Novel freed isobar analysis of D-meson decays

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PWA11/ATHOS6

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Rio de Janeiro, Brazil

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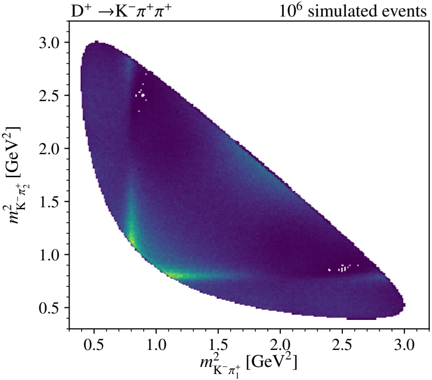
September 5th 2019

Partial-Wave Analysis (PWA)

Some basics

- Understand the processes leading to the final state



Inspired by CLEO:
Phys. Rev. **D78** 052001 (2008).

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  Describe the complex-valued amplitude of the process:

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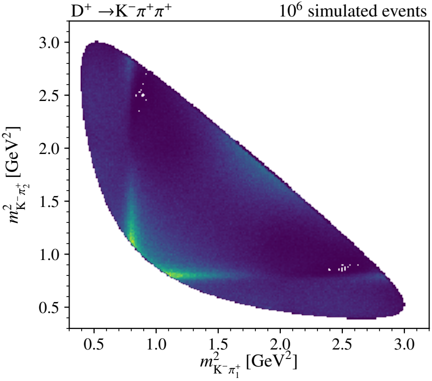

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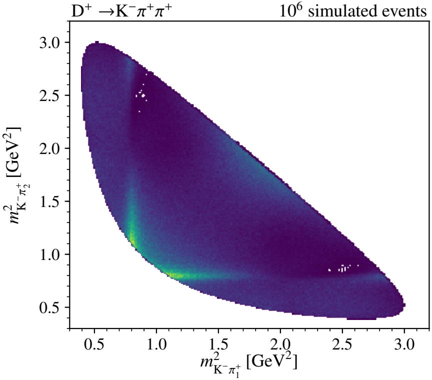

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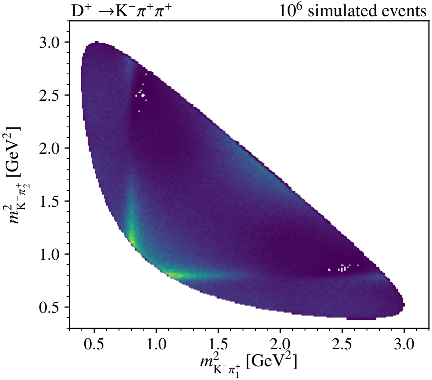
- Goal: Learn about the amplitude
- Fit intensity distribution to the data (extended unbinned log-likelihood)



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Data-set

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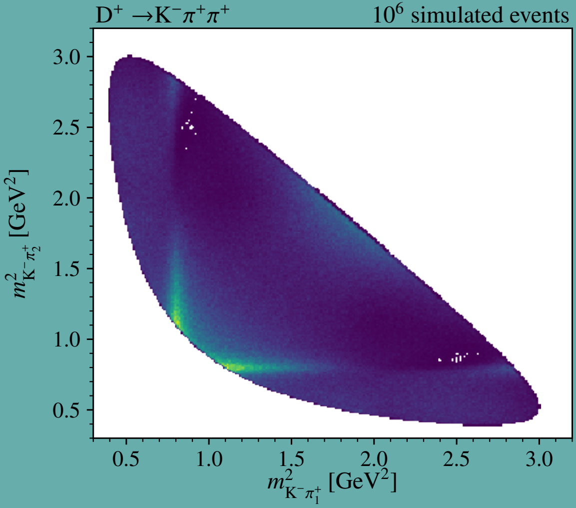

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  - Have to be known beforehand

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Motivation

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- Effects neglected or falsely attributed
  - “Leakage”

Freed-isobar method
Step-like isobar amplitudes

- Total intensity as function of phase-space variables $\vec{\tau}$:

$$\mathcal{I}(\vec{\tau}) = \left| \sum_{i}^{\text{waves}} \mathcal{T}_i \left[\psi_i\left(\vec{\tau}\right) \Delta_i\left(m_{\text{isob}}\right) + \text{Bose sym.} \right] \right|^2$$

Fit parameters: Production amplitudes $\mathcal{T}_i$

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- Fixed isobar amplitudes $\rightarrow$ Sets of $m_{\pi^-\pi^+}$ bins: (**MIPWA**)

$$\Delta_i(m_{\text{isob}}) \rightarrow \sum_{\text{bins}} \mathscr{T}_i^{\text{bin}} \Delta_i^{\text{bin}}(m_{\text{isob}}) \equiv [\pi\pi]_{J^{PC}}$$

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- Each $m_{\pi^-\pi^+}$ bin behaves like an independent partial wave with $\mathcal{T}_i^{\text{bin}} = \mathcal{T}_i \mathscr{T}_i^{\text{bin}}$:

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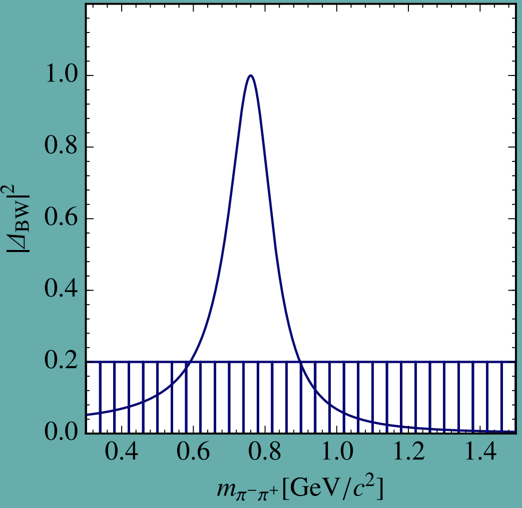

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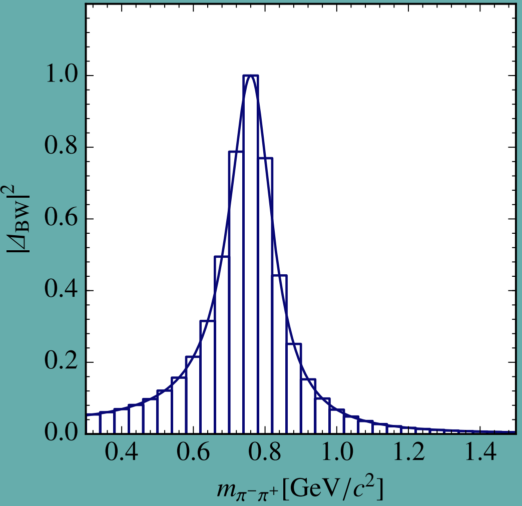

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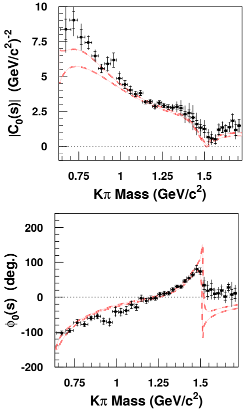

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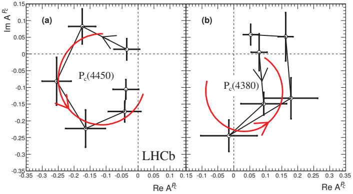

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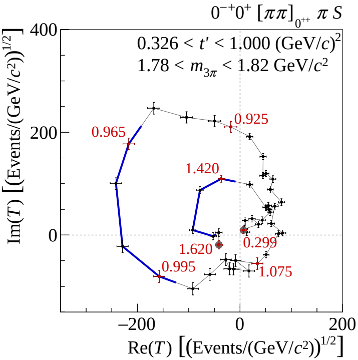

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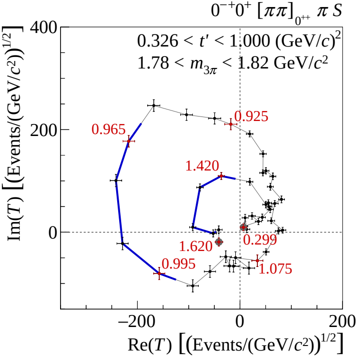

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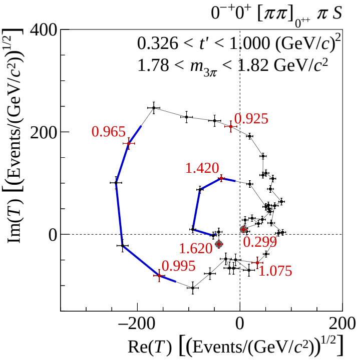

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- Fully freed fit?
- Simply replace all isobars and fit?

$[\pi\pi]_S$: $f_0(\ldots)$ $[\pi\pi]_P$: $\rho(770)$ $[\pi\pi]_D$: $f_2(1270)$
$[K\pi]_S$: κ, $K_0^*(1410)$ $[K\pi]_P$: $K^*(\ldots)$ $[K\pi]_D$: $K_2^*(1430)$

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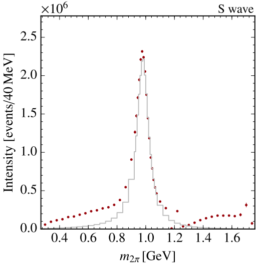

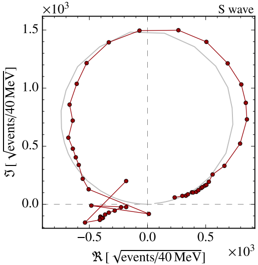

D meson decays

$D \to 3\pi$

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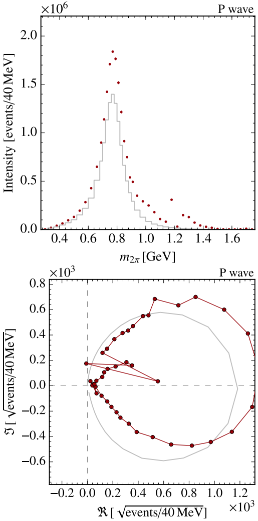

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FK, D. Greenwald, D. Ryabchikov, B. Grube, S. Paul, PRD **97** (2018) 114008

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$$\underbrace{\psi_{0^{++}}\left(\vec{\tau}\right)\Delta^0_{0^{++}}\left(m_{12}\right) + \psi_{1^{--}}\left(\vec{\tau}\right)\Delta^0_{1^{--}}\left(m_{12}\right)}_{=2\mathcal{C}\left(m_{23}^2-m_{12}^2\right)} + \underbrace{\text{Bose symm.}}_{+2\mathcal{C}\left(m_{12}^2-m_{23}^2\right)} = 0$$

at **every point** $\vec{\tau}$ in phase space (arbitrary m_{12} and m_{23})

- $\psi_{0^{++}}\left(\vec{\tau}\right) = 1; \quad \psi_{1^{--}}\left(\vec{\tau}\right) = \left(m_{12}^2 + 2m_{23}^2 - m_{3\pi}^2 - 3m_\pi^2\right)/4$

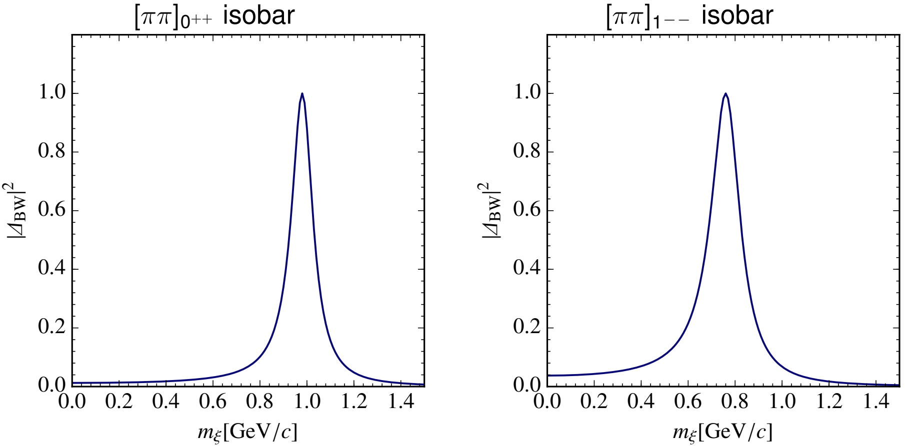
$$\Rightarrow \Delta^0_{0^{++}}\left(m_{12}\right) = \mathcal{C}\left(m_{3\pi}^2 + 3m_\pi^2 - 3m_{12}^2\right); \quad \Delta^0_{1^{--}}\left(m_{12}\right) = 4\mathcal{C}$$

FK, D. Greenwald, D. Ryabchikov, B. Grube, S. Paul, PRD **97** (2018) 114008

Zero mode in the $J^{PC} = 0^{-+}$ waves
Effects on dynamic isobar amplitudes

$$\Delta_{\text{BW}}\left(m_{\pi^-\pi^+}\right) + \mathcal{C}\Delta^0\left(m_{\pi^-\pi^+}\right)$$

$$\mathcal{C} = 0.00 + 0.00i$$



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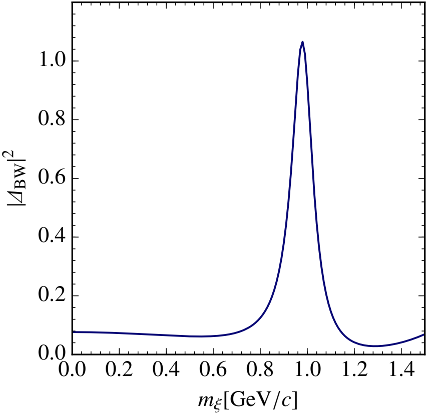
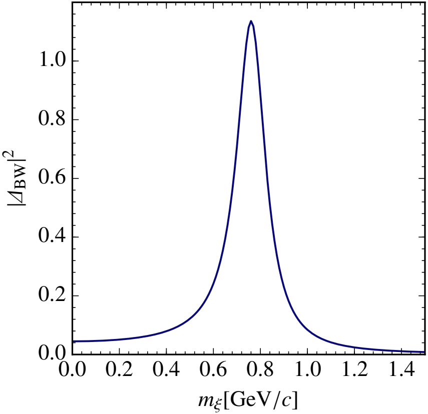


$[\pi\pi]_{1^{--}}$ isobar



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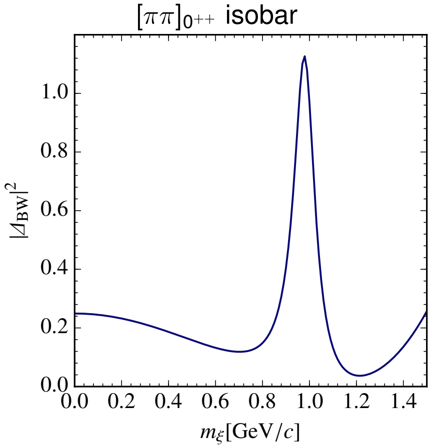

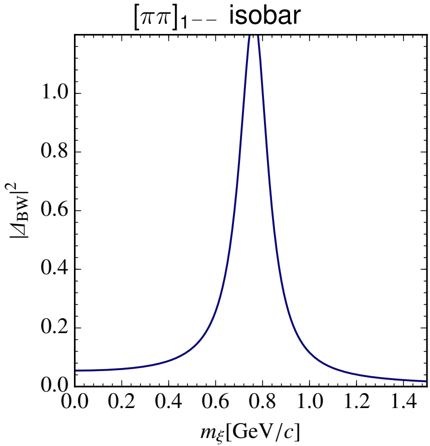

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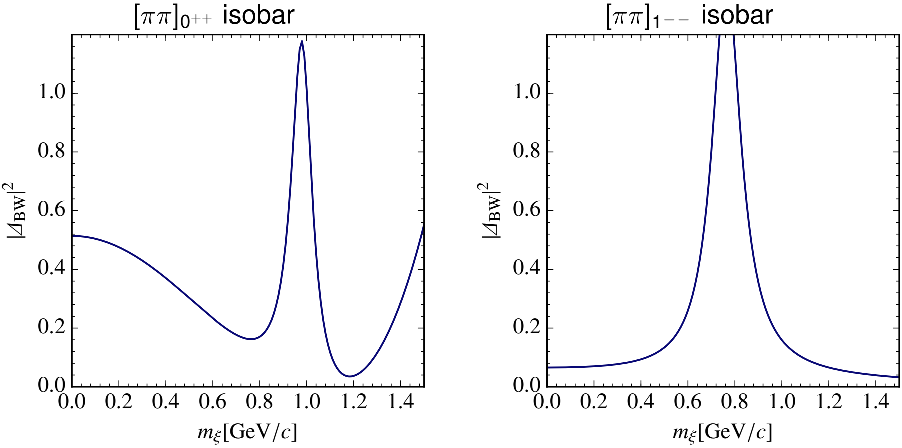

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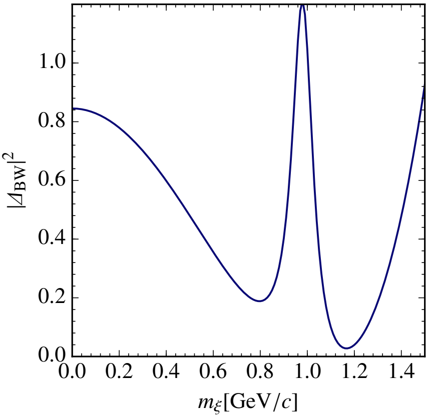

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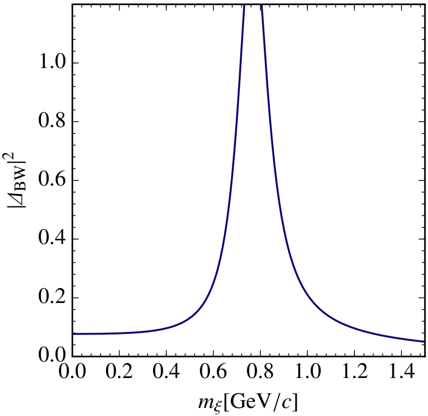

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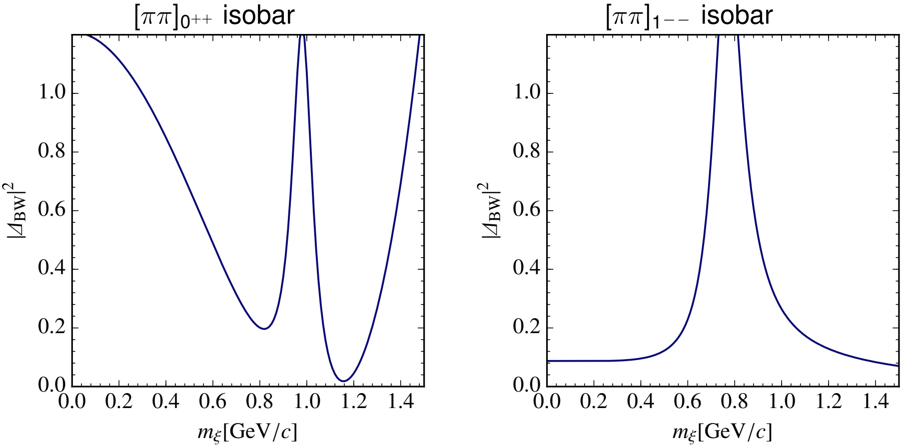

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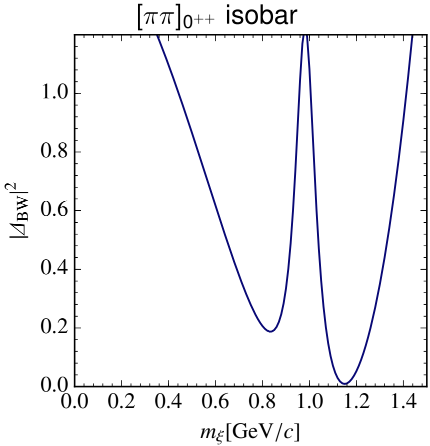

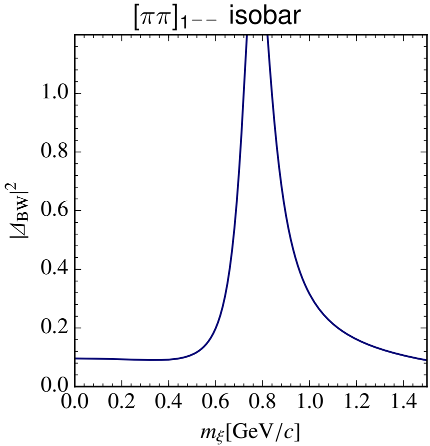

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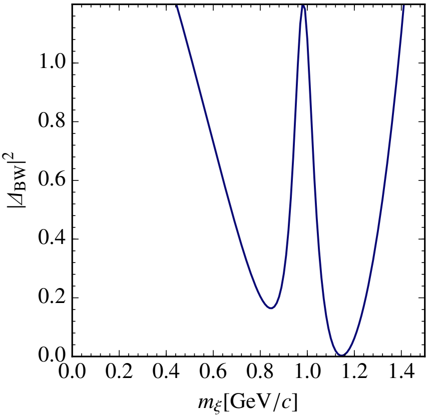
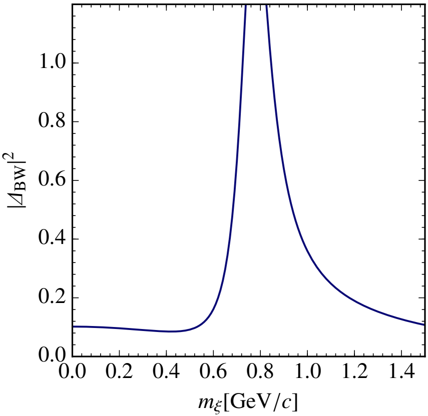


$[\pi\pi]_{1^{--}}$ isobar



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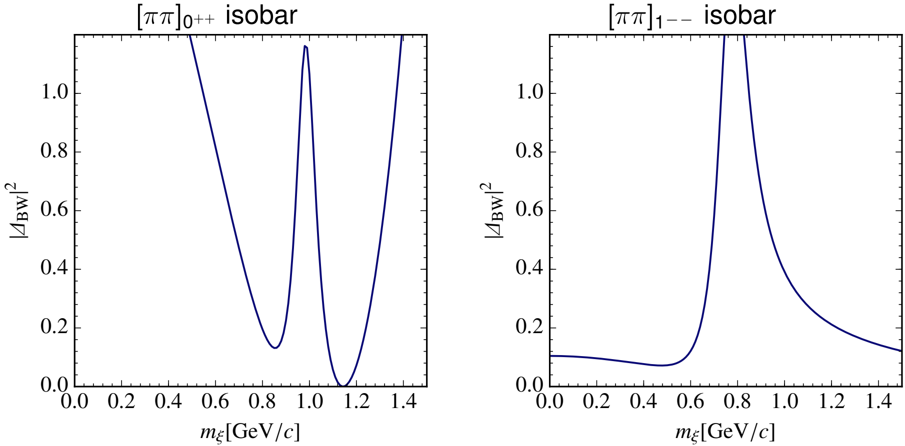

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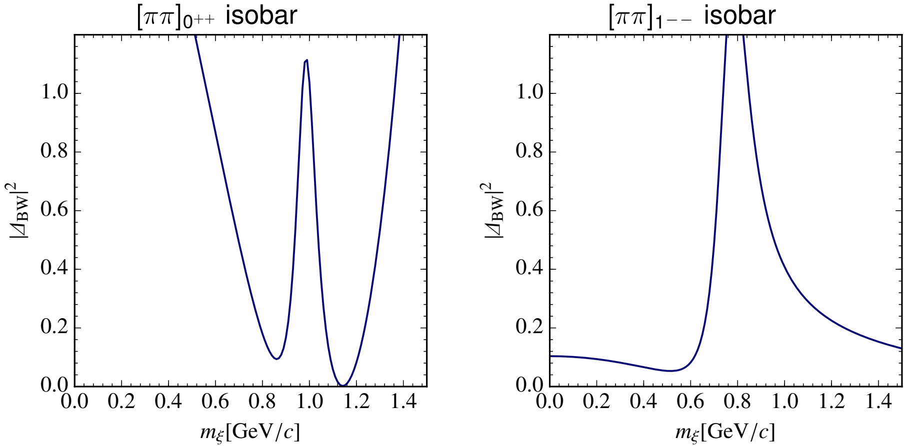

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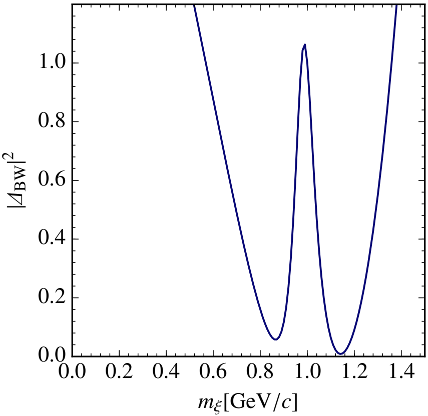

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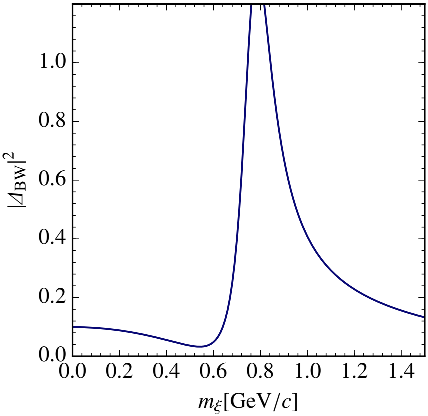

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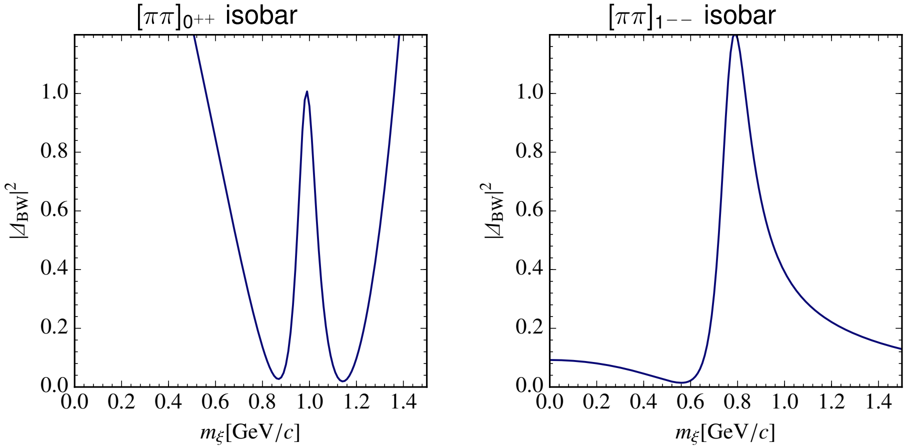

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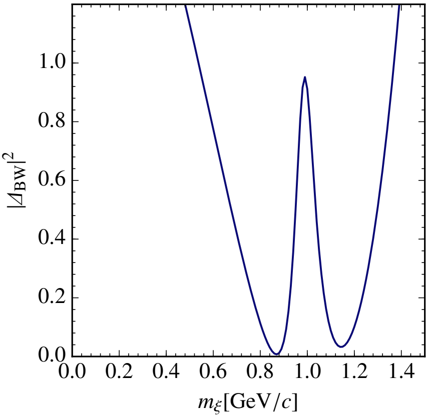

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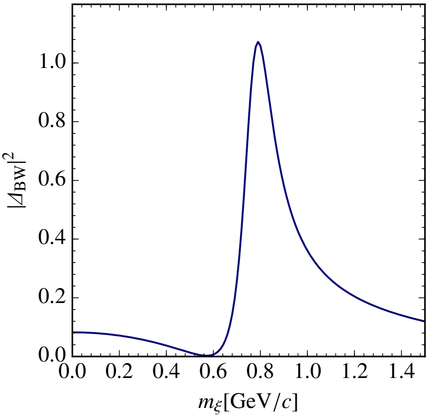

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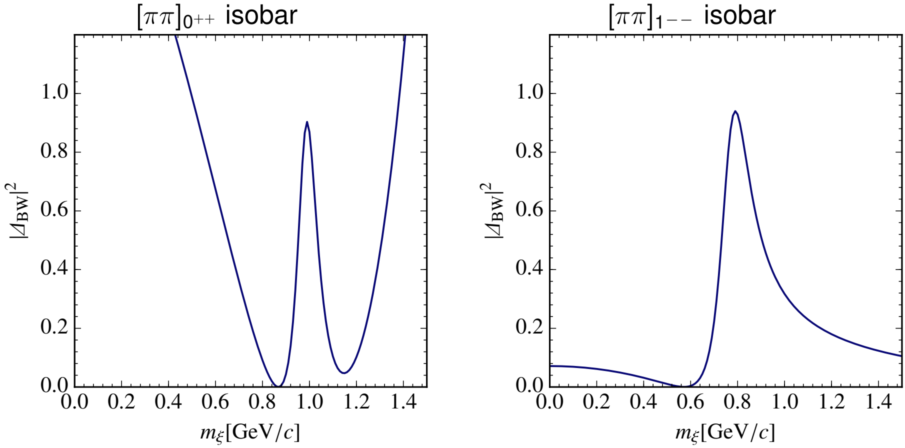

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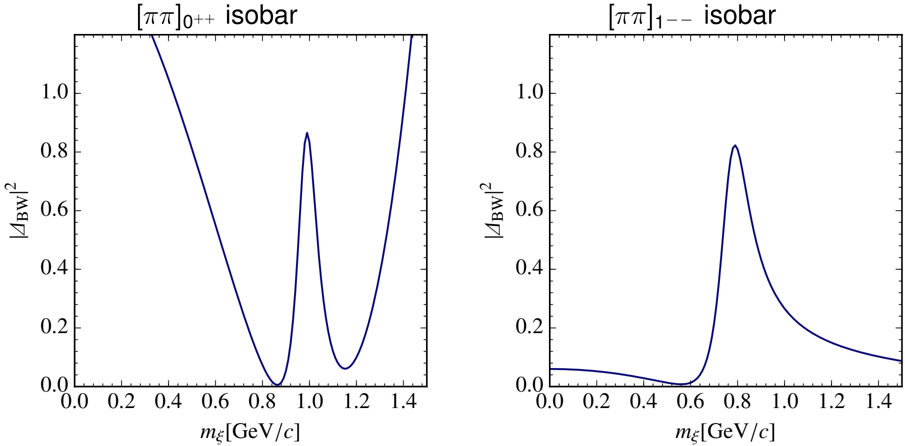

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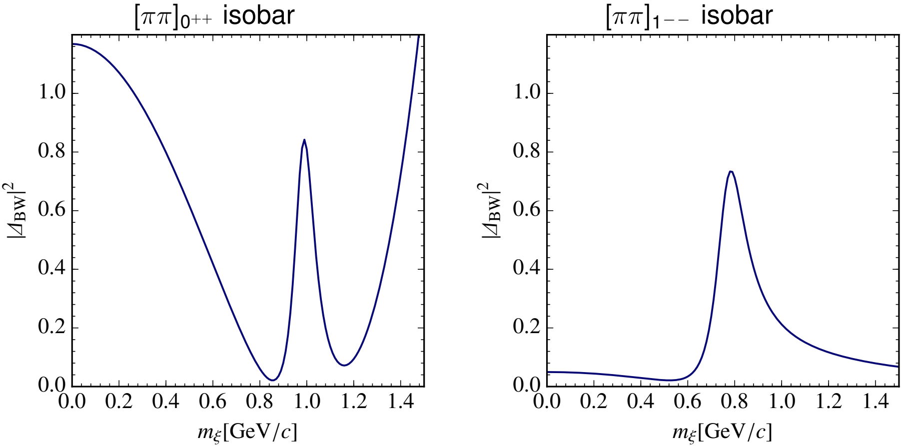

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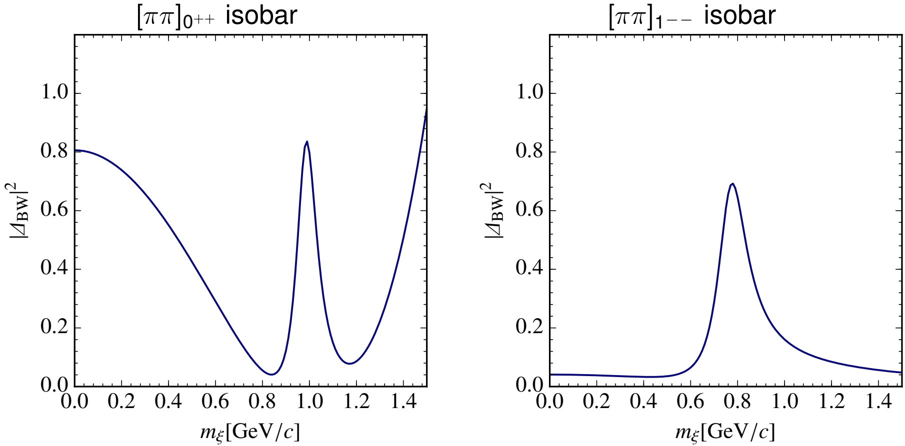

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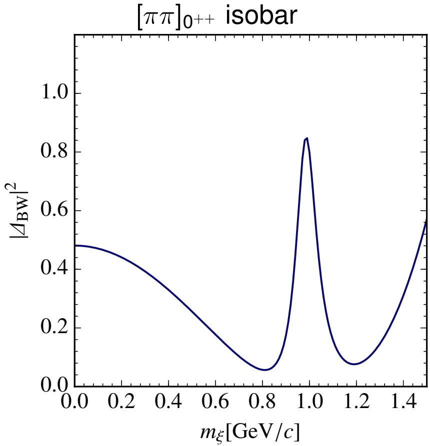

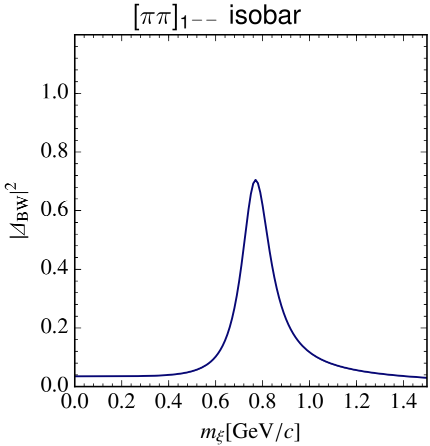

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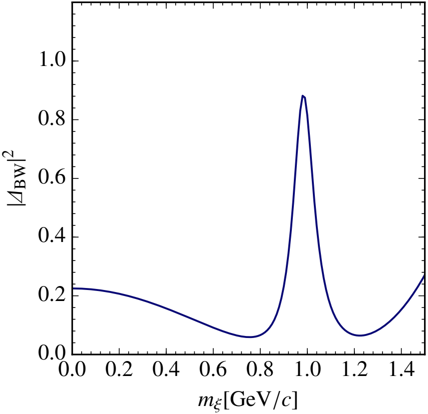

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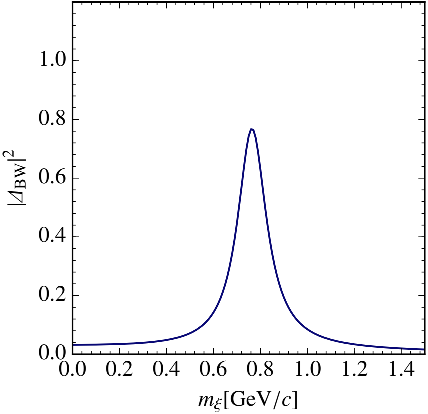

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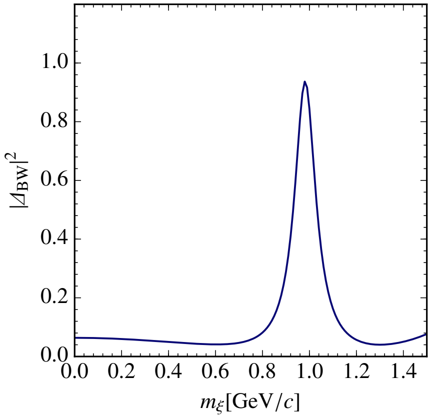

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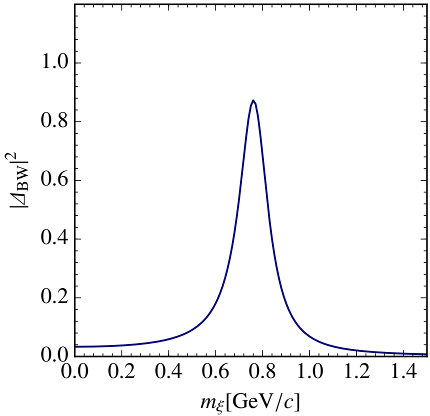

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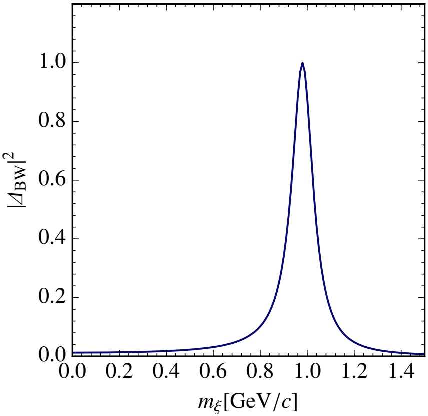

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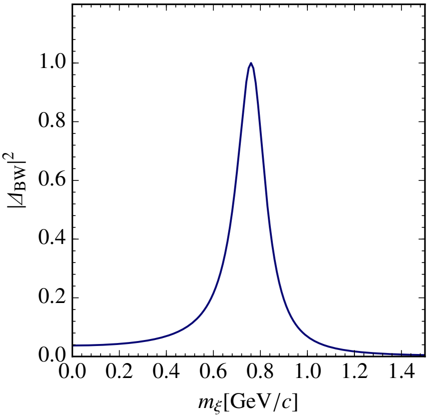

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Zero modes: 3π example

Resolving the ambiguity

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  - Intensity and likelihood also invariant: Continuous ambiguity $\mathcal{C}$

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- **Note:** Resolving the ambiguity fixes only a single complex-valued degree of freedom. $n_{\text{bins}} - 1$ complex-valued degrees of freedom remain free.

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- In the cas
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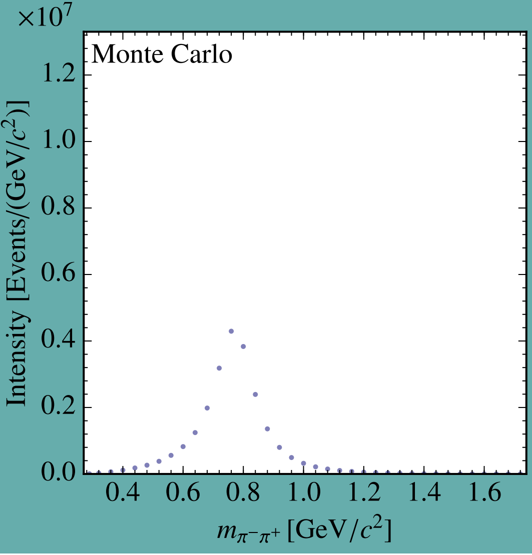
Verification of the method



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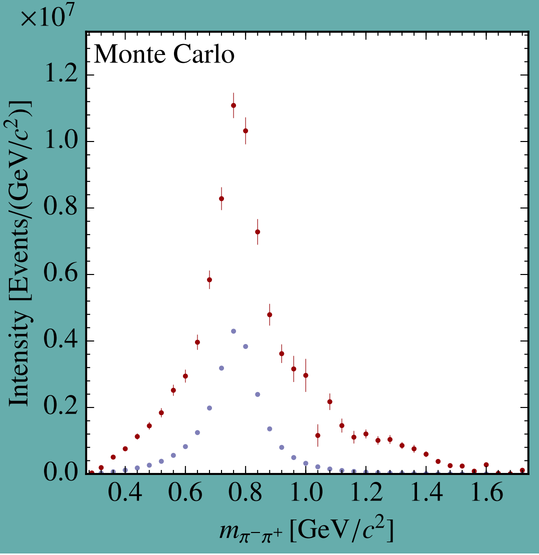

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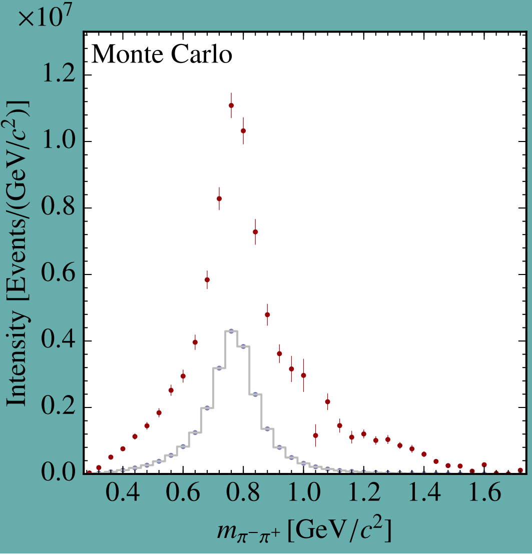

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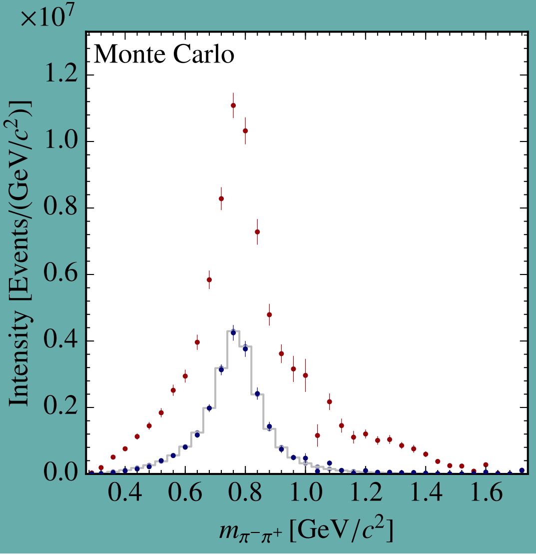

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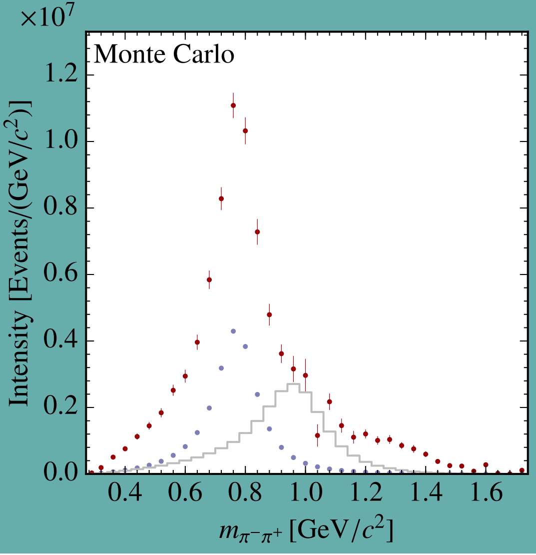

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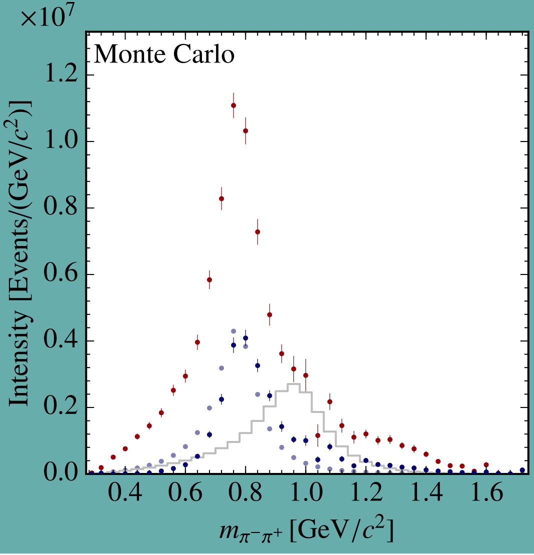
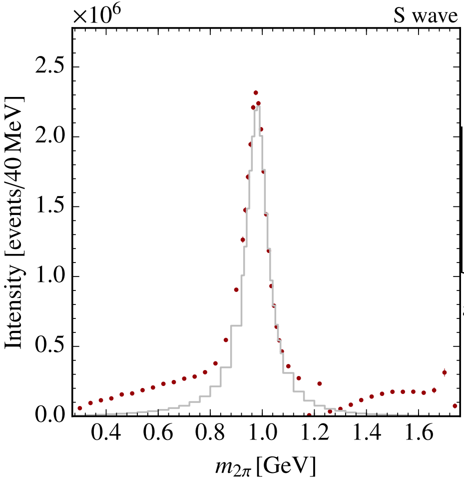
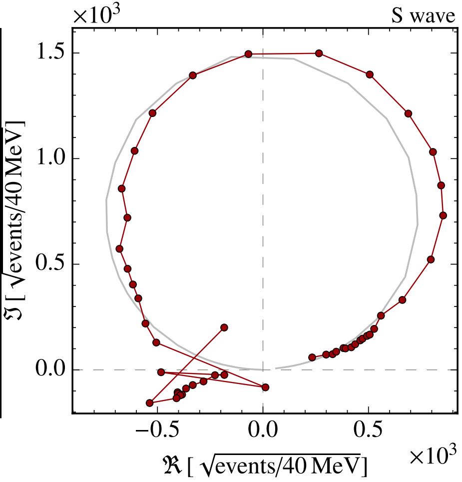
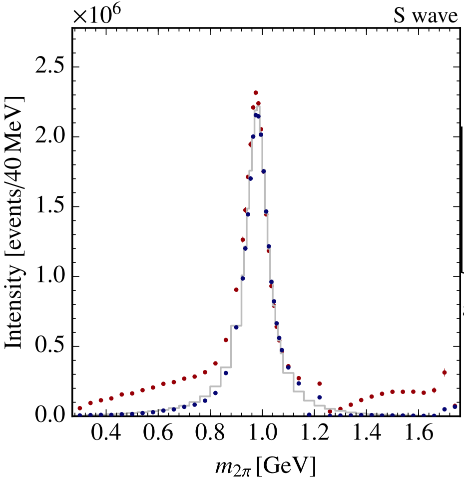


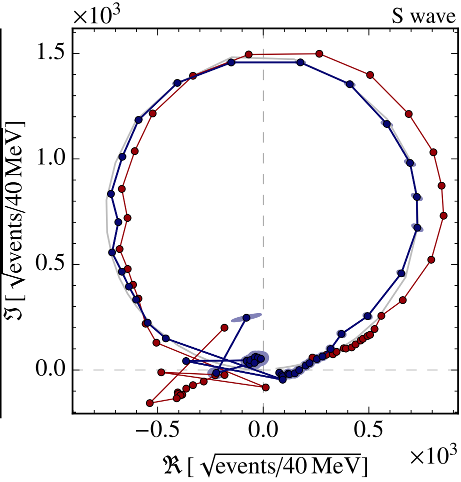
$D^- \to \pi^-\pi^+\pi^-$
Ambiguity resolved

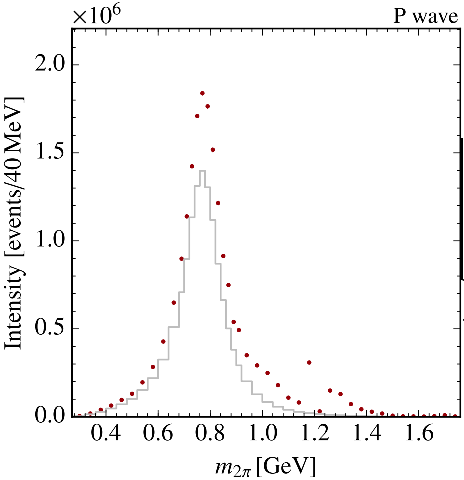


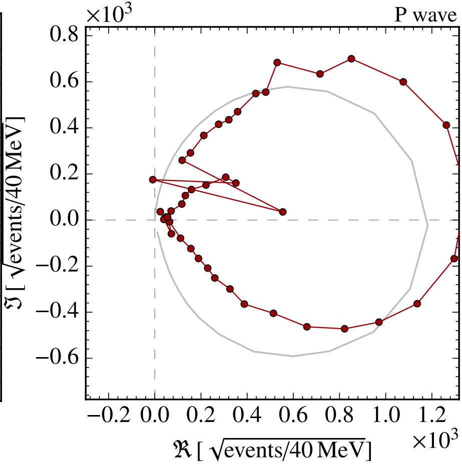


$D^- \to \pi^-\pi^+\pi^-$
Ambiguity resolved






×10^6
P wave
2.0
1.5
1.0
0.5
0.0
Intensity [events/40 MeV]
0.4
0.6
0.8
1.0
1.2
1.4
1.6
$m_{2\pi}$ [GeV]


×10^3
P wave
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
$\Im$ [$\sqrt{\text{events/40 MeV}}$]
−0.2
0.0
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0.4
0.6
0.8
1.0
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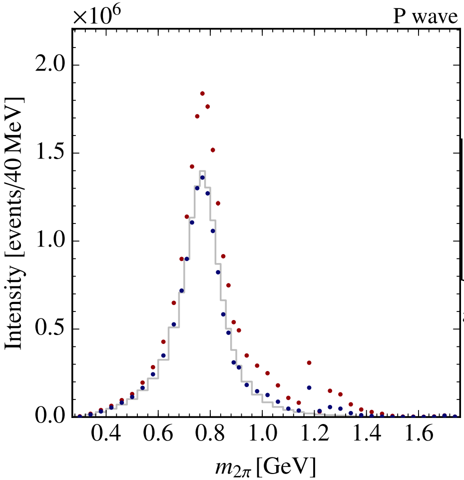
×10^3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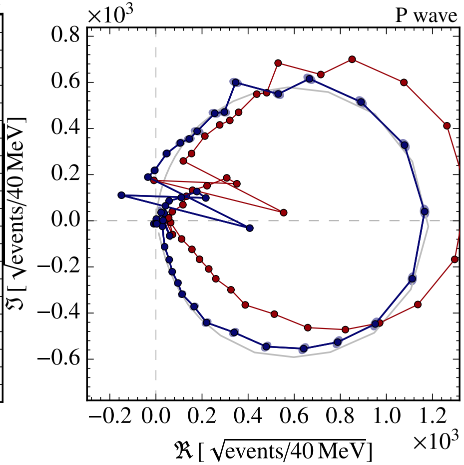
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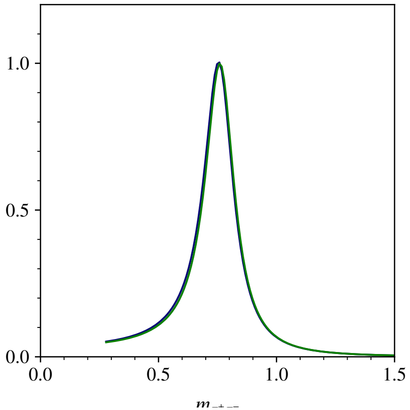
Zero modes
Single freed waves

- Small difference in fixed wave:
  - Fixed-width Breit-Wigner
  - Relativistic Breit-Wigner



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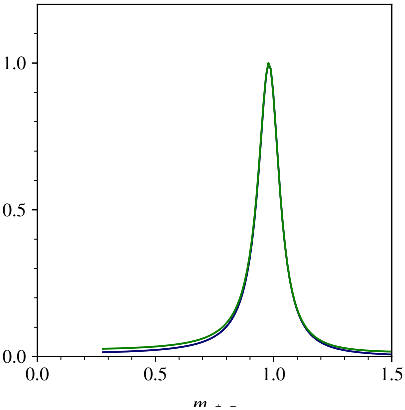
- Difference projects into zero mode
- Affects $[\pi\pi]_S$ wave



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- Difference projects into zero mode
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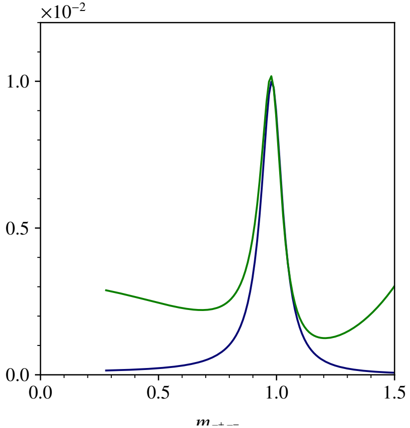
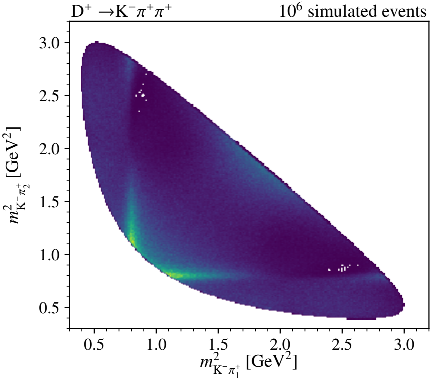
- Size of effect alway the same
- S intensity often smaller than P wave



- Next case:

$$D^+ \to K^-\pi^+\pi^+$$

- Again one million events
- Model inspired by CLEO
  Phys. Rev. **D78** 052001 (2008).
  - Also including D wave



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  - Bin width: 40 MeV$/c^2$ in $m_{K\pi}$
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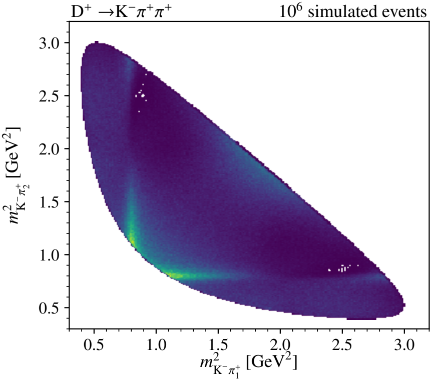

Second test case
$D^+ \to K^-\pi^+\pi^+$

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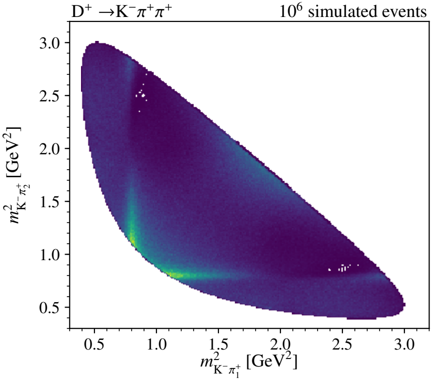
- 104 waves in total
  - 208 fit parameters



- Fully freed fit of
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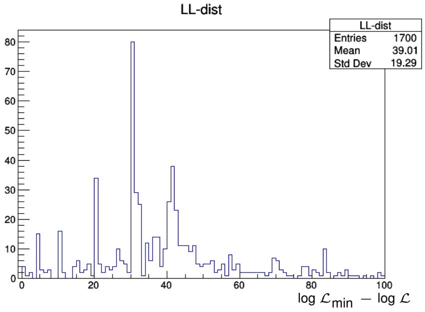

- Fully freed fit of $[K\pi]_S$, $[K\pi]_P$, $[K\pi]_D$
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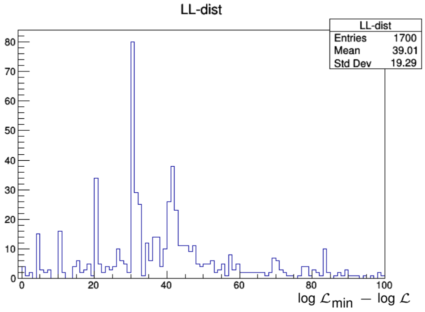
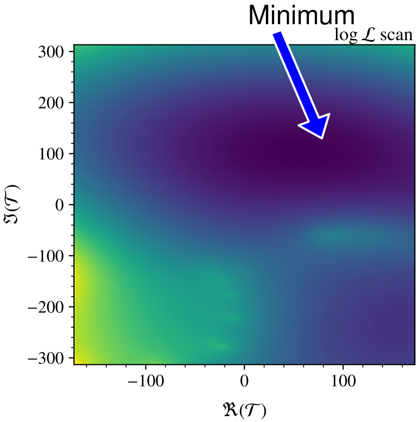
- How to find best minimum?



Fully free fit
Multimodality

- Start values for all bins independent
- Most of the fits end up in local minima
- Plot $-\log \mathcal{L}$ around minimum



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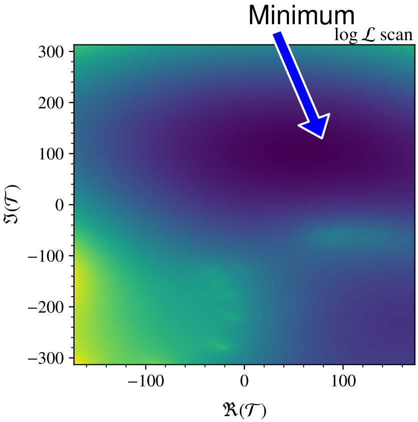

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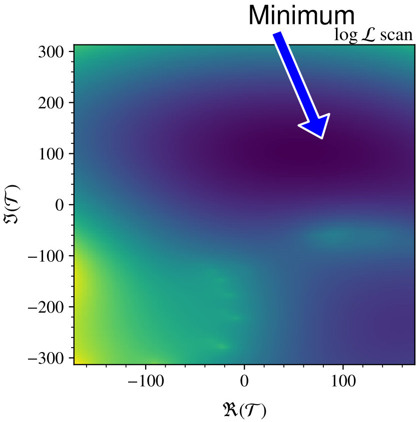

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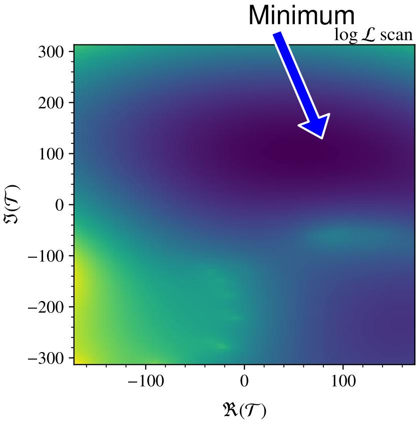

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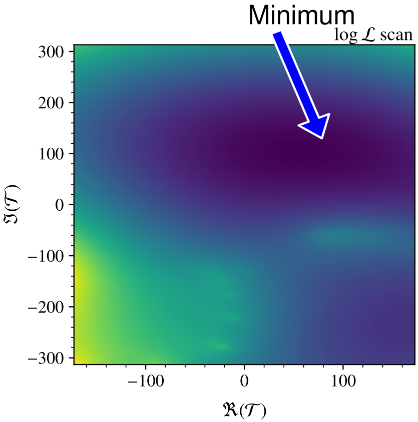
- Similar for other bins
- Approximately $2^{n_{\text{bins}}}$ local minima



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Fourier basis

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- Change the fit basis to Fourier modes:
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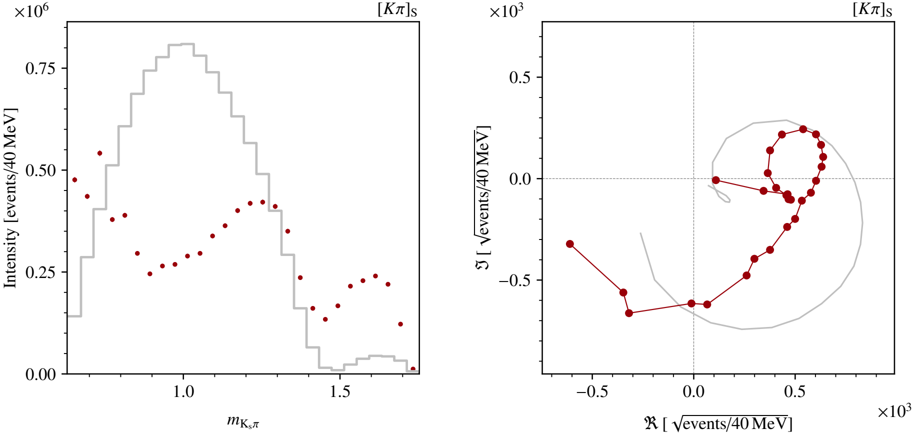
- Still multimodal
  - Likelihood gap ≈ 1000
- Best result consistently found
  - 6% of attempts

Fully freed fit
Best result



Fully freed fit

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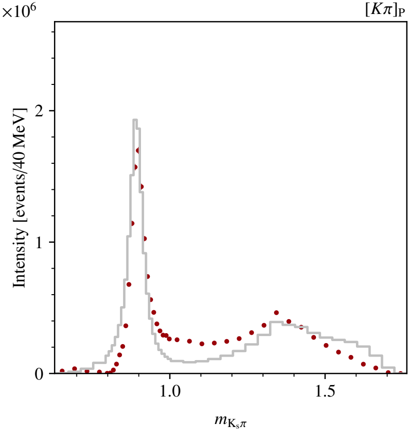

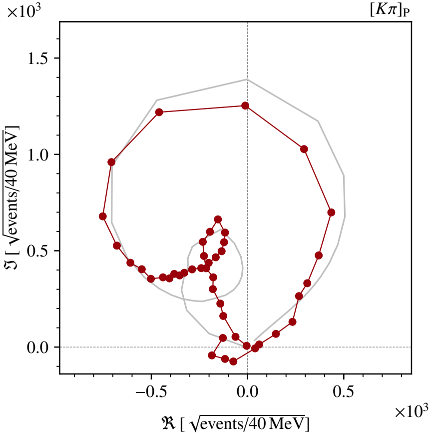

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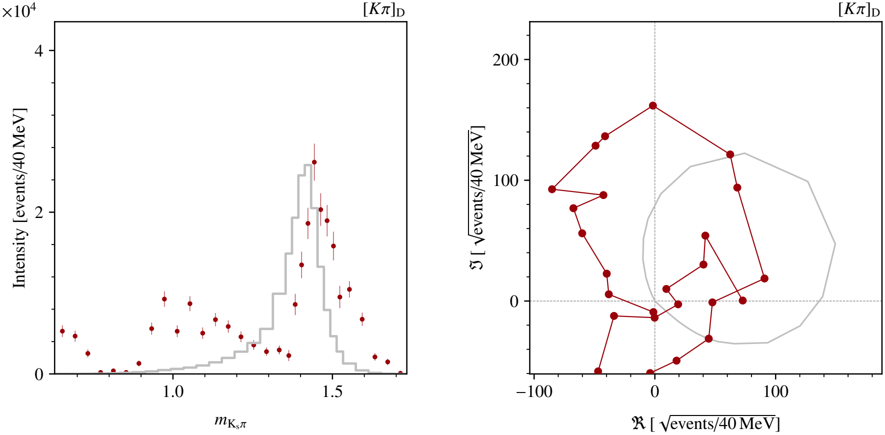

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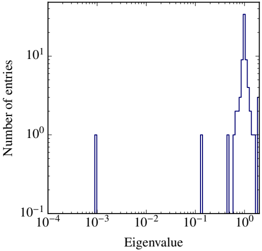

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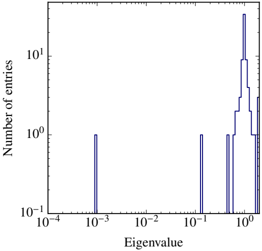

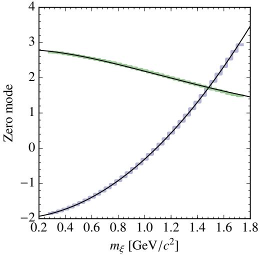

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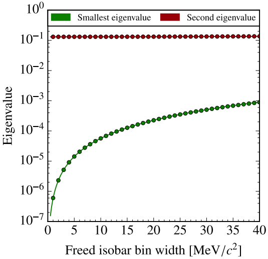

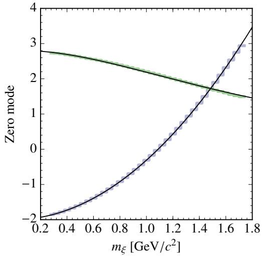

Back to $D^+ \rightarrow K^-\pi^+\pi^+$

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Fully freed fit
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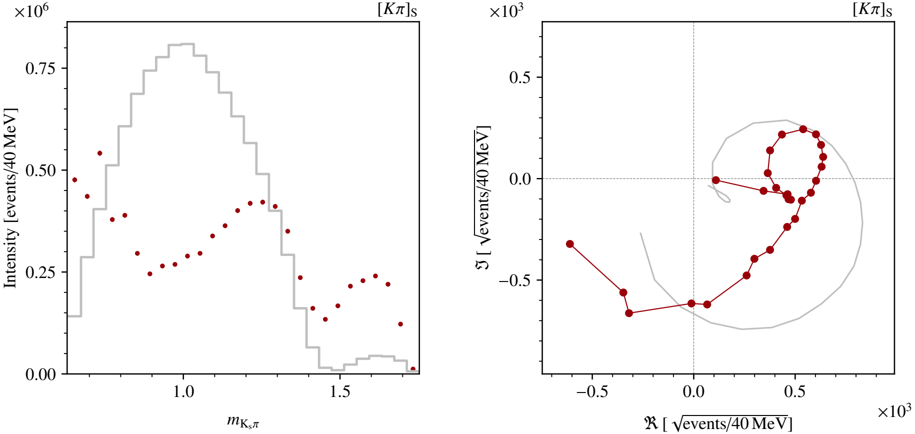

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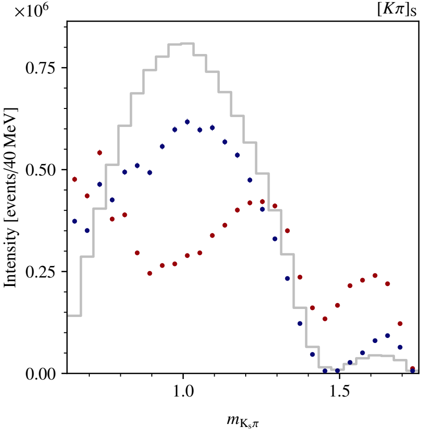

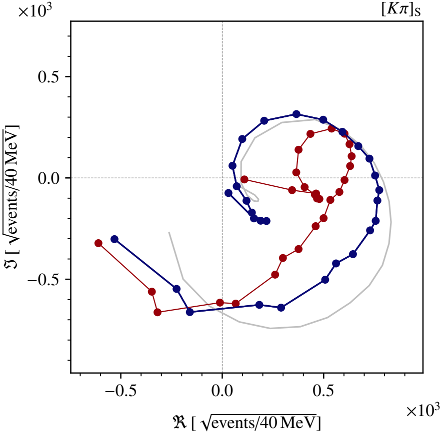

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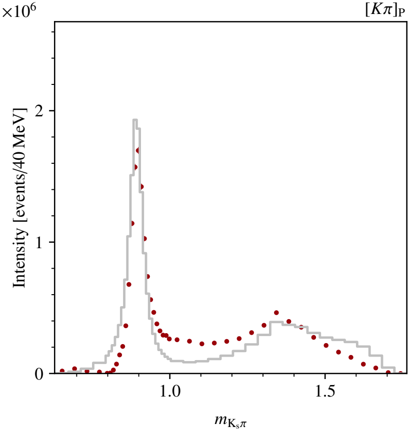

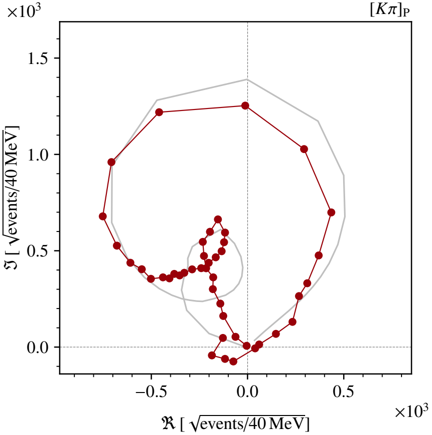

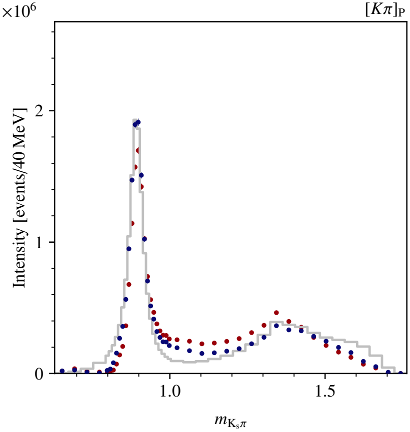

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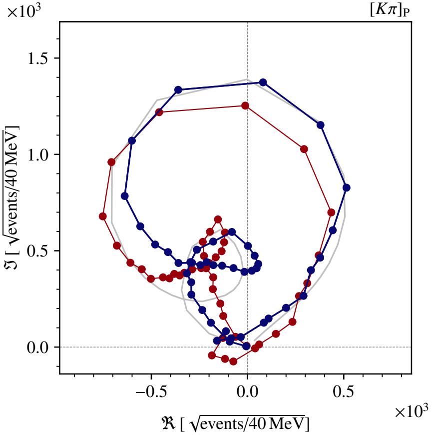

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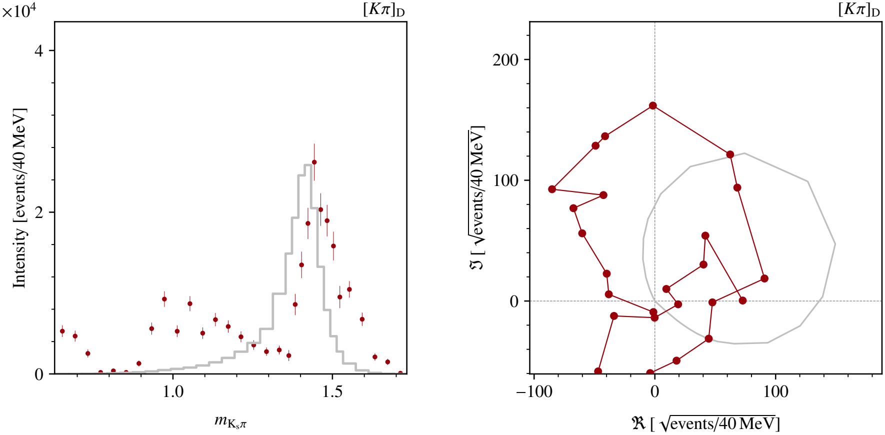

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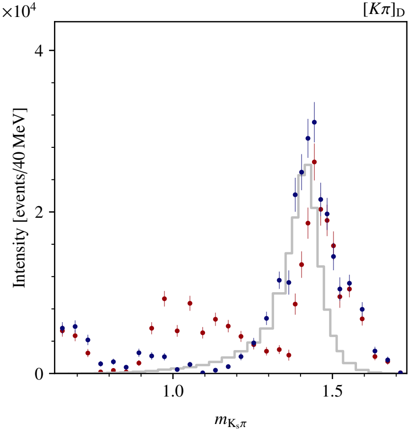

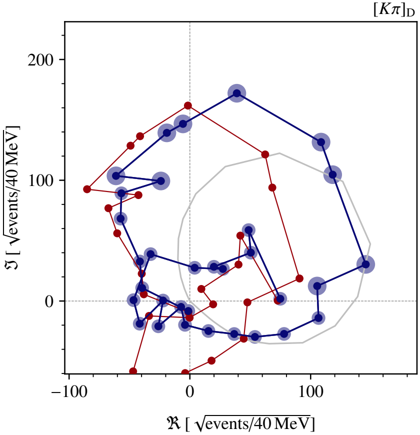

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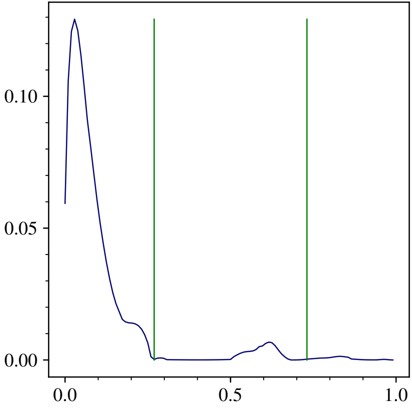

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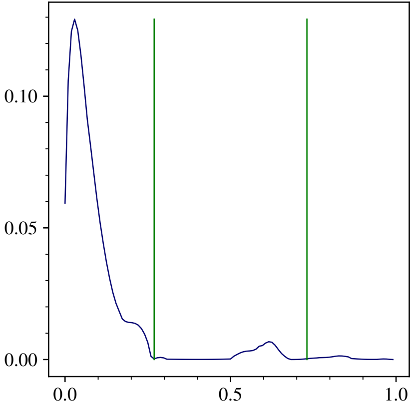
- Improve function basis



Further steps

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