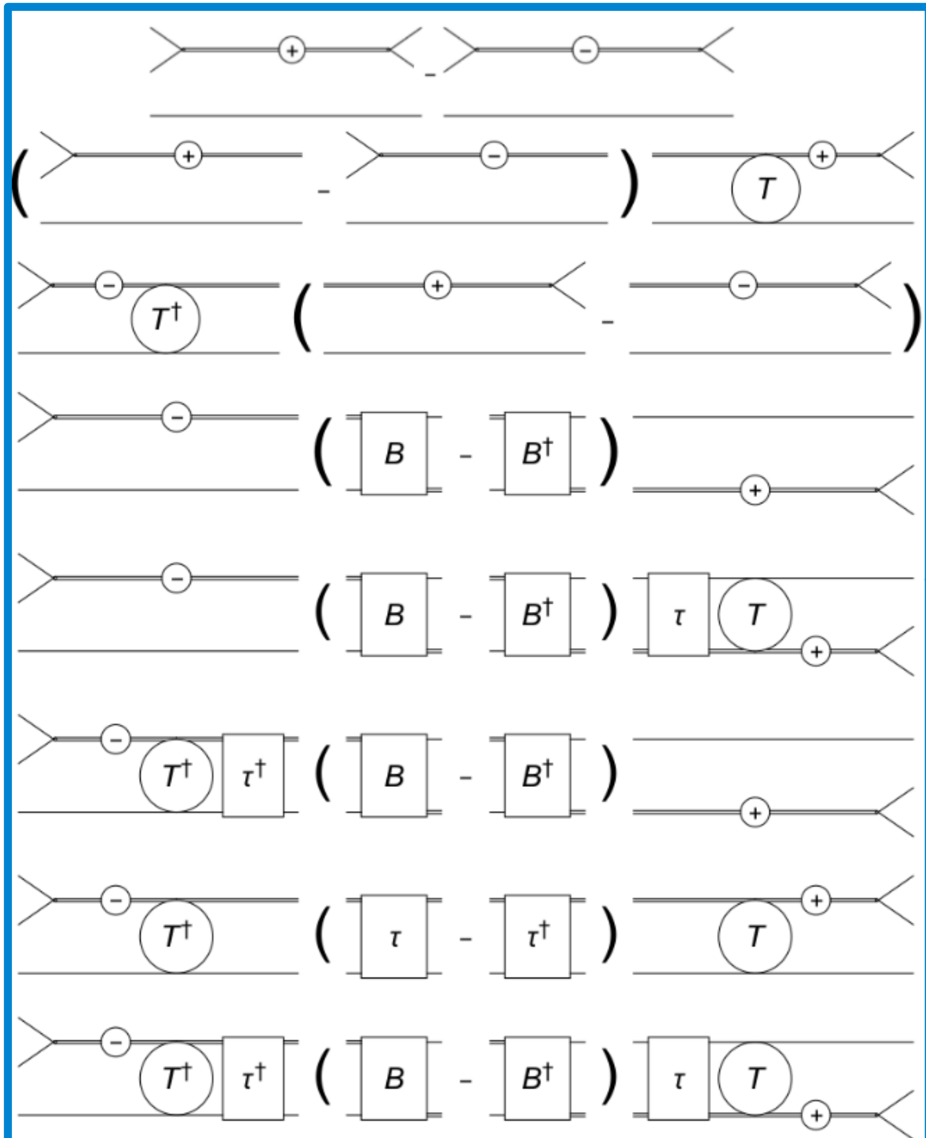

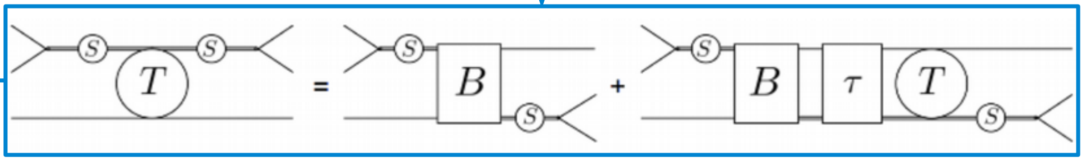

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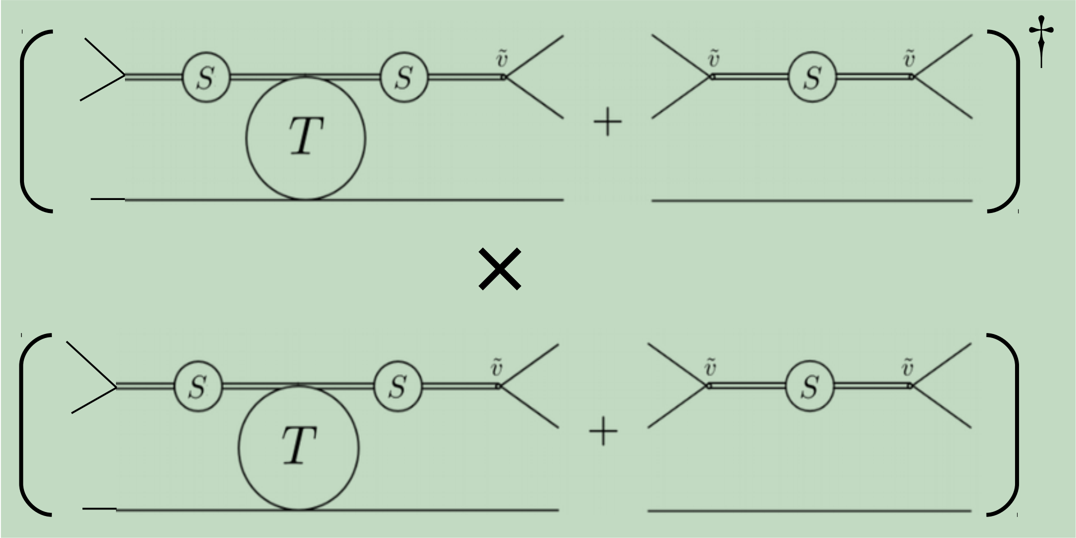

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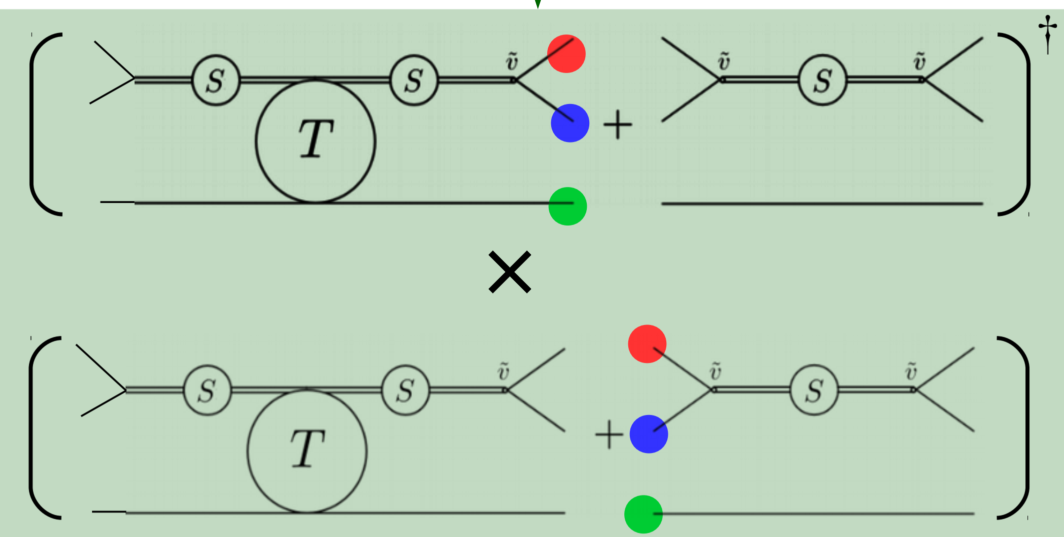

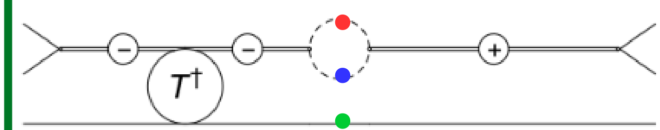

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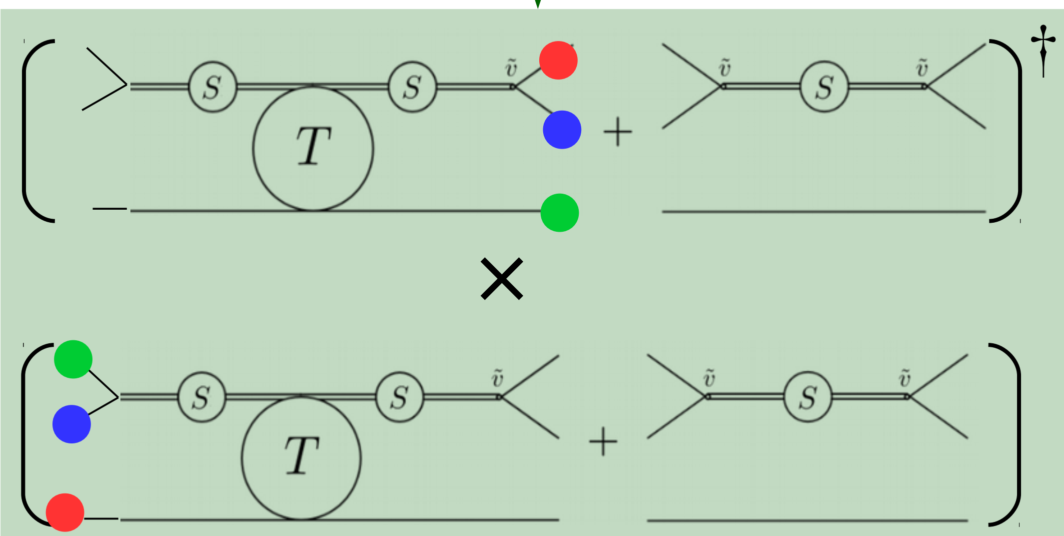

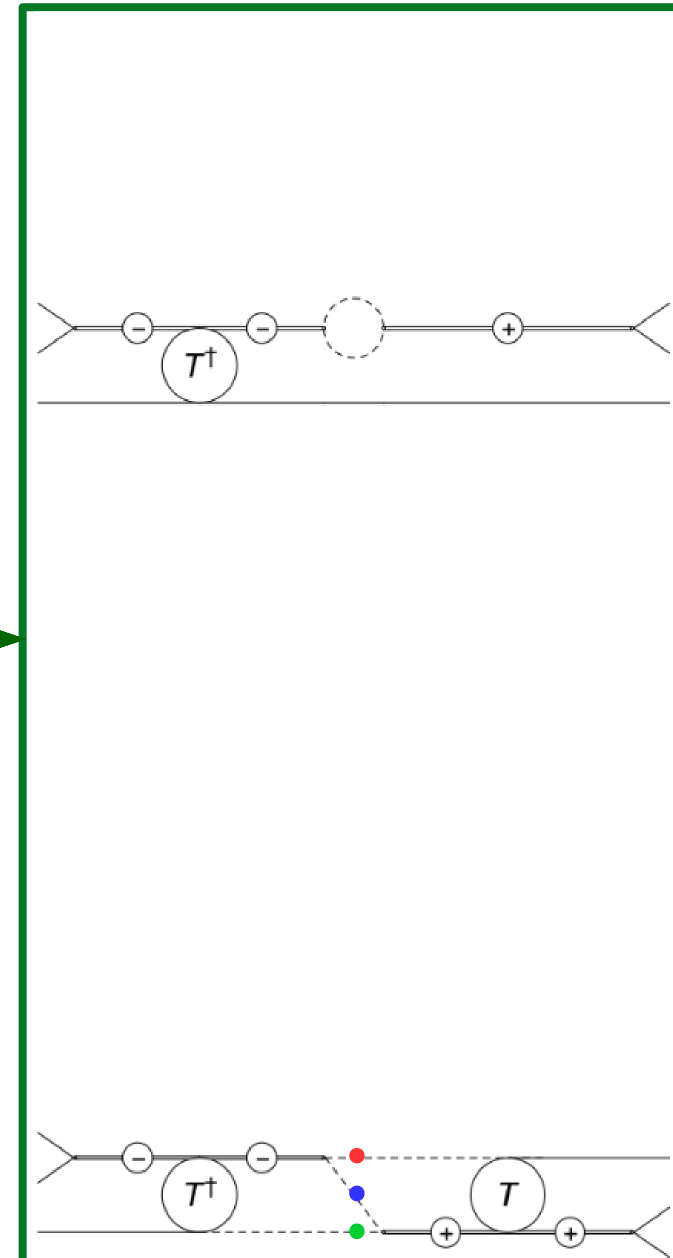

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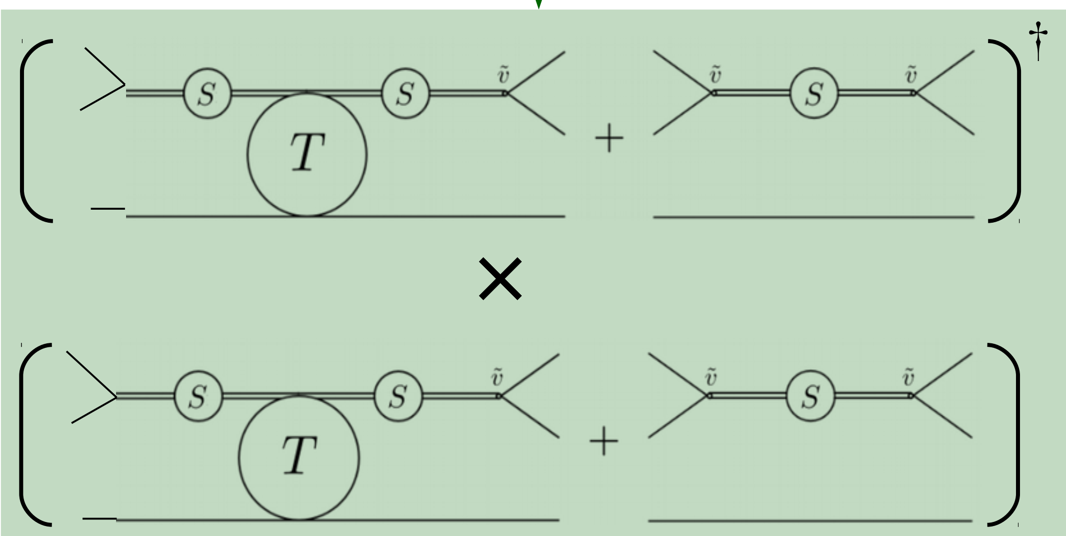
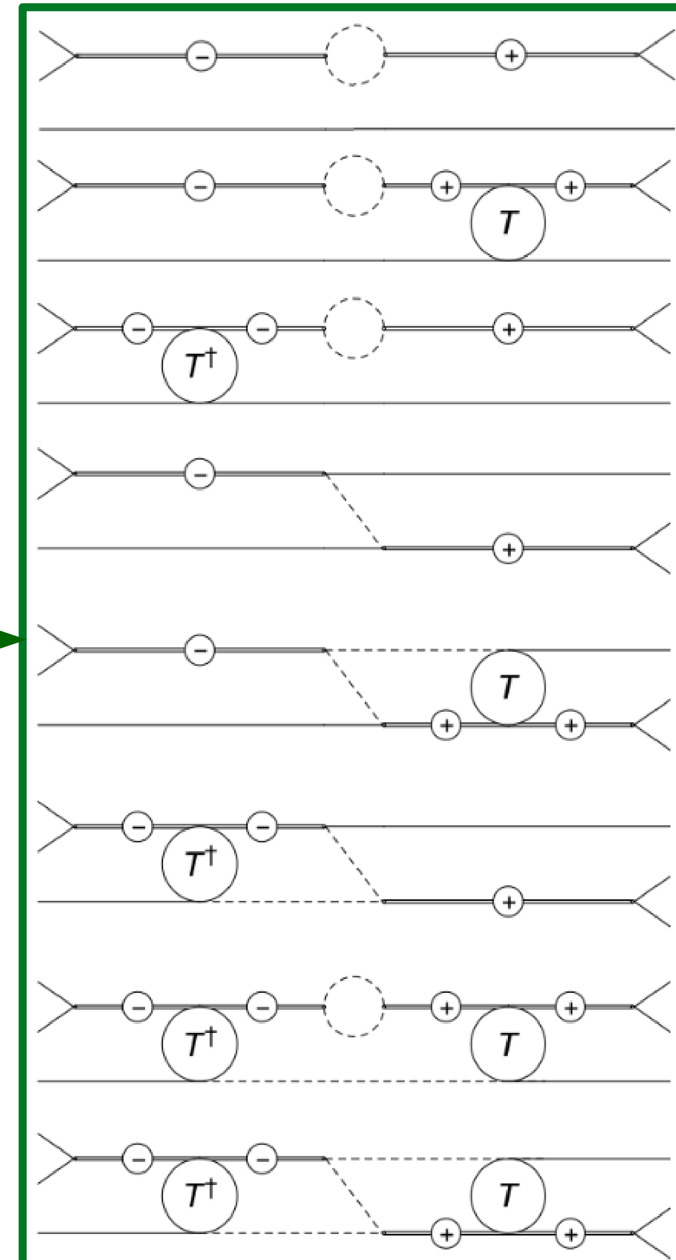


8 top.



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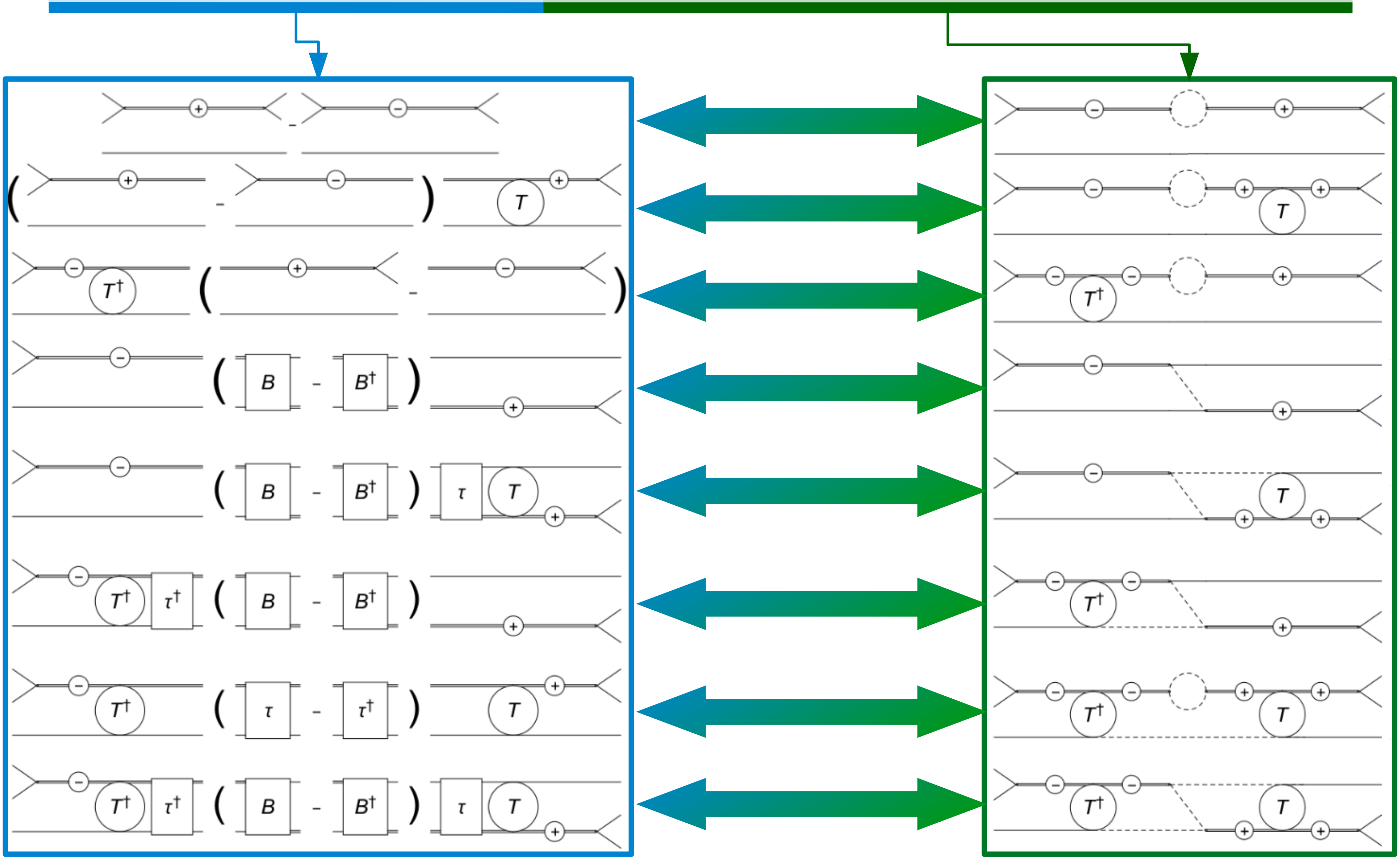
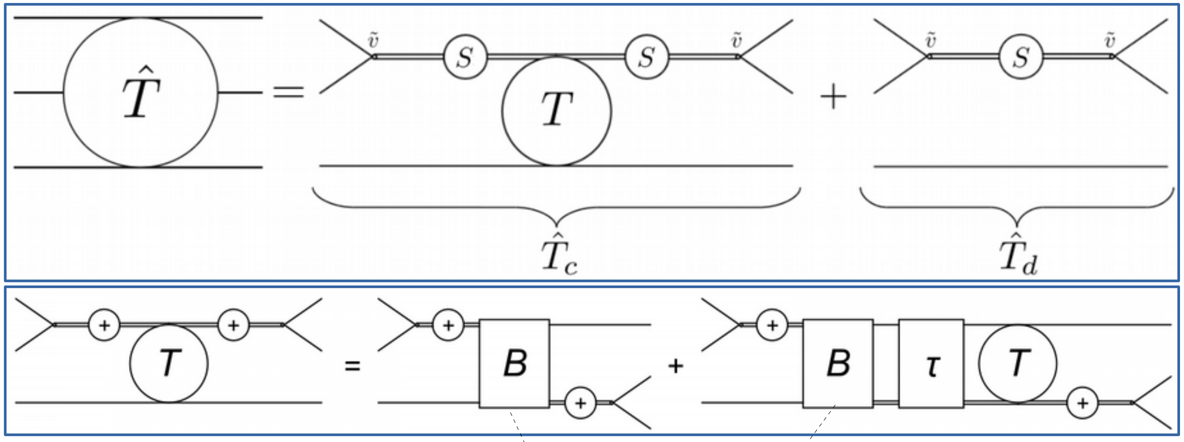


Scattering amplitude

3 → 3 scattering amplitude is a 3-dimensional integral equation



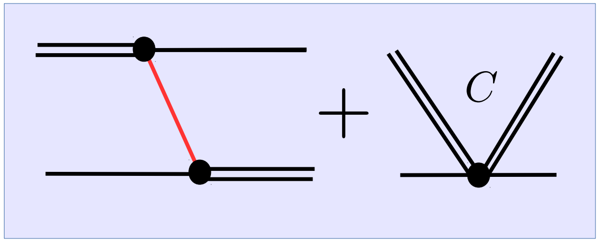
– Imaginary parts of ***B*, *S*** are fixed by **unitarity/matching**

– For simplicity ***v*=*λ*** (full relations available)

$$\mathrm{Disc}\, B(u) = 2\pi i \lambda^2 \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2+\mathbf{Q}^2}}$$

- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q - \sqrt{m^2+\mathbf{Q}^2}+i\epsilon\right)} + C$$



- one-**π** exchange in TOPT → *RESULT, NOT INPUT !*

- One can map to field theory, but does not have to. Result is a-priori dispersive.

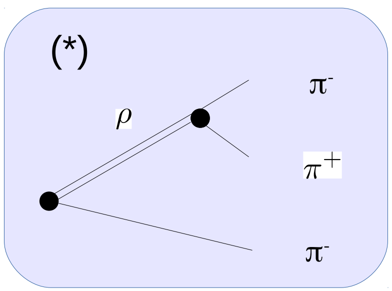
Application: The $a_1(1260)$ lineshape

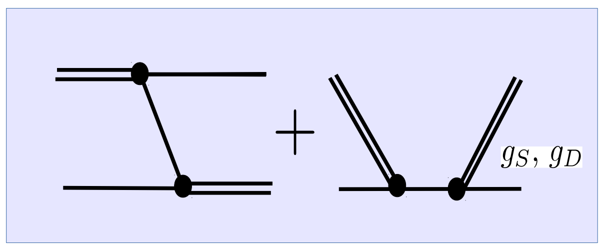
Sadasivan, M.D., Mai, in preparation

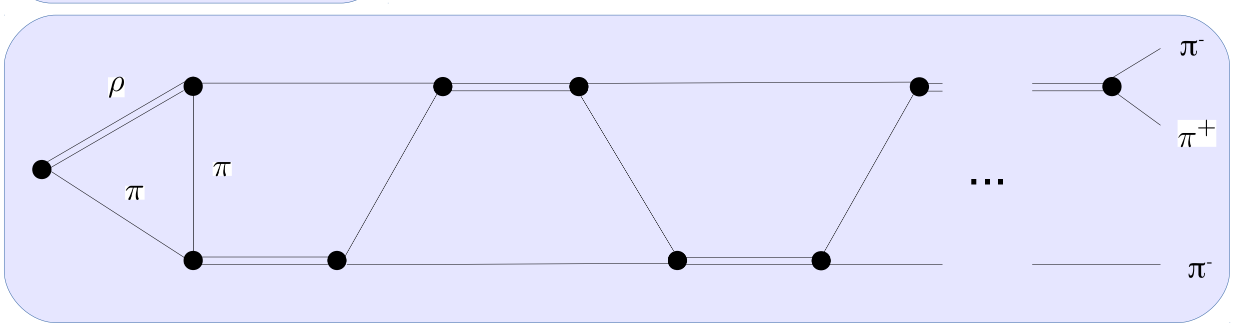
- Recent efforts to study 3-body production beyond the "isobar approximation" (*)

  P. Magalhães, A. C. dos Reis et al., PRD84 (2011); Khmechandani, Martinez, Oset, PRC77 (2008); <u>JPAC</u>: Mikhasenko, Wunderlich et. al., JHEP (2019); Mikhasenko, Pilloni et. al., PRD98 (2018); A. Jackura et al., EPJC79 (2019); <u>Jülich</u>: Janssen et al., PRL (1993)

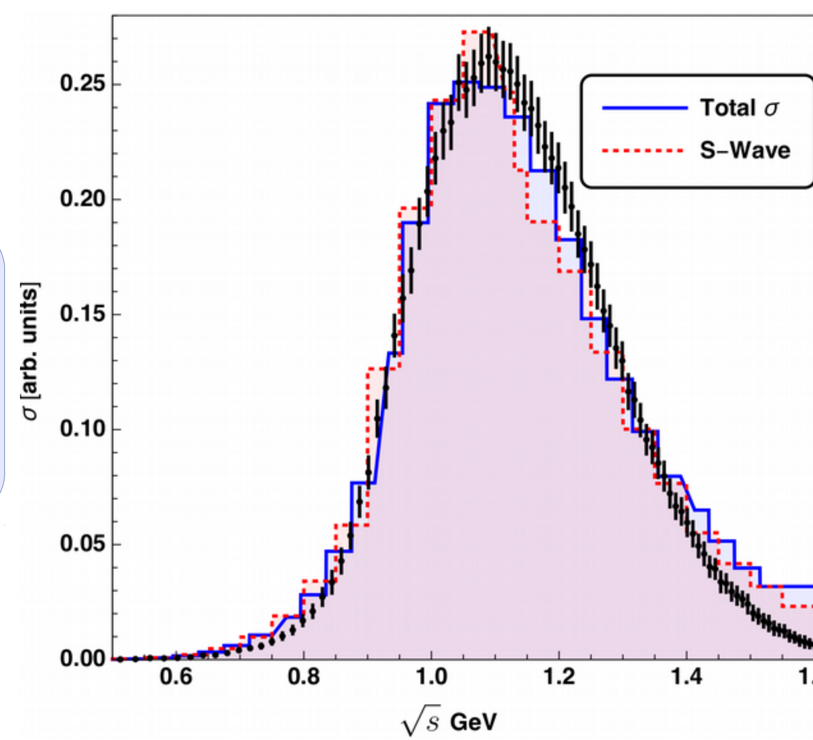
- Here: Full solution of three-body equation with **exact** three-body unitarity
- S- and D-waves included







+ symmetrization final states

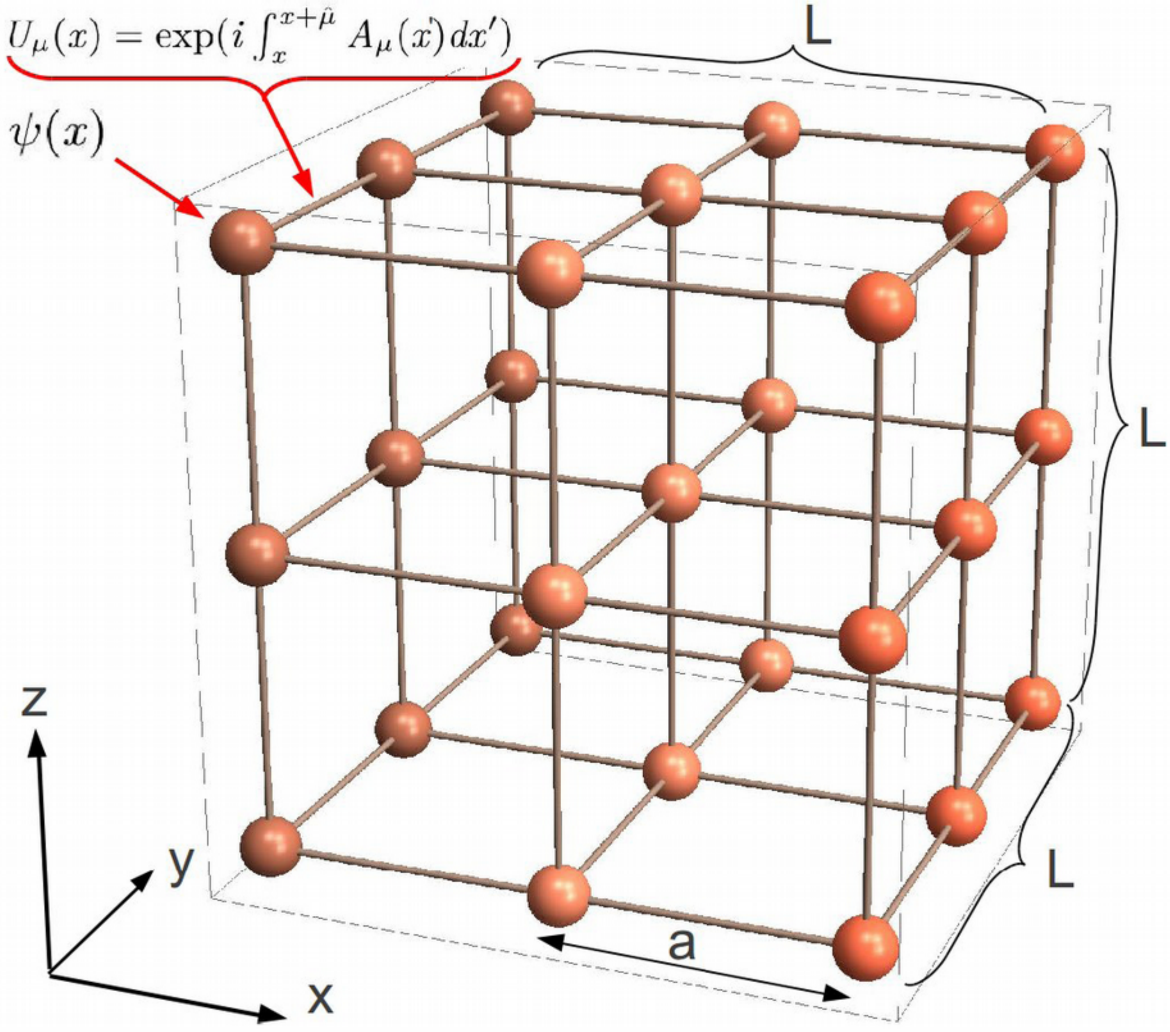


Data: ALEPH coll. hep-ex/0506072

(*) here meant in the sense of "no rescattering", "no three-body unitarity"

From two to three particles in finite volume

The cubic lattice



- Side length L, periodic boundary conditions $\Psi(\vec{x}) \stackrel{!}{=} \Psi(\vec{x} + \hat{\mathbf{e}}_i\, L)$ $\rightarrow$ finite volume effects $\rightarrow$ Infinite volume $L \rightarrow \infty$ extrapolation
- Lattice spacing a $\rightarrow$ finite size effects Modern lattice calculations: $a \simeq 0.07$ fm $\rightarrow p \sim 2.8$ GeV $\rightarrow$ (much) larger than typical hadronic scales;

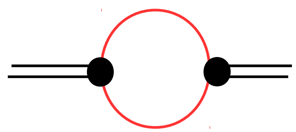
  not considered here.
- Unphysically large quark/hadron masses $\rightarrow$ (chiral) extrapolation required.

How to derive the 2-body quantization condition

Two-body unitarity

↓

On-shell condition



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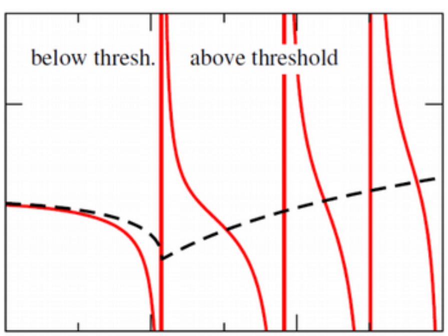
Imaginary parts



Infinite → Fin. Vol

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3} \to \frac{1}{L^3}\sum_s \sum_{i=1}^{\vartheta(s)}$$

↓

Power-law fin-vol. effects

↓

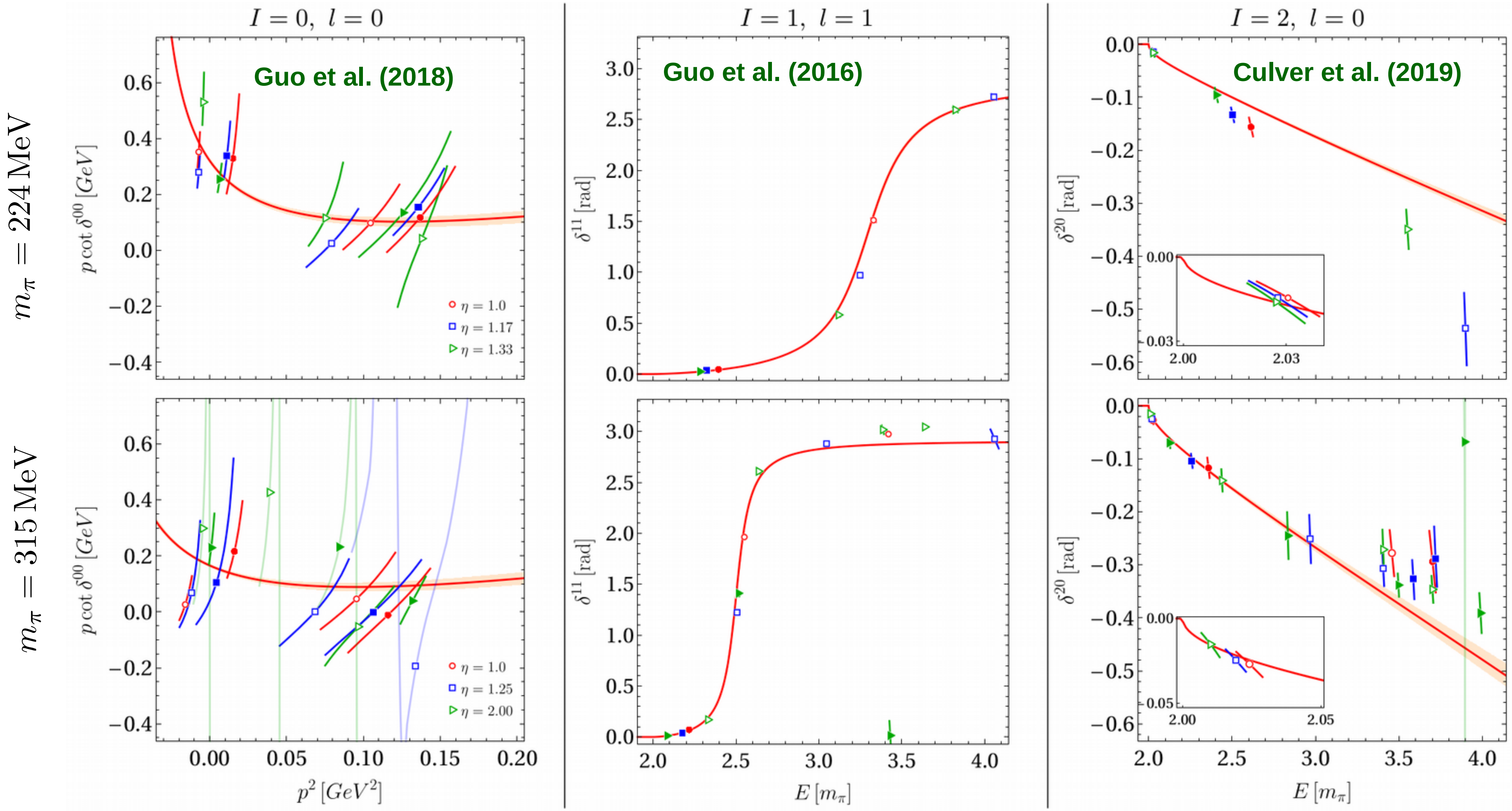
Lüscher

$$p\cot\delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \text{Re}\, G(E)\right)$$

.

GWU lattice group: All Isospins

[Culver et al., PRD100 (2019); Mai et al., arXiv:1908.01847 [hep-lat]]

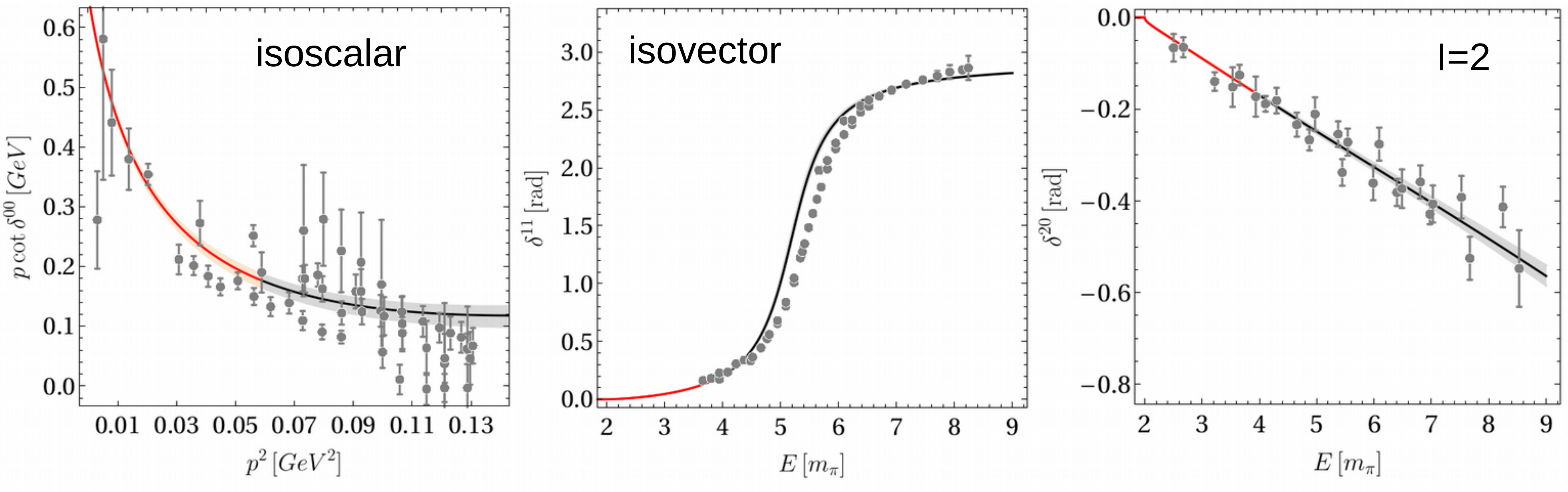


- Simultaneous fit with Inverse Amplitude Method (more later)
- Including correlation between energy eigenvalues, pion masses and pion decay constants
- Including correlations across energy eigenvalues & isospins

GWU lattice group: Chiral Extrapolation

(Optional & model depedent)

[Mai et al., arXiv:1908.01847 [hep-lat]]



Scattering lengths and resonance poles:

m_π [MeV]	~ 315	~ 224	139
$m_\pi\, a_0^{I=0}$	$+1.9008^{+0.0521}_{-0.0593}$	$+0.6985^{+0.0010}_{-0.0015}$	$+0.2132^{+0.0008}_{-0.0009}$
$m_\pi\, a_0^{I=2}$	$-0.1538^{+0.0021}_{-0.0018}$	$-0.0952^{+0.0010}_{-0.0009}$	$-0.0433^{+0.0002}_{-0.0002}$
m_σ [MeV]	$+591^{+6}_{-5} - i109^{+4}_{-4}$	$+502^{+4}_{-4} - i175^{+6}_{-5}$	$+443^{+3}_{-3} - i221^{+6}_{-6}$
$g_{\sigma\pi\pi}$ [MeV]	533^{+2}_{-2}	426^{+2}_{-2}	$397.8^{+0.6}_{-0.6}$
m_ρ [MeV]	$+789^{+1}_{-1} - i20^{+0}_{-0}$	$+738^{+2}_{-1} - i43^{+1}_{-1}$	$+724^{+2}_{-4} - i67^{+1}_{-1}$
$g_{\rho\pi\pi}$ [MeV]	226^{+2}_{-2}	282^{+3}_{-2}	323^{+5}_{-3}

THREE-BODY AMPLITUDE IN A BOX

M. Mai, MD, EPJA 2017 [arXiv: 1709.08222]

Overview

Lüscher-like formalism in 3 → 3 case is under investigation

Polejaeva/Rusetsky (2012)

Briceño/Hansen/Sharpe (2014-)

Non-relativistic approaches based on dimer picture & effective field theory

Kreuzer, Griesshammer(2012), Hammer et al. (2016)

F. Romero, Rusetsky, Urbach et. al. (2018)

Equivalence of various 3-body formalisms; three-body unitarity for Hansen/Sharpe

Requirements

Jackura et al. (2019) [JPAC], Briceño et al. (2019)

- 3-body systems involve (resonant) two-body sub-amplitudes: Construct such that 2-body information can be included
- Need extrapolations between different energies (problem of underdetermination)
- Allow for systematic improvement by allowing more and more quantum numbers as lattice data improve (problem of underdetermination)
- At least, **all** possible intermediate on-shell configurations must be identified and included to ensure all power-law finite-volume effects are taken account of.

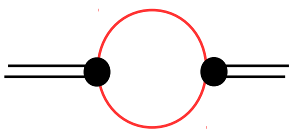
⟹ This work: **Quantization condition from 3-body unitarity in isobar formulation**

How to derive the 2-body quanti-zation condition

Two-body unitarity

↓

On-shell condition



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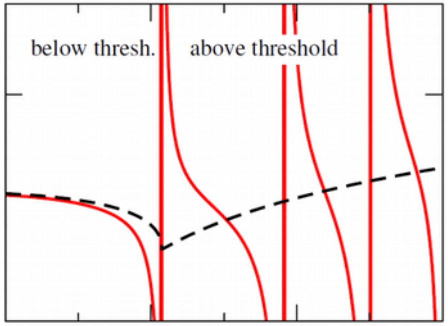
Imaginary parts



Infinite → Fin. Vol

↓

Power-law fin-vol. effects

↓

Lüscher

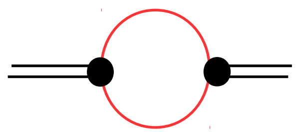
$$p\cot\delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\,G(E)\right)$$

Three-body?

Analogously!

Two-body unitarity

On-shell condition



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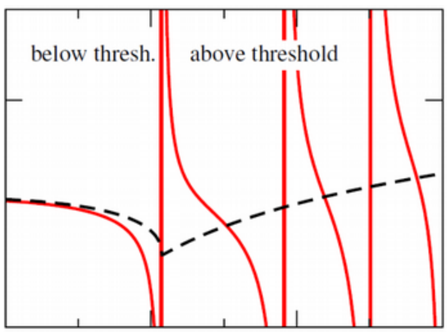

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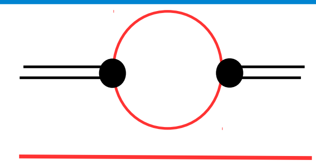

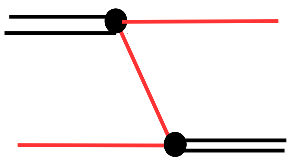

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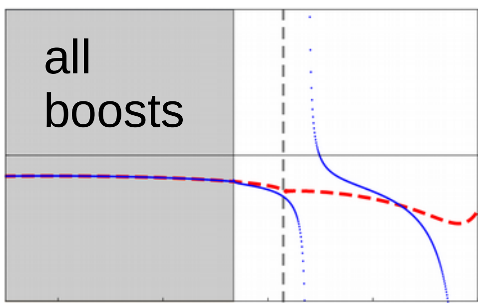
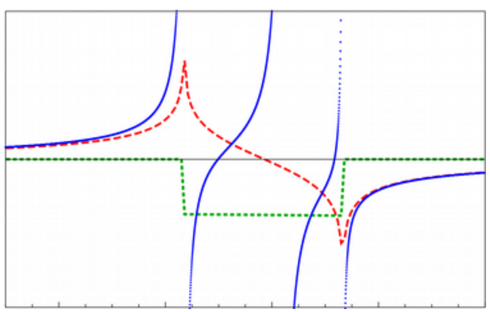




Power-law fin-vol. effects

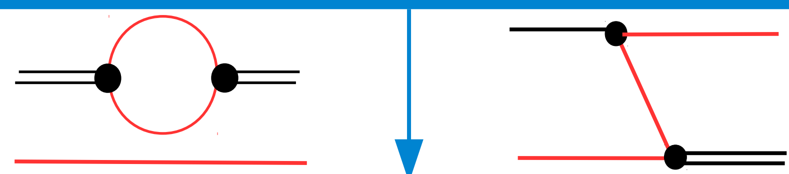
Quantization Condition

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma \mathbf{ss}'}_{\mathbf{uu}'}(\mathbf{W^2}) + \frac{\mathbf{2E_s\, L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss}'}\delta_{\mathbf{uu}'}\right) = \mathbf{0}$$

Only exact three-body unitarity guarantees the cancellation of unphysical 1st and 2nd order poles

Three-body unitarity

On-shell condition



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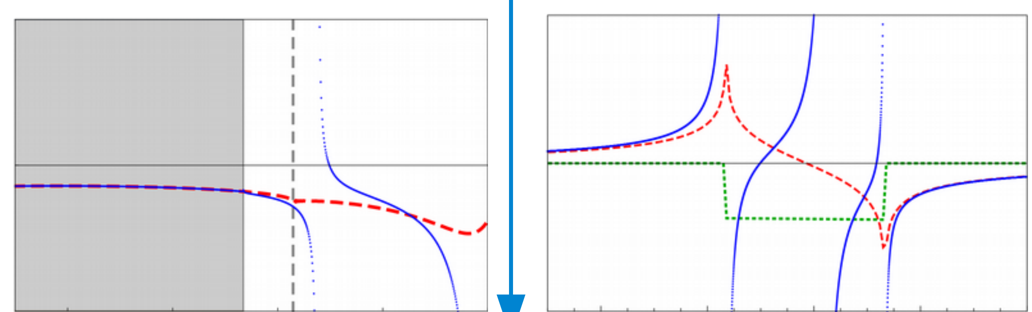

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.

A physical system:

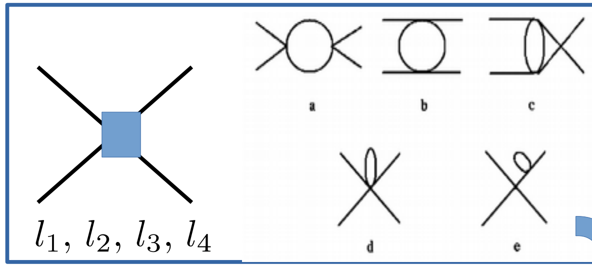
$\pi^+\pi^+\pi^+$

Mai, M.D., PRL 122 (2019), 062503

Three positive pions

- Maximal isospin: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$
  - Repulsive channel
  - ***L=2.5 fm, m_π=291/352/491/591 MeV***

NPLQCD, Detmold et al.(2008)

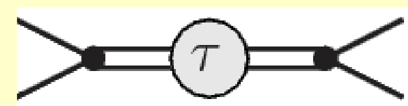


Inverse Amplitude method

Truong(1988), Peláez (1999), Gómez Nicola, Peláez (2002),...

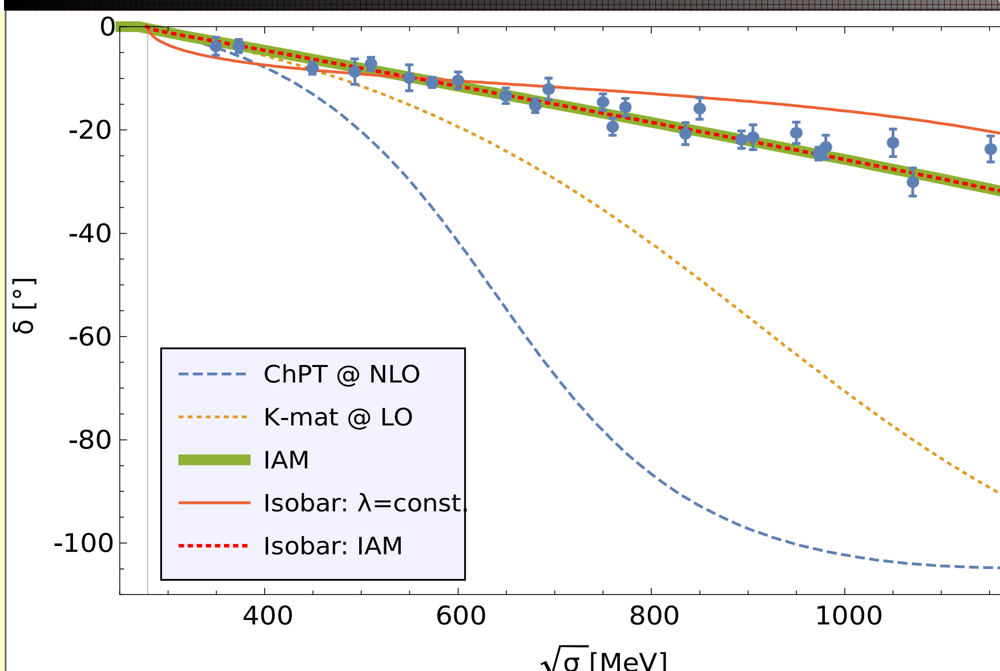
I. 2-body subchannel:

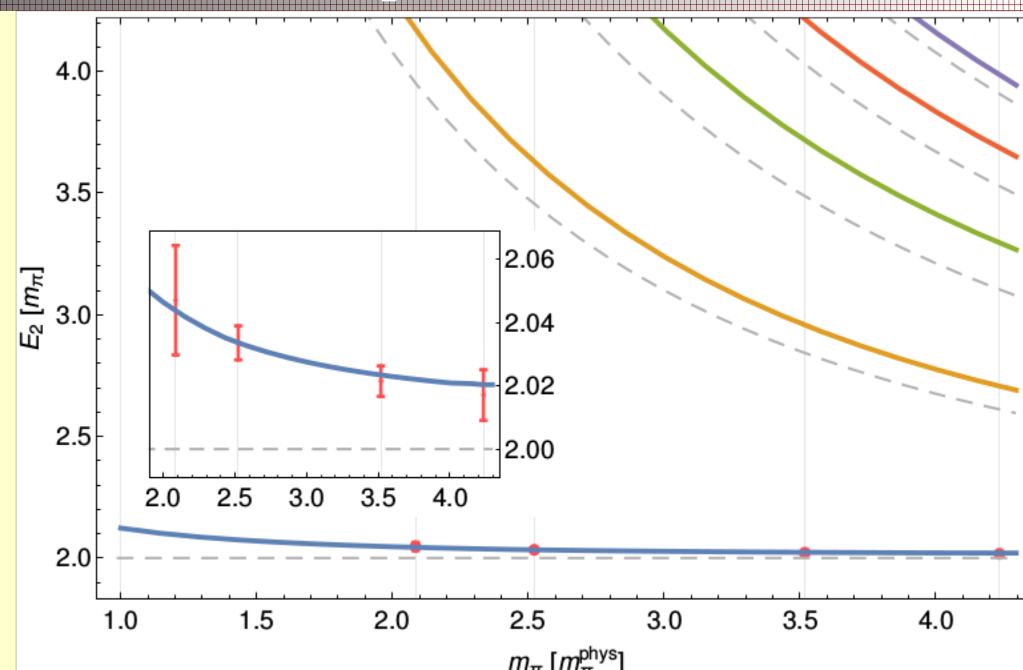
- one-channel problem: $\pi\pi$-system in S-wave, I=2
- 2-body amplitude consistent with 3-body one



$$\frac{T_{\rm LO}^2}{T_{\rm LO}-T_{\rm NLO}}$$

discretize (Lüscher) → predicted fin-vol. spectrum





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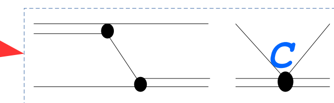
NPLQCD, Detmold et al.(2008)

II. 3-body spectrum

QUANTIZATION CONDITION

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma \mathbf{ss'}}_{\mathbf{uu'}}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-\mathbf{1}}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right) = \mathbf{0}$$

Remaining unknown: ***C***



- *genuine (momenta-dependent) 3-body "force"*
- *simplest case:* $C_{\boldsymbol{qp}} = c\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$

- Maximal isospin: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$
  - Repulsive channel
  - ***L=2.5 fm, m_π=291/352/491/591 MeV***

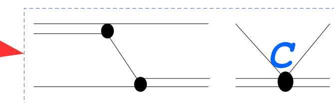
NPLQCD, Detmold et al.(2008)

II. 3-body spectrum

Remaining unknown: ***C***

QUANTIZATION CONDITION

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma \mathbf{ss}'}_{\mathbf{uu}'}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss}'}\delta_{\mathbf{uu}'}\right) = \mathbf{0}$$



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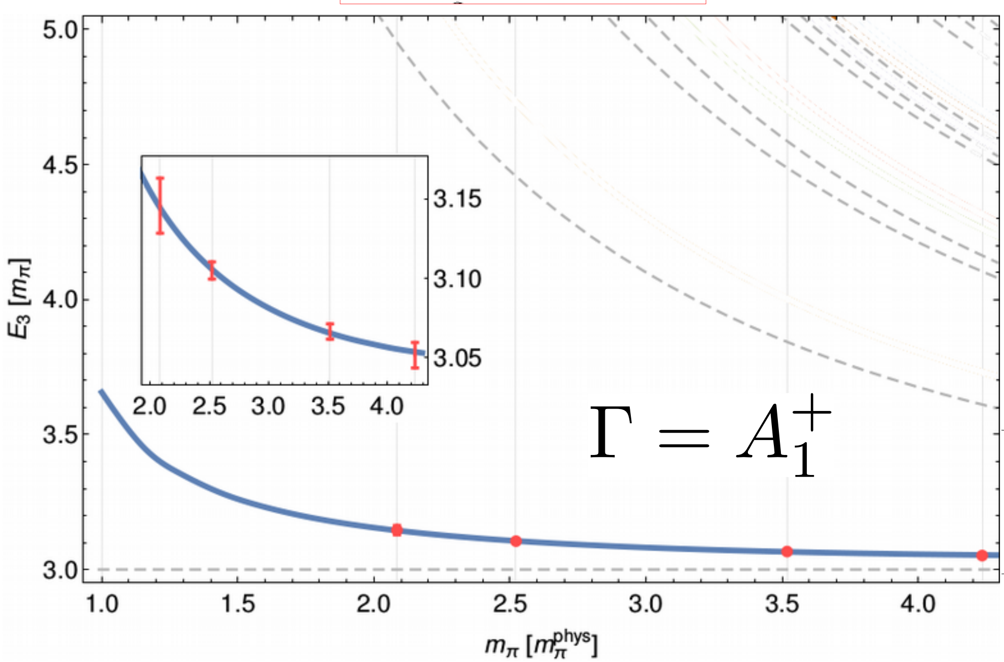


Fit *C* to NPLQCD ground state level
→ ***C=(0.2 ± 1.5)·10^-10***

First prediction of excited levels for physical system

- Maximal isospin: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$
  - Repulsive channel
  - ***L=2.5 fm, m_π=291/352/491/591 MeV***

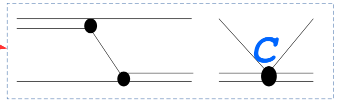
NPLQCD, Detmold et al. (2008)

II. 3-body spectrum

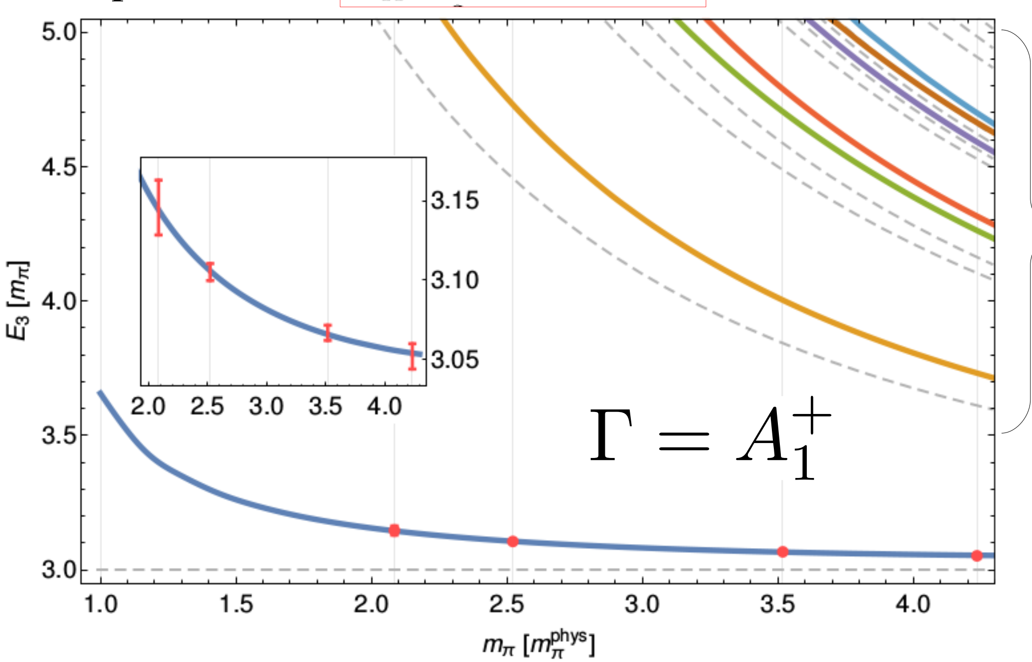
Remaining unknown: c

QUANTIZATION CONDITION

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma \mathbf{ss}'}_{\mathbf{uu}'}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss}'}\delta_{\mathbf{uu}'}\right) = \mathbf{0}$$



- *genuine (momenta-dependent) 3-body "force"*
- *simplest case:* $C_{qp} = c\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$



Predict excited spectrum:

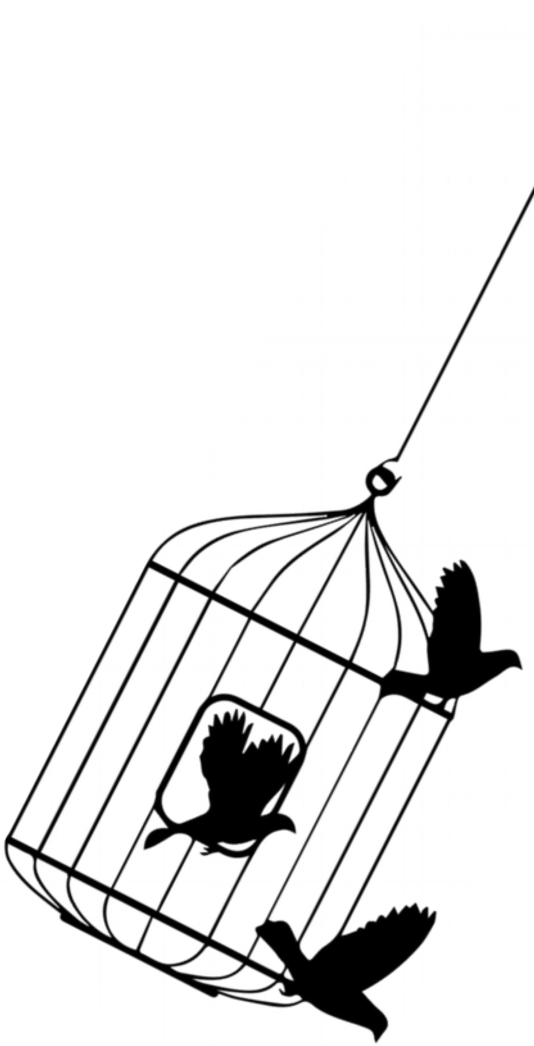
→ novel pattern

1/1 of interacting/non-interacting lvls

→ all QC-poles are simple

→ chiral extrapolation to phys point

(under assumptions)



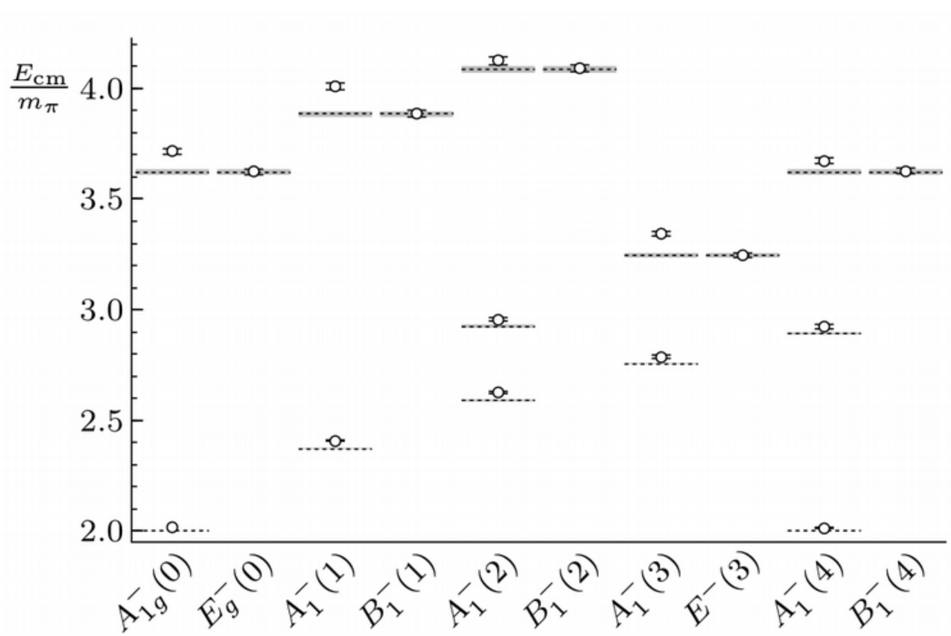
The Moving $\pi^+\pi^+\pi^+$ System

Mai, M.D., Alexandru, Culver
(in preparation)

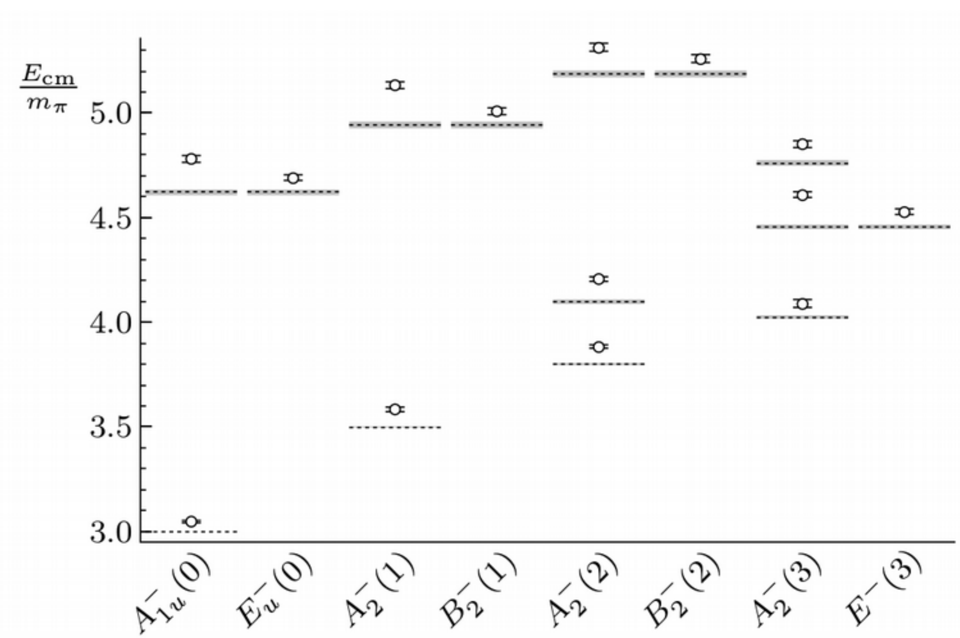
New Lattice Data

Hörz, Hanlon, arXiv:1905.04277 [hep-lat]

Two-body spectrum $\pi^+\pi^+$



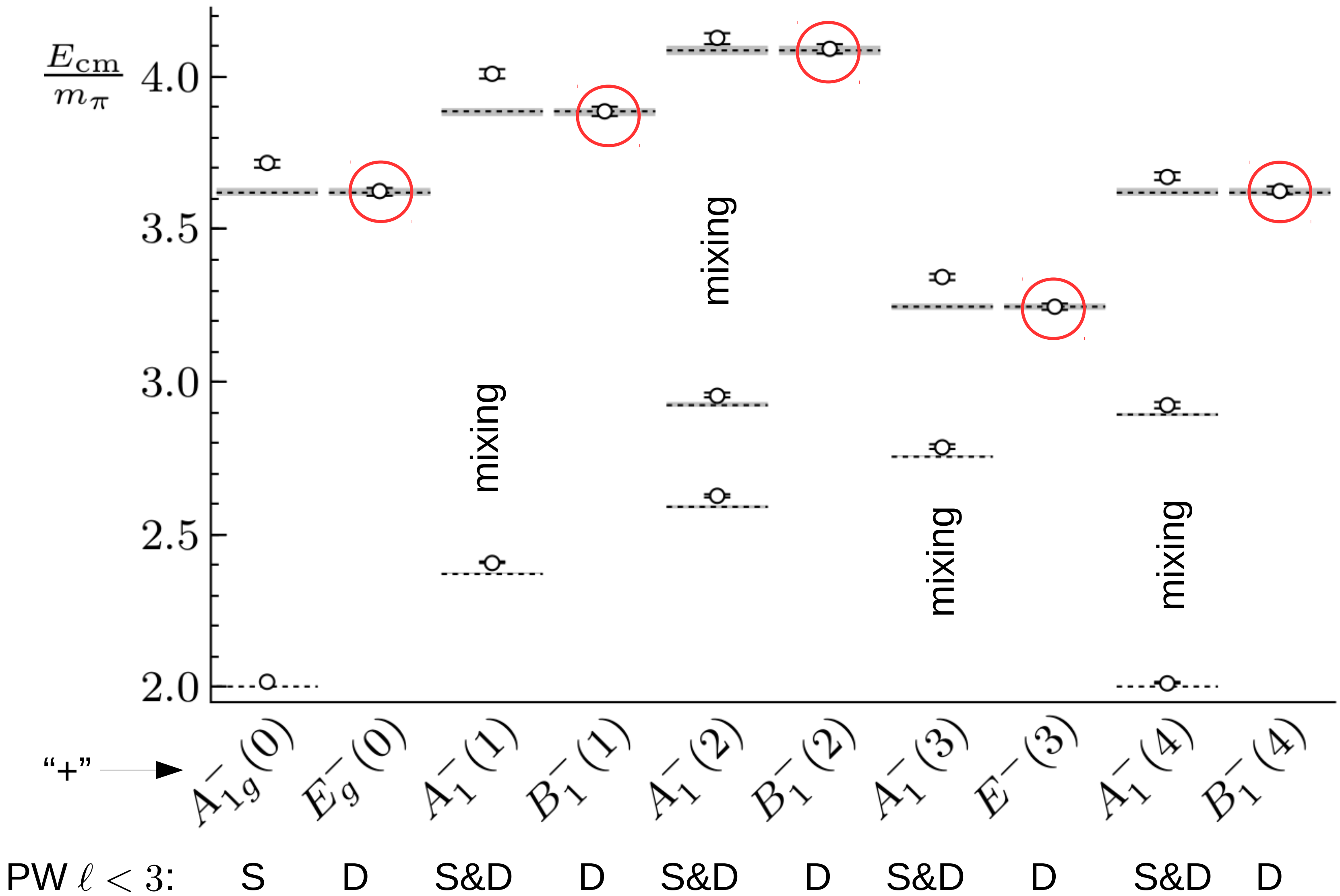
Three-body spectrum $\pi^+\pi^+\pi^+$



- **First lattice data on excited energy eigenvalues** from multi-pion operators → More reliable extraction of scattering eigenvalues
- D200 CLS ensemble (2+1) with improved Wilson fermions and tree-level Lüscher–Weisz gauge action; stochastic LapH method; m_π=200 MeV; **L=4.1 fm**
- High number of Wick contraction (20,679,840 diagrams) managed with novel method from quantum chemistry

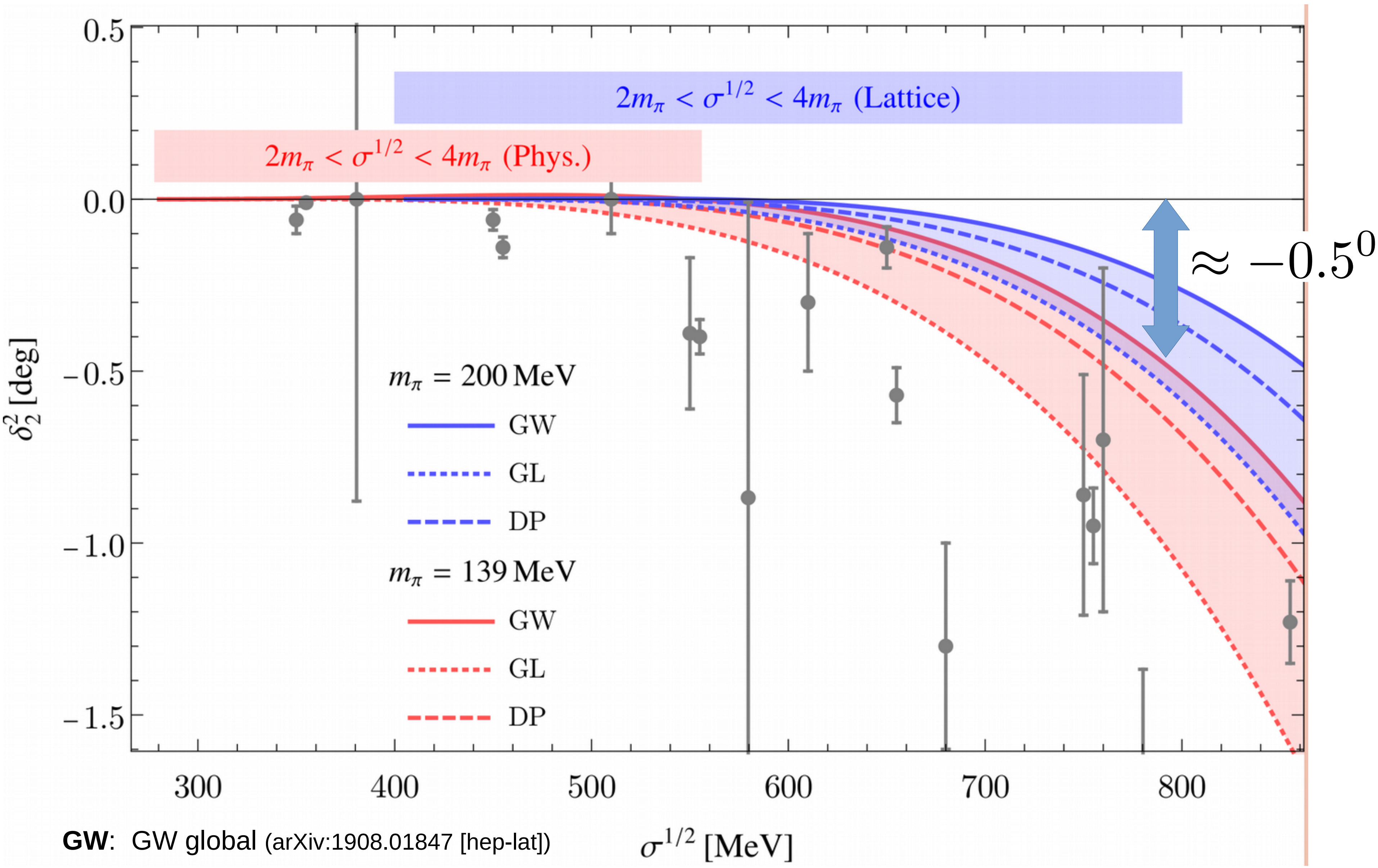
Two-body spectrum: D-wave (I)

Hörz, Hanlon, arXiv:1905.04277 [hep-lat]



→ I=2 D-wave vanishes within uncertainties – what does IAM predict?

D-wave (II): prediction



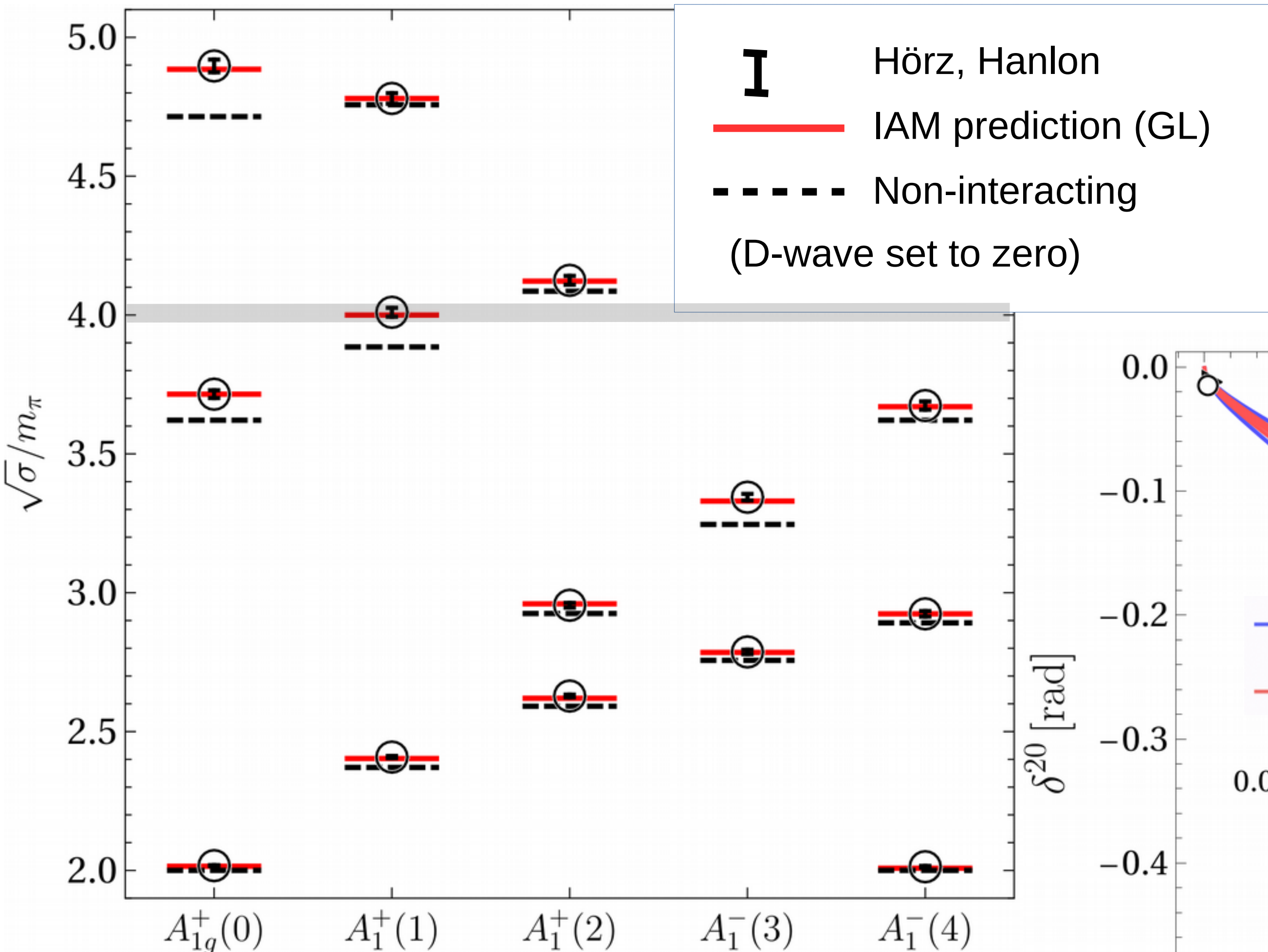
GW: GW global (arXiv:1908.01847 [hep-lat])

GL: Gasser, Leutwyler (Annals Phys. 158, 1984)

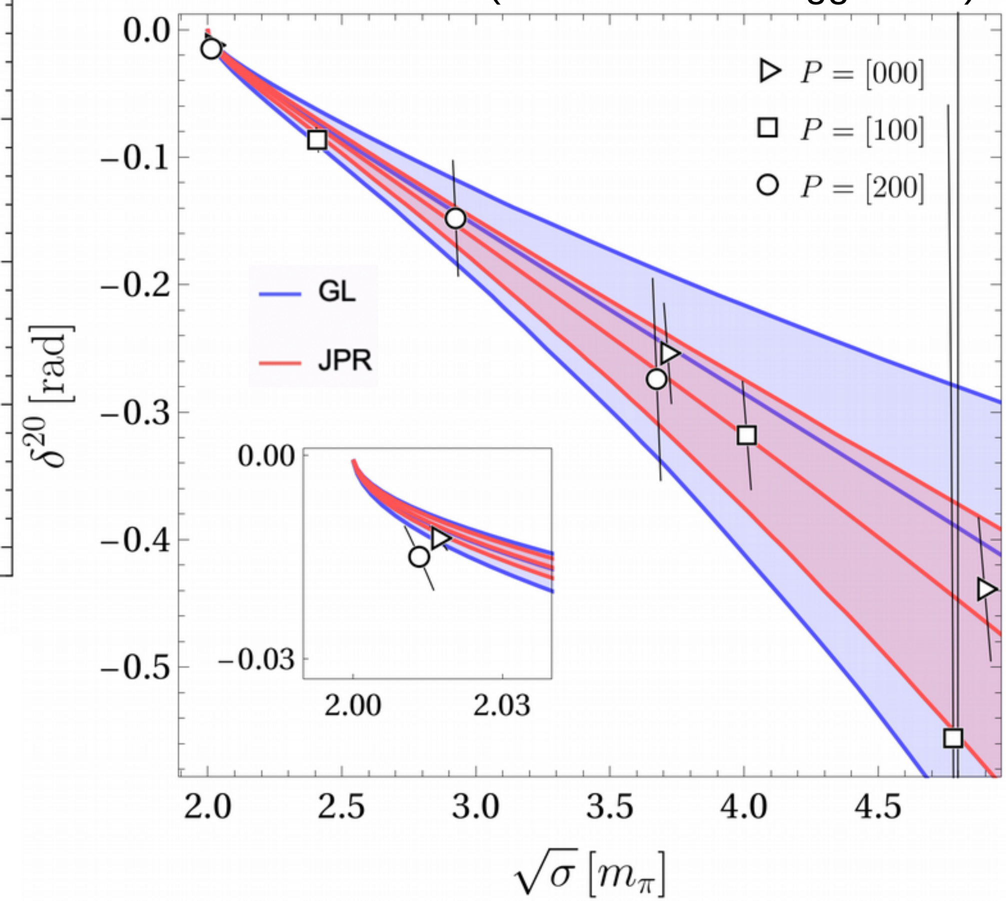
DP: Dobado, Peláez (PRD 56 (1997))

See also [Nebreda, Peláez, Ríos, PRD83 (2011)]

IAM predictions 2-body spectrum



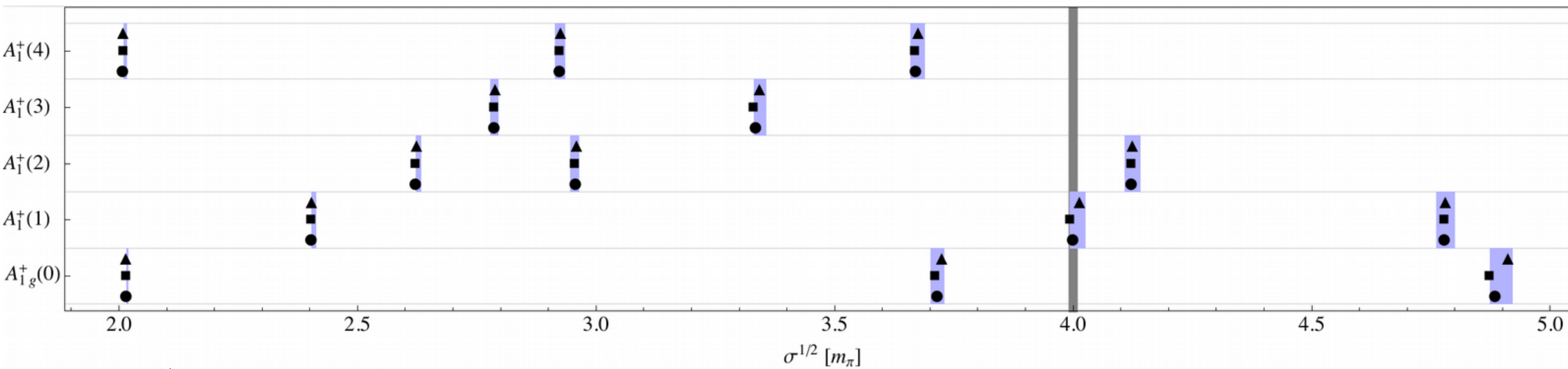
(uncertainties exaggerated)



→ We may consider this as *any* suitable 2-body Parametrization (like, e.g., K-matrix with conformal mapping)

IAM predictions: Different LECs

(D-wave set to zero)

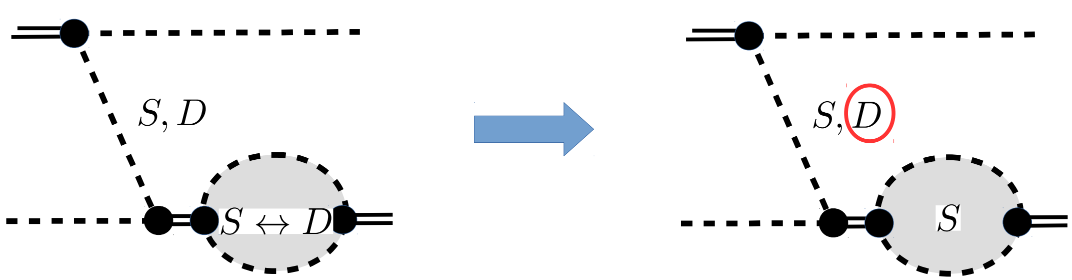


▲ Nebrada, Peláez, Rios (PRD88, 2013)

■ GW global (arXiv:1908.01847 [hep-lat])

● Gasser, Leutwyler (Annals Phys. 158, 1984)

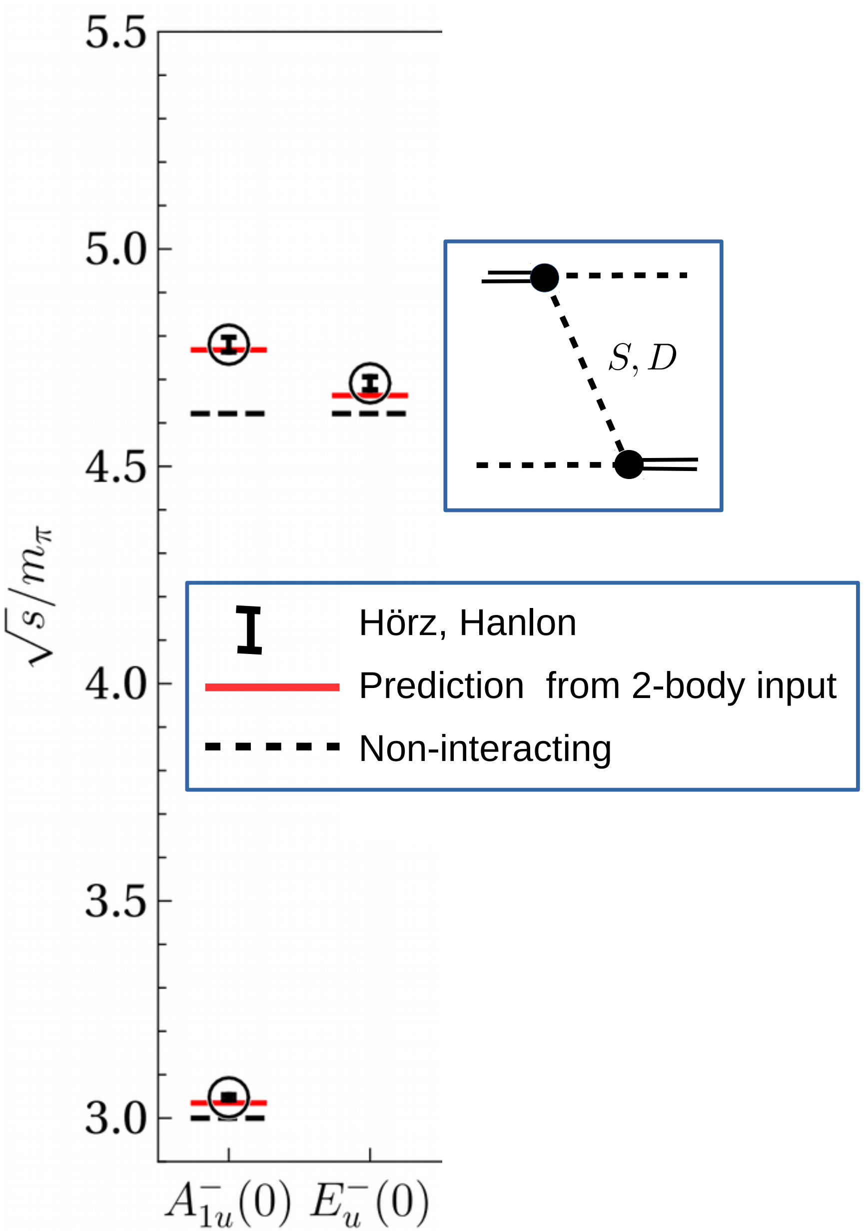
- Robust predictions of the 2-body spectrum irrespective of used LECs
- No sign of D-wave up to very high energies in irreps with S&D-wave mixing
- Ignore the vanishing $\pi^+\pi^+$ – D-wave, but keep the important π^+ – isobar D-wave



“in-flight transitions”

3-body Spectrum: Predictions (I)

Data: Hörz, Hanlon, arXiv:1905.04277 [hep-lat]



S **D** (lowest participating wave)

- S-wave prediction good at threshold (like for NPLQCD data)
- S-wave prediction good at high energies
  → **Energy dependence matched**
- No sign of 3-body force (like for NPLQCD data)
- D-wave prediction qualitatively good
  → **Relative* strength between S- and D-wave matched**
  → **Consequence that 3-body interaction dominated by exchange**
  → **Consequence of 3-body Unitarity**

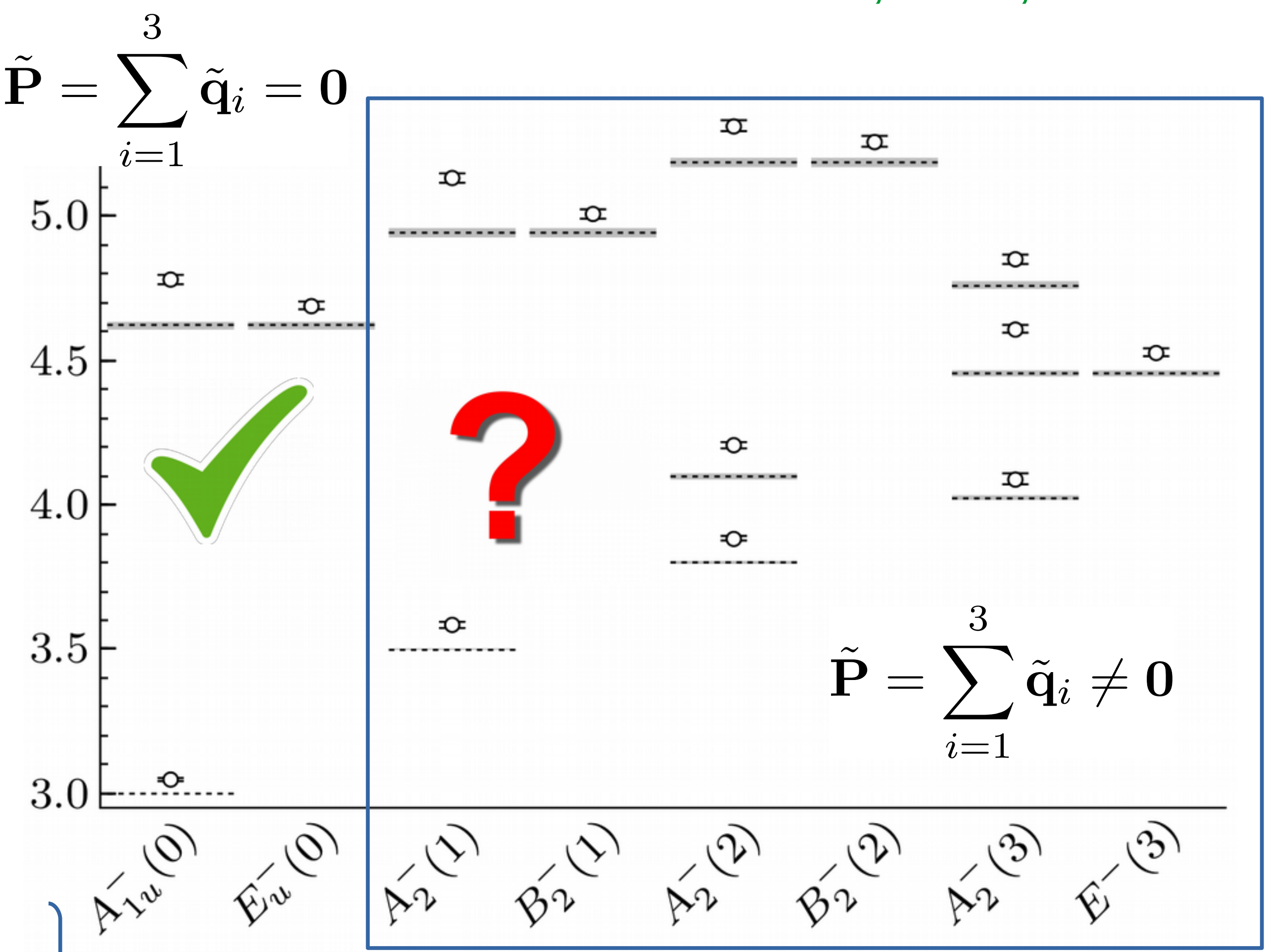
- **Three-body unitarity directly visible in the eigenvalue spectrum of lattice QCD**

*and absolute

Technical note: Projection technique for 3-body systems to irreps from M.D., Hammer, Mai, Pang, Rusetsky, Wu PRD97 (2018)

3-body spectrum: Moving frames

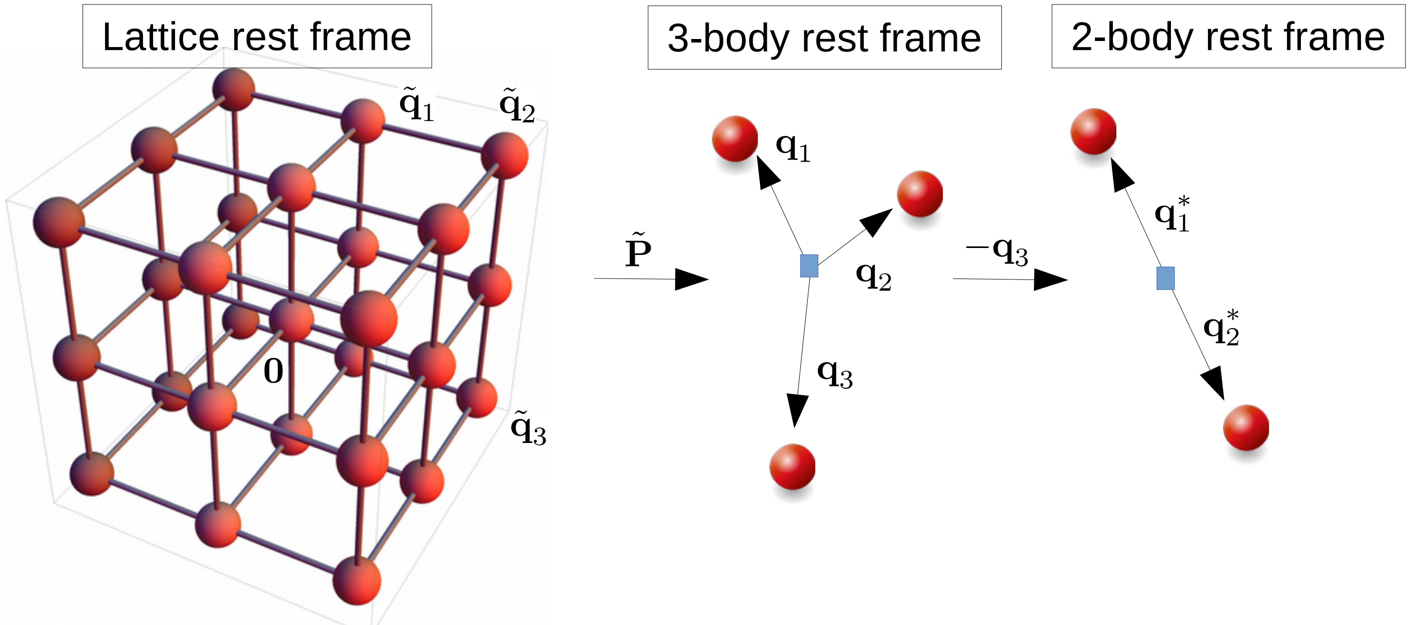
Data: Hörz, Hanlon, arXiv:1905.04277 [hep-lat]



Low participating isobar-π^+ waves: S, D, S&D, S&D, S&D, S&D, S&D, S&D

→ Need to develop a framework for moving 3-body systems!

Moving frames for 3-body systems



Usually: Explicit S- and D-wave projected parameterizations in coupled channels

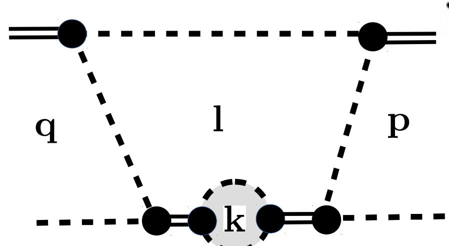
Here: Boost of unprojected 3-body amplitude. A-posteriori projections with suitable Clebsch-Gordan coefficients → Requires plane-wave solution of scattering

$$\tilde{\mathbf{P}} = \tilde{\mathbf{q}}_1 + \tilde{\mathbf{q}}_2 + \tilde{\mathbf{q}}_3 = \tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2 + \tilde{\mathbf{p}}_3$$

$$(L/2\pi)\tilde{\mathbf{P}} \in \{(0,0,1),\ (0,1,1),\ (1,1,1)\}$$

$$\mathbf{q} = \tilde{\mathbf{q}} + \left[\left(\frac{\tilde{P}^0}{\sqrt{s}} - 1\right)\frac{\tilde{\mathbf{q}}\tilde{\mathbf{P}}}{|\tilde{\mathbf{P}}^2|} - \frac{\tilde{q}^0}{\sqrt{s}}\right]\tilde{\mathbf{P}}$$

3-body summation:
$$\int \frac{d^3\mathbf{l}}{(2\pi)^3} g(\mathbf{l}) \to \int \frac{d^3\tilde{\mathbf{l}}}{(2\pi)^3} g(\mathbf{l}(\tilde{\mathbf{l}}))\, \tilde{J}(\tilde{\mathbf{l}}) \to \frac{1}{L^3}\sum_{\mathbf{n}} g(\mathbf{l}(\tilde{\mathbf{l}}))\, \tilde{J}(\tilde{\mathbf{l}})$$

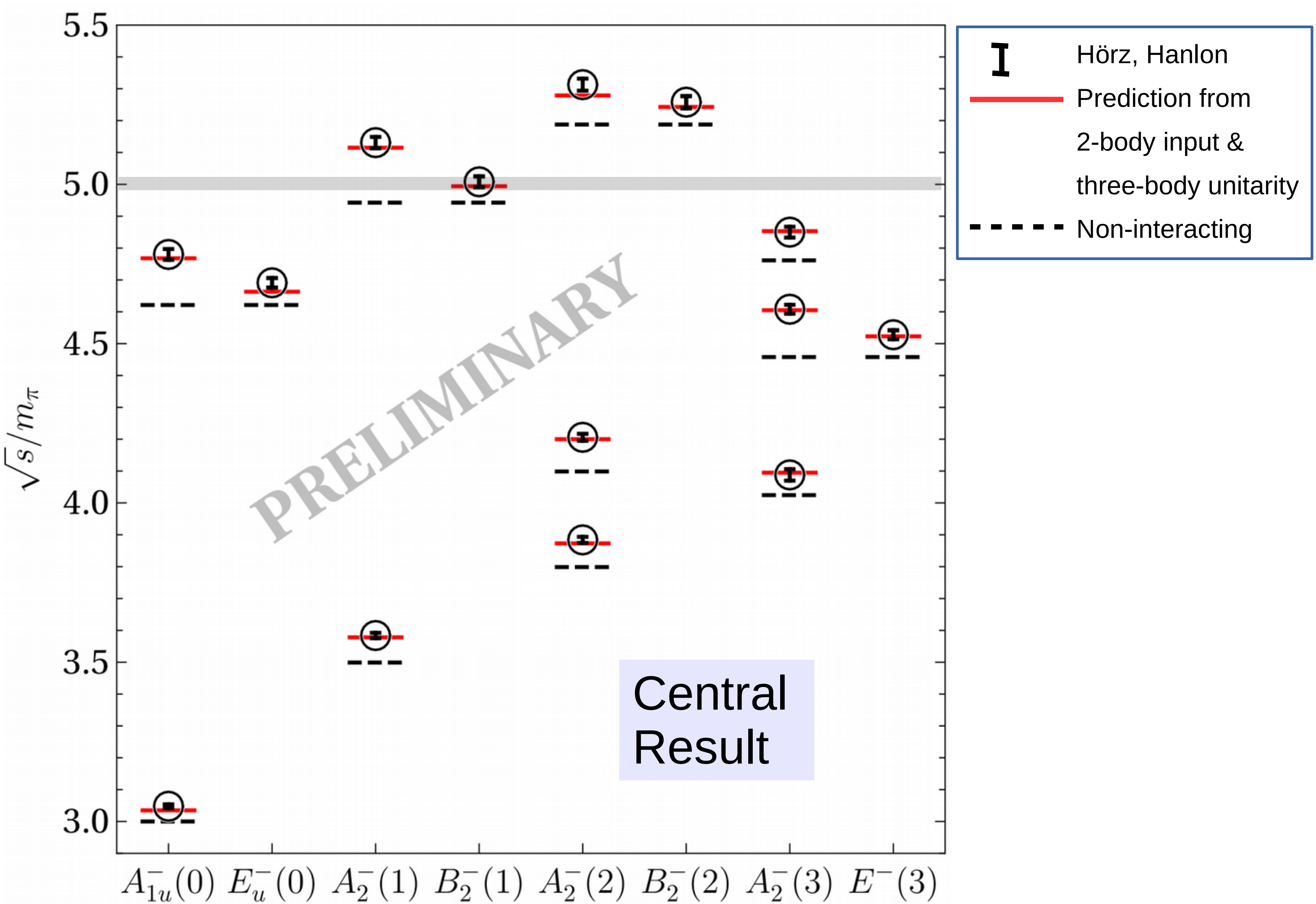


$\hat{T}(\mathbf{q}(\tilde{\mathbf{q}}), \mathbf{p}(\tilde{\mathbf{p}}))$ 3→3 boosted plane-wave amplitude

Poles → Eigenvalues

3-body spectrum: Complete Predictions

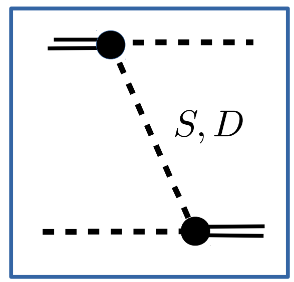
Data: Hörz, Hanlon, arXiv:1905.04277 [hep-lat]



Summary

3-body Unitarity

- 3-body unitarity dictates on-shell condition (exchange term & isobar propagator)
- On-shell condition dictates leading, power-law finite-volume effects
- "Bare-bone"infinite-volume extrapolation tool (in spirit of Lüscher equation)
- Optional: Pion-mass extrapolation



The $\pi^+\pi^+\pi^+$ System

- First application to physical 3-body system [PRL 2019]
- NPLQCD threshold data well predicted, excited levels predicted
- First explanation of excited 3-body levels (data from Hörz/Hanlon)
- Consequences of three-body unitarity directly visible in data (S vs. D waves)
- First development and application of moving frames for 3-body systems

OUTLOOK

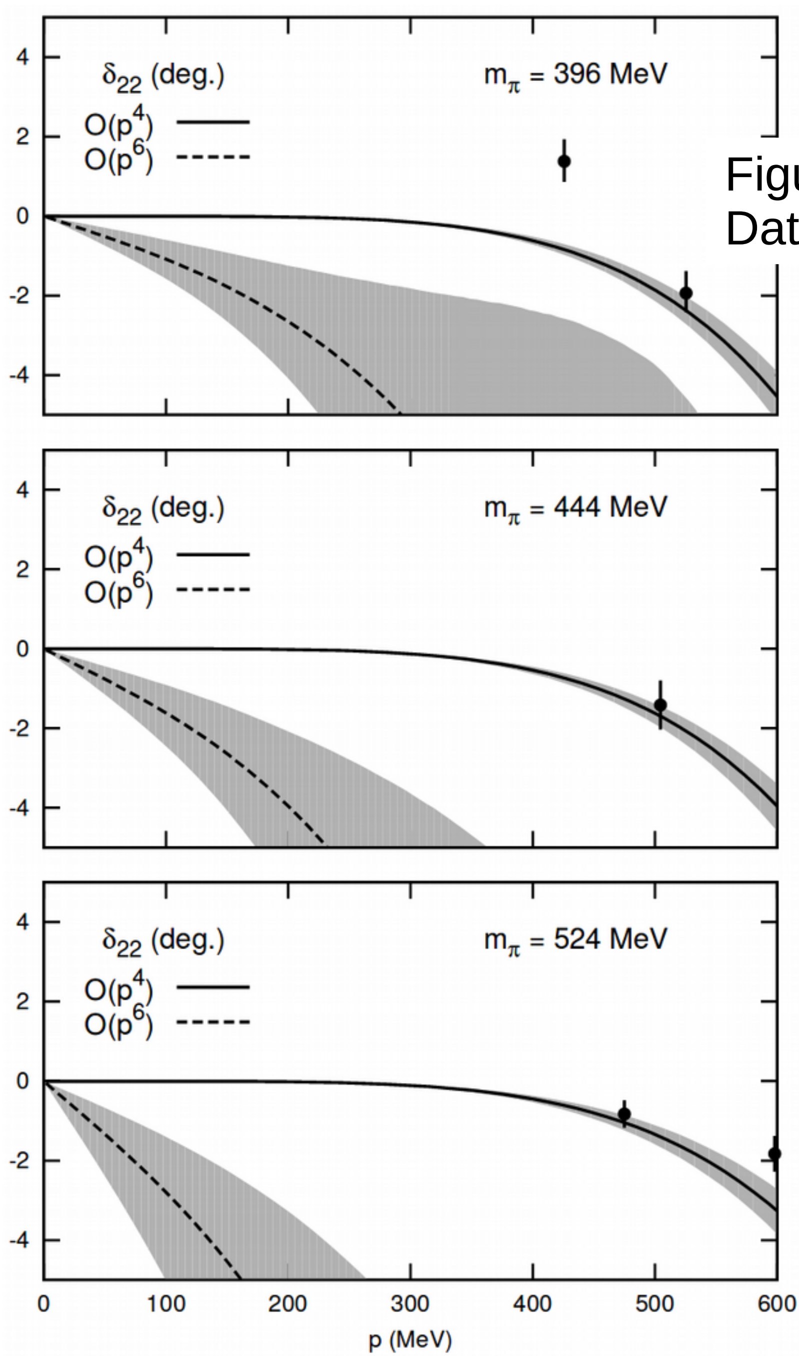
→ Implementation of spin isobars & multiple isobars

→ unequal masses

→ practical studies: $a_1(1260)$, Roper, exotics...

SPARES

I=2 D-wave at HadSpec Pion Masses



Perturbative O(p4), O(p^6) calculation

Figure from [Nebreda, Peláez, Ríos, PRD83 (2011)]
Data: [Dudek, Edwards, Peardon, Richards, Thomas, PRD83 (2011)]

Scattering amplitude – analytic expression

Recasting in on-shell 2→2 amplitudes + real 3-body forces

$$\langle q_1, q_2, q_3 | \hat{T}_c(s) | p_1, p_2, p_3 \rangle =$$

$$\frac{1}{3!} \sum_{n=1}^{3} \sum_{m=1}^{3} T_{22}(\sigma(q_n)) \langle q_n | T(s) | p_m \rangle T_{22}(\sigma(p_m))$$

External on-shell 2-body interaction

with

$$\langle q | T(s) | p \rangle = \langle q | C(s) | p \rangle + \frac{1}{m^2 - (P - p - q)^2 - i\epsilon}$$

$$- \int \frac{\mathrm{d}^3 \boldsymbol{\ell}}{(2\pi)^3} \frac{1}{2E_\ell} T_{22}(\sigma(\ell)) \Bigg(\langle \ell | C(s) | q \rangle$$

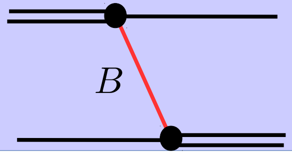
$$+ \frac{1}{m^2 - (P - p - \ell)^2 - i\epsilon} \Bigg) \langle \ell | T(s) | p \rangle$$

Real three-body force

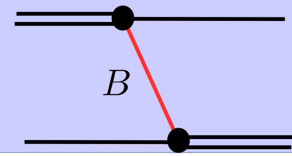
Exchange force

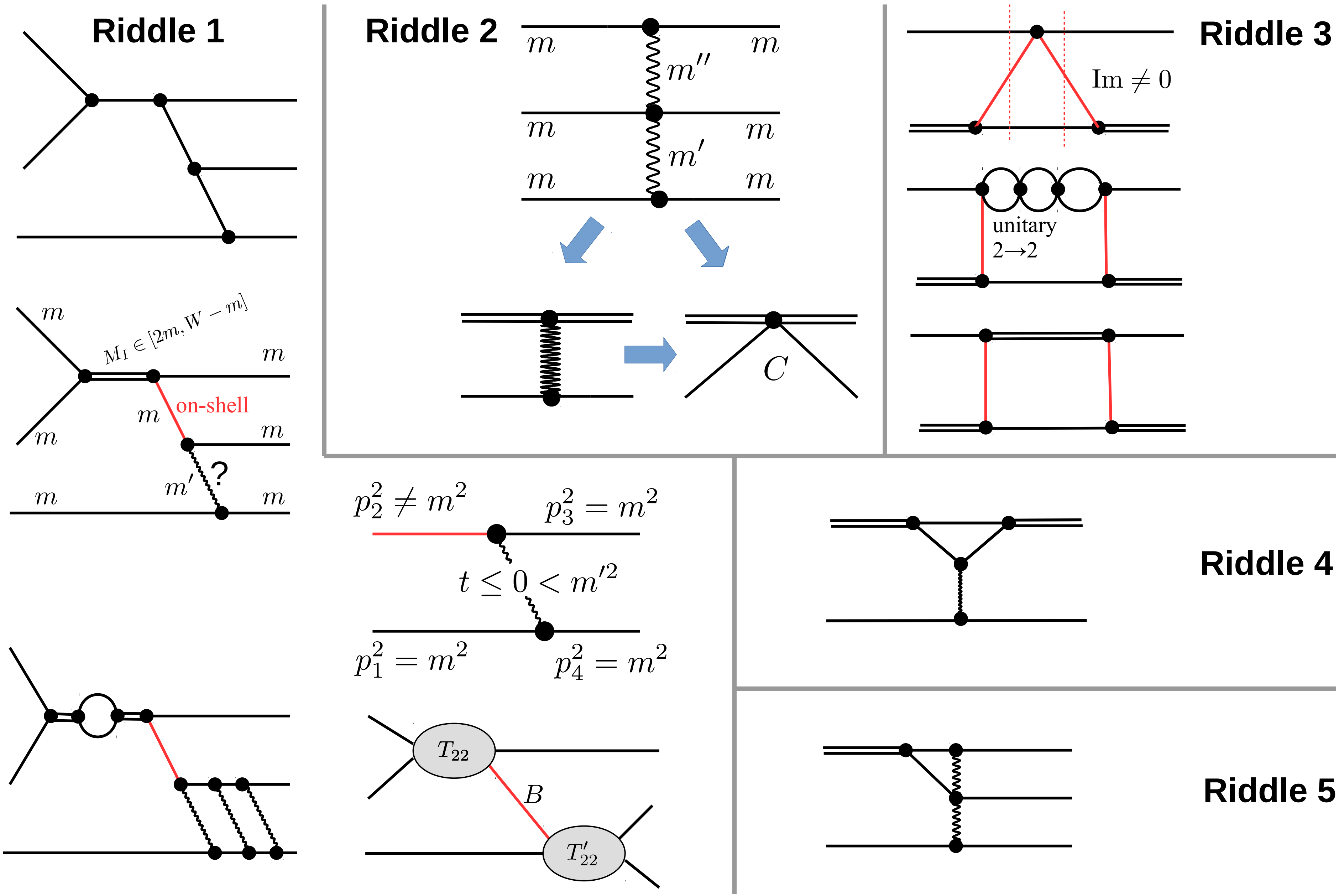
On-shell 2→ 2 interaction (even within integral, but without left-hand cuts)

The Power of Unitarity

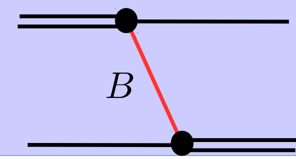
Question: Does  provide full imaginary part of all possible 3$\rightarrow$ 3 transitions?

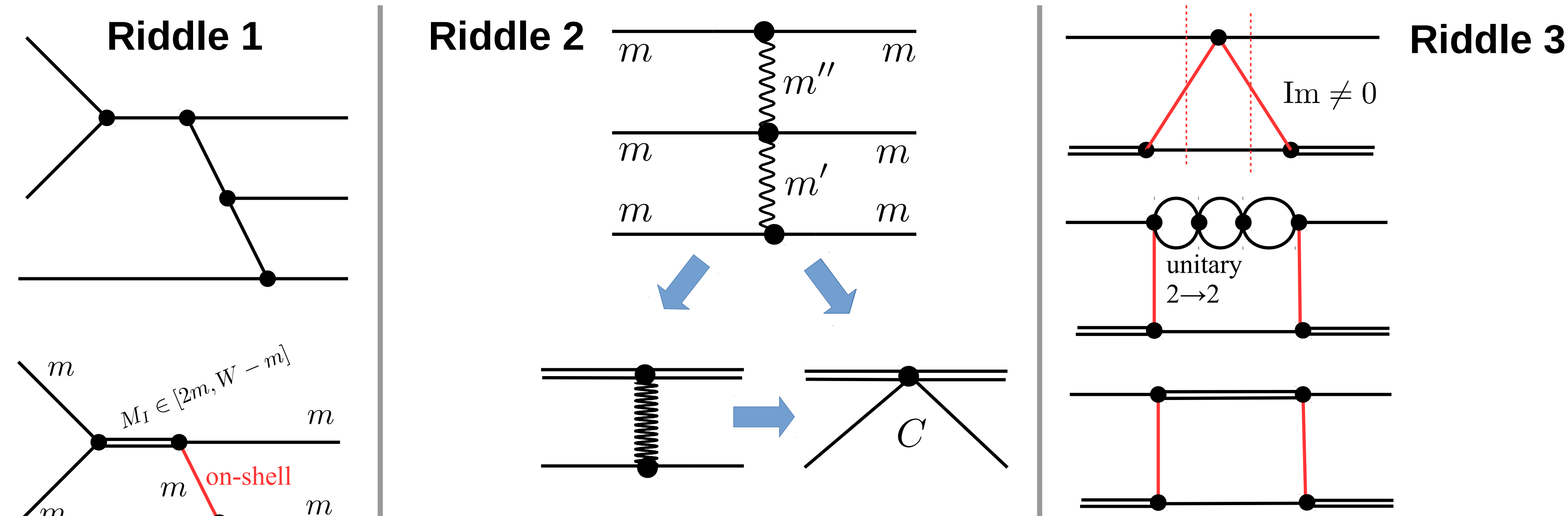
The Power of Unitarity

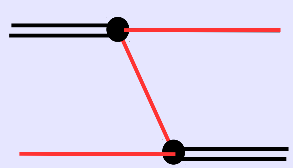
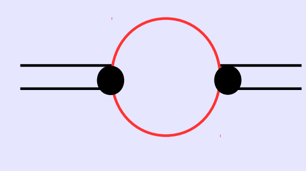
Question: Does  provide full imaginary part of all possible 3→ 3 transitions?



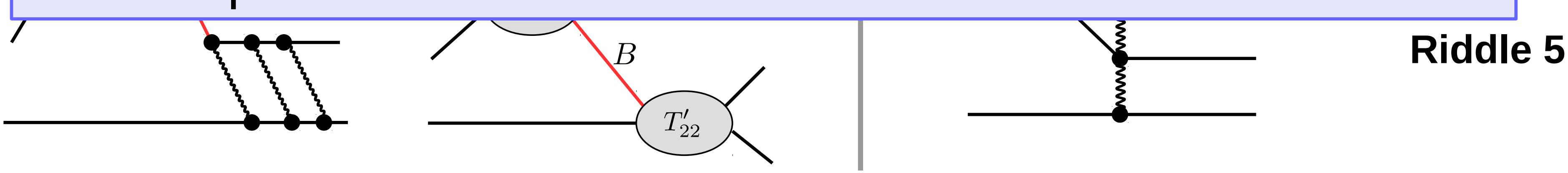
The Power of Unitarity

Question: Does  provide full imaginary part of all possible 3→ 3 transitions?



Answer: Yes.  and 

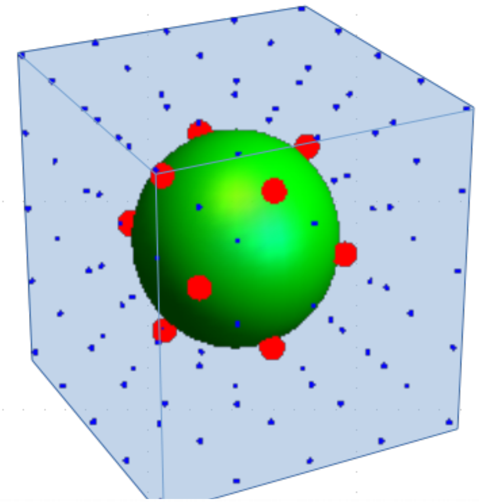
are the only on-shell configurations in physical region. Three-body unitarity avoids many artificial complications of diagrammatic expansions.



Projection to irreps

[M.D.,Hammer,Mai,Pang,Rusetsky,Wu(2018)]

- **Lüscher formalism relies on regular 2→ 2 potentials**
  - Now: manifestly singular interactions
  - Find generalization that projects also the interactions to the irreps of cubic symmetry, not only propagation
- **Separation of variables**
  - shells = sets of points related by O_h
  - Analogous to radial coordinate in infinite volume
- **Find the orthonormal basis for arbitrary functions defined on each point of a given shell.**



$$q_i = \frac{2\pi}{L} n_i, \quad n_i \in \mathbb{Z}, \quad i = 1,2,3$$

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3} \to \frac{1}{L^3} \sum_s \sum_{i=1}^{\vartheta(s)}$$

- **J (inf. volume) → irreps (finite volume):** $\Gamma \in \{A_1^\pm, A_2^\pm, E^\pm, T_1^\pm, T_2^\pm\}$
- **Partial wave projection (inf. Volume) → Irrep. projection (fin.)**

$$f(\mathbf{p}) = \sqrt{4\pi} \sum_{\ell m} Y_{\ell m}(\hat{p}) f_{\ell m}(p)$$

$$f_{\ell m}(p) = \frac{1}{\sqrt{4\pi}} \int d\Omega Y^*_{\ell m}(\hat{p}) f(\mathbf{p})$$

→

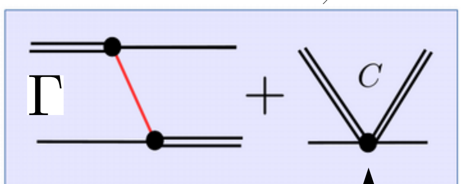
$$f^s(\hat{p}_j) = \sqrt{4\pi} \sum_{\Gamma\alpha} \sum_a f_a^{\Gamma\alpha s} \chi_a^{\Gamma\alpha s}(\hat{p}_j)$$

$$f_a^{\Gamma\alpha s} = \frac{\sqrt{4\pi}}{\vartheta(s)} \sum_{j=1}^{\vartheta(s)} f^s(\hat{p}_j) \chi_a^{\Gamma\alpha s}(\hat{p}_j)$$

(a is index u in quantization condition; Quantization condition has projection in incoming AND outgoing basis states with indices u, u')

Quantization Condition

$$\mathrm{Det}\left(\mathbf{B}_{\mathbf{uu'}}^{\mathbf{\Gamma ss'}}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right) = \mathbf{0}$$



W – total energy

s/s' - shell index

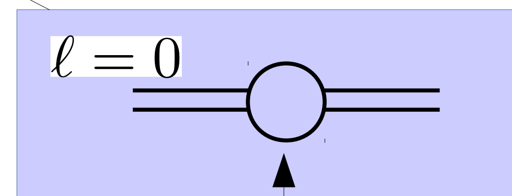
u/u' - basis index

ϑ – multiplicity

L – lattice volume

E_s – spect. energy

Determinant of (s,u) x (s',u') matrix at fixed W, Γ, L



Fix to 3→3 data

Fix to 2→2 data:

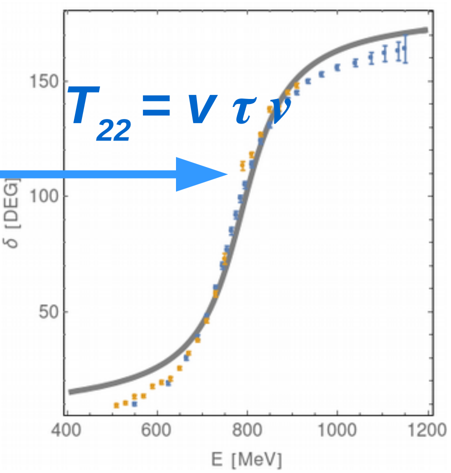
$T_{22} = v\,\tau\,v$

- Not a Lüscher-like equation ("left": infinite volume, "right": finite volume)
- Instead: Fix parameters to lattice eigenvalues
- With parameters fixed, evaluate infinite-volume amplitude
- Same workflow as in many 2-body coupled-channel fits (see, e.g., M.D., Meißner, Oset, Rusetsky, EPJA (2012))

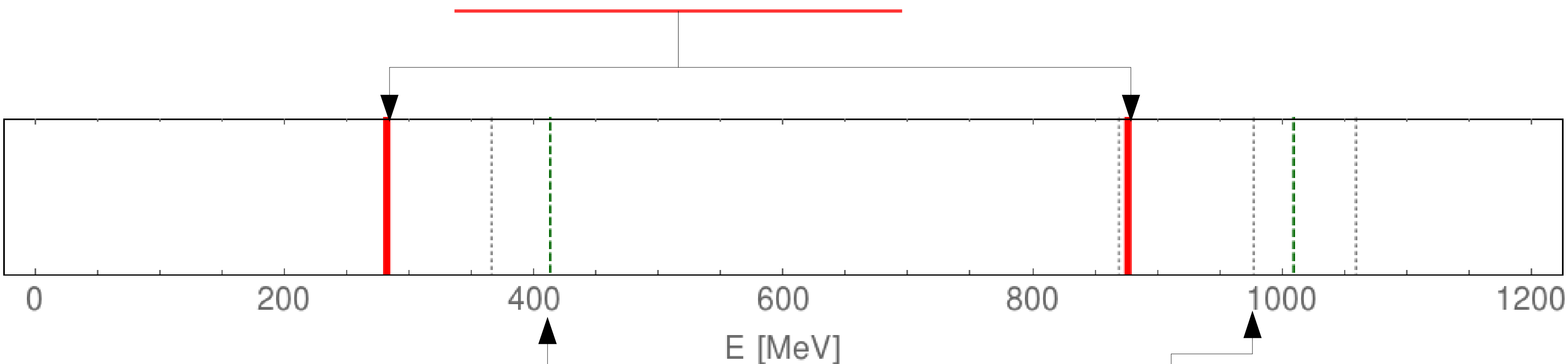
Numerical demonstration

[M. Mai, MD, EPJA 2017 [arXiv: 1709.08222]]

- Numerical demonstration of three-body finite volume formalism
- 3 particles in finite volume: ***m=138 MeV, L=3 fm***
- one S-wave isobar → two unknowns:
  - vertex(Isobar→2 stable particles)
  - subtraction constant (~mass)



- **Project to $\Gamma = A^{1+}$**
  - **→ prediction of 3body energy-eigenlevels (C=0)**



unphysical lvls cancel out (exact proof available)

Two-body scattering on lattice

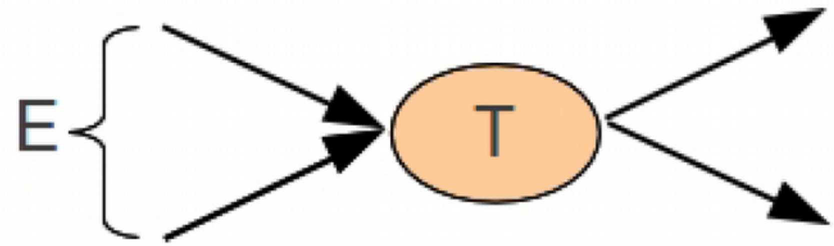
Input for 3-body

Two body scattering

In the infinite volume

- Unitarity of the scattering matrix S: $SS^\dagger = \mathbb{1}$ $\quad [S = \mathbb{1} - i\,\frac{p}{4\pi E}\,T]$.

$$\mathrm{Im}\, T^{-1}(E) = \sigma \equiv \frac{p}{8\pi\, E}$$



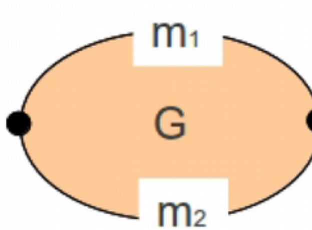
- $\rightarrow$ Generic (Lippman-Schwinger) equation for unitarizing the T-matrix:

$$T = V + V\,G\,T \qquad \mathrm{Im}\, G = -\sigma$$

V: (Pseudo)potential, σ: phase space.

- G: Green's function:

$$G = \int \frac{d^3\vec{q}}{(2\pi)^3}\,\frac{f(|\vec{q}|)}{E^2 - (\omega_1 + \omega_2)^2 + i\epsilon},$$
$$\omega_{1,2}^2 = m_{1,2}^2 + \vec{q}^{\;2}$$



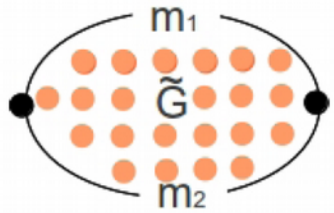
Discretization

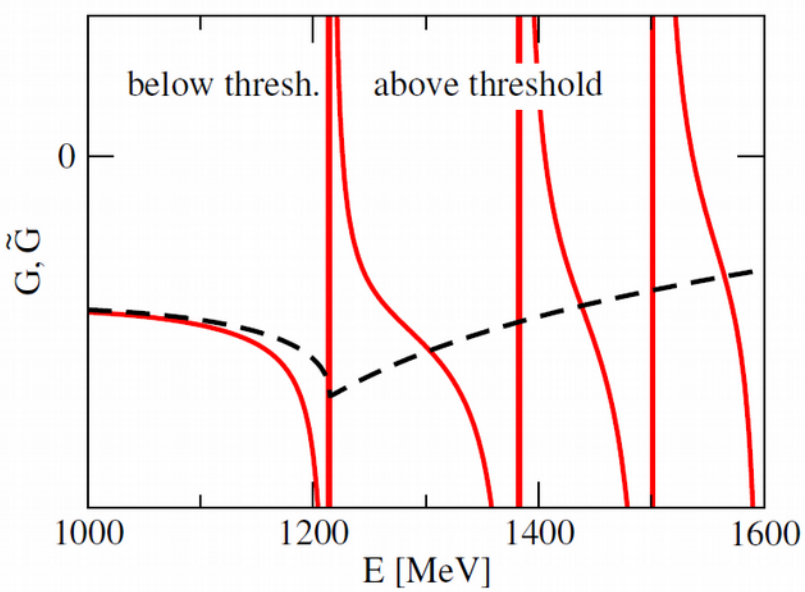
Discretized momenta in the finite volume with periodic boundary conditions

$$\Psi(\vec{x}) \stackrel{!}{=} \Psi(\vec{x} + \hat{\mathbf{e}}_i\, L) = \exp\left(i\, L\, q_i\right) \Psi(\vec{x}) \quad \Longrightarrow \quad q_i = \frac{2\pi}{L}\, n_i, \quad n_i \in \mathbb{Z}, \quad i = 1, 2, 3$$

$$\boxed{\int \frac{d^3\vec{q}}{(2\pi)^3}\, g(|\vec{q}\,|^2) \to \frac{1}{L^3} \sum_{\vec{n}} g(|\vec{q}|^2), \quad \vec{q} = \frac{2\pi}{L}\,\vec{n}, \quad \vec{n} \in \mathbb{Z}^3}$$

$$G \to \tilde{G} = \frac{1}{L^3} \sum_{\vec{q}} \frac{f(|\vec{q}|)}{E^2 - (\omega_1 + \omega_2)^2}$$





- $E > m_1 + m_2$: $\tilde{G}$ has poles at *free energies in the box,* $E = \omega_1 + \omega_2$
- $E < m_1 + m_2$: $\tilde{G} \to G$ exponentially with L (regular summation theorem).

Finite → infinite volume: the Lüscher equation

Warning: rather crude re-derivation

- Measured eigenvalues of the Hamiltonian (tower of *lattice levels* $E(L)$)
  → Poles of scattering equation $\tilde{T}$ in the finite volume → determines V:

$$\tilde{T} = (1 - V\tilde{G})^{-1} V \;\rightarrow\; V^{-1} - \tilde{G} \overset{!}{=} 0 \;\rightarrow\; V^{-1} = \tilde{G}$$

- The interaction V determines the T-matrix in the infinite volume limit:

$$T = \left(V^{-1} - G\right)^{-1} = \left(\tilde{G} - G\right)^{-1}$$

- Re-derivation of Lüscher's equation (T determines the phase shift δ):

$$\boxed{p \cot \delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\, G(E)\right)}$$

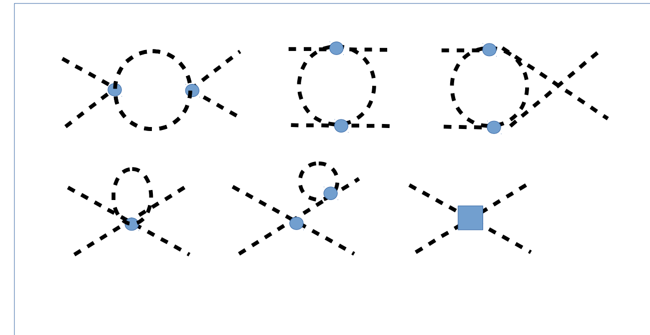
  - V and dependence on renormalization have disappeared (!)
  - p: c.m. momentum
  - E: scattering energy
  - $\tilde{G} - \mathrm{Re}\,G$: known kinematical function
    ($\simeq \mathcal{Z}_{00}$ up to exponentially suppressed contributions)
  - One phase at one energy.

Finite-volume & chiral extrapolations

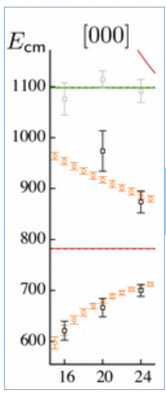
QCD calculations in finite volume

- unphysical pion mass
- (periodic) boundary conditions

  → discrete momenta & discrete spectrum

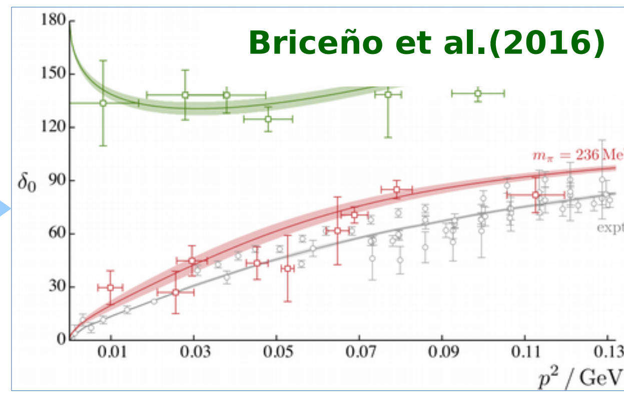


Recipe for 2 → 2 scattering (e.g. $I=J=0$ ππ scattering)



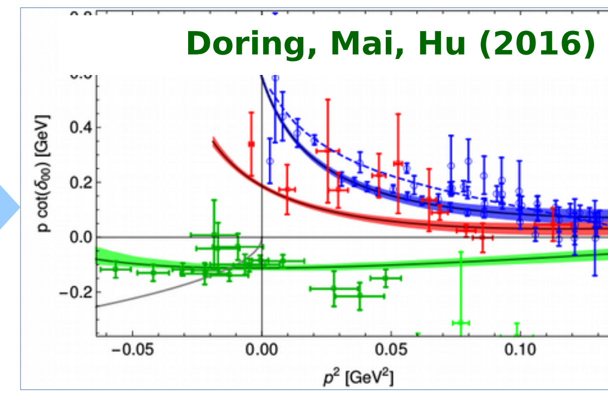
HSC(2016)

step 1



(This step can be skipped)

step 2



LÜSCHER(1986)

- 1 eigenenergy ↔ 1 phase-shift in infinite volume
- also with coupled channels

He et al. (2005)

Doring, Prelovsek, HSC

CHIRAL EXTRAPOLATIONS

- M_π dependence from NLO ChPT (IAM)

Gasser, Leutwyler(1981)

Dobado, Pelaez (1997)

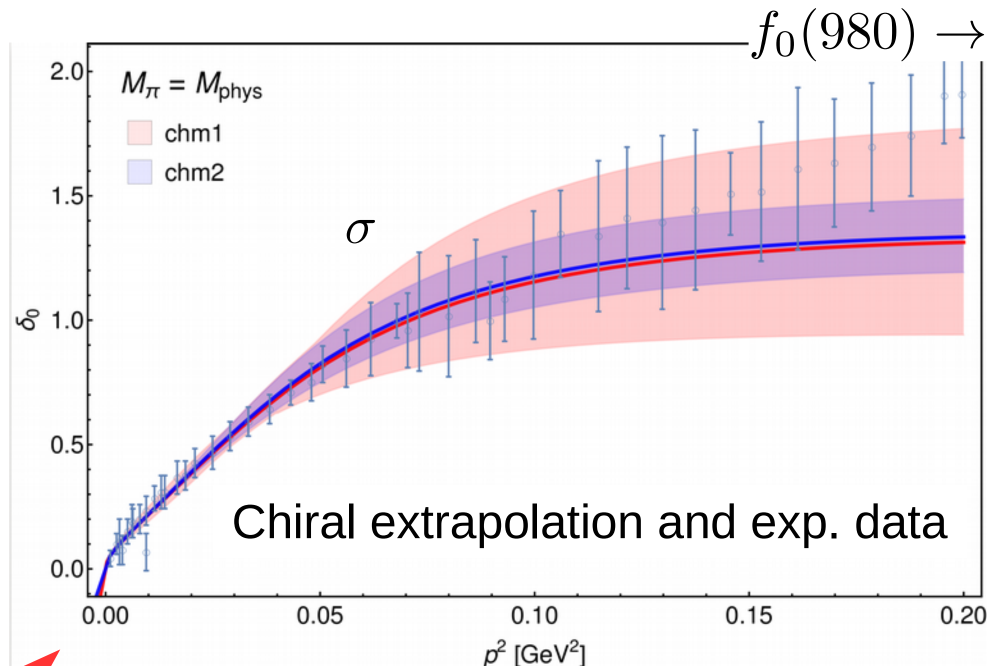
- Extrapolation in flavor

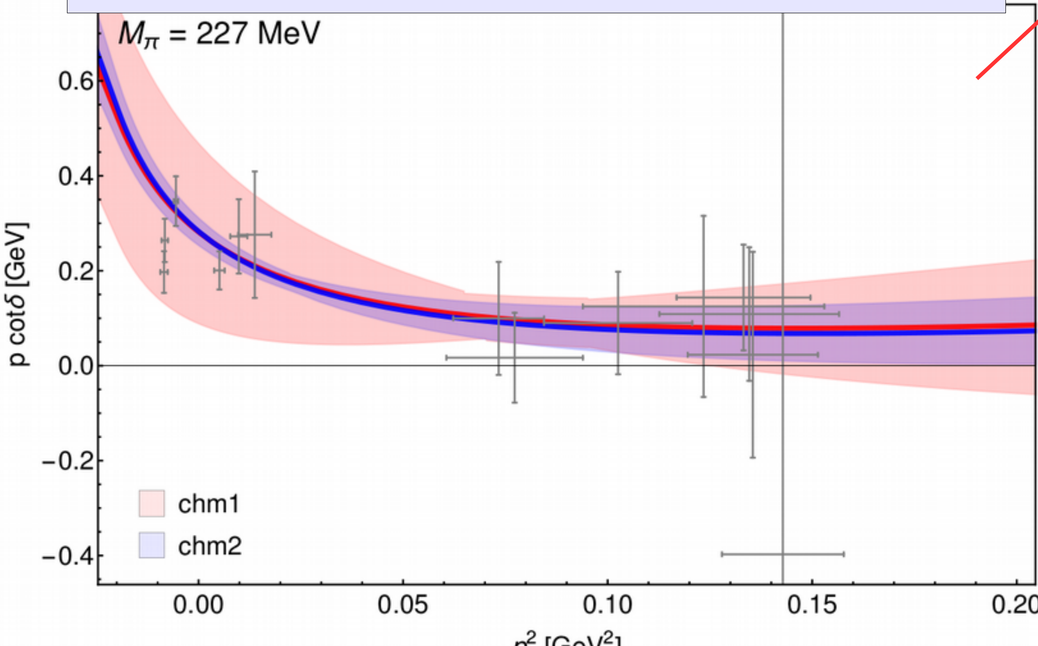
B. Hu, MD, R. Molina M. Mai et al. (2016)

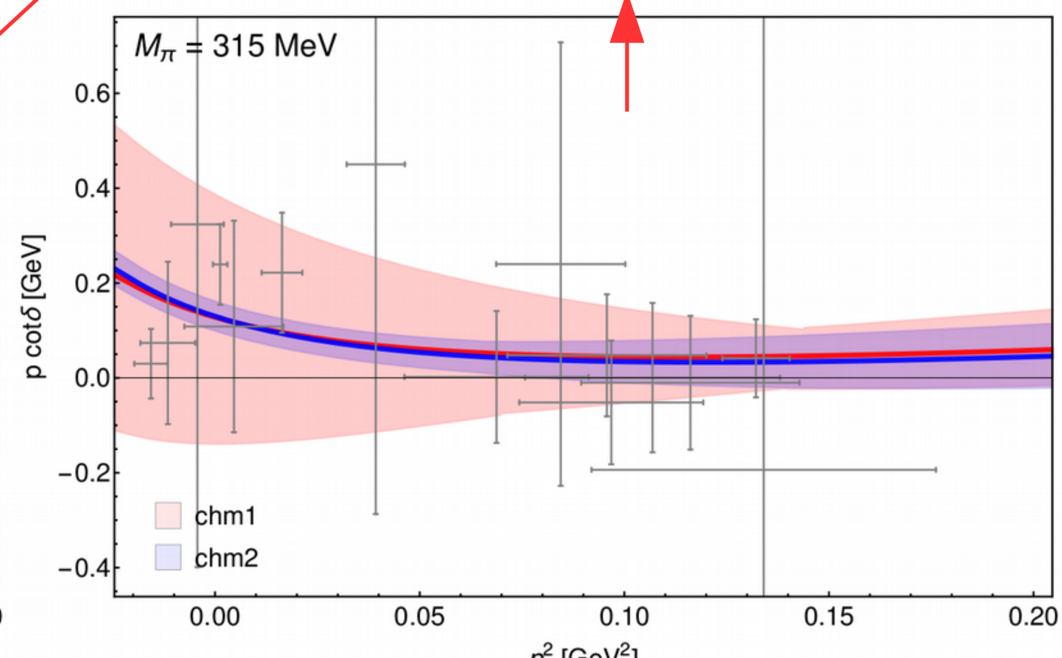
GWU lattice group: the isoscalar sector

[Guo, Alexandru, Molina, M.D., M. Mai, PRD (2018)]

- nHYP-smeared clover fermions with mass-degenerate quark flavors (N_f = 2)
- M_π=227 MeV and 315 MeV
- 3 elongated boxes
- Large variational basis including several meson-meson operators
- Moving frames
- Conformal mapping for σ pole extraction
- Unitarized Chiral Perturbation Theory fits for chiral extrapolation:
  chm1: $I = L = 0,\ M_\pi = 227, 315\,\mathrm{MeV}$
  chm2: $I = L = 0,\ 1,\ M_\pi = 227, 315\,\mathrm{MeV}$







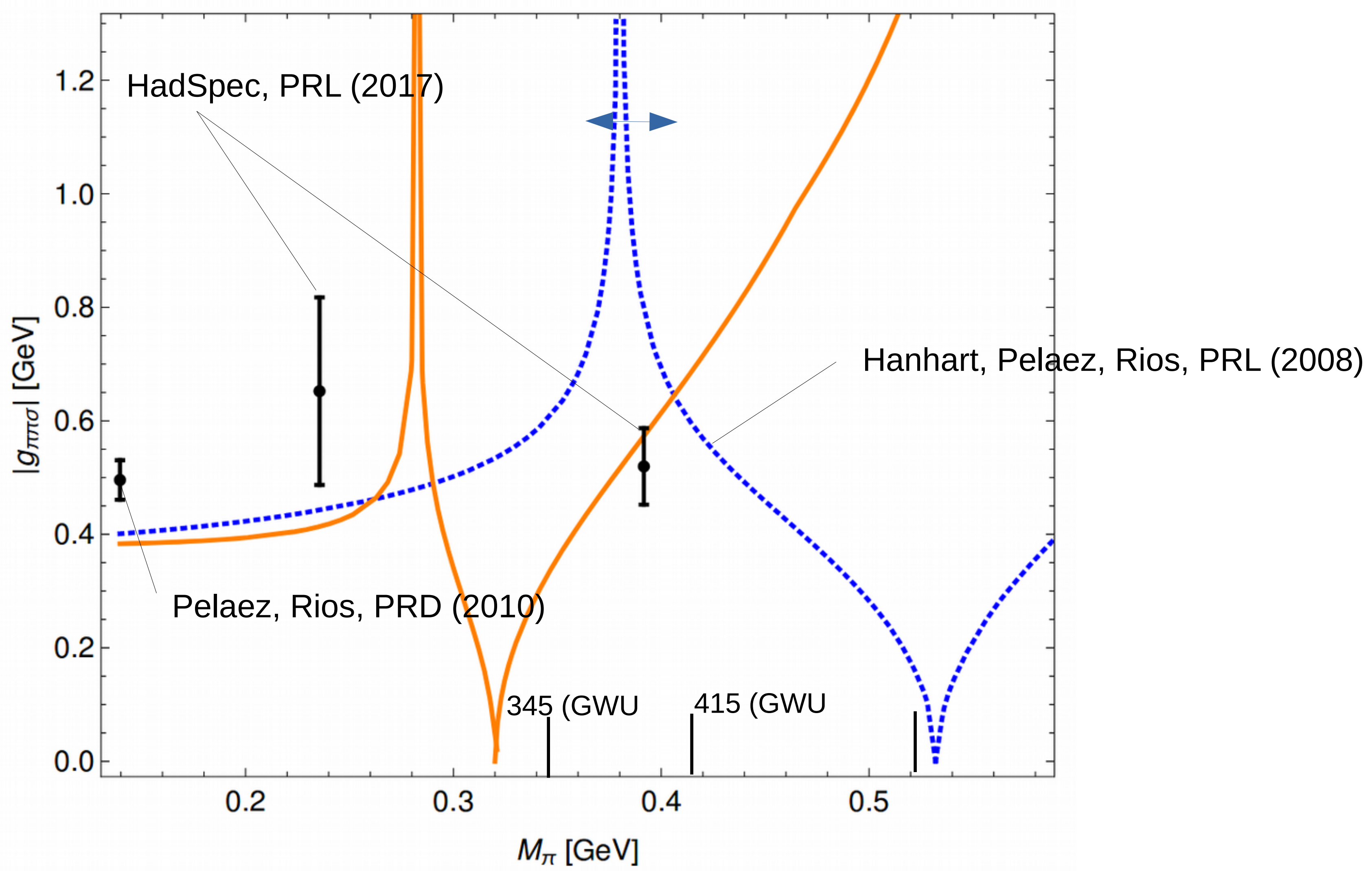
Chiral extrapolation of σ pole

		$M_\pi = 138$ MeV		
Parametrization	Fitted data	Re z^*	$-$ Im z^*	$\lvert g \rvert$
chm1	$\sigma_{227,315}$	440^{+60}_{-90}	240^{+20}_{-50}	$3.0^{+0.2}_{-0.6}$
chm2	σ_{227} ρ_{227}	430^{+20}_{-30}	250^{+30}_{-30}	$3.0^{+0.1}_{-0.1}$
chm2	σ_{315} ρ_{315}	460^{+10}_{-15}	210^{+40}_{-30}	$3.0^{+0.1}_{-0.1}$
chm2	$\sigma_{227,315}$ $\rho_{227,315}$	440^{+10}_{-16}	240^{+20}_{-20}	$3.0^{+0.0}_{-0.0}$
Ref. [1]	experimental	449^{+22}_{-16}	275^{+12}_{-12}	$3.5^{+0.3}_{-0.2}$

[1] J. R. Pelaez, Phys. Rept. **658**, 1 (2016), arXiv:1510.00653 [hep-ph].

[Consistent with conformal-mapping amplitude parametrization (model-independent, not shown)]

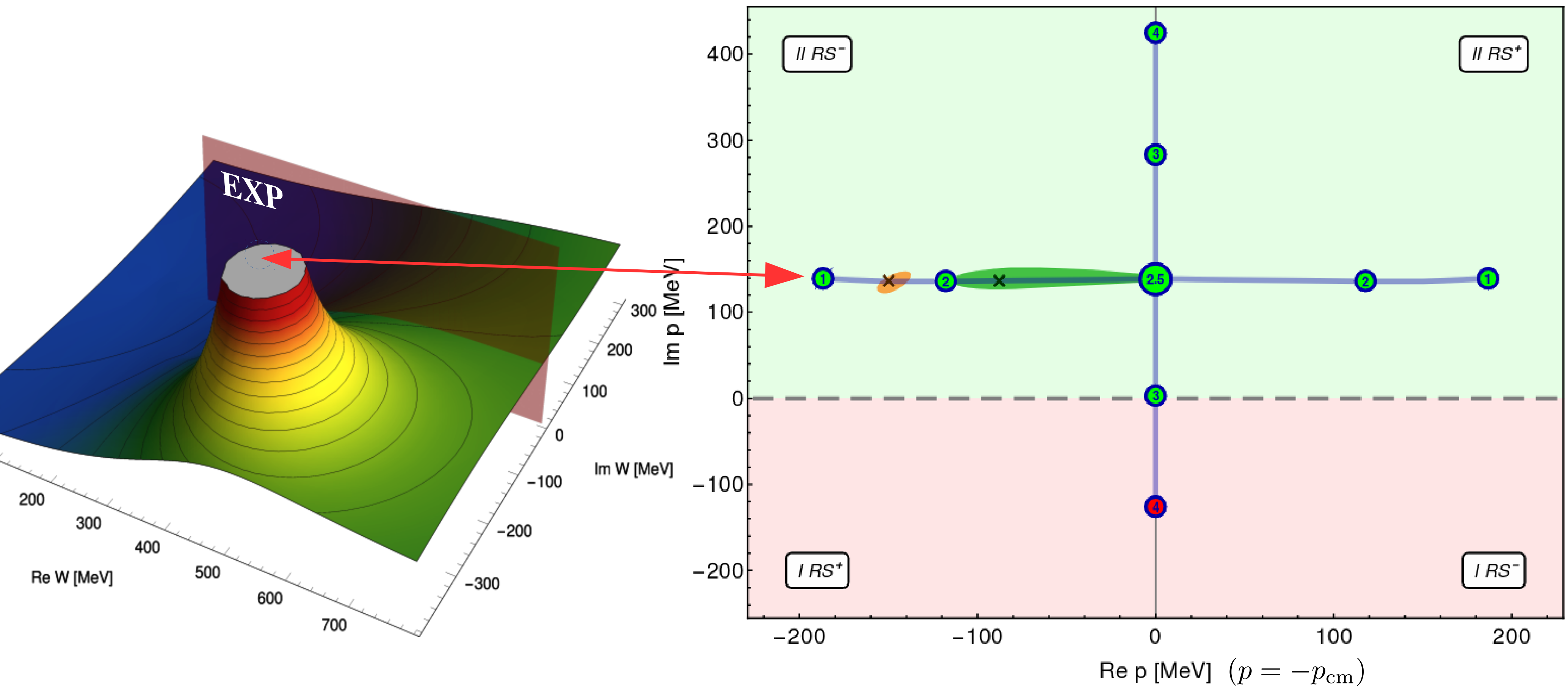
Residues



HadSpec, PRL (2017)
Hanhart, Pelaez, Rios, PRL (2008)
Pelaez, Rios, PRD (2010)
345 (GWU
415 (GWU
$|g_{\pi\pi\sigma}|$ [GeV]
M_π [GeV]
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0.2
0.3
0.4
0.5

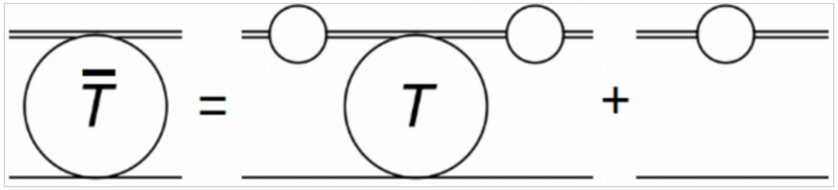
Pole trajectory

First prediction: Hanhart, Pealez, Rios, PRL (2008)



→ σ becomes a (virtual) bound state @ M_π = (345) 415 MeV

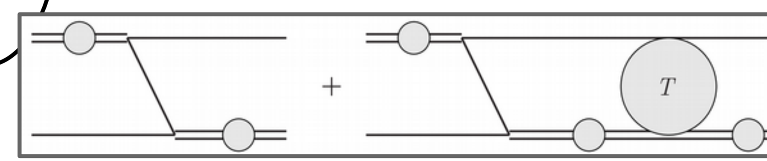
Cancellations



→ fin. vol. normalization of δ-distribution!

$$\bar{T}^{A_1^+}_{nm}(s) = \tau_n(s) T^{A_1^+}_{nm}(s) \tau_m(s) - 2E_n \tau_n(s) \frac{L^3}{\vartheta(n)} \delta_{nm}$$

$$T^{A_1^+}_{nm}(s) = B^{A_1^+}_{nm}(s) - \frac{1}{L^3} \sum_{x \in sets_8} \vartheta(x) B^{A_1^+}_{nx}(s) \frac{\tau_x(s)}{2E_x} T^{A_1^+}_{xm}(s)$$



$B^{A_1^+}$ singular at $W^+ = E_m + E_n + E(\boldsymbol{q}_{nj} + \boldsymbol{p}_{mi})$

τ_m^{-1} singular at $W^{\pm\pm} = E_m \pm E((2\pi/L)\boldsymbol{y}) \pm E((2\pi/L)\boldsymbol{y} + \boldsymbol{p}_{mi})$ for $\boldsymbol{y} \in \mathbb{Z}^3$

– when isobar-momenta are discretized in the 3-body cms momenta

$$\tau = \sigma(k) - M_0^2 - \frac{1}{(2\pi)^3} \int d^3\boldsymbol{\ell} \frac{\lambda^2}{2E_\ell(\sigma(k) - 4E_\ell^2 + i\epsilon)}$$

Also: all 2nd order singularities in determinant cancel → All consequence of Manifest three-body unitarity