

# The role of charged exotic states in $e^+e^- \rightarrow \psi'(J/\psi) \pi^+\pi^-$

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Collaborators:

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## 1 Exotic States

- Naive Quark Model
- Exotic Explanations
- Experimental Results

## 2 Formalism

- Cross Section
- Rescattering  $s$ -channel
- Unitarity in the  $s$ -channel
- Dispersion Relation
- Left-Hand Cuts
- Anomalous Threshold
- Inverse Amplitude Method

## 3 $\psi(2S) \pi^+ \pi^-$

- $Z_c(3900)$
- No Intermediate State
- $Z_c(4016)$  or  $Z_c(4020)$ ?

## 4 Perspectives

- $e^+ e^- \rightarrow J/\psi \pi^+ \pi^-$
- Strange Partner of  $Z_c(3900)$ ?
- Preliminary results for  $e^+ e^- \rightarrow J/\psi \pi^+ \pi^-$
- Predictions for  $e^+ e^- \rightarrow J/\psi K^+ K^-$

## 5 Summary

# 1 Exotic States

# Quark Model

- Mesons:



$$P = (-1)^{L+1} \quad C = (-1)^{L+S}$$

L	S	$J^{PC}$
0	0	$0^{-+}$
	1	$1^{--}$
1	0	$1^{+-}$
	1	$0^{++} \quad 1^{++} \quad 2^{++}$
2	0	$2^{-+}$
	1	$1^{--} \quad 2^{--} \quad 3^{--}$

## Forbidden Quantum Numbers:

$$0^{--}, 0^{+-}, 1^{-+}, 2^{+-}, \dots$$

## Quarks

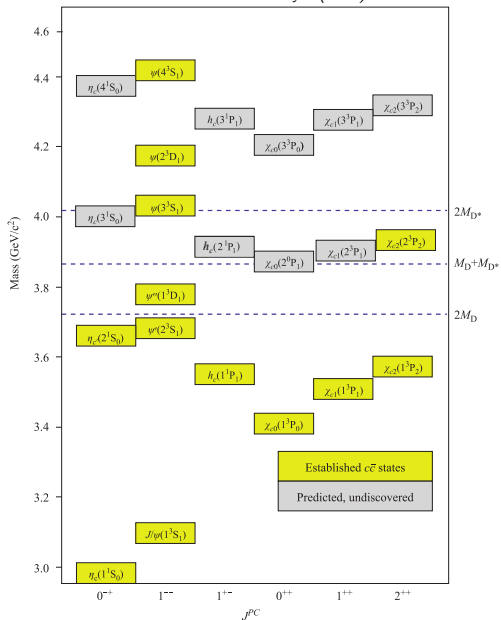
	I	II	III
mass	$\approx 2.4 \text{ MeV}/c^2$	$\approx 1.275 \text{ GeV}/c^2$	$\approx 172.44 \text{ GeV}/c^2$
charge	$2/3$	$2/3$	$2/3$
spin	$1/2$	$1/2$	$1/2$
	<b>u</b> up	<b>c</b> charm	<b>t</b> top
QUARKS	$\approx 4.8 \text{ MeV}/c^2$	$\approx 95 \text{ MeV}/c^2$	$\approx 4.18 \text{ GeV}/c^2$
	$-1/3$	$-1/3$	$-1/3$
	$1/2$	$1/2$	$1/2$
	<b>d</b> down	<b>s</b> strange	<b>b</b> bottom

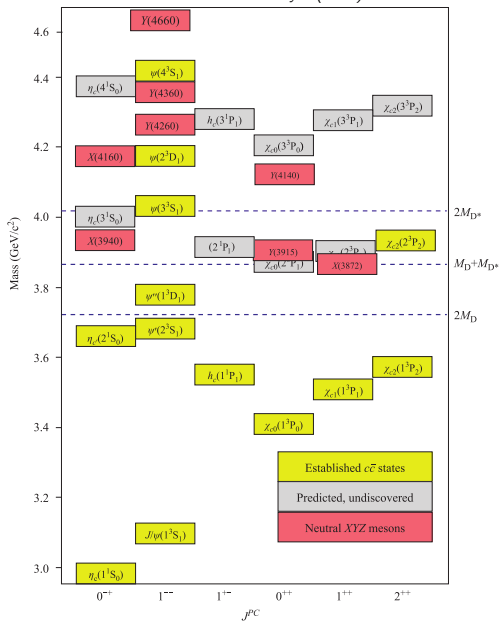
- Heavy Quarks:

$$\frac{d^2}{dr^2} u(r) + 2\mu \left[ E - V(r) - \frac{l(l+1)}{2\mu r^2} \right] u(r) = 0$$

- Phenomenological Potential:

$$V(r) \simeq -\frac{\kappa}{r} + br + \text{spin-terms}$$

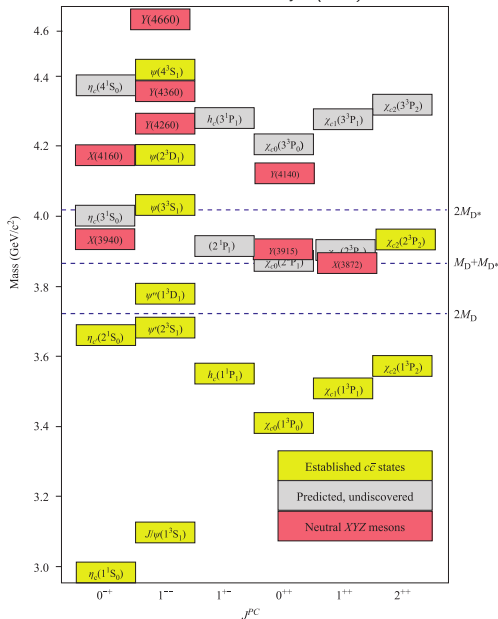
S. L. Olsen *Front.Phys.* (2015)

S. L. Olsen *Front.Phys.* (2015)

## Alternative explanations:

- Hybrids and Glueballs
- Tetraquark
- Molecular state
- Hadrocharmonium
- Kinematics effects

S. L. Olsen *Front.Phys.* (2015)



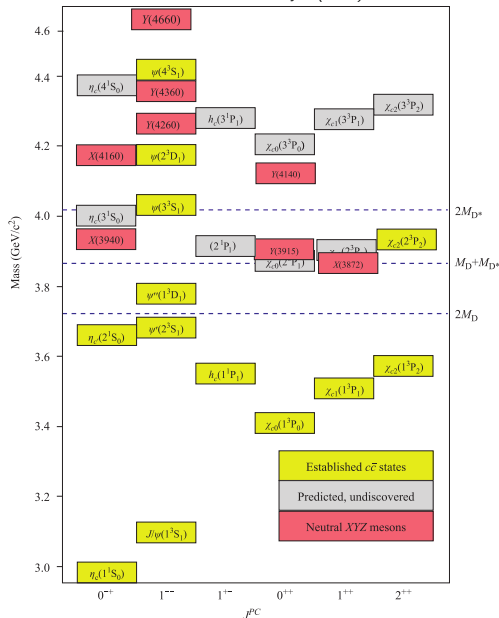
## Alternative explanations:

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- Kinematics effects

## No Unique Structure

No new physics! All interpretations are based on QCD.

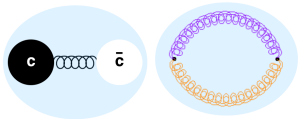
S. L. Olsen *Front.Phys.* (2015)





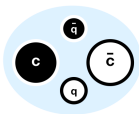


## Hybrids, Glueballs



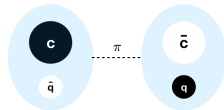
- ↓ Charged exotic states
- ↓ Search for the missing  $J^{PC}$  (GlueX, Compass and Panda)

## Tetraquark



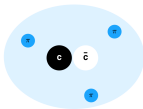
- ↓ Too many states predicted

## Molecular State



- ↓ Not enough to explain radiative decays
- ↓ Production at high- $p_T$

## Hadrocharmonium



- ↓ Decays predominantly into D-mesons

If the force  $\left\{ \begin{array}{l} \downarrow, \text{ not state} \\ \uparrow, \text{ D-mesons} \end{array} \right.$

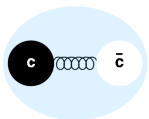
## Non-Resonance Interpretation

**Threshold effects:** Kinematics effects due to the opening of a channel producing a cusp effect.

**Triangle singularities:** when particles can go simultaneously on shell in a triangle loop.

Sufficient to explain the signals in the final observable?

## Hybrids, Glueballs



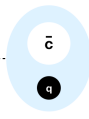
↓ Charged exotic  
 ↓ Search for the  
 (GlueX, Compa

## Tetraquark

## Recent Reviews

- Y-R Liu *et al.*, Progress in Particle and Nuclear Physics 04, 003 (2019)
- R.M. Albuquerque *et al.*, J.Phys G: Nucl. Part. Phys. 46, 093002 (2019)
- S.L. Olsen, T. Skwarnicki, D. Zieminska, Rev. Mod. Phys. 90, 15003 (2018)
- F-K, Guo *et al.*, Rev. Mod. Phys. 90 014004 (2018)
- R.F. Lebed, R.E. Mitchell, E. S. Swanson, Progress in Particle and Nuclear Physics 93, 143 (2017)
- A. Esposito, A. Pilloni, A.D. Polosa, Physics Reports 668, 1 (2017)
- H-X. Chen *et al.* Physics Reports 639, 1-121 (2016)
- M.R. Shepherd, J.J. Dudek, R.E. Mitchell, Nature 534 (2016) no.7608, 487-493

## Molecular State



to explain  
 S  
 at high- $p_T$

## Hadroch



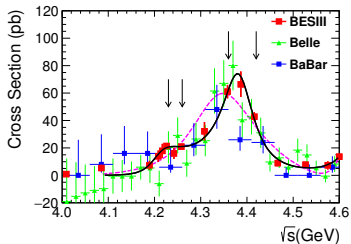
↓ Decays predom  
 D-mesons

If the force { ↓  
 ↑, D-mesons

observable?

$$e^+e^- \rightarrow \psi(2S) \pi^+ \pi^-$$

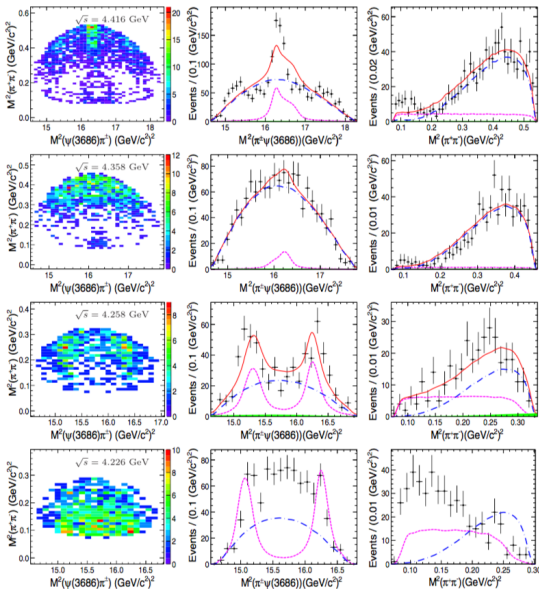
## ● $Y(4220) + Y(4390)$



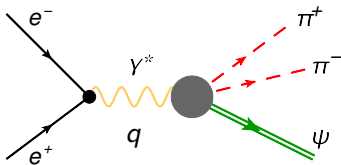
## $Z_c(3900) + Z_c(4030)$

- Two charged exotic states!
- No consistent description
- Below  $K\bar{K}$  threshold

*BESIII PRD (2017)*



## 2 Formalism



$$s = (p_{\pi^+} + p_{\pi^-})^2$$

$$t = (p_{\psi} + p_{\pi^-})^2$$

$$u = (p_{\psi} + p_{\pi^+})^2$$

### Double Differential Cross Section

$$\frac{\partial^2 \sigma}{\partial s \partial t} = \frac{1}{3} \frac{e^2}{(2\pi)^3} \frac{1}{2^5} \frac{(q^2 + 2m_e^2)}{\sqrt{q^2(q^2 - 4m_e^2)}} \frac{1}{(q^2)^3} \sum_{\lambda_1 \lambda_2} |\mathcal{H}_{\lambda_1 \lambda_2}|^2$$

$$\langle \pi\pi\psi(\lambda_2) | \mathcal{T} | \gamma^*(\lambda_1) \rangle = (2\pi)^4 \delta(q - p_{\psi} - p_{\pi^+} - p_{\pi^-}) \mathcal{H}_{\lambda_1 \lambda_2}$$

### Independent Helicity Amplitudes

P-symmetry:  $\mathcal{H}_{++}$ ,  $\mathcal{H}_{+-}$ ,  $\mathcal{H}_{+0}$ ,  $\mathcal{H}_{0+}$  and  $\mathcal{H}_{00}$

• CP-symmetry:  $J_{\pi\pi} \rightarrow$  even

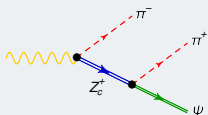
• Bose-symmetry:  $I_{\pi\pi} = 0, 2$

•  $I_{\psi} = 0$ ;  $I_{\gamma} = 0, 1 \implies I_{\pi\pi} = 0$

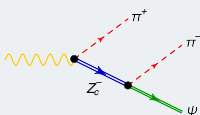
# Rescattering s-channel

$$\mathcal{H}_{\lambda_1\lambda_2}(s, t, u) = H_{\lambda_1\lambda_2}^{(Z_c)}(t, u) + H_{\lambda_1\lambda_2}^R(s)$$

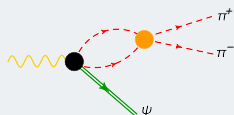
t-channel



u-channel



s-channel



$$h_{\lambda_1\lambda_2}^{(0)}(s) = \frac{1}{2} \int_{-1}^{+1} dz \mathcal{H}_{\lambda_1\lambda_2}(s, t, u) = \underbrace{h_{\lambda_1\lambda_2}^R(s)}_{\text{r.h.c.}} + \underbrace{h_{\lambda_1\lambda_2}^{(0),L}(s)}_{\text{l.h.c.}}$$

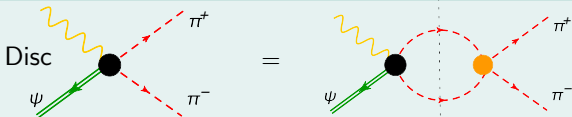
$$h_{\lambda_1\lambda_2}^{(0),L}(s) \equiv \frac{1}{2} \int_{-1}^{+1} dz H_{\lambda_1\lambda_2}^{(Z_c)}(t, u)$$

$$h_{\lambda_1\lambda_2}^{(0),R}(s) \equiv \frac{1}{2} \int_{-1}^{+1} dz H_{\lambda_1\lambda_2}^R(s)$$

## Unitarity in the s-channel

$$J_{\pi\pi} = 0$$

$$I_{\pi\pi} = 0$$



$$\text{Disc } h_{\lambda_1\lambda_2}(s) \equiv \frac{1}{2i} (h_{\lambda_1\lambda_2}(s + i\epsilon) - h_{\lambda_1\lambda_2}(s - i\epsilon)) = t_{\pi\pi}^*(s) \rho(s) h_{\lambda_1\lambda_2}(s) \theta(s > 4m_\pi^2)$$

- The pions interaction amplitude can be written in terms of the phase shift:

$$t_{\pi\pi}^*(s) = \frac{e^{-i\delta_{\pi\pi}(s)} \sin \delta_{\pi\pi}(s)}{\rho(s)}$$

- The  $\pi\pi$ -rescattering can be parametrized in terms of  $\Omega$ :  $\text{Disc } \Omega(s) = t_{\pi\pi}^*(s) \rho(s) \Omega(s)$
- Therefore, we can use the  $\delta_{\pi\pi}(s)$  through the **Omnès Function**:

$$\Omega(s) = \exp \left[ \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\delta_{\pi\pi}(s')}{s' - s} \right]$$

Omnès, Nuovo Cim. (1958)  
Muskhelishvili, (1953)



# Dispersion Relation

- Assuming no kinematic constraints, we look for a solution in terms of the Omnès function:

$$h_{\lambda_1\lambda_2}^R(s) = \Omega(s) G_{\lambda_1\lambda_2}(s)$$

- The unitarity relation for the Omnès function is

$$\text{Disc } \Omega(s) = t_{\pi\pi}^*(s) \rho(s) \Omega(s) \theta(s > 4m_\pi^2)$$

- Since  $\text{Disc } h_{\lambda_1\lambda_2}(s) = \text{Disc } h_{\lambda_1\lambda_2}^R(s)$ , one can write down a DR for  $G_{\lambda_1\lambda_2}$ :

$$G_{\lambda_1\lambda_2} = - \int_{4m_\pi^2}^{\infty} \frac{ds'}{\pi} \frac{\text{Disc } (\Omega)^{-1}(s') h_{\lambda_1\lambda_2}^L(s')}{s' - s}$$

## Helicity Amplitude with Rescattering

$$\mathcal{H}_{\lambda_1\lambda_2}(s, t, u) = H_{\lambda_1\lambda_2}^{(Z_c)}(t, u) + \Omega(s) \left\{ a + bs - \frac{s^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2} \frac{\text{Disc } (\Omega)^{-1}(s') h_{\lambda_1\lambda_2}^L(s')}{s' - s} \right\}$$

- 2 subtraction constants to reduce the sensitive to high energy.

# Left-Hand Cuts

## Invariant Amplitudes

- The helicity amplitude  $\mathcal{H}^{\mu\nu}$  can be written in the general form as

$$\mathcal{H}^{\mu\nu} = \sum_{i=1}^5 F_i L_i^{\mu\nu}$$

where  $F_i$  are the invariant amplitudes and  $L_i^{\mu\nu}$  is a complete set of Lorenz structures.

- For the S-wave:

$$h_{++}^{(0)}(s) = \frac{s - q^2 - m_\psi^2}{2} f_1(s) - q^2 m_\psi^2 f_4(s)$$

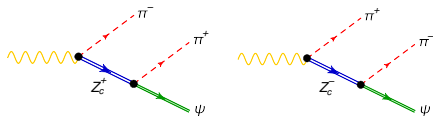
$$h_{00}^{(0)}(s) = -qm_\psi \left( f_1(s) - \frac{s - q^2 - m_\psi^2}{2} f_4(s) \right)$$

with  $f_i$  the partial wave expansion of  $F_i$ .

- The helicity amplitudes are correlated

$$h_{++}^{(0)}(s) \pm h_{00}^{(0)}(s) \sim \mathcal{O}(s - (q \pm m_\psi)^2)$$

## $Z_c$ Exchange Mechanism



$$\mathcal{H}_{\lambda_1 \lambda_2}^{Z_c} = (V_{Z_\pm \psi \pi})^{\beta\nu} S_{\nu\mu}(Q_{z_\pm}) (V_{\gamma^* \pi Z_\pm})^{\mu\alpha} \times \epsilon_\alpha(\mathbf{p}_{\gamma^*}, \lambda_1) \epsilon_\beta^*(\mathbf{p}_\psi, \lambda_2)$$

- We observe that

$$\mathcal{H}_{+-}^{Z_c} = \mathcal{H}_{+0}^{Z_c} = \mathcal{H}_{0+}^{Z_c} \approx 0$$

$$\sum_{\lambda_1 \lambda_2} |\mathcal{H}_{\lambda_1 \lambda_2}^{Z_c}|^2 = 2 |\mathcal{H}_{++}^{Z_c}|^2 + |\mathcal{H}_{00}^{Z_c}|^2 \approx 3 |\mathcal{H}_{++}^{Z_c}|^2$$

$\therefore$  the effects of the kinematical constraints can be ignored!

- pole contribution:  $t = m_z^2$  and  $u = m_z^2$  in the numerators.

# Anomalous Threshold

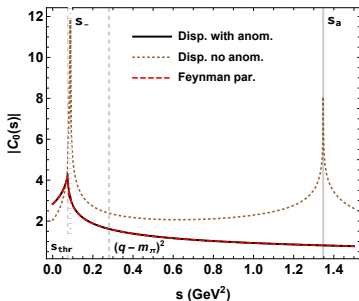
- Depending on the kinematics new nonphysical singularities might appear ( $q^2 > 2m_\pi^2 + 2m_z^2 - m_\psi^2$ ).
- The anomalous piece that emerges because the anomalous branch point moves onto the first Riemann sheet distorting the integration contour. Effectively, that can be written as

$$\int_{-1}^1 \frac{dz}{t - m_z^2} = \int_{-1}^1 \frac{dz}{u - m_z^2} = -\frac{2}{k(s)} \log \left( \frac{X(s) + 1}{X(s) - 1} \right) - i \frac{4\pi}{k(s)} \theta(s_- < s < s_a)$$

- $s_a = 2m_\pi^2 + m_\psi^2 + q^2 - 2m_z^2$  is the position where the argument of the logarithm changes sign.
- Analytical continuation:  $q \rightarrow q + i\epsilon$

## Cross-Check

- Comparison of the DR with the scalar triangle loop calculated via traditional method.



# Inverse Amplitude Method

## Modified-IAM for (J=0):

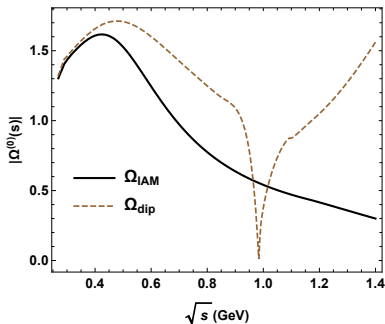
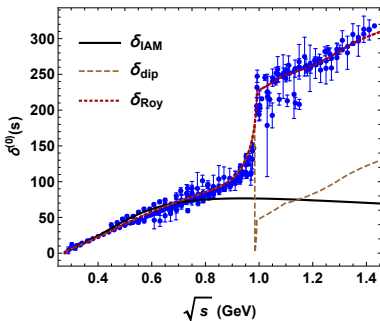
$$t_{\pi\pi}(s) = \frac{|t_{\text{LO}}(s)|^2}{t_{\text{LO}}(s) - t_{\text{NLO}}(s) + A^{\text{mIAM}}(s)}$$

- Correct positions of Adler zeros;
- Consistent description of  $f_0(500)$ .

GomezNicola et al. PRD (2008)

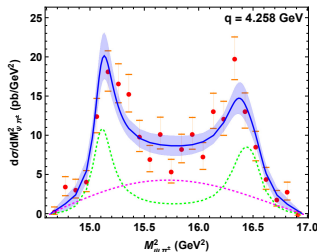
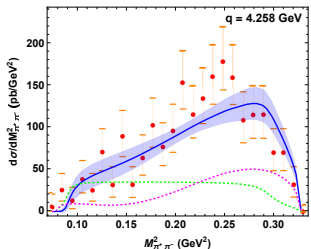
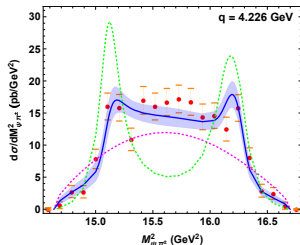
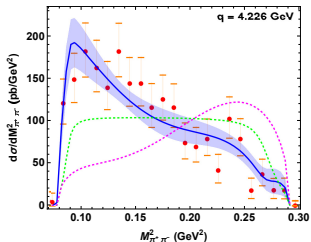
## Omnès Function

$$\Omega(s) = \exp \left[ \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\delta_{\pi\pi}(s')}{s' - s} \right]$$



3  $\psi(2S) \pi^+ \pi^-$

# $Z_c(3900)$



$$\chi_{red}^2 = 1.16$$

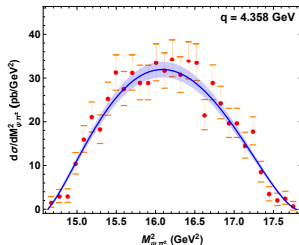
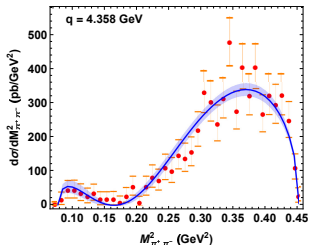
- Combined Fit
- Only  $Z_c$
- Only  $\pi\pi$ -FSI

- $a(q^2)$  and  $b(q^2)$  are complex numbers
- Global normalization:  $N(q^2, C_{\gamma^* Z_c \pi}, C_{Z_c \psi \pi})$

$$\chi_{red}^2 = 1.01$$

Data is normalized using the total cross section

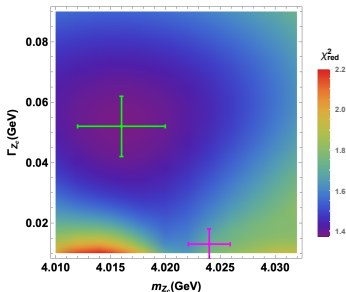
# No $Z_c$ Intermediate State



$$\chi_{red}^2 = 0.83$$

- No intermediate state is required;
- Two real subtraction constants and the  $\pi\pi$  Omnès function describe well the data;
- The left-hand cuts are dominated by the contact interaction or possible D-meson loops (absorbed in the subtraction constants).

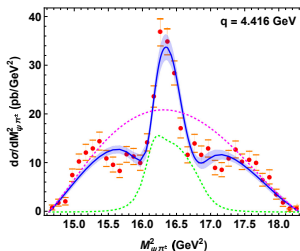
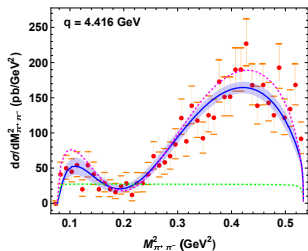
# $Z_c(4016)$ or $Z_c(4020)$ ?



$m_{Z_c}(\text{GeV})$	$\Gamma_{Z_c}(\text{MeV})$
4.016(4)	52(10)
4.024(2)	13(5)

$Z_c(4020)$  observed by BESIII in

- $e^+e^- \rightarrow D^* \bar{D}^* \pi$
- $e^+e^- \rightarrow h_c \pi \pi$

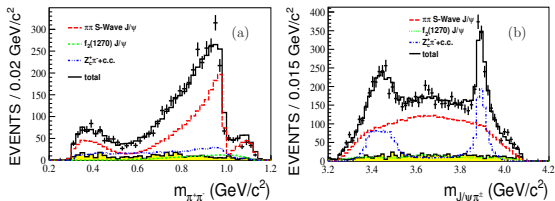


$$\chi^2_{red} = 1.38$$

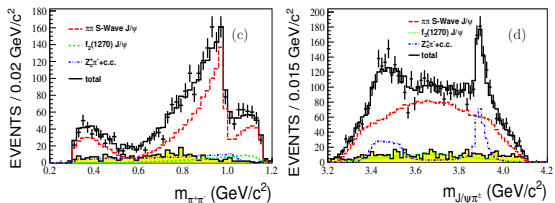
- Combined Fit
- Only  $Z_c$
- Only  $\pi\pi$ -FSI



## 4 Perspectives

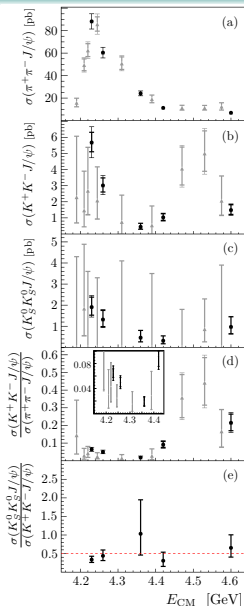
$e^+e^- \rightarrow J/\psi \pi^+\pi^-$ 

BESIII PRL (2017)



BESIII PRD (2018)

- Above  $K\bar{K}$  threshold  $\rightarrow$  couple channel rescattering;
- No evidence of the strange partner of  $Z_c(3900)$ ;
- $\sigma(J/\psi K^+K^-)$  is suppressed compared to  $\sigma(J/\psi \pi^+\pi^-)$ .



## Strange Partner of $Z_c(3900)$ ?

$$\begin{bmatrix} \mathcal{H}_{\psi\pi\pi} \\ \mathcal{H}_{\psi KK} \end{bmatrix} = \begin{bmatrix} h_{Z_c}(t, u) \\ h_{Z_c^s}(t, u) \end{bmatrix} + \underbrace{\begin{bmatrix} \Omega_{\pi\pi, \pi\pi} & \Omega_{\pi\pi, K\bar{K}} \\ \Omega_{K\bar{K}, \pi\pi} & \Omega_{K\bar{K}, K\bar{K}} \end{bmatrix}}_{\bar{\Omega}} \left\{ \begin{bmatrix} a + bs \\ c + ds \end{bmatrix} - \frac{s^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2} \frac{\text{Disc}(\bar{\Omega})^{-1}(s')}{s' - s} \begin{bmatrix} h_{Z_c}^L(s') \\ h_{Z_c^s}^L(s') \end{bmatrix} \right\}$$

## Strange Partner of $Z_c(3900)$ ?

$$\begin{bmatrix} \mathcal{H}_{\psi\pi\pi\pi} \\ \mathcal{H}_{\psi KK} \end{bmatrix} = \begin{bmatrix} h_{Z_c}(t, u) \\ h_{Z_c^*}(t, u) \end{bmatrix} + \underbrace{\begin{bmatrix} \Omega_{\pi\pi, \pi\pi} & \Omega_{\pi\pi, K\bar{K}} \\ \Omega_{K\bar{K}, \pi\pi} & \Omega_{K\bar{K}, K\bar{K}} \end{bmatrix}}_{\tilde{\Omega}} \left\{ \begin{bmatrix} a + bs \\ c + ds \end{bmatrix} - \frac{s^2}{4m_\pi^2} \int \frac{ds'}{s'^2} \frac{\text{Disc}(\tilde{\Omega})^{-1}(s')}{s' - s} \begin{bmatrix} h_{Z_c}^L(s') \\ h_{Z_c^*}^L(s') \end{bmatrix} \right\}$$

### Naive Insights:

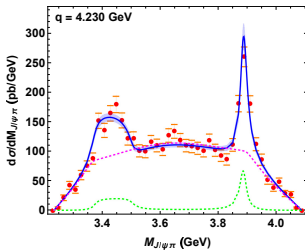
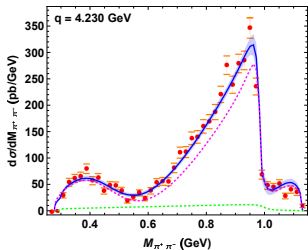
- ( $m_s \approx 95 \text{ MeV}$ )  $\gg$  ( $m_u \approx 2.2 \text{ MeV}$ ) and ( $m_d \approx 4.7 \text{ MeV}$ )

$$\implies m_{Z_c^{(s)}} > m_{Z_c}$$

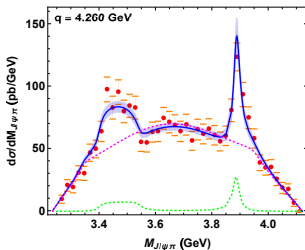
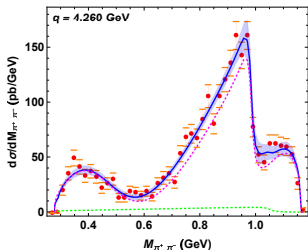
- For  $q = 4.26(4.23) \text{ GeV}$ ,  $\gamma^*(q) \rightarrow K Z_c^{(s)}$  and  $Z_c^{(s)} \rightarrow J/\psi K$  would constrain the  $Z_c^{(s)}$  mass:

$$3.59 < m_{Z_c^{(s)}} < 3.77(3.74) \text{ GeV}$$

# $e^+e^- \rightarrow J/\psi \pi^+ \pi^-$ (Preliminary)



(4.230 GeV)



(4.260 GeV)

$$\chi_{red}^2 = 2.17$$

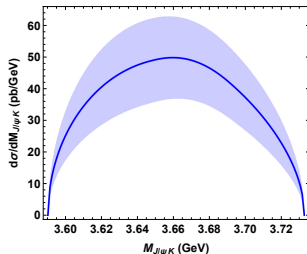
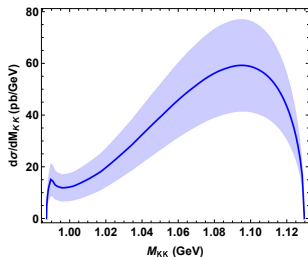
**Extra Constrain:**

$$e^+e^- \rightarrow J/\psi K^+K^-$$

- $a(q^2)$ ,  $b(q^2)$ ,  $c(q^2)$  and  $d(q^2)$  are complex numbers
- Global normalization:  $N(q^2, C_{\gamma^* Z_c \pi}, C_{Z_c \psi \pi})$

$$\chi_{red}^2 = 1.03$$

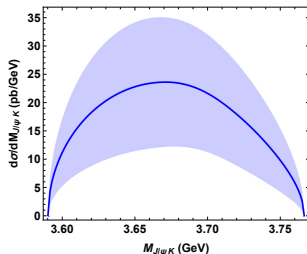
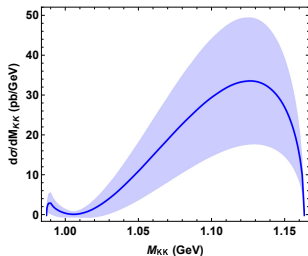
# Predictions for $e^+e^- \rightarrow J/\psi K^+K^-$ (Preliminary)



(4.230 GeV)

$$\sigma_{4.23}^{\text{pred}} = 5.29 \pm 1.47 \text{ pb}$$

$$\sigma_{4.23}^{\text{BES}} = 5.27 \pm 0.75 \text{ pb}$$



(4.260 GeV)

$$\sigma_{4.26}^{\text{pred}} = 3.08 \pm 1.54 \text{ pb}$$

$$\sigma_{4.26}^{\text{BES}} = 3.08 \pm 0.44 \text{ pb}$$

## 5 Summary

## Summary

$$e^+e^- \rightarrow \psi(2S)\pi^+\pi^-$$

- We perform an amplitude analysis of the reaction  $e^+e^- \rightarrow \psi(2S)\pi^+\pi^-$  at different  $e^+e^-$ -CM energies  $q$ ;
- The  $\pi\pi$ -FSI is treated using the dispersion theory and we studied quantitatively the contribution of the charged exotic mesons as intermediate states;
- The exotic state  $Z_c(3900)$  plays an important role to explain the invariant mass distribution at both  $q = 4.226$  and  $q = 4.258$  GeV;
- To explain the enhancement in the experimental data at  $q = 4.416$  GeV a heavier charged state is needed instead, with  $m_Z = 4.016(4)$  GeV and  $\Gamma_Z = 52(10)$  MeV. However, this state could be the already known  $Z_c(4020)$ ;
- For  $q = 4.358$  GeV no intermediate  $Z_c$  state is necessary for left-hand cuts in order to describe both  $\psi\pi$  and  $\pi\pi$  line shapes. It points to another left-hand contribution which we absorbed in the subtraction constants;
- The  $\pi\pi$ -FSI is the main mechanism to describe the  $\pi\pi$ -line shape for all the energies.



Thank you for listening!



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# Appendix

# Inverse Amplitude Method

- partial wave elastic unitarity

$$\text{Im}[t_{\pi\pi}(s)] = \rho_{\pi\pi}(s) |t_{\pi\pi}(s)|^2 \implies \text{Im}\left[\frac{1}{t_{\pi\pi}(s)}\right] = -\rho_{\pi\pi}(s)$$

- The ChPT amplitude only satisfy the unitarity condition perturbatively:

$$\text{Im}[t_{\text{LO}}] = 0; \quad \text{Im}[t_{\text{NLO}}] = \rho_{\pi\pi}(s) |t_{\text{LO}}(s)|^2 \quad \text{with } t_{\pi\pi} = t_{\text{LO}} + t_{\text{NLO}} + \dots$$

- One can write down a dispersion relation for the ChPT amplitudes as

$$t_{\text{LO}}(s) = \sum_{l=0}^k a_l s^l; \quad t_{\text{NLO}}(s) = \sum_{l=0}^k b_l s^l + \frac{s^k}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^k} \frac{\text{Im}[t_{\text{NLO}}]}{s' - s - i\epsilon} + I_{LC}[t_{\text{NLO}}]$$

- Same analytic structure for  $t$  and  $t^{-1}$ :  $G(s) \equiv t_{\text{LO}}^2/t_{\pi\pi} \implies \text{Im}[G(s)] = -\text{Im}[t_{\text{NLO}}]$
- Thus  $G(s) \simeq t_{\text{LO}}(s) - t_{\text{NLO}}(s)$ , considering that  $I_{LC}[G(s)] = -I_{LC}[t_{\text{NLO}}]$  and ignoring pole contributions:

$$t_{\pi\pi}(s) \simeq \frac{|t_{\text{LO}}(s)|^2}{t_{\text{LO}}(s) - t_{\text{NLO}}(s)}$$

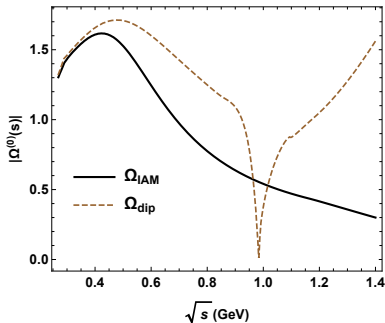
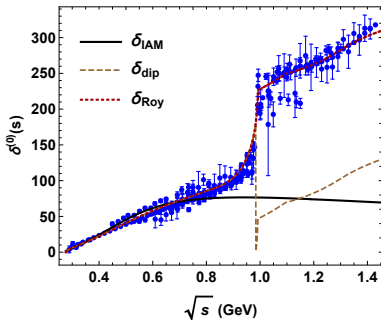
# Inverse Amplitude Method

- Spurious poles emerges below threshold for the scalar waves ( $J=0$ ), thus to reproduce correctly the Adlers zeros the IAM must be modified as

$$t_{\pi\pi}(s) = \frac{|t_{LO}(s)|^2}{t_{LO}(s) - t_{NLO}(s) + A^{mIAM}(s)}$$

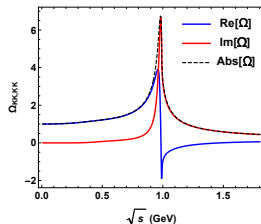
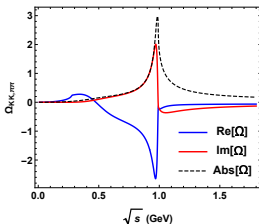
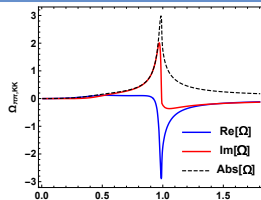
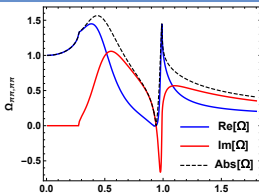
GomezNicola et al.  
PRD (2008)

with the adler zero  $s_A = s_{LO} + s_{NLO} + \mathcal{O}(p^6)$  and  $t_{LO}(s_{LO} + s_{NLO}) + t_{NLO}(s_{LO} + s_{NLO}) = 0$



# Couple Channel

$$\begin{bmatrix} \mathcal{H}_{\psi\pi\pi} \\ \mathcal{H}_{\psi KK} \end{bmatrix} = \begin{bmatrix} h_{Z_c}(t, u) \\ h_{Z_c^s}(t, u) \end{bmatrix} + \underbrace{\begin{bmatrix} \Omega_{\pi\pi, \pi\pi} & \Omega_{\pi\pi, K\bar{K}} \\ \Omega_{K\bar{K}, \pi\pi} & \Omega_{K\bar{K}, K\bar{K}} \end{bmatrix}}_{\bar{\Omega}} \left\{ \begin{bmatrix} a + bs \\ c + ds \end{bmatrix} - \frac{s^2}{4m_\pi^2} \int \frac{ds'}{s'^2} \frac{\text{Disc}(\bar{\Omega})^{-1}(s')}{s' - s} \begin{bmatrix} h_{Z_c}^L(s') \\ h_{Z_c^s}^L(s') \end{bmatrix} \right\}$$



Danilkin & Vanderhaeghe, PLB (2019)