lattice QCD for precision flavour physics

⇒ precision prospects for CKM determination

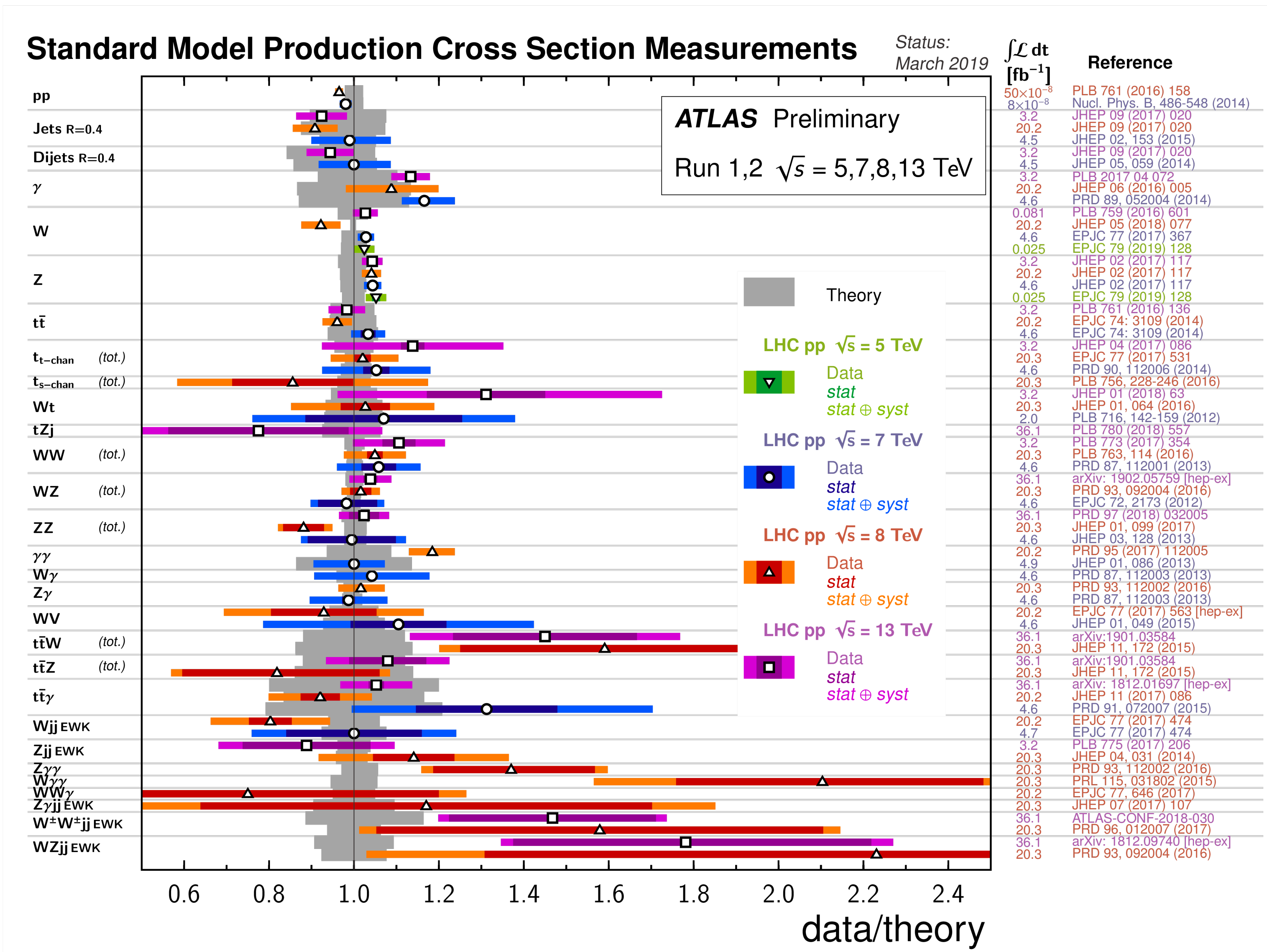
Carlos Pena

ift UAM

CERN Council Open Symposium on the Update of the
European Strategy for Particle Physics - Granada, May 2019

why we care

- NP: energy frontier has revealed the/a Higgs boson+barren (?) land
  - exquisite control of SM predictions needed to dig up possible new Physics
  - hadronic sector: α_s, quark masses, ...



[ATLAS 2019]

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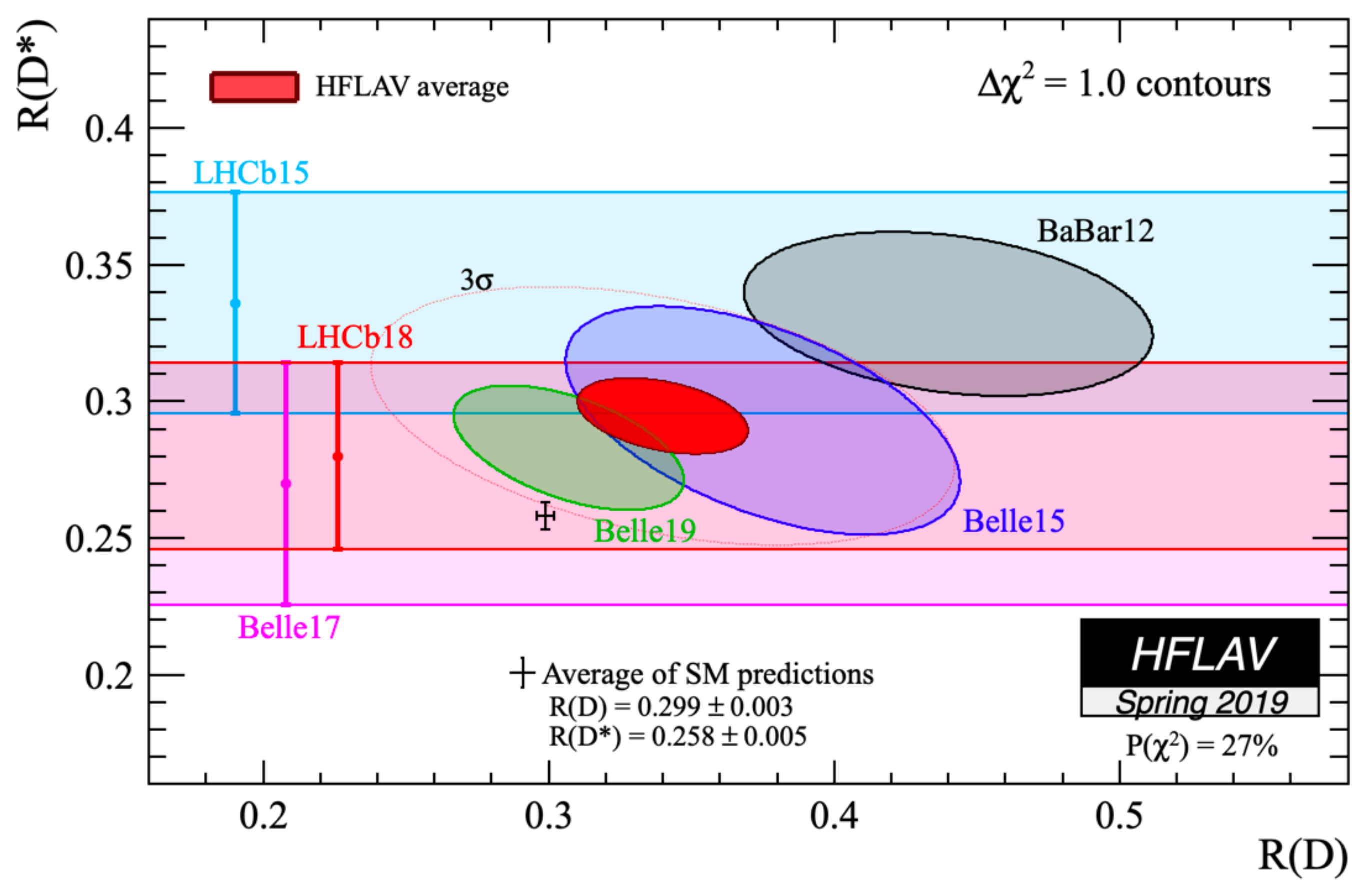
  - land of opportunity (LHCb, Belle II, BESIII; NA62, KOTO; $(g\text{-}2)_\mu$ programme; nEDM; ...)
  - **strong interaction effects key**



[HFLAV 2019]

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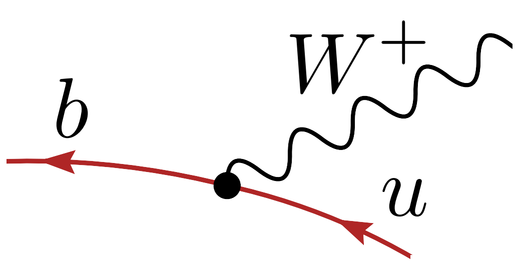
- **is the SM's CKM mechanism the only source of flavour-changing interactions, CP violation? [and: is LFU preserved?]**



$\propto g_{\rm w} V_{ub}$

$$V_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

[Cabibbo PRL 10 (1963) 531]
[Kobayashi, Maskawa Prog. Theor. Phys. 49 (1973) 652]

meeting the challenge from experiment

extremely active experimental programme in coming decade(s):

- heavy quark physics: LHCb, Belle II, BESIII (charm), ...
- kaon physics: NA62, KOTO, ...

lattice QCD needs to keep up with experimental precision — and make an effort to deliver PREdictions (including new physics).

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projections — including reduction in theory (lattice) uncertainty:

Observables	Belle (2017)	Belle II 5 ab^{-1}	Belle II 50 ab^{-1}
$\lvert V_{cb}\rvert$ incl.	$42.2 \cdot 10^{-3} \cdot (1 \pm 1.8\%)$	1.2%	–
$\lvert V_{cb}\rvert$ excl.	$39.0 \cdot 10^{-3} \cdot (1 \pm 3.0\%_{\text{ex.}} \pm 1.4\%_{\text{th.}})$	1.8%	1.4%
$\lvert V_{ub}\rvert$ incl.	$4.47 \cdot 10^{-3} \cdot (1 \pm 6.0\%_{\text{ex.}} \pm 2.5\%_{\text{th.}})$	3.4%	3.0%
$\lvert V_{ub}\rvert$ excl. (WA)	$3.65 \cdot 10^{-3} \cdot (1 \pm 2.5\%_{\text{ex.}} \pm 3.0\%_{\text{th.}})$	2.4%	1.2%
$\mathcal{B}(B \to \tau\nu)$ $[10^{-6}]$	$91 \cdot (1 \pm 24\%)$	9%	4%
$\mathcal{B}(B \to \mu\nu)$ $[10^{-6}]$	< 1.7	20%	7%
$R(B \to D\tau\nu)$ (Had. tag)	$0.374 \cdot (1 \pm 16.5\%)$	6%	3%
$R(B \to D^*\tau\nu)$ (Had. tag)	$0.296 \cdot (1 \pm 7.4\%)$	3%	2%

[Belle II Physics Book, arXiv:1808.10567]

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extremely active experimental programme in coming decade(s):

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- kaon physics: NA62, KOTO, ...

to do list:

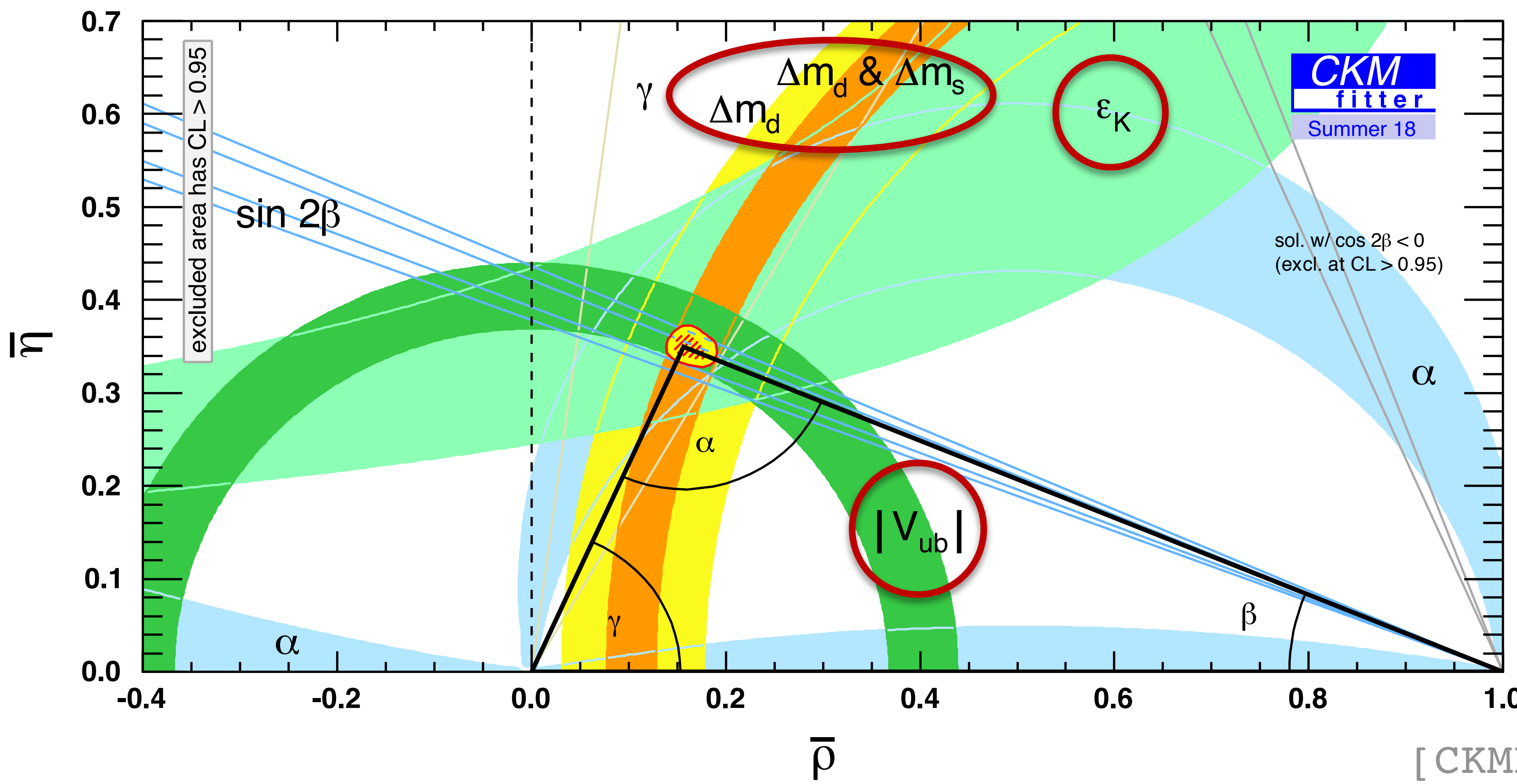
- bring precision standards of lattice B-physics to (or below) 1% for (semi)leptonic meson decay, **as already achieved in kaon sector**.
- ditto, few % in baryon channels, neutral meson mixing.
- make inroads in multihadron/(broad) resonance final states.
- long-distance OPE: rare decays, charm CP violation, ...

where we stand

global fit:

$$|V_{\rm CKM}| = \begin{pmatrix} 0.97446(10) & 0.22452(44) & 0.00365(12) \\ 0.22438(44) & 0.97359^{(10)}_{(11)} & 0.04214(76) \\ 0.00896^{(24)}_{(23)} & 0.04133(74) & 0.999105(32) \end{pmatrix}$$

[PDG 2018]



[CKMFitter 2018]

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[PDG 2018]

exclusive determination with lattice input: errors between few permille (light, strange) and few percent (charm, bottom)

$$|V_{\mathrm{CKM}}| = \begin{pmatrix} 0.97437(16) & 0.2249(7) & 0.00373(14) \\ 0.2166(7)(50) & 1.004(2)(16) & 0.0401(10) \\ & & \end{pmatrix}$$

[FLAG 2018]

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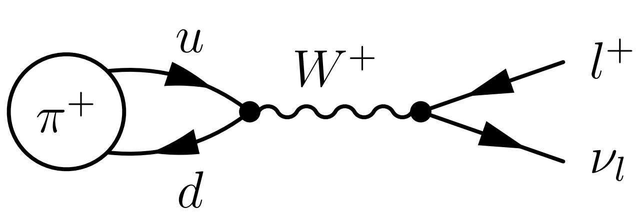
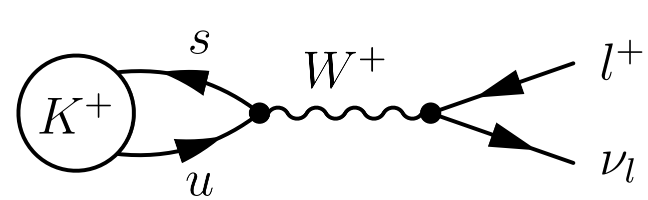
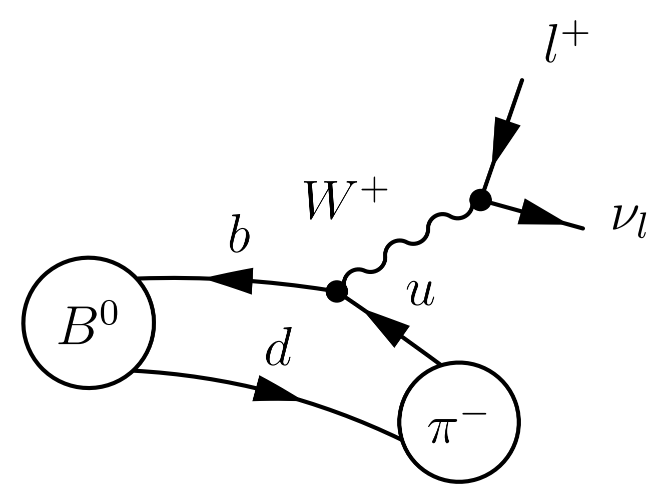
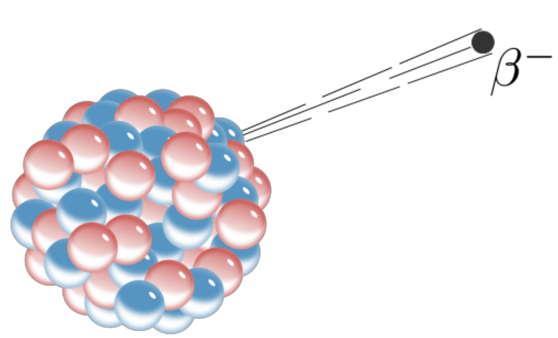
[FLAG 2018]

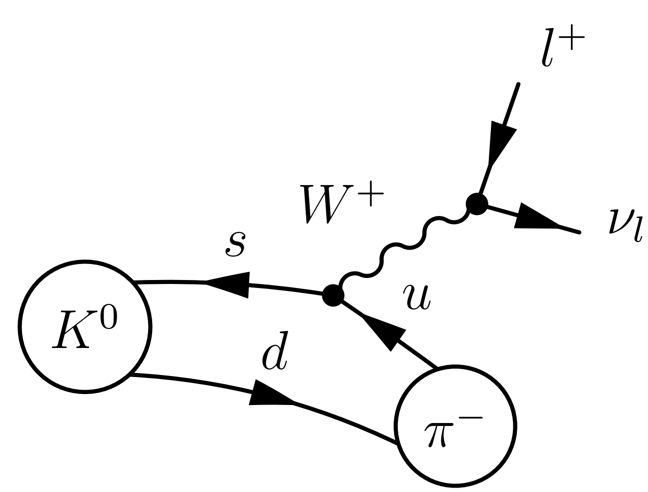
1st row:











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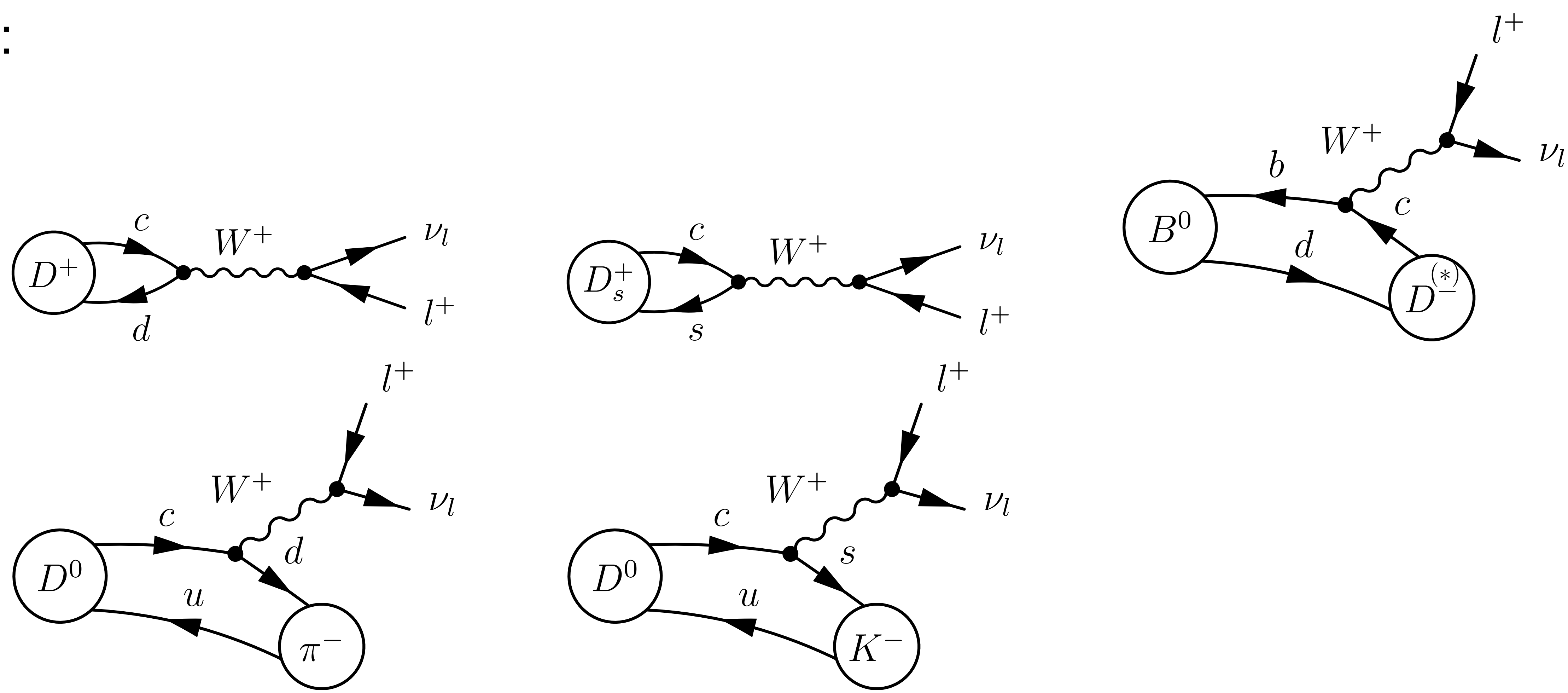
[FLAG 2018]

2nd row:



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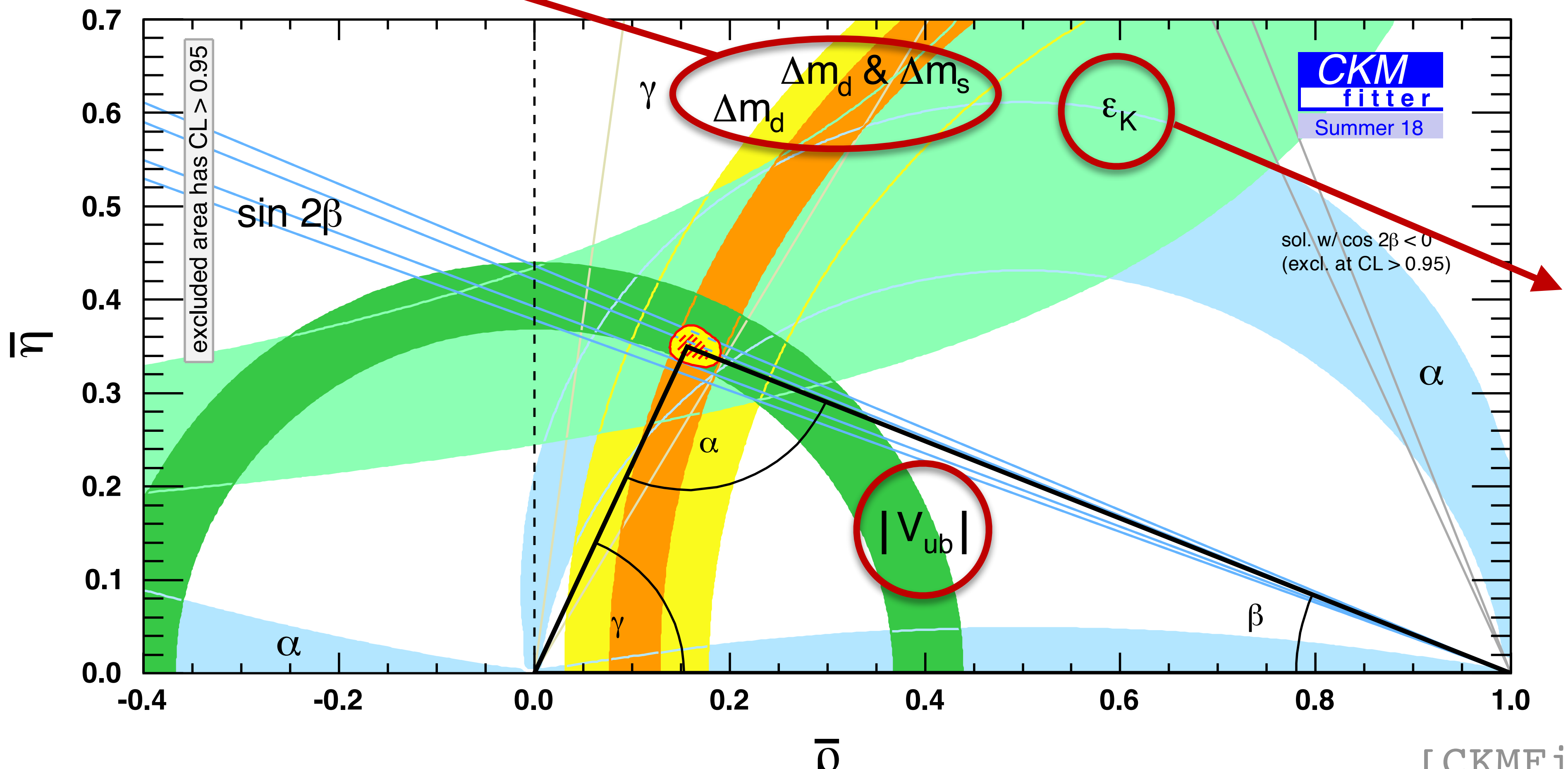
[FLAG 2018]

$f^2_{B_{(s)}} B_{B_{(s)}}$

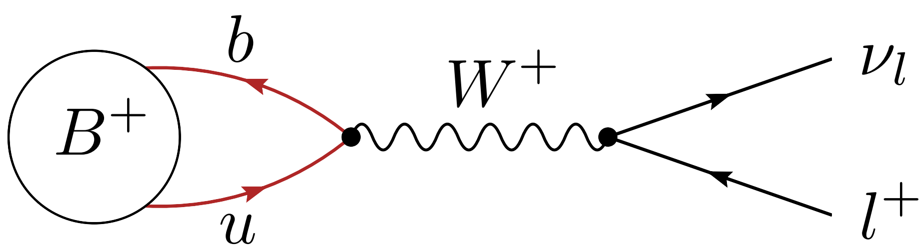


$B_K, |V_{cb}|^4$

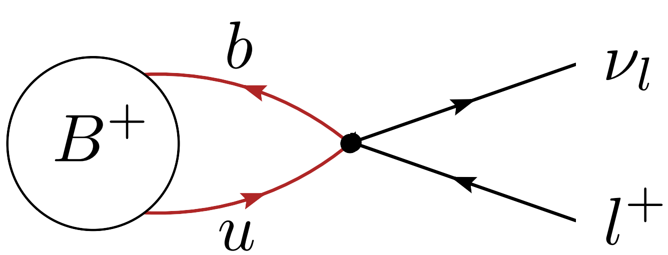
[CKMFitter 2018]

OPE for weak decays of hadrons

electromagnetic corrections to hadronic weak matrix elements traditionally neglected in lattice studies.



$$\frac{g_{\mathrm{w}}^2}{p^2 - M_W^2} = -2\sqrt{2}G_{\mathrm{F}}\left[1 + \mathcal{O}\left(\frac{p^2}{M_W^2}\right)\right]$$



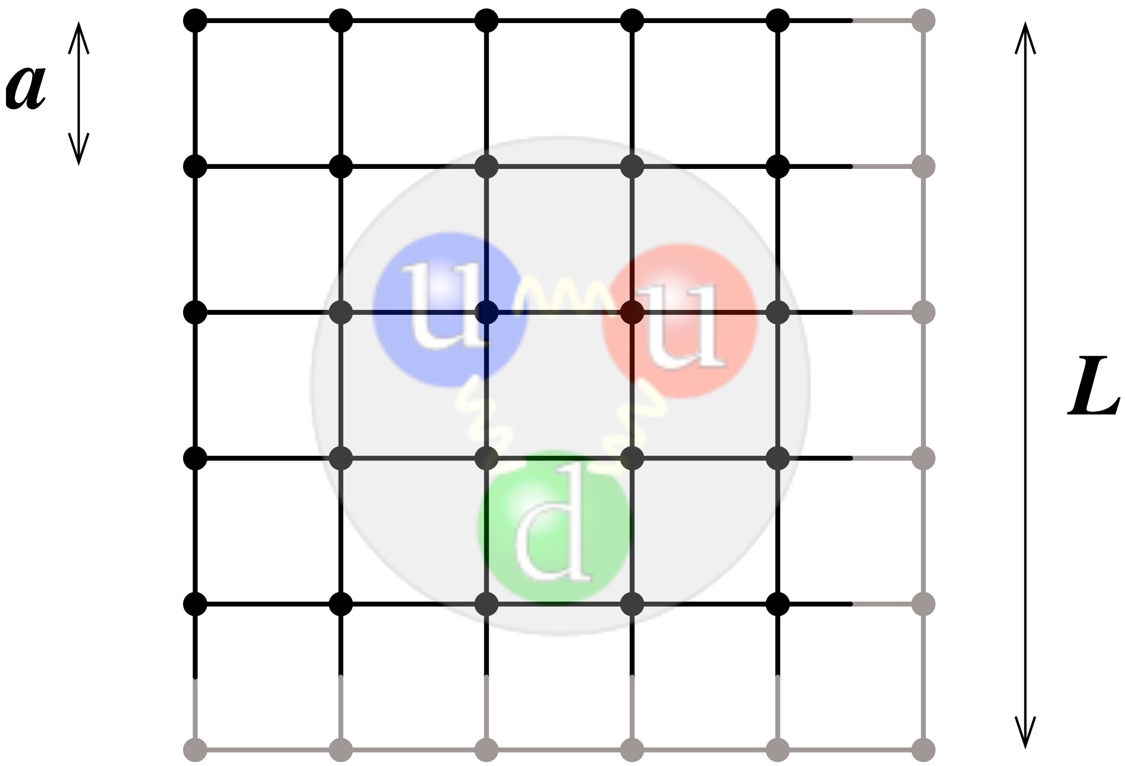
$$\propto \langle l^+\nu_l|(\bar{b}_{\mathrm{L}}\gamma_\mu u_{\mathrm{L}})(\bar{l}_{\mathrm{L}}\gamma^\mu \nu_{\mathrm{L}})|B^+\rangle_{\mathrm{QCD+QED}}$$
$$= \langle 0|\bar{b}\gamma_\mu\gamma_5 u|B^+\rangle_{\mathrm{QCD}} \times \;(\mathrm{EM})\; + \;\mathcal{O}(\alpha_{\mathrm{em}})$$

as precision has started to approach percent levels, estimation of e.m. effects has become an issue.

lattice QCD

$$\mathcal{L}_{\rm QCD} = -\frac{1}{2g^2}\mathrm{tr}\left[F_{\mu\nu}F^{\mu\nu}\right] + \sum_{q=1}^{N_{\rm f}} \bar{\psi}_q \left[i\not{D} - m_q\right]\psi_q + \underbrace{\frac{i\theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\left[F_{\mu\nu}F_{\rho\sigma}\right]}_{\not{C}\not{P}}$$

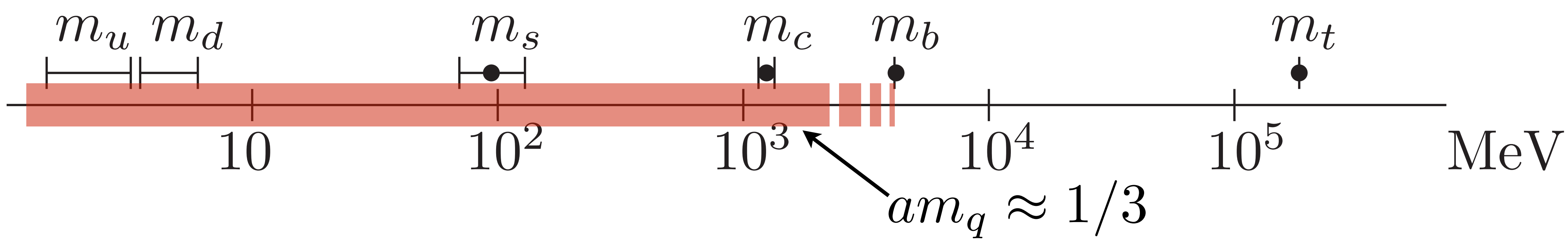
first-principles approach = control all systematic uncertainties

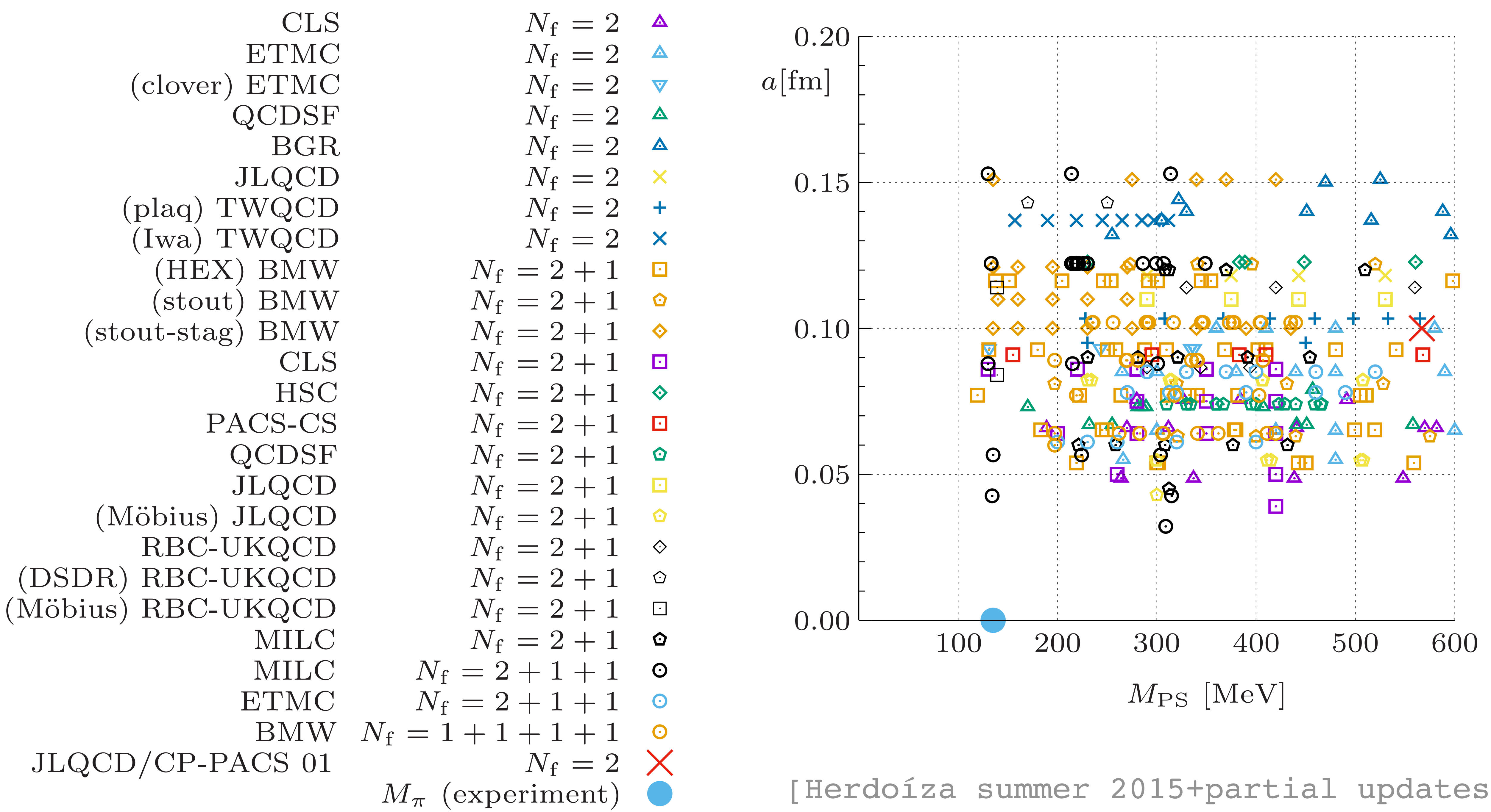


[Wilson 1974]

- spacetime = Euclidean lattice
- allows to define path integral rigorously *and* compute it via Monte Carlo methods
- QCD recovered by removing cutoffs at physical kinematics
- values of Lagrangian parameters fixed by $N_{\rm f}$+1 hadron masses/decay constants — everything else are **predictions**

physics reach





[Herdoíza summer 2015+partial updates]

lattice QCD for phenomenology: FLAG

Flavour Lattice Averaging Group: your one-stop repository of lattice results, world averages / estimates

FLAG2019 4th edition: results up to 2018/09/30 [Aoki et al., arXiv:1902.08191]

advisory board: S Aoki, M Golterman, R Van de Water, A Vladikas

editorial board: G Colangelo, S Hashimoto, A Jüttner, S Sharpe, U Wenger

working groups:

quark masses	T Blum, A Portelli, A Ramos
V_{ud}, V_{us}	S Simula, T Kaneko, JN Simone
LECs	S Dürr, H Fukaya, UM Heller
kaon mixing	P Dimopoulos, G Herdoíza B Mawhinney
heavy leptonic + mixing	D Lin, Y Aoki, M Della Morte
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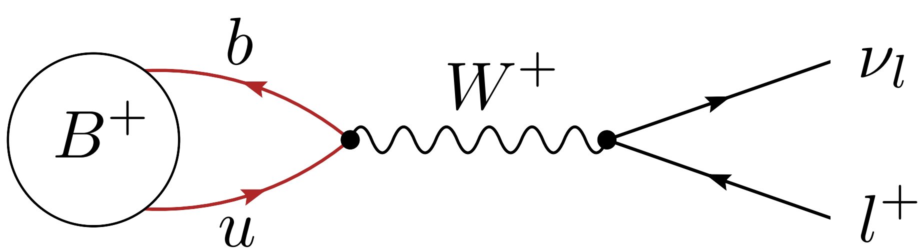
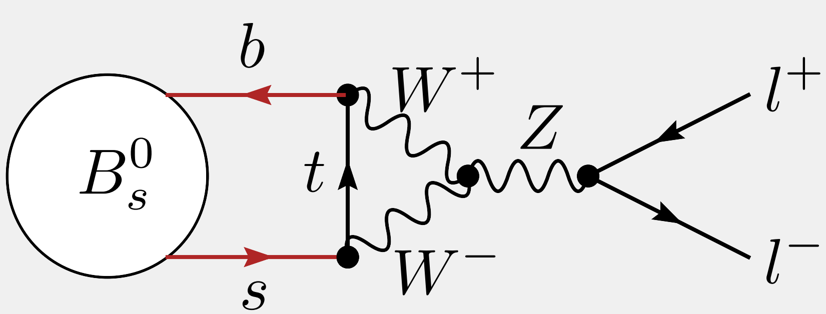
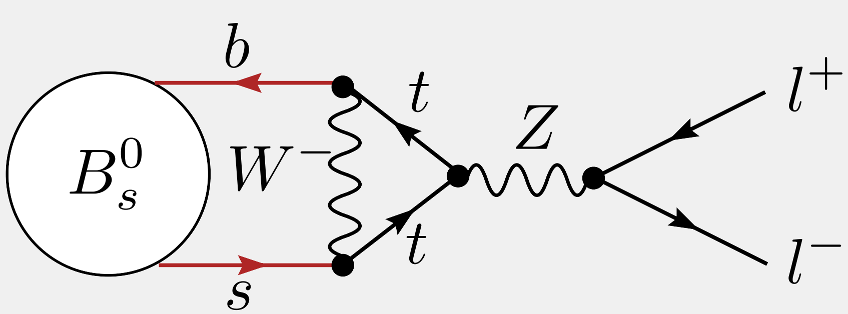
leptonic decay

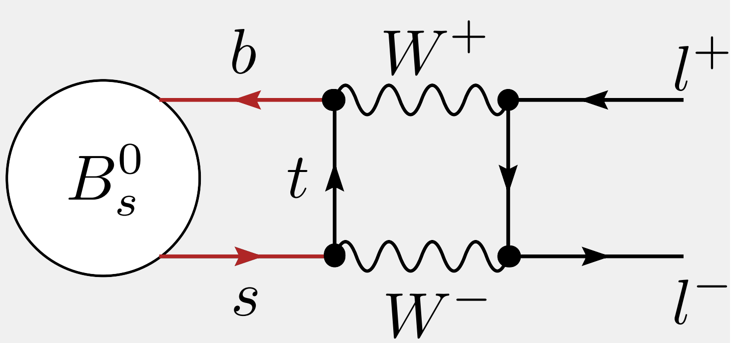


$$\frac{\mathcal{B}(B^+ \to l^+\nu_l)}{\tau_{B^+}} = \frac{G_{\mathrm{F}}^2}{8\pi} m_l^2 m_{B^+}^2 \left(1 - \frac{m_l^2}{m_{B^+}^2}\right)^2 |V_{ub}|^2 f_{B^+}^2$$

$$\langle 0|\bar{b}\gamma^\mu\gamma^5 q|B^+(p)\rangle = f_{B^+} p^\mu$$

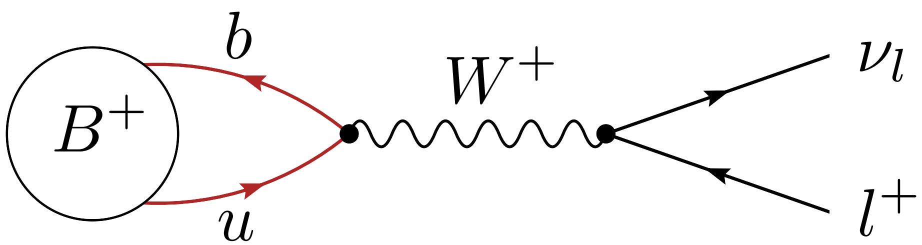






$$\frac{\mathcal{B}(B_q \to l^+l^-)}{\tau_{B_q}} = \frac{G_{\mathrm{F}}^2}{\pi} Y\left(\frac{\alpha}{4\pi\sin^2\theta_{\mathrm{W}}}\right)^2 m_{B_q} m_l^2 \sqrt{1 - 4\frac{m_l^2}{m_{B_q}^2}}\, |V_{tb}^* V_{tq}|^2 f_{B_q}^2$$

leptonic decay



$$\frac{\mathcal{B}(B^+ \to l^+\nu_l)}{\tau_{B^+}} = \frac{G_F^2}{8\pi} m_l^2 m_{B^+}^2 \left(1 - \frac{m_l^2}{m_{B^+}^2}\right)^2 |V_{ub}|^2 f_{B^+}^2$$

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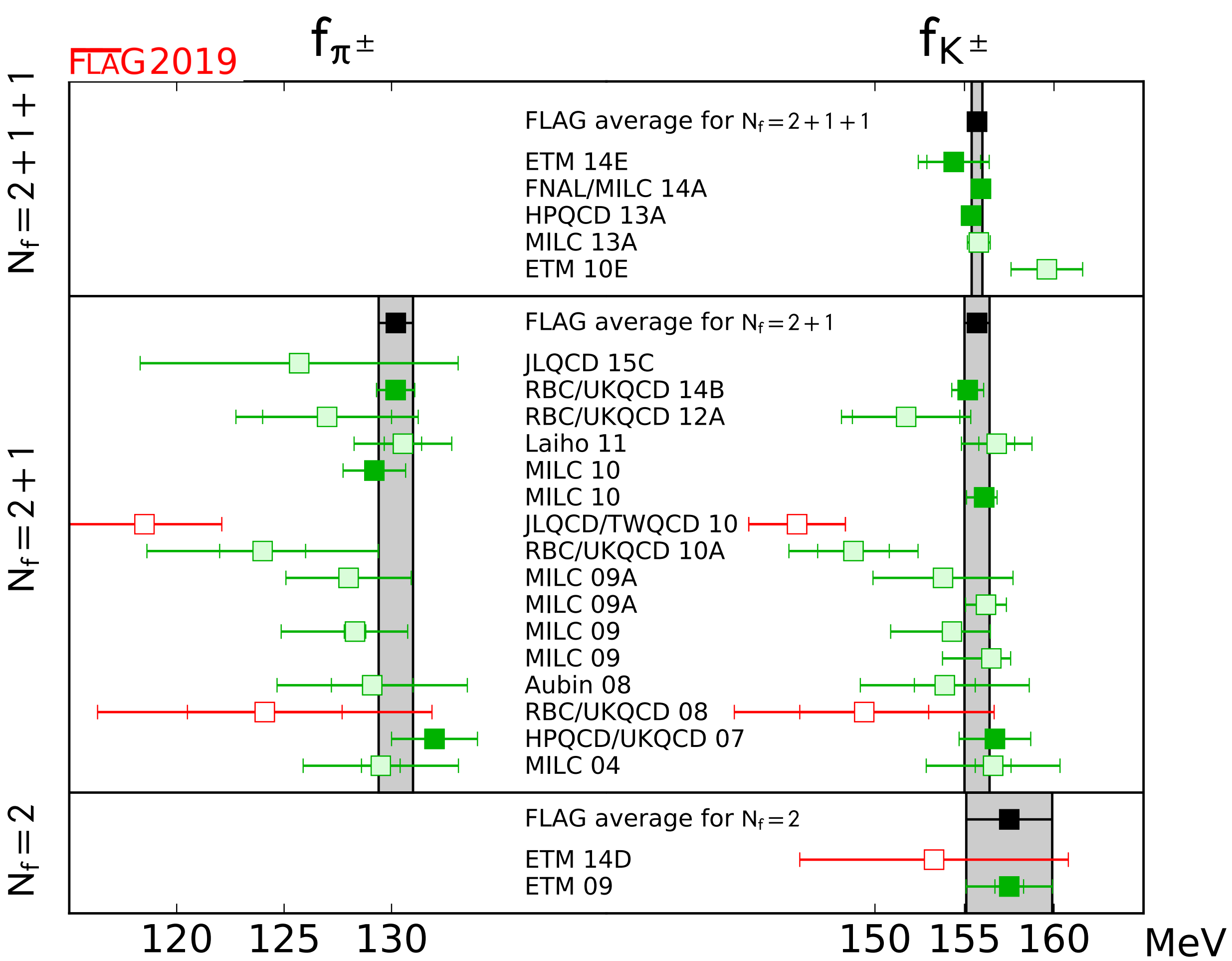
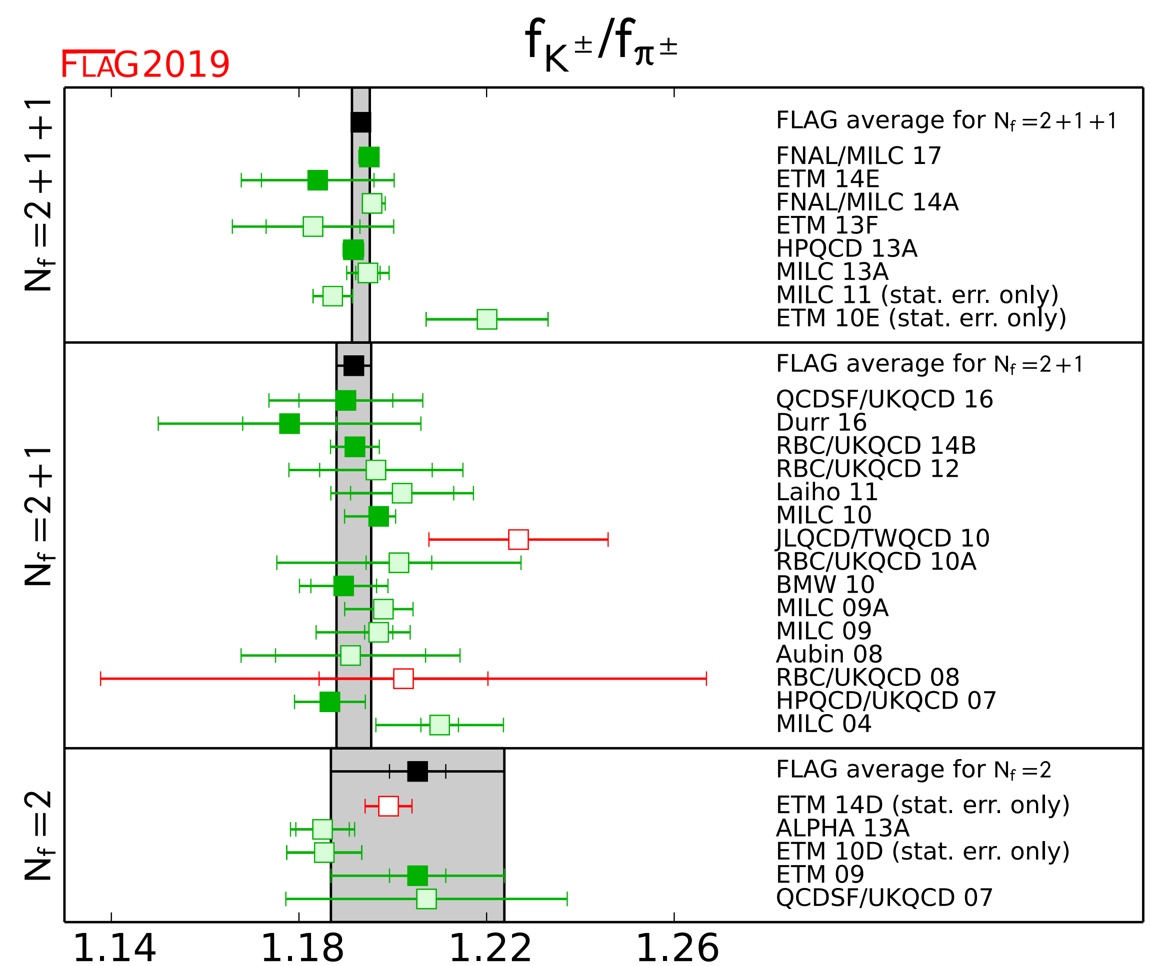
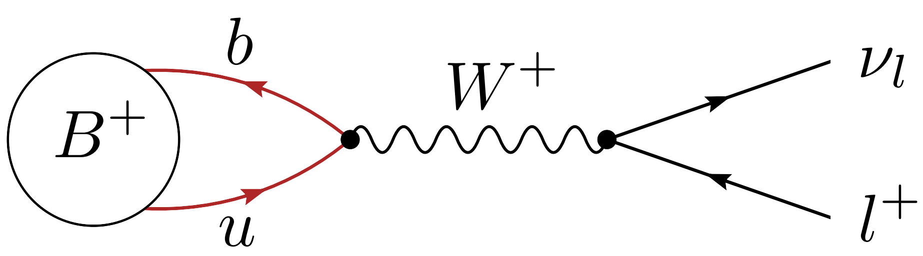




precision at **few-per-mille**, QED+IB corrections crucial for next stage

leptonic decay



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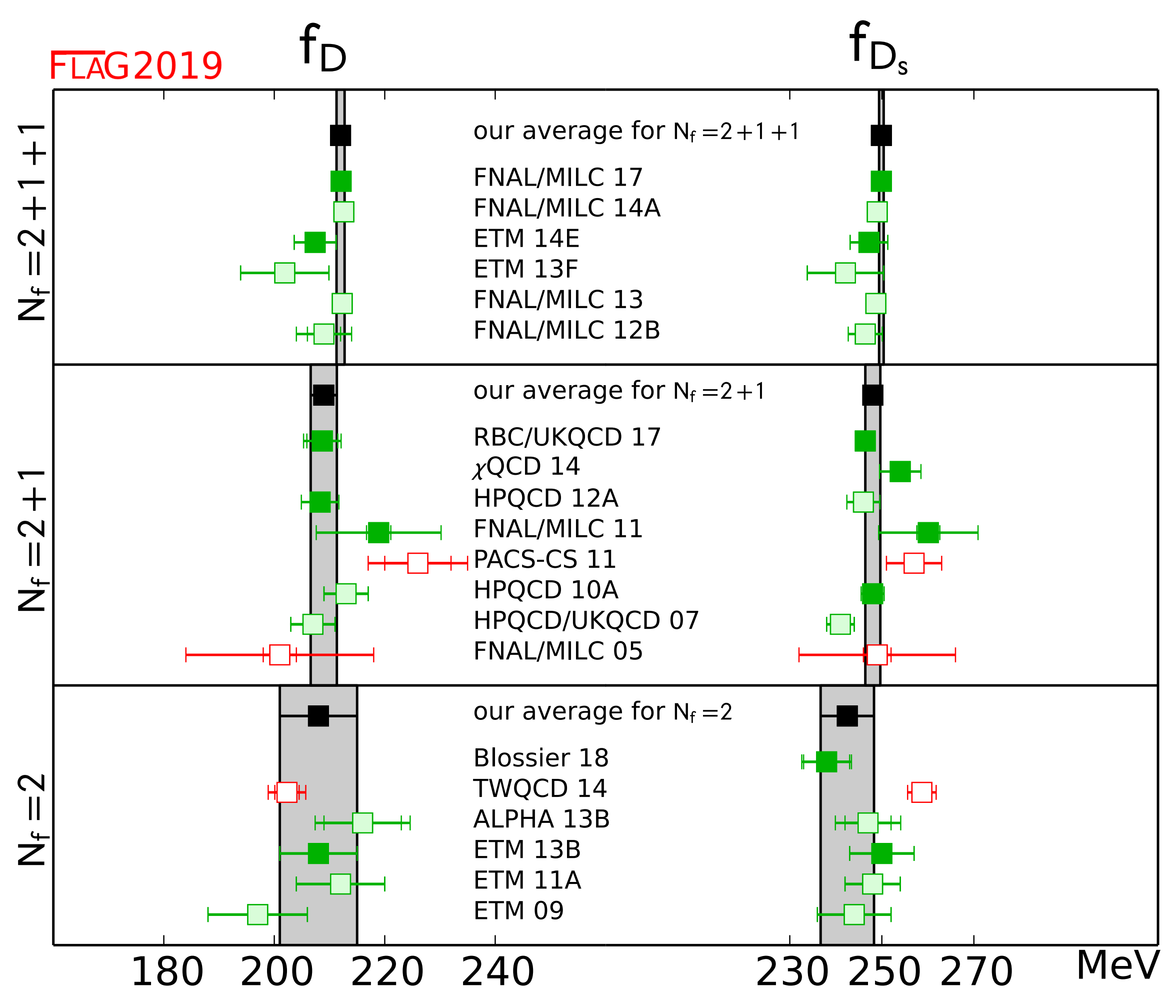

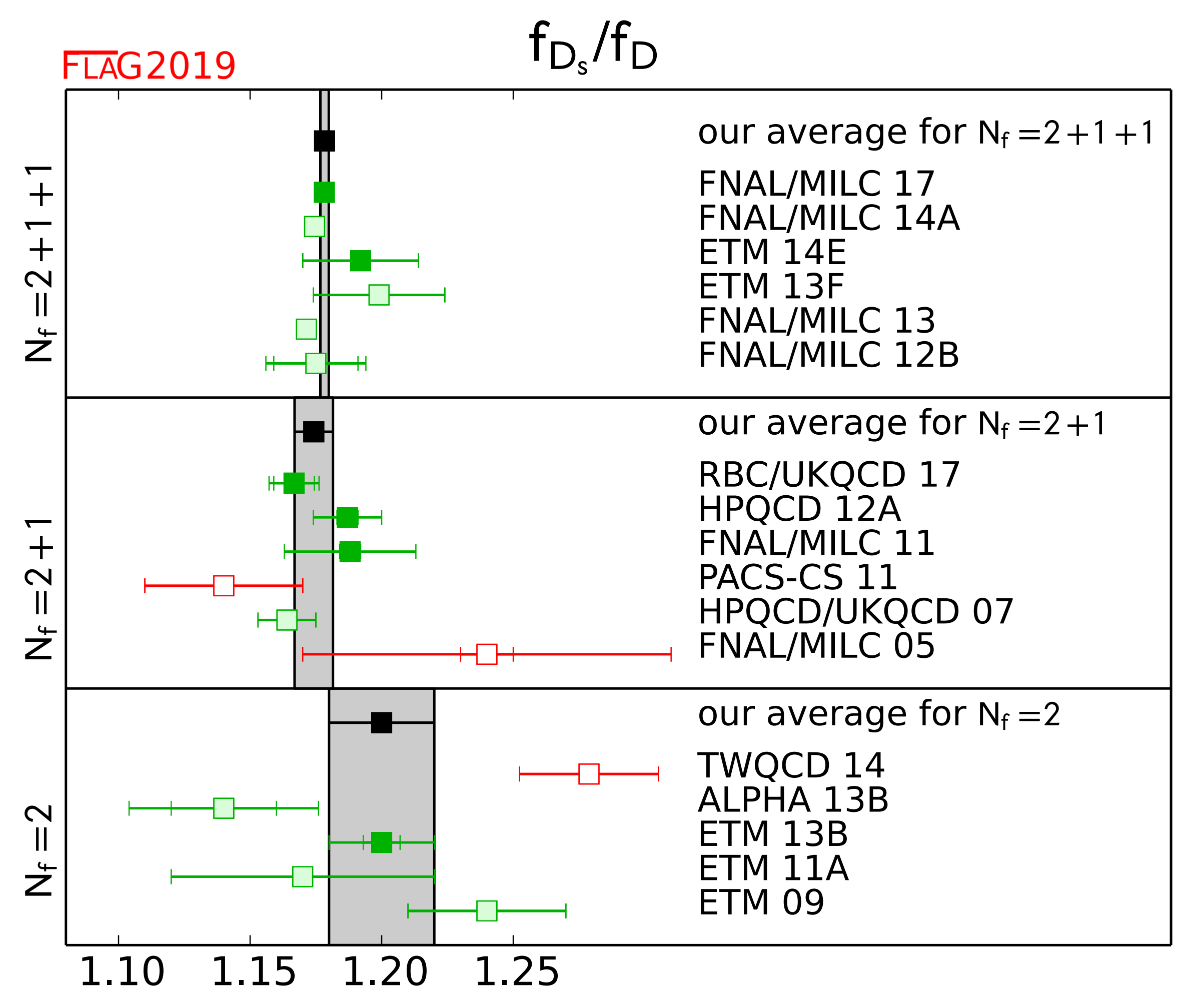

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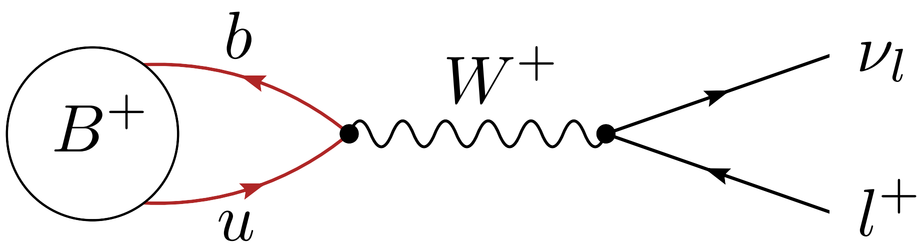
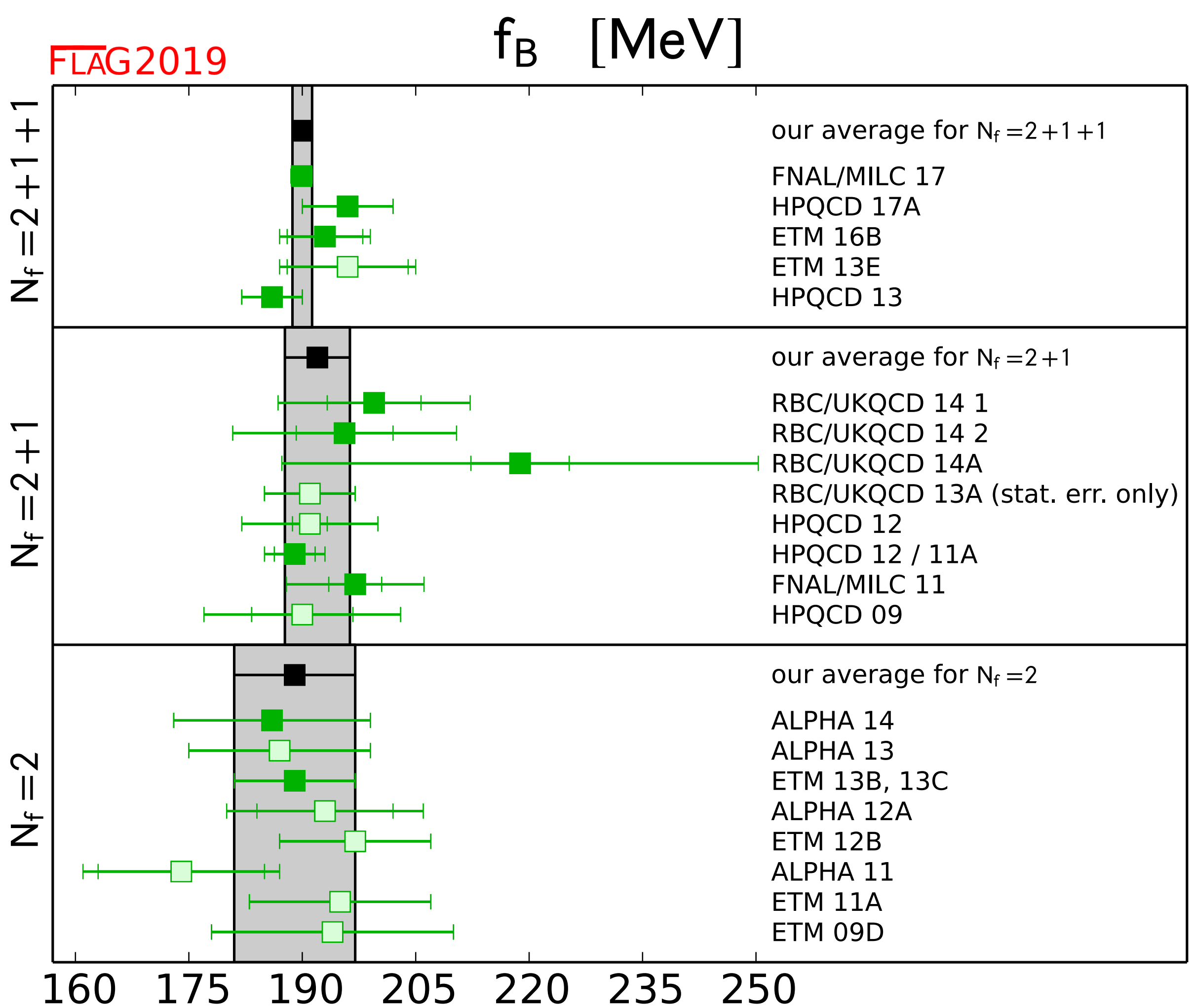
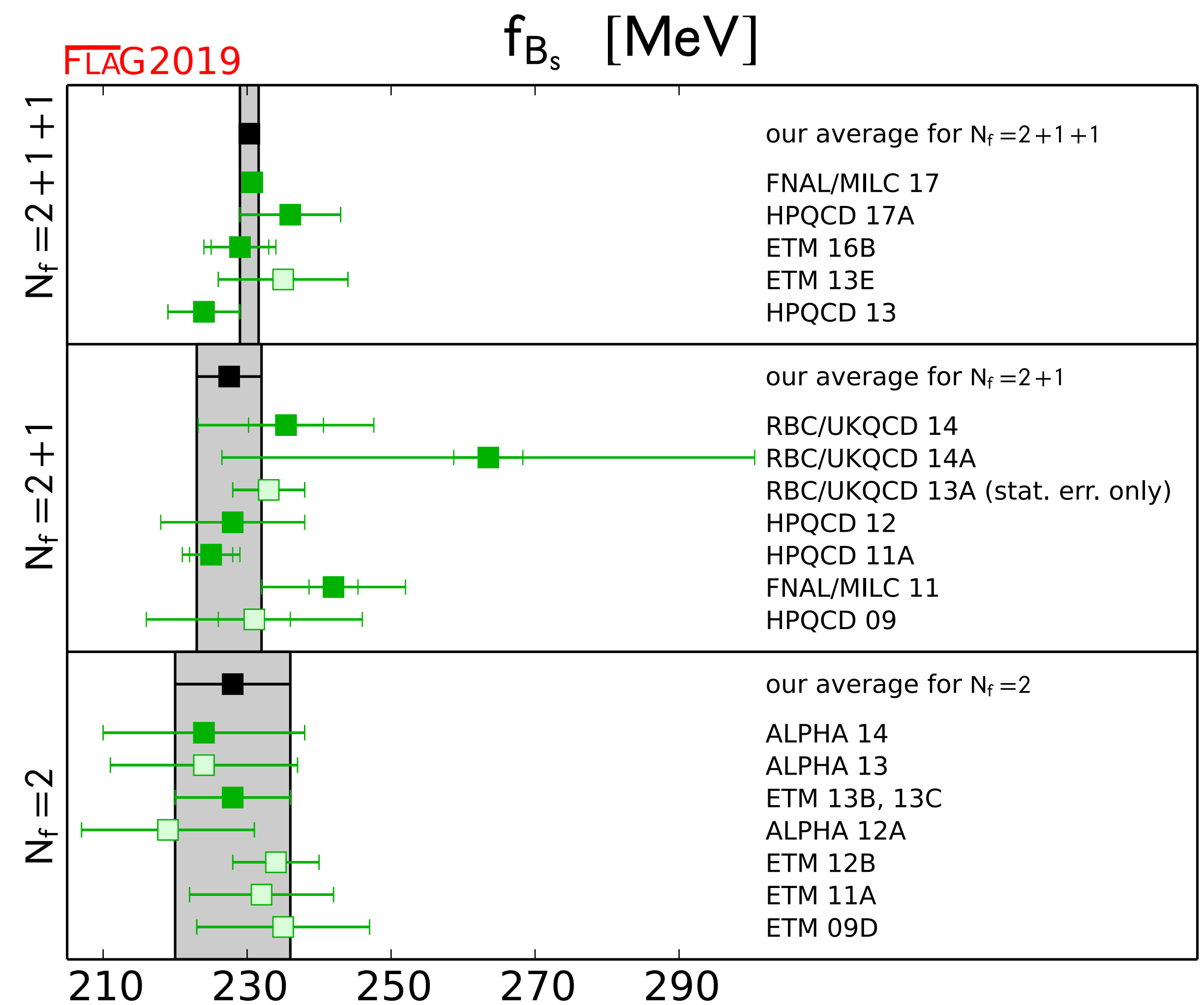
leptonic decay



$$\frac{\mathcal{B}(B^+ \to l^+ \nu_l)}{\tau_{B^+}} = \frac{G_F^2}{8\pi} m_l^2 m_{B^+}^2 \left(1 - \frac{m_l^2}{m_{B^+}^2}\right)^2 |V_{ub}|^2 f_{B^+}^2$$

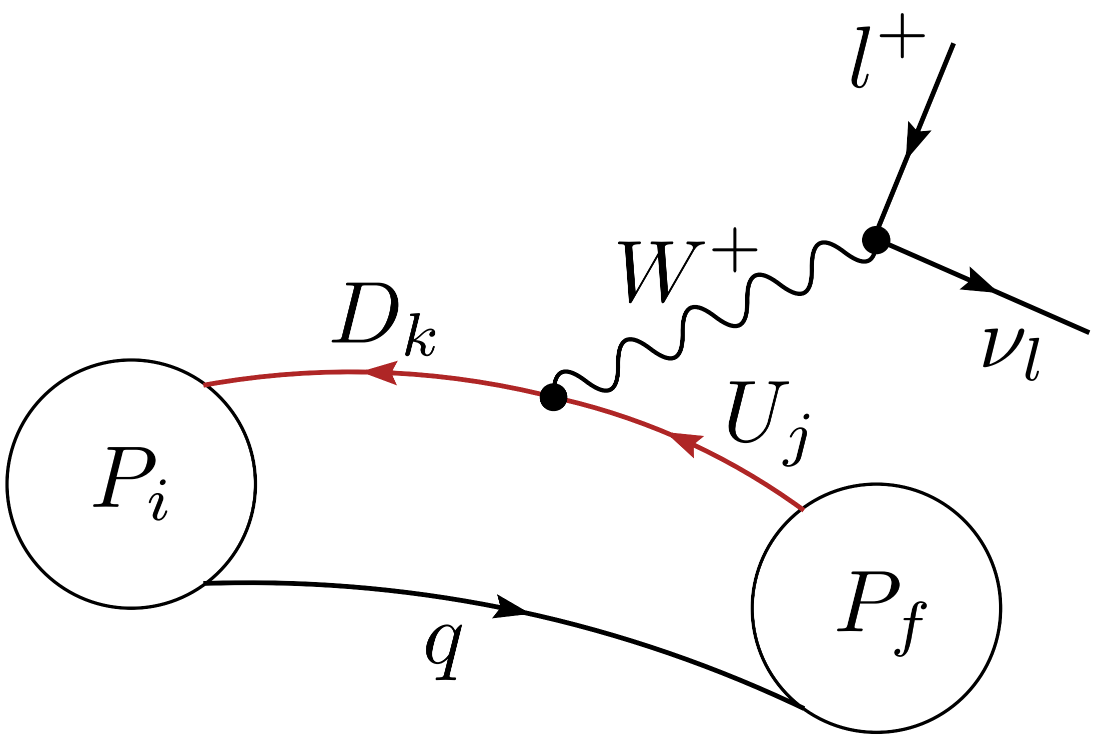
$$\langle 0 | \bar{b} \gamma^\mu \gamma^5 q | B^+(p) \rangle = f_{B^+} p^\mu$$





precision reaching **sub-percent**, QED+IB corrections crucial for next stage

semileptonic decay

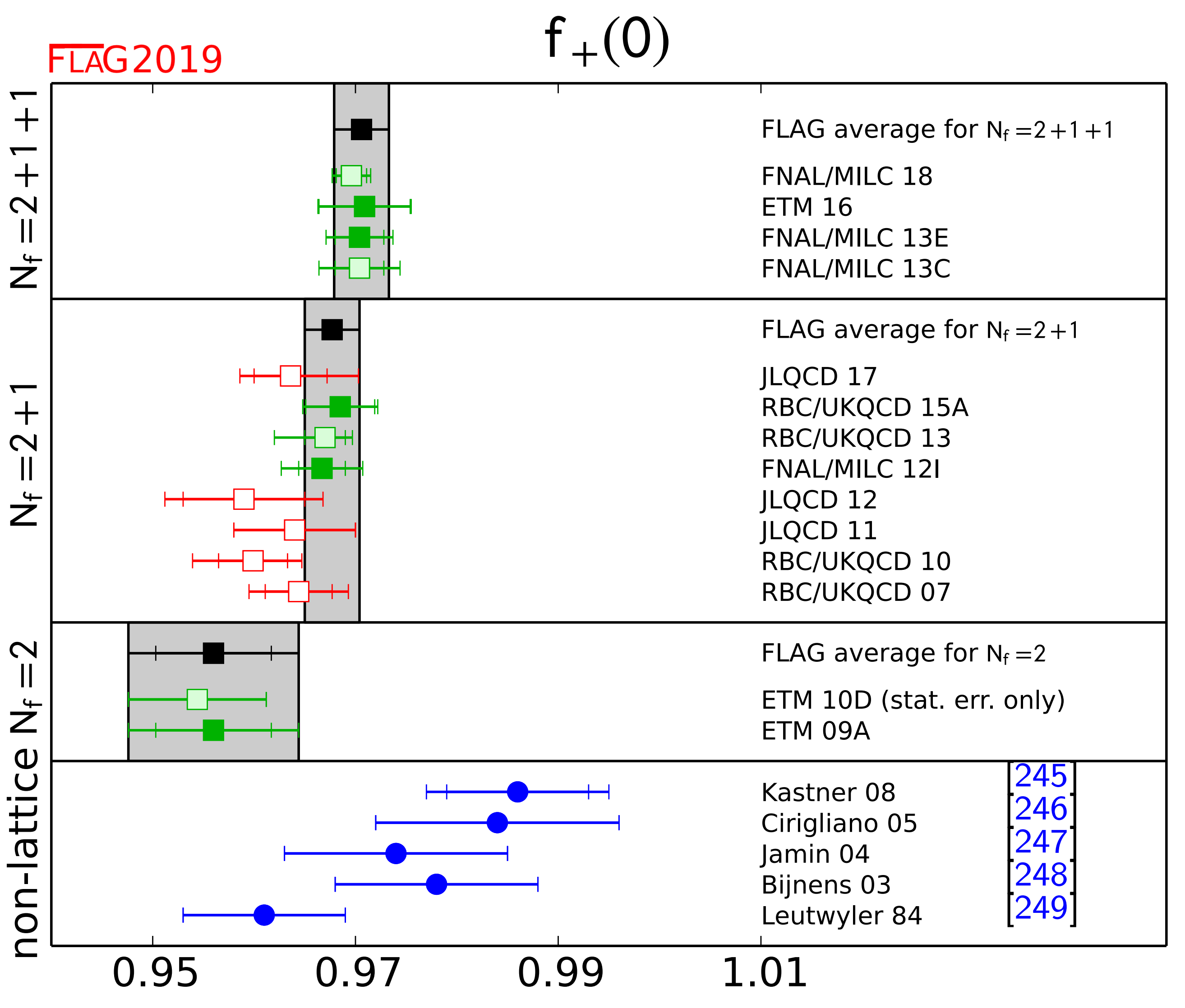


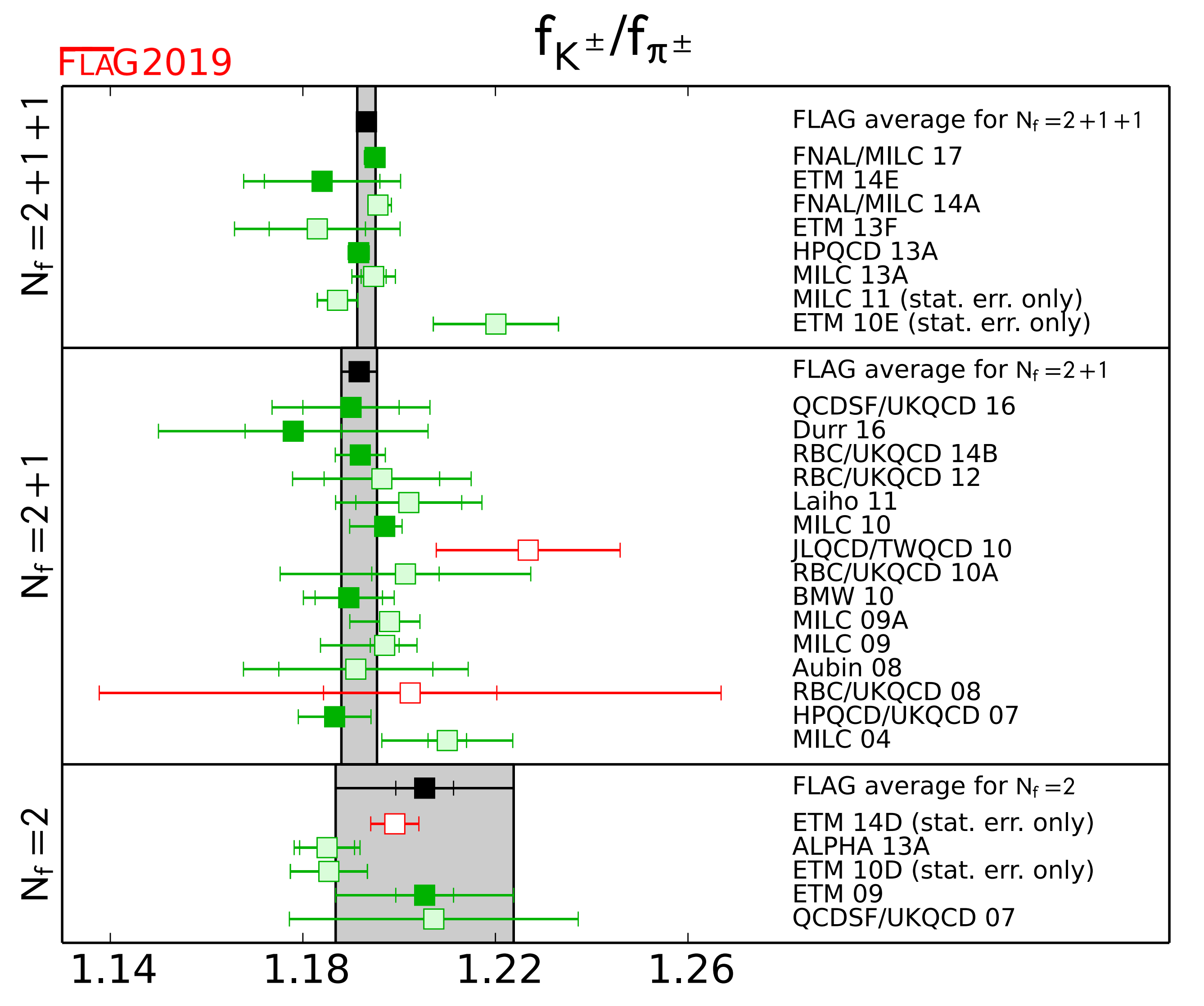
$$\frac{\mathrm{d}\Gamma(P_i \to P_f l \nu)}{\mathrm{d}q^2} = \frac{G_{\mathrm{F}}^2 |V_{jk}|^2}{24\pi^3} \frac{(q^2 - m_l^2)^2 \sqrt{E_f^2 - m_f^2}}{q^4 m_i^2}$$

$$\times \left[\left(1 + \frac{m_l^2}{2q^2}\right) m_i^2 (E_f^2 - m_f^2) |f_+(q^2)|^2 + \frac{3 m_l^2}{8 q^2} (m_i^2 - m_f^2)^2 |f_0(q^2)|^2 \right]$$

$$\langle P_f(p') | \bar{D}_k \gamma_\mu U_j | P_i(p) \rangle = f_+(q^2) \left(p_\mu + p'_\mu - \frac{m_i^2 - m_f^2}{q^2} q_\mu \right) + f_0(q^2) \frac{m_i^2 - m_f^2}{q^2} q_\mu \,, \qquad q = p - p'$$

semileptonic decay: $K \to \pi$

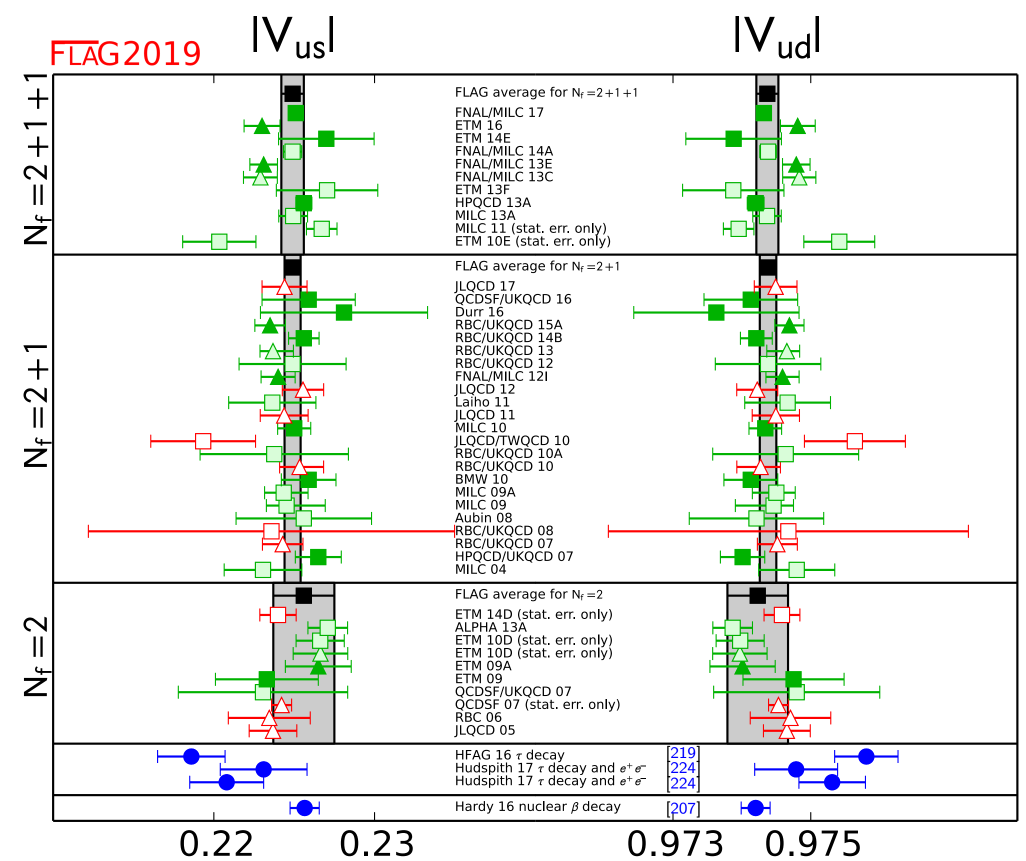




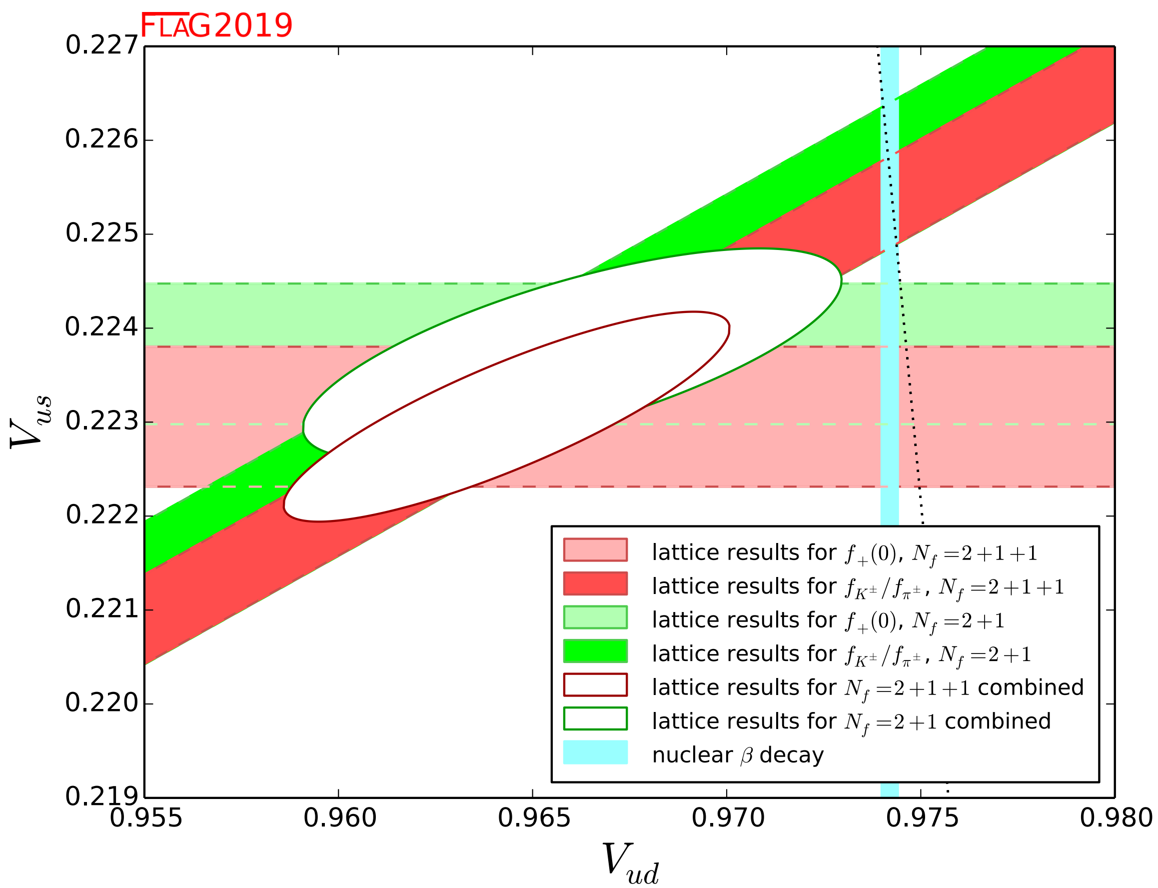
precision at **few-per-mille**, QED+IB corrections crucial for next stage

semileptonic decay: $K \to \pi$

precision for CKMs still theory-dominated (exp results for K decay much more precise)



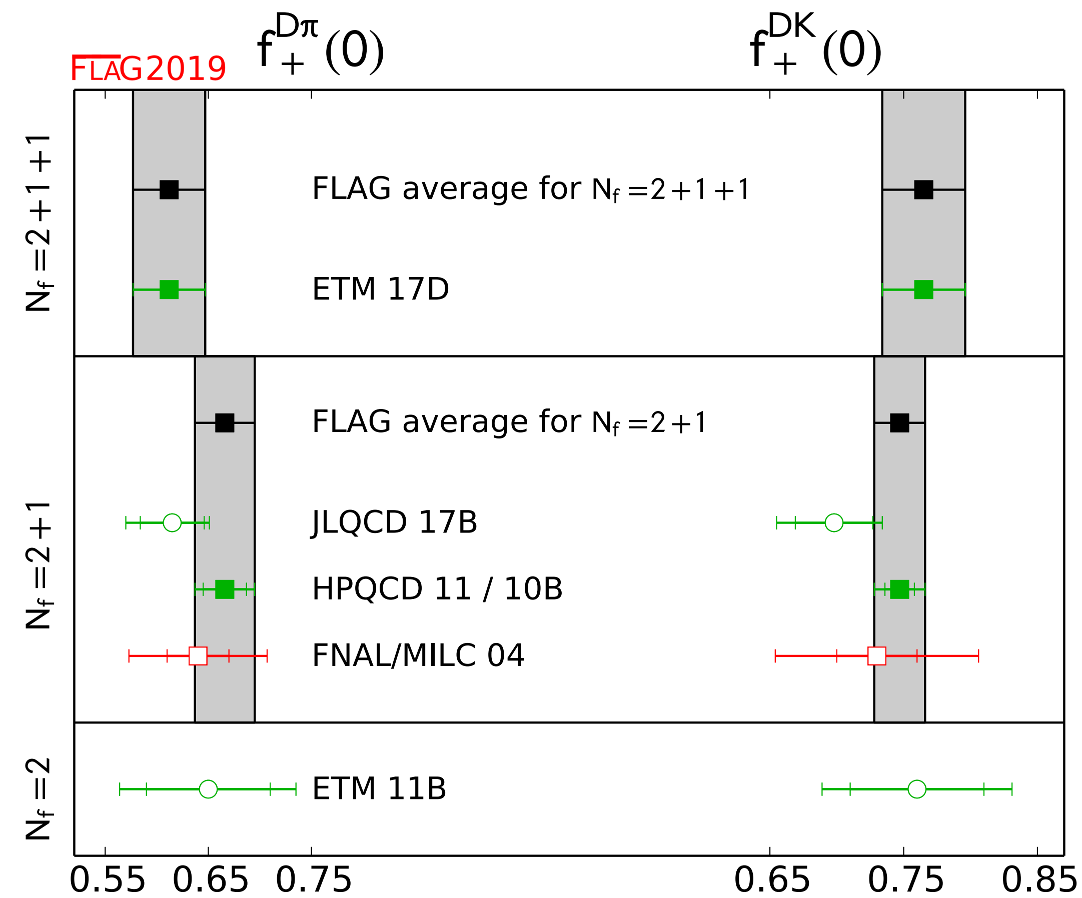
negligible dependence on charm mass, good agreement among various determinations.

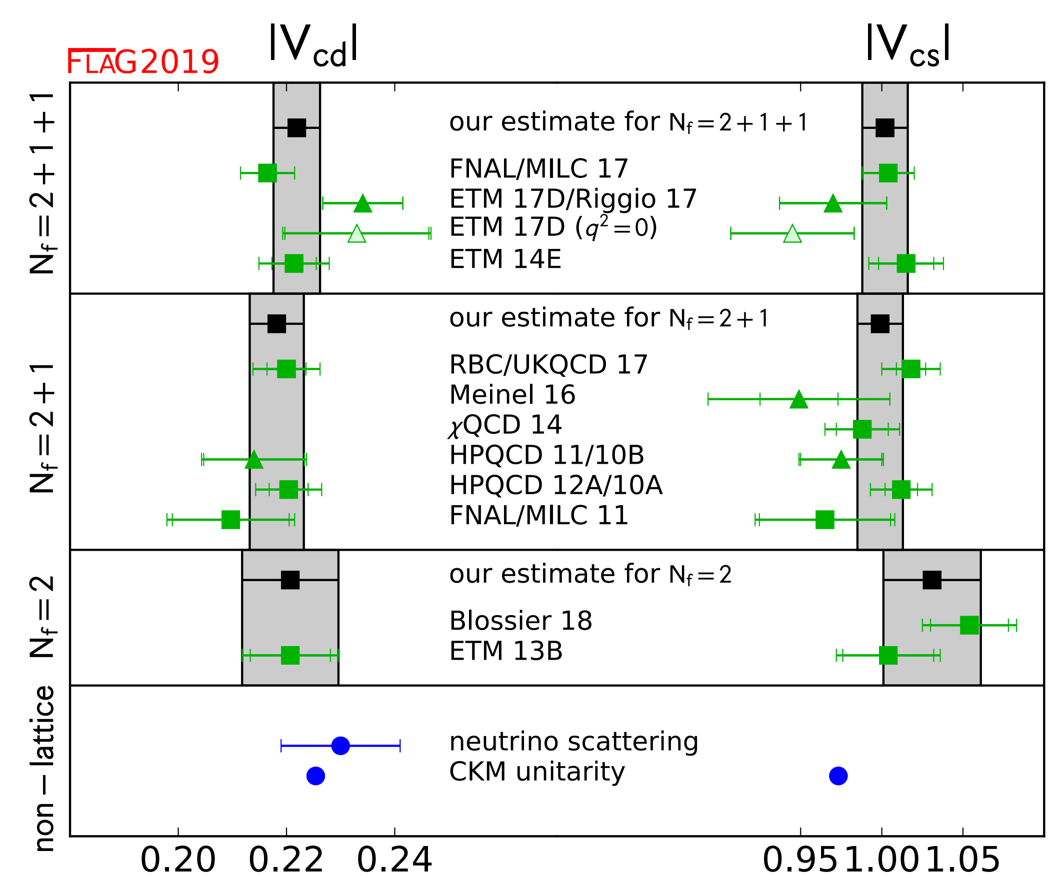


semileptonic decay: $D \to \pi$, $D \to K$

very few results, although ETMC (+ ongoing FNAL/MILC) has the first computation of the q^2 dependence of all form factors

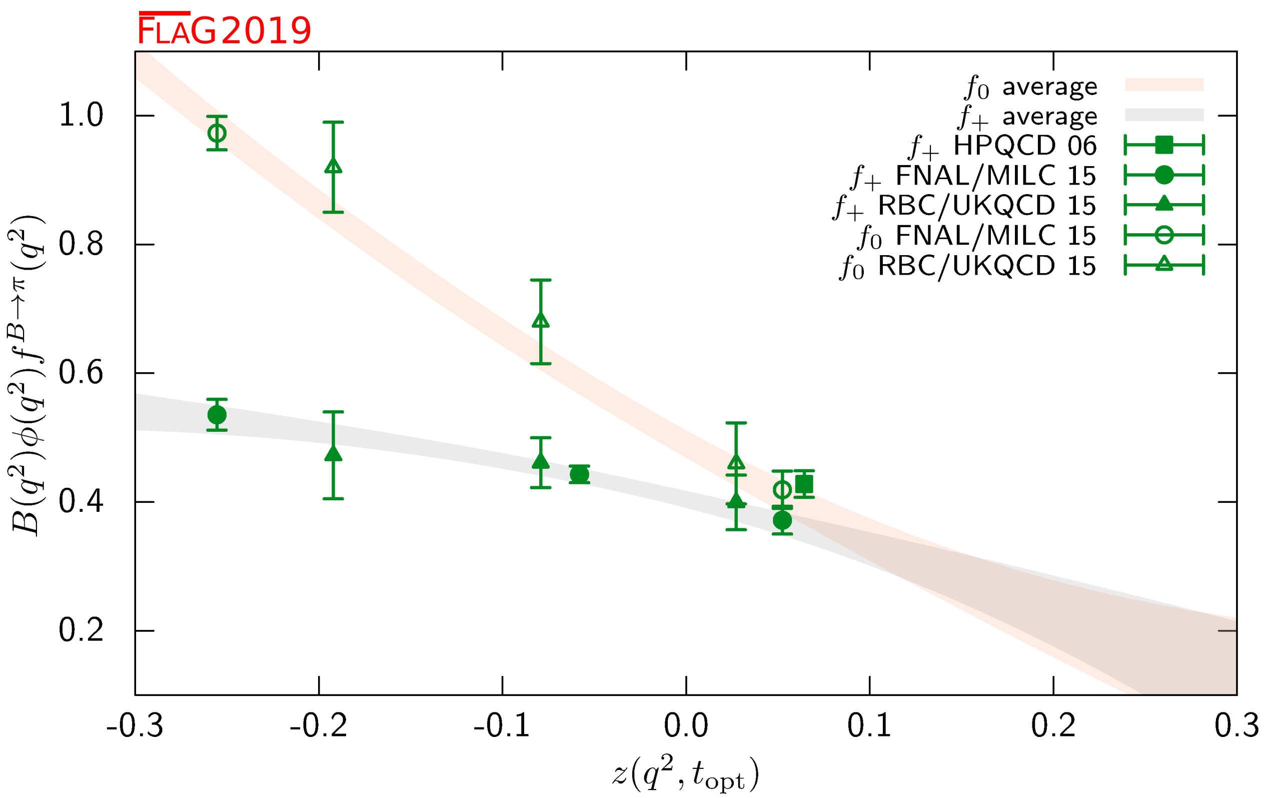
(**relevant:** extrapolation of exp rates to q^2=0 already sensitive to parametrisation)

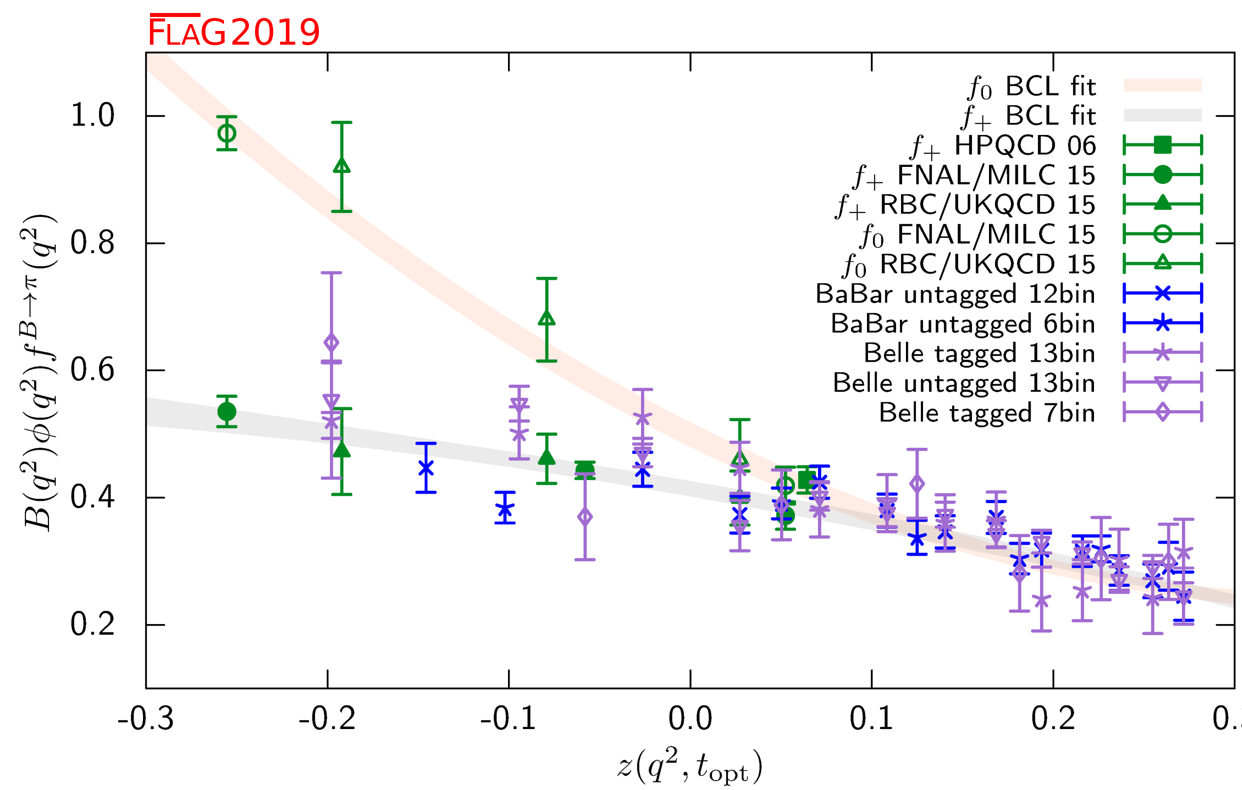


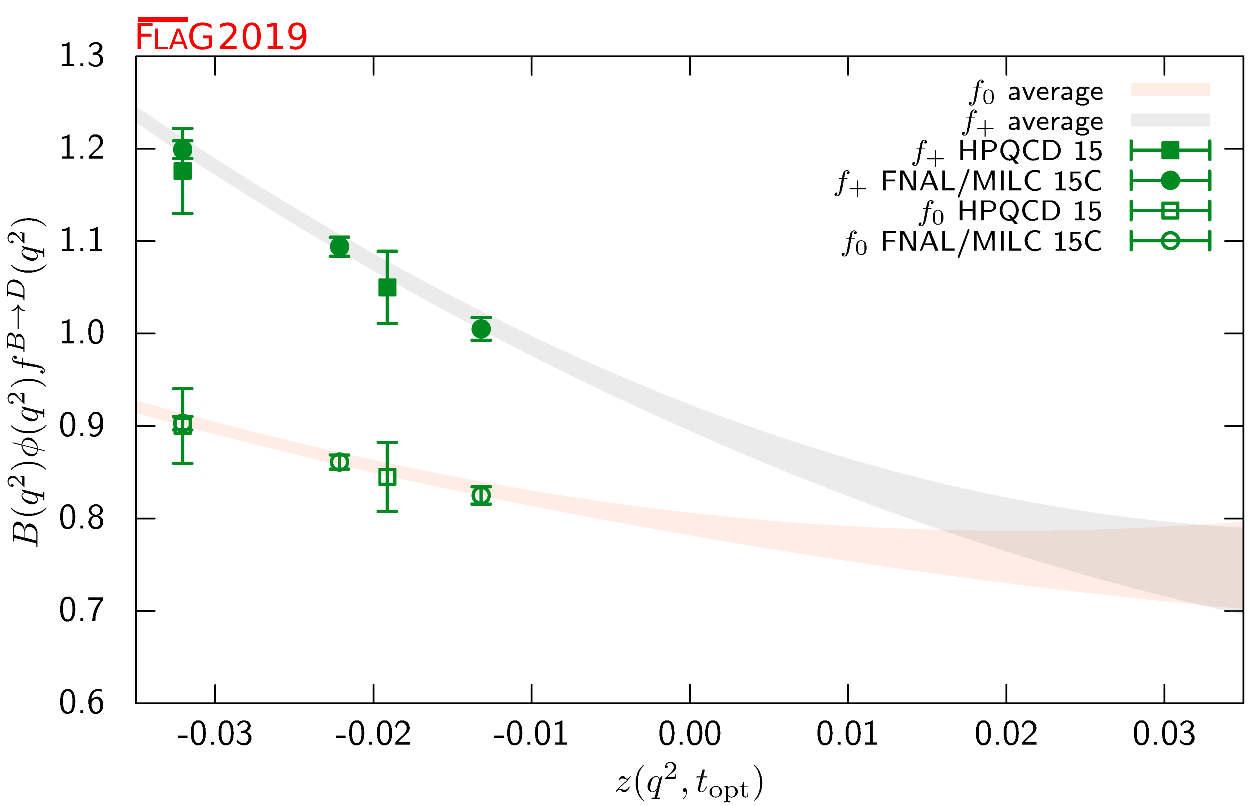


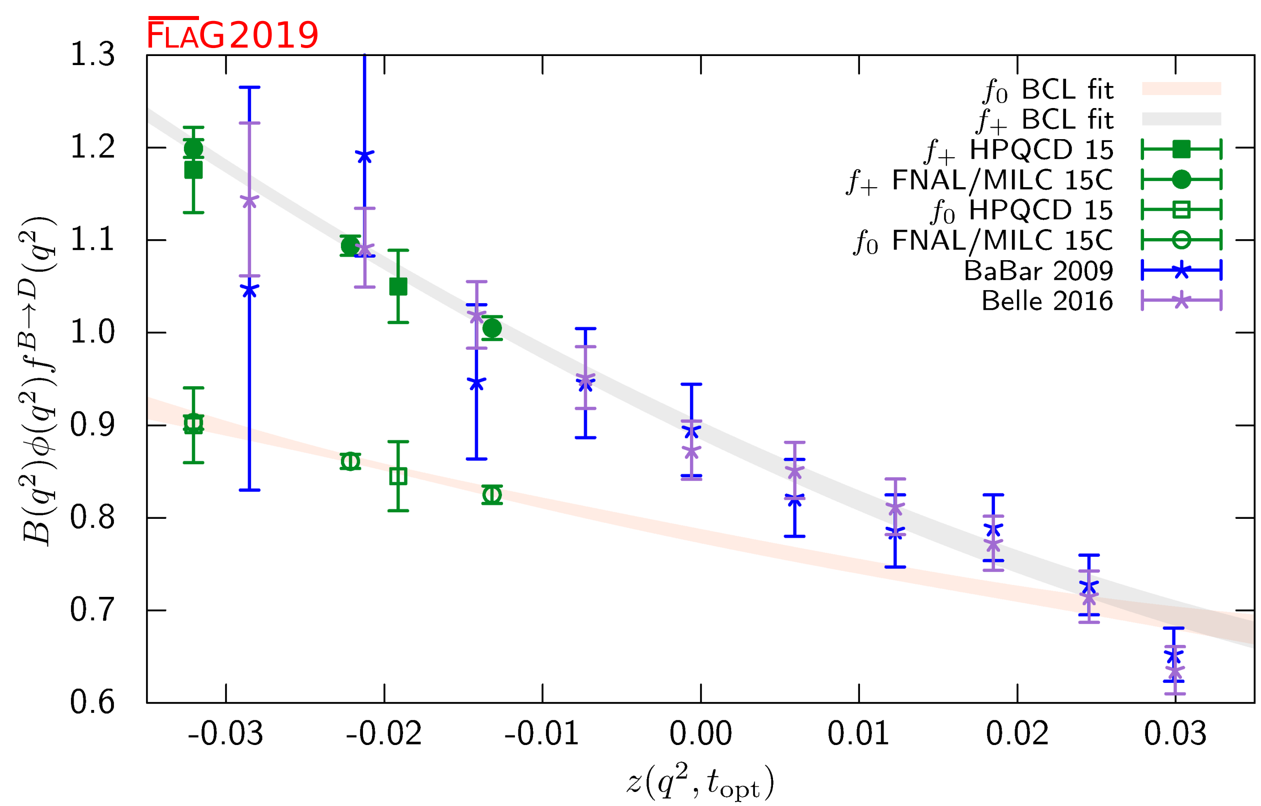
semileptonic decay: $B \to D$, $B \to \pi$

parametrisation of q^2 dependence plays a key role



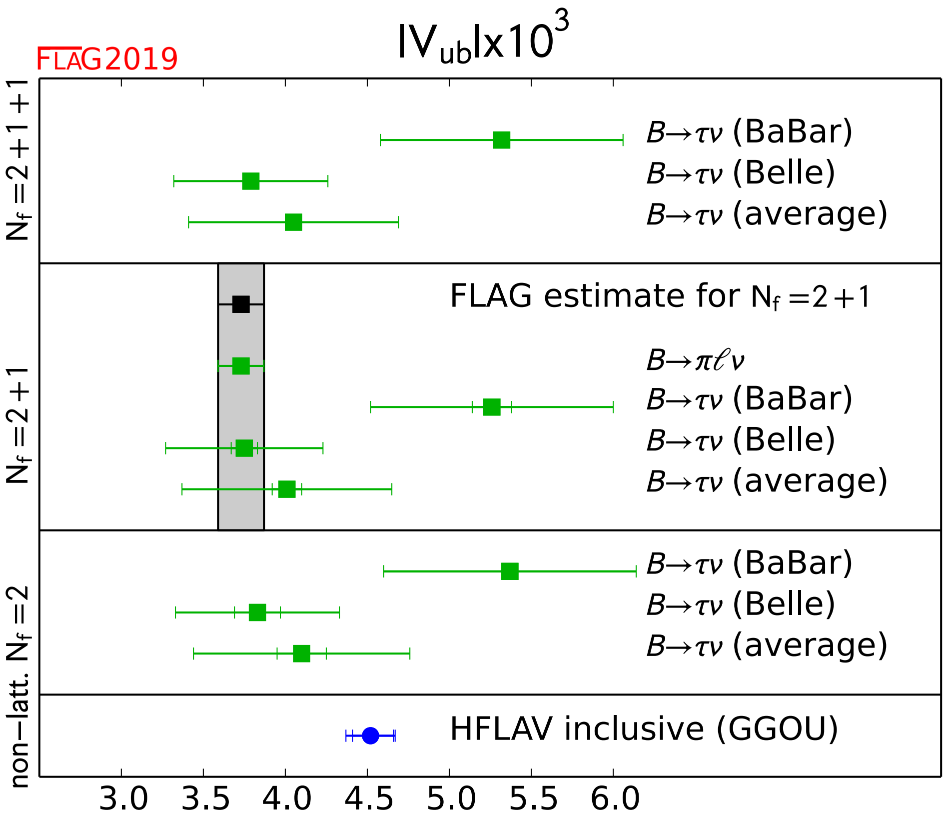


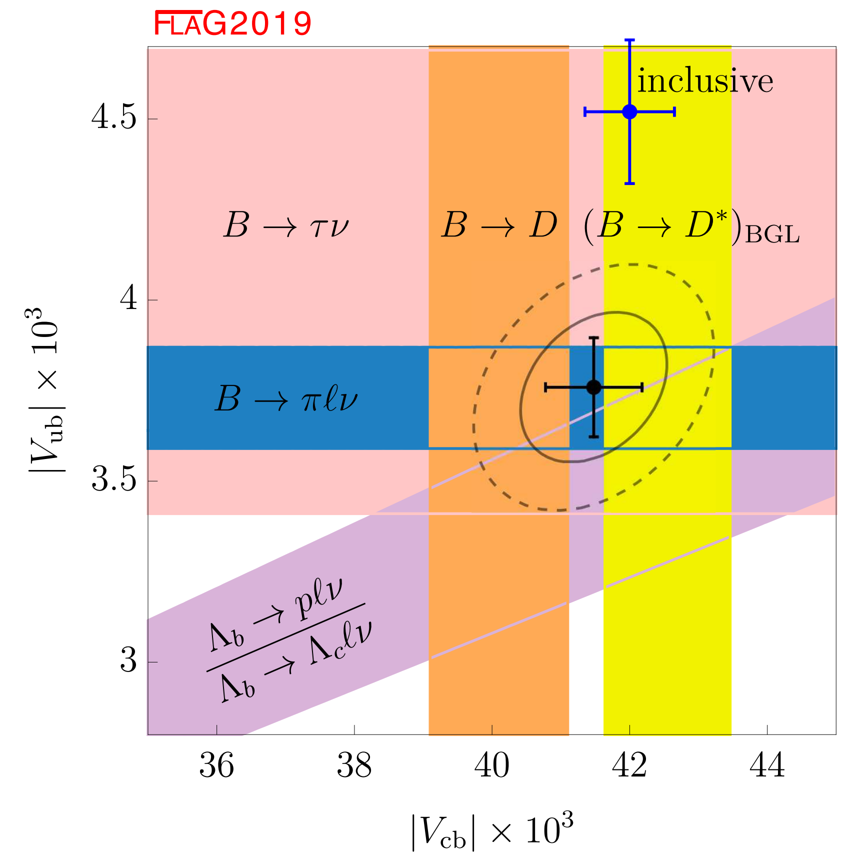


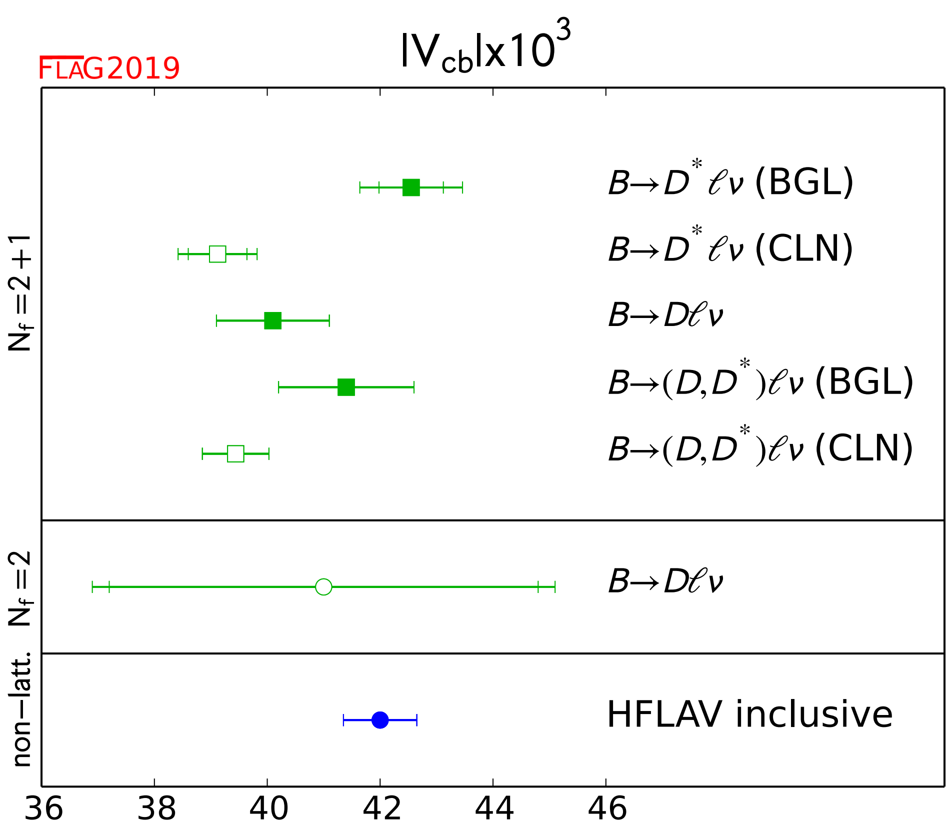


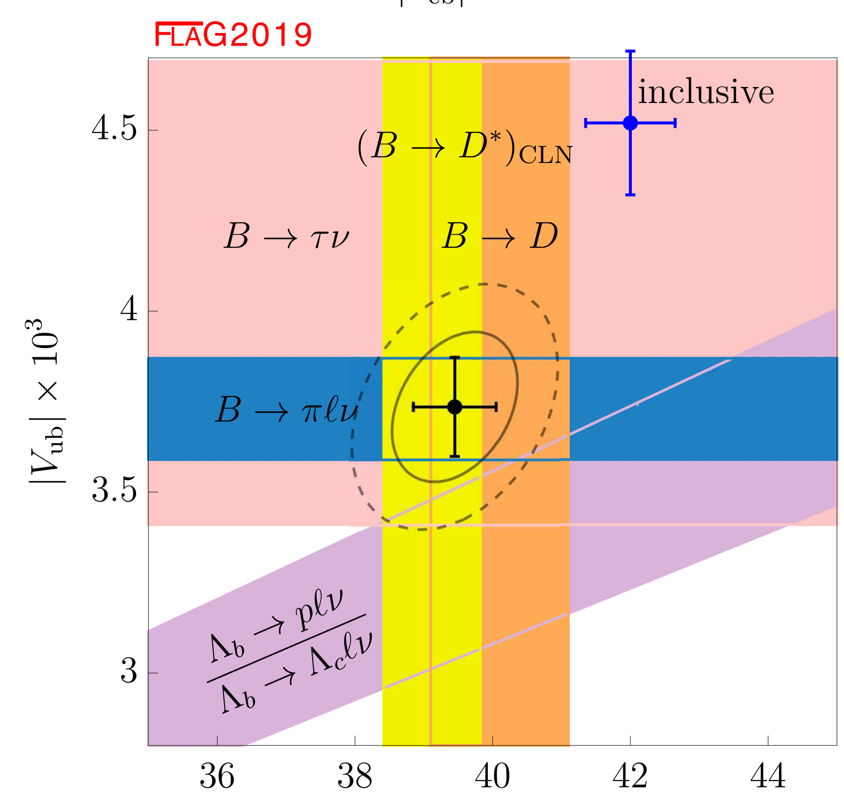
semileptonic decay: $B \to D$, $B \to \pi$

CKMs: few % errors



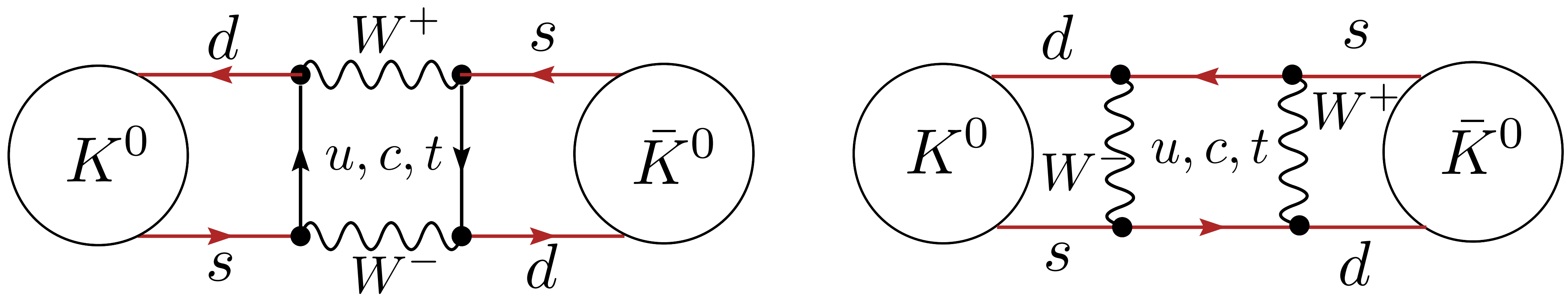






[cf. Bigi, Gambino, Schacht 2016-17]

neutral meson mixing



$$\epsilon_K = \frac{\mathcal{A}[K_L \to (\pi\pi)_{I=0}]}{\mathcal{A}[K_S \to (\pi\pi)_{I=0}]} = \exp(i\phi_\epsilon)\sin(\phi_\epsilon)\left[\frac{\mathrm{Im}(M_{12}^{\mathrm{SD}})}{\Delta M_K} + \frac{\mathrm{Im}(M_{12}^{\mathrm{LD}})}{\Delta M_K} + \frac{\mathrm{Im}(A_0)}{\mathrm{Re}(A_0)}\right]$$

$$\mathrm{Im}(M_{12}^{\mathrm{SD}}) = \frac{G_{\mathrm{F}}^2 M_W^2}{12\pi^2}\left[\lambda_c^2 S_0(x_c)\eta_1 + \lambda_t^2 S_0(x_t)\eta_2 + 2\lambda_c\lambda_t S_0(x_c,x_t)\eta_3\right] f_K^2 m_K \hat{B}_K$$

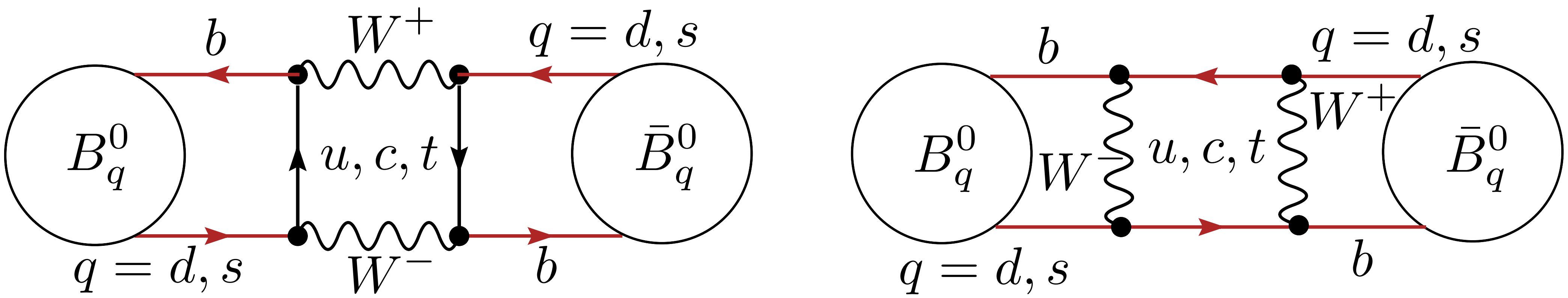
$$= \frac{G_{\mathrm{F}}^2 M_W^2}{12\pi^2}|V_{cb}|^2\lambda^2\bar{\eta}\left[S_0(x_c)\eta_1 + |V_{cb}|^2(1-\bar{\rho})S_0(x_t)\eta_2 + S_0(x_c,x_t)\eta_3\right] f_K^2 m_K \hat{B}_K$$

$$\lambda_q = V_{qs}^* V_{qd}\,, \qquad x_q = \frac{m_q^2}{M_W^2}$$

$$B_K = \frac{\langle \bar{K}^0|(\bar{s}_{\mathrm{L}}\gamma^\mu d_{\mathrm{L}})(\bar{s}_{\mathrm{L}}\gamma_\mu d_{\mathrm{L}})|K^0\rangle}{\frac{8}{3} f_K^2 m_K^2}$$

long-distance contribution relevant at ~1% precision

neutral meson mixing



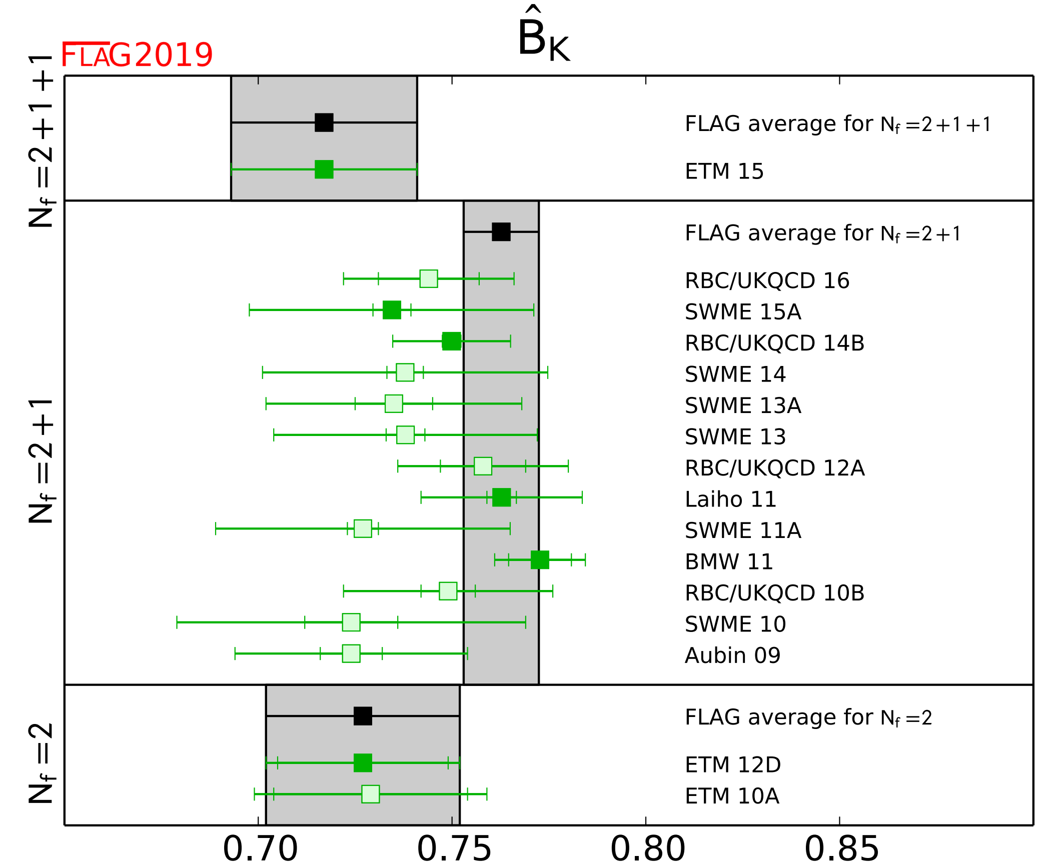
$$\Delta m_q = 2|M_{12}|$$

$$M_{12} = \frac{G_F^2 M_W^2}{12\pi^2} (V_{tq}^* V_{tb})^2 S_0(m_t^2/M_W^2) \eta_B m_{B_q} f_{B_q}^2 \hat{B}_{B_q}$$

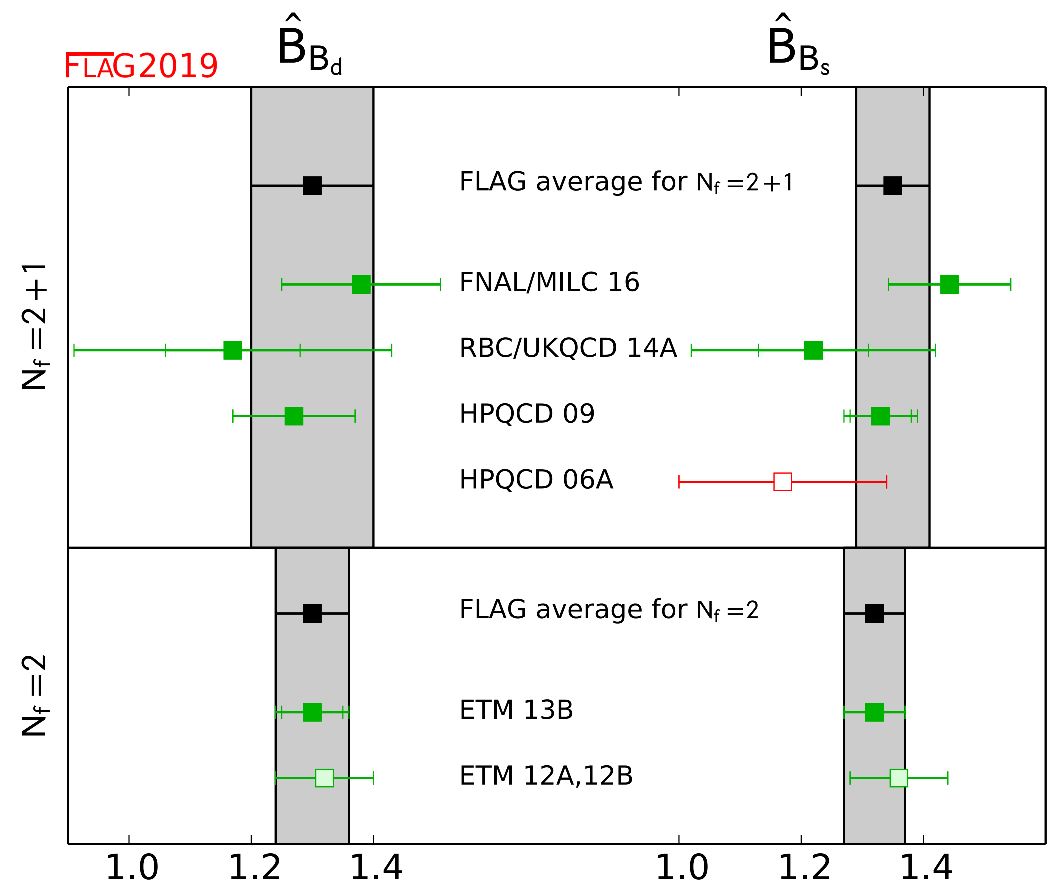
$$B_{B_q} = \frac{\langle \bar{B}_q^0 | (\bar{b}_L \gamma^\mu q_L)(\bar{b}_L \gamma_\mu q_L) | B_q^0 \rangle}{\frac{8}{3} f_{B_q}^2 m_{B_q}^2}$$

neutral meson mixing

FLAG average sports 1.3% error — work out long-distance contribution, QED corrections



paucity of results wrt kaon sector glaring, 5-8% errors: largest room for improvement among basic CKM quantities



the present

- CKMs from pion/kaon physics receive **permille** uncertainties from the lattice; **few %** in charm, bottom CKMs. Kaon mixing at %.

- several exclusive channels allow for crosschecks
  - pion, kaon, charm: leptonic+semileptonic (including Λ_c).
  - bottom: baryon decay (Λ_b, p, ...); $B \to D^* l\nu$; **predictions** for $B_s \to K l\nu$, $B_s \to D_s^{(*)} l\nu$, $B_c \to (M) l\nu$, ...; first information on channels with other vector resonances.
  - **bonus:** same techniques provide equally-precise **BSM input**.

- largest room for bread-and-butter improvement: charm SL, B mixing

- developing: multihadron/resonances in final state

the (short-term) future

- fully tame the B sector: fully relativistic b quarks

- systematically add electromagnetic + strong isospin breaking
  - QCD+QED
  - working examples

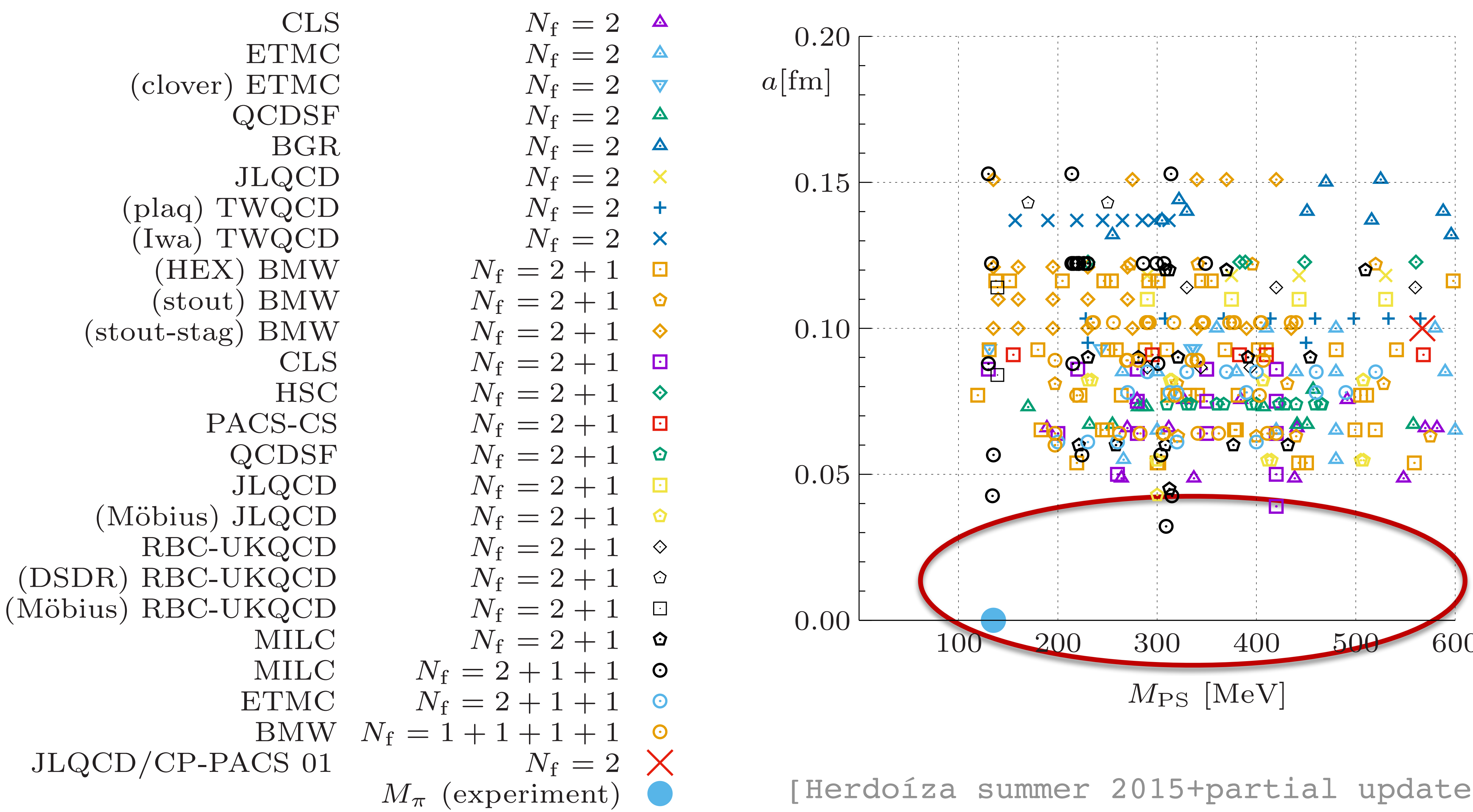
- work out long-distance OPE contributions
  - **bonus:** open new channels (rare K decays, charm CP violation, ...)

- improve channels with resonances / >1 hadron in final state

- ....

fully relativistic *b* quarks

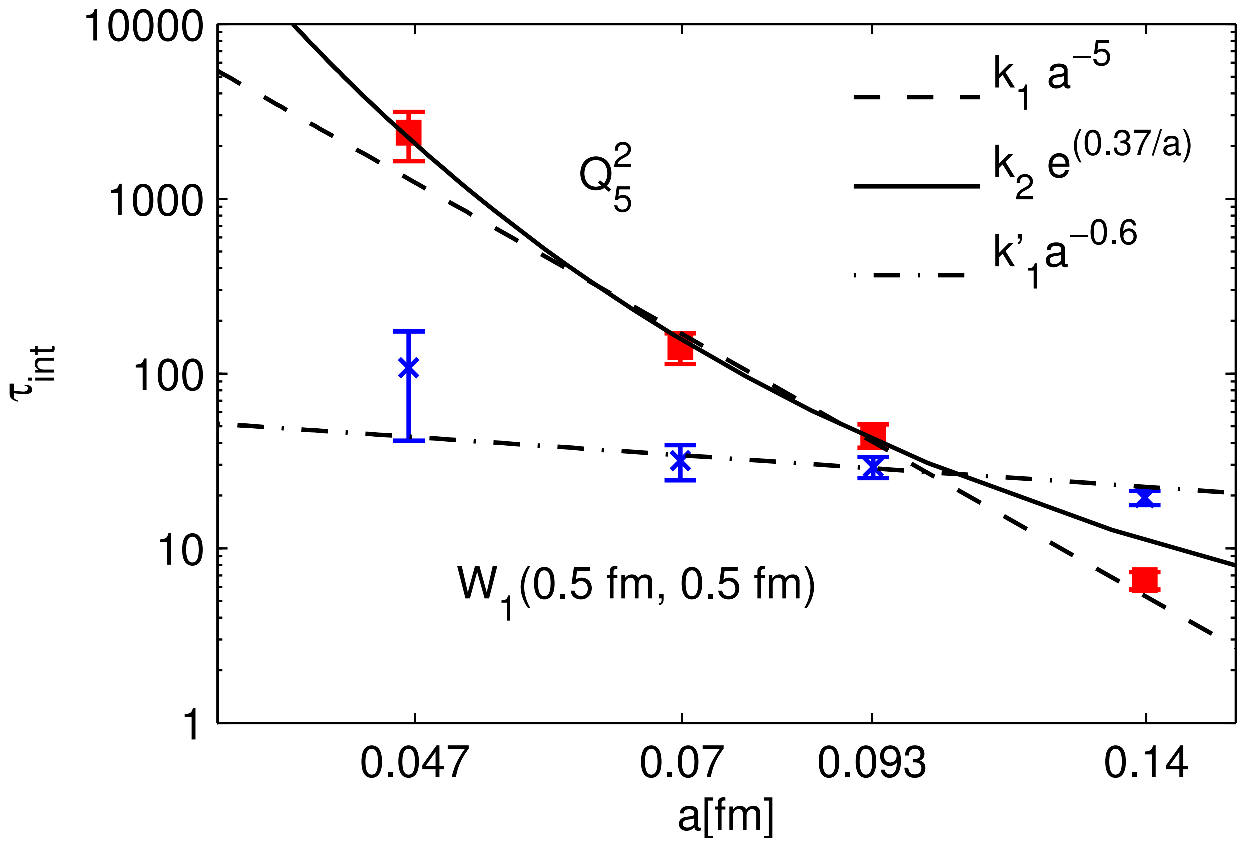
$(am_b)^2 \lesssim \frac{1}{3} \leftrightarrow a \lesssim 0.03\ \mathrm{fm}$ ⇒ populate lower lattice spacings in simulation landscape

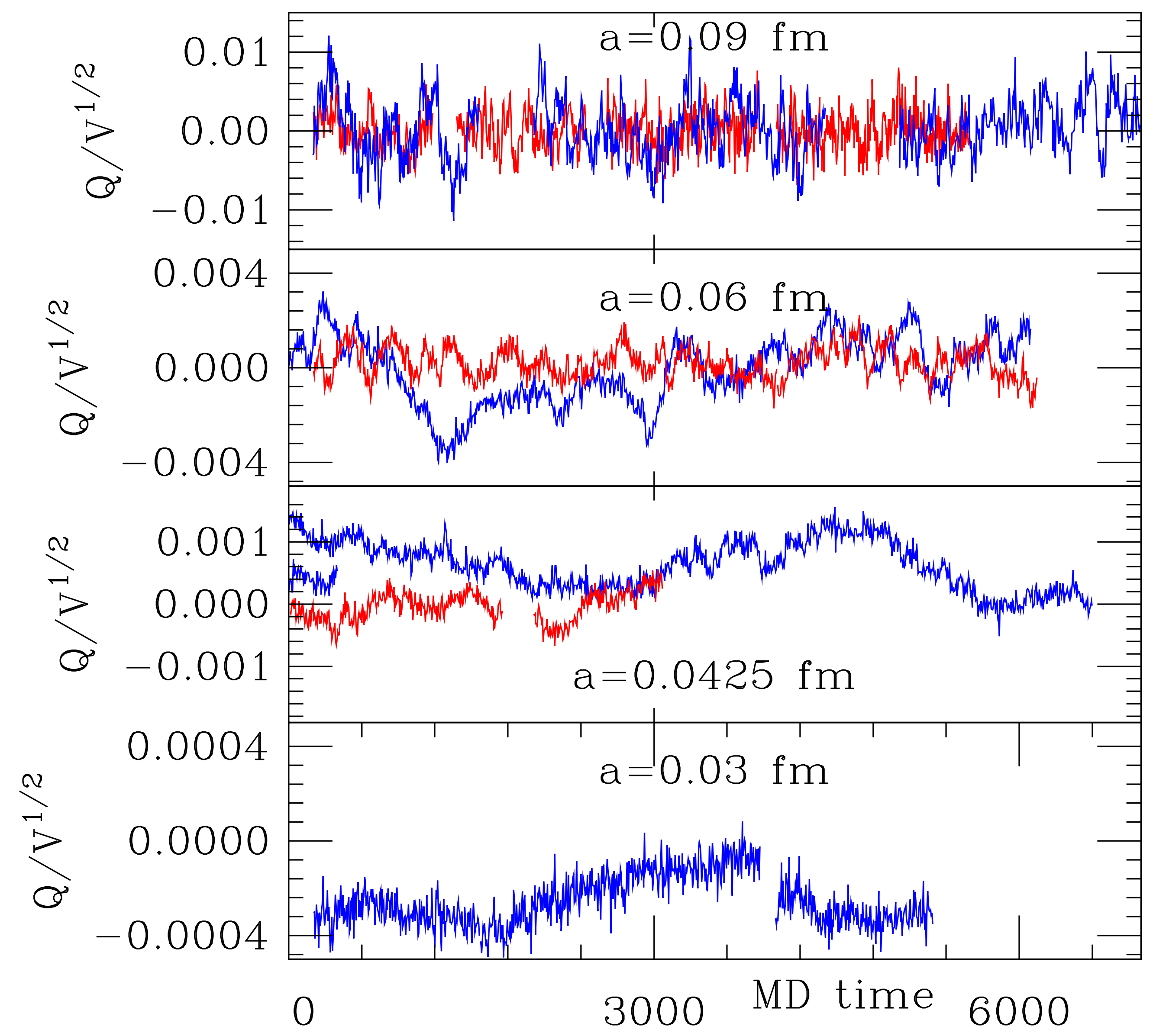


[Herdoíza summer 2015+partial updates]

fully relativistic *b* quarks

algorithmic issue: strong lattice space dependence of autocorrelations





[Del Debbio, Panagopoulos, Vicari 2002]
[Schaefer, Sommer, Virotta 2010]

[Lüscher, Schaefer 2011; CLS N_f=2+1 obc programme]
[Mages et al. 2015; Laio et al. 2015; Brower et al. 2015; Detmold, Endres 2016]

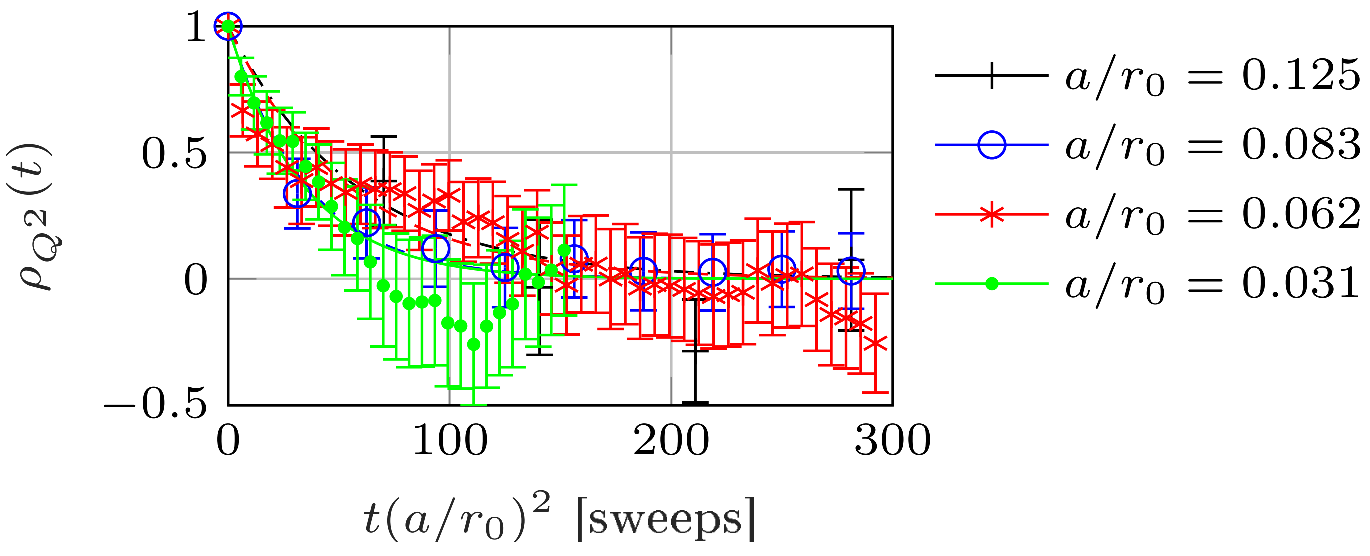
[MILC N_f=2+1+1 ensembles]

fully relativistic *b* quarks

algorithmic issue: strong lattice space dependence of autocorrelations

- improve algorithmic performance by simulating with non-trivial boundary conditions.
  [Lüscher, Schaefer 2011; CLS N_f=2+1 programme]

[Husung et al. Lattice 2017]

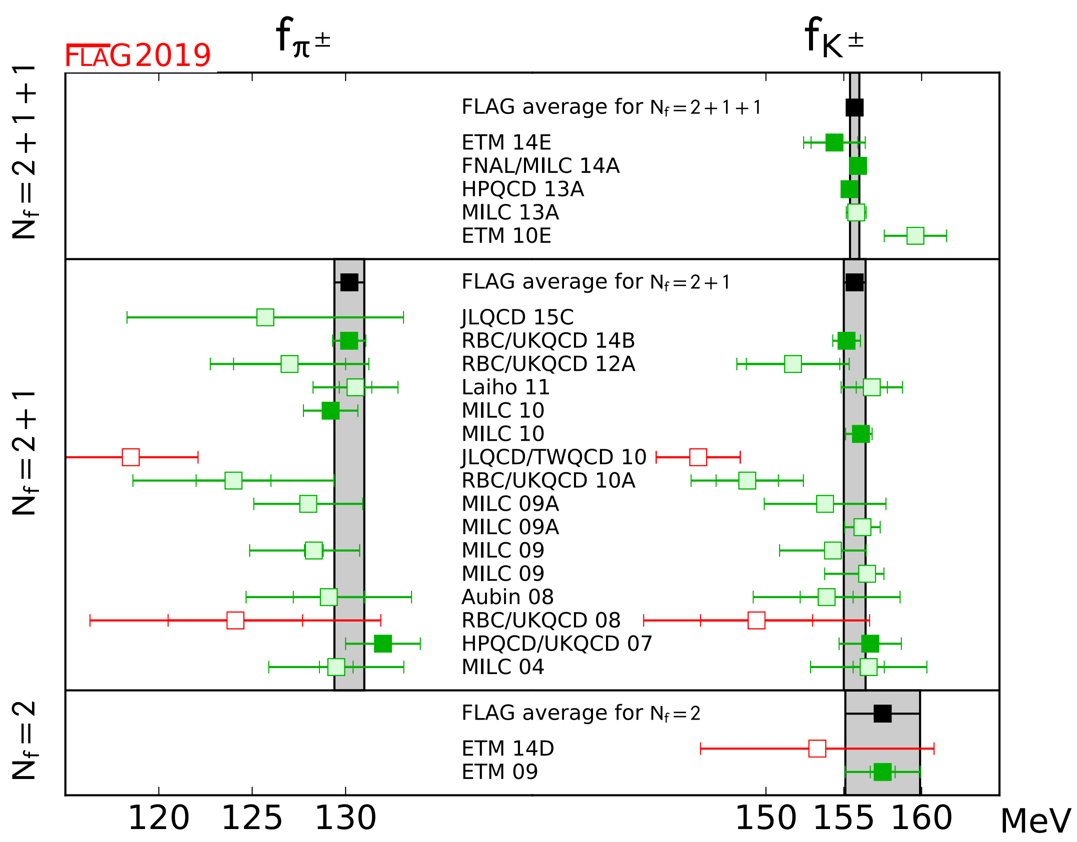


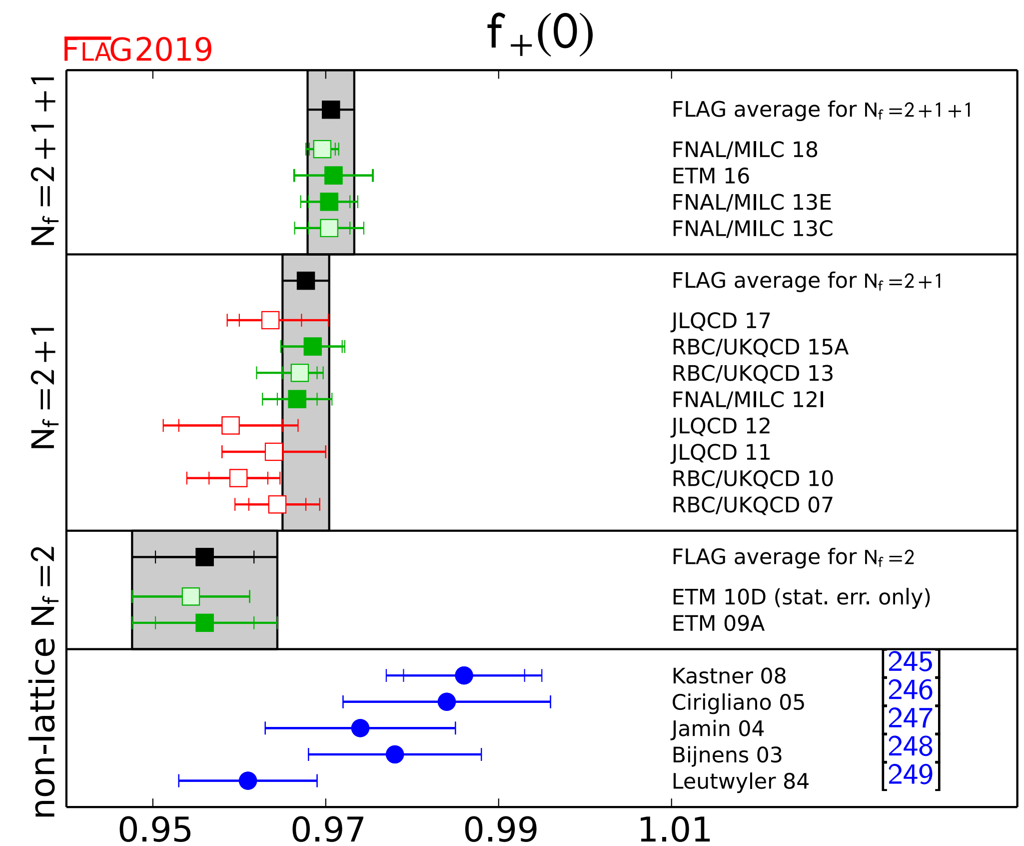
- estimate finite-volume corrections stemming from long autocorrelations (MILC's quark masses, decay constants).
  [Bernard, Toussaint PRD 97 (2018) 074502; MILC N_f=2+1+1 programme]

	m_l = physical
$Q^2_{\text{sample}}/Q^2_{\chi\text{PT}}$	0.65
f_K/f_π	1.19680(0.00114)[0.00015]
aM_π	0.028964(0.000020)[0.000008]
af_D	0.045389(0.000245)[0.000006]
aM_D	0.400678(0.000258)[0.000001]
af_{D_s}	0.053582(0.000025)[0.000000]
aM_{D_s}	0.422041(0.000037)[0.000000]

reliance on effective theory being rapidly eroded

QED (+ isospin breaking)





$$f_{\pi^\pm} = 130.2(0.8)\ \text{MeV} \qquad (0.6\ \%) \qquad \delta^{\chi\text{PT}}_{\text{e.m.}}(\pi^- \to l^- \bar{\nu}) = 1.8\%$$

$$f_{K^\pm} = 155.7(0.3)\ \text{MeV} \qquad (0.2\ \%) \qquad \delta^{\chi\text{PT}}_{\text{e.m.}}(K^- \to l^- \bar{\nu}) = 1.1\%$$

$$f_+(0) = 0.9706(27) \qquad (0.3\ \%) \qquad \delta^{\chi\text{PT}}_{\text{e.m.}}(K \to \pi l \bar{\nu}) = 0.5\text{—}3.0\%$$

[Aoki et al. arXiv:1902.08191; Cirigliano et al. RMP 84 (2012) 399]

precision of standalone QCD computation in isospin limit well below the size of e.m.+IB corrections

QED (+ isospin breaking)

no mass gap in QED $\Rightarrow$ massless photons in physical spectrum $\Rightarrow$ not easy to work in finite volume; two ways out:

- expand observables in $\alpha_{\rm em}$ and m_u-m_d, compute coefficients of expansion non-perturbatively in QCD

  [de Divitiis et al. (RM123) PRD 87 (2013) 114505]

- simulate QCD+QED directly, including isolated charges — possibly at unphysically large values of $\alpha_{\rm em}$ and m_u-m_d + extrapolation.
  - formulate QED in finite volume, treat zero modes by hand

    [Hayakawa, Uno Prog. Theor. Phys. 120 (2008) 413]
  - introduce photon mass (fixed gauge), extrapolate to massless photon limit

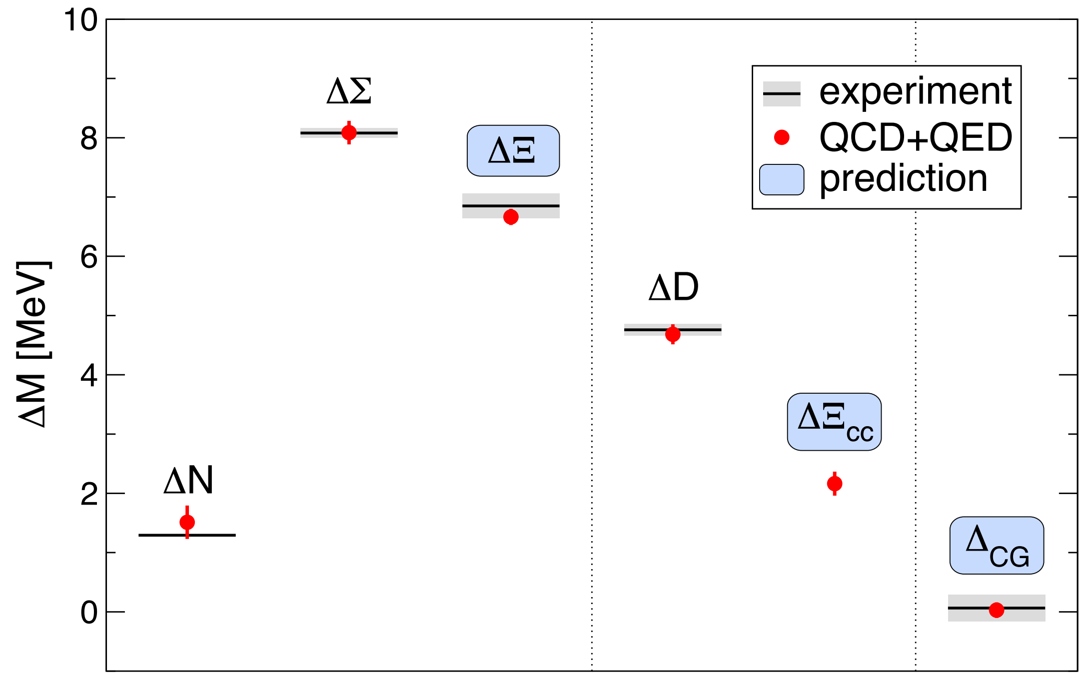
    [Endres et al. PRL 117 (2016)]
  - introduce non-trivial C* boundary conditions

    [Wiese NPB 375 (1992) 45; Lucini et al. JHEP 1602 (2016) 076]

QED+IB: illustrative pioneering results

- ab-initio computation of baryon mass splittings

  [BMW Collab. Science 347 (2015) 1452]



- light-meson leptonic rates

  [RM123 PRL 120 (2018) 072001; arXiv:1904.08731]

  [also: RBC/UKQCD arXiv:1902.00295]

$$\Gamma(\pi^\pm \to \mu^\pm \nu_\ell[\gamma]) = (1.0153 \pm 0.0019)\,\Gamma^{(0)}(\pi^\pm \to \mu^\pm \nu_\ell)$$

$$\Gamma(K^\pm \to \mu^\pm \nu_\ell[\gamma]) = (1.0024 \pm 0.0010)\Gamma^{(0)}(K^\pm \to \mu^\pm \nu_\ell)$$

- meson masses and HVP

  [RBC/UKQCD JHEP 1709 (2017) 153]

- strong IB in $(g\text{-}2)_\mu$

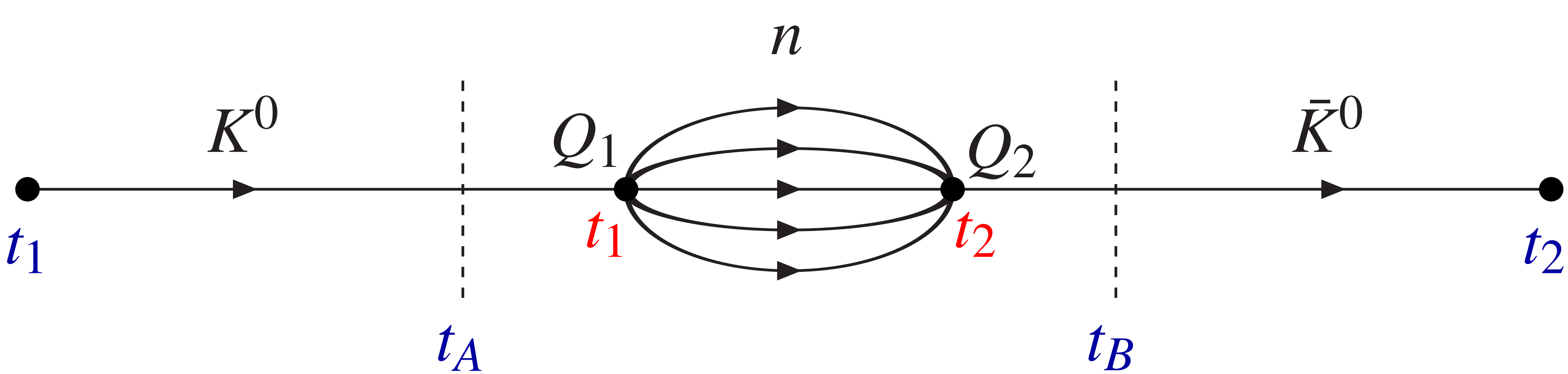
  [FNAL/MILC+HPQCD PRL 120 (2018) 152001]

- .....

OPE long-distance contributions
(+ rare decays/charm $\not{CP}$)

"long-distance" contributions appear when loops involve exchanges of light d.o.f. in the effective weak theory description.

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 2\mathcal{P}\sum_\alpha \frac{\langle \bar{K}^0 \,|\, \mathcal{H}_W \,|\, \alpha\rangle \langle \alpha \,|\, \mathcal{H}_W \,|\, K^0\rangle}{m_K - E_\alpha} = 3.483(6) \times 10^{-12}\,\mathrm{MeV}.$$



practical implementation on the lattice worked out by RBC/UKQCD

[Christ et al. PRD88 (2013) 014508]

preliminary result: $\Delta m_K = 5.5(1.7) \times 10^{-12}$ MeV

[Bai et al. Lattice 2017]

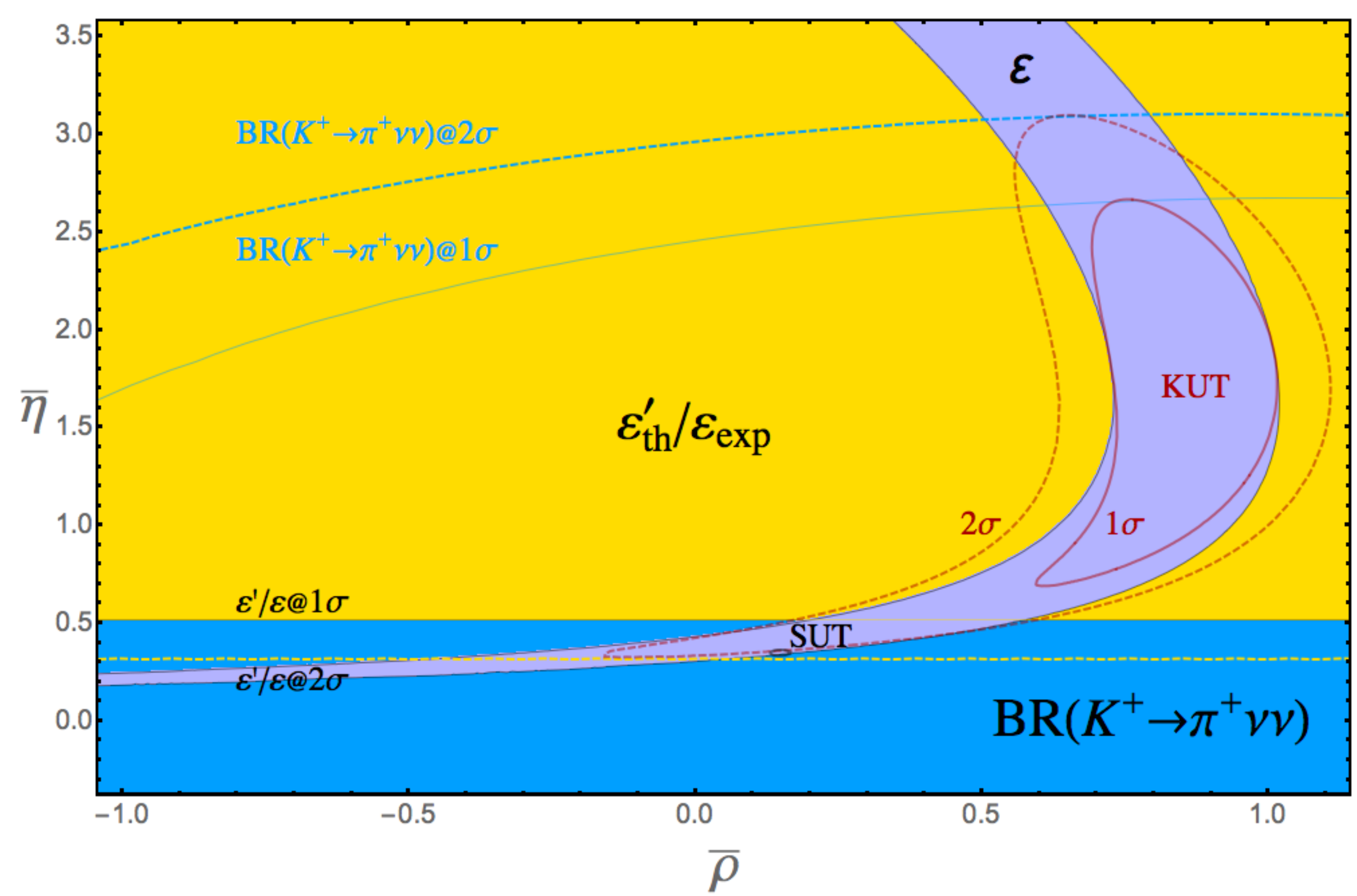
OPE long-distance contributions
(+ rare decays/charm $\not{CP}$)

with this technique in place, other similar problems can be attacked.

- rare kaon decays: $K \to \pi l^+ l^-,\ K \to \pi \nu \bar{\nu}$

[RBC/UKQCD PoS Lattice2016 (2017) 303; PRD 98 (2018) 074509]

"emerging kaon UT"



[Lehner, Lunghi, Soni PLB 759 (2016) 82]

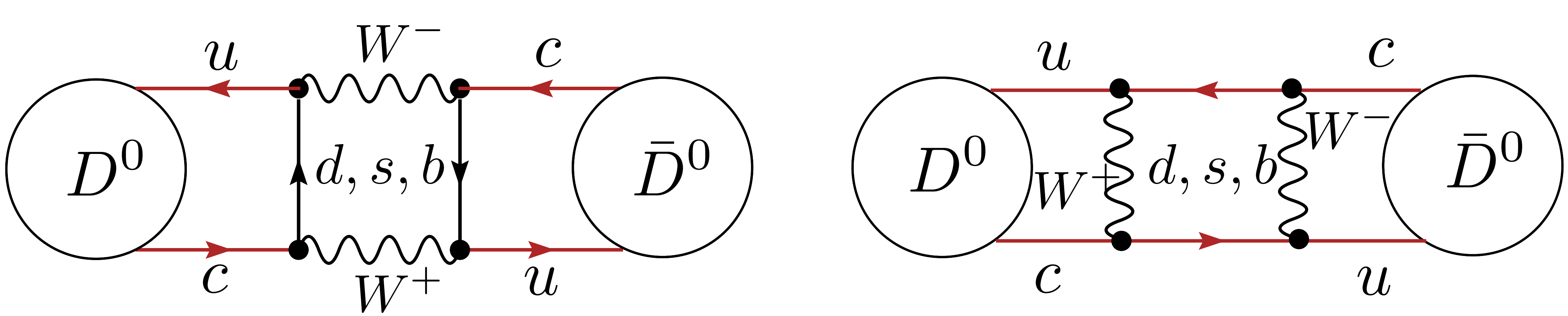
OPE long-distance contributions
(+ rare decays/charm $\not{C}\!P$)

with this technique in place, other similar problems can be attacked.

- CP-conserving rare kaon decays: $K \to \pi l^+ l^-, \; K \to \pi \nu \bar{\nu}$

[RBC/UKQCD PoS Lattice2016 (2017) 303; PRD 98 (2018) 074509]

- charm CP violation???



conclusions & outlook

- lattice flavour phenomenology has long reached its age of maturity, keeping apace with/abreast of experiment.

- upcoming era will require sub-percent precision in staple observables. tools are in place.
  - finer lattice spacings for precision B-physics
  - quantitative control of e.m. and strong isospin breaking corrections

- new avenues being open for lattice studies.
  - baryon decay
  - long-distance contributions to OPE
  - multihadron/resonances in final state
  - inclusive rates

- lattice collaborations have become large and resource-intensive, in both human and computational terms; sustained support is needed to keep synergy with experimental efforts.

conclusions & outlook

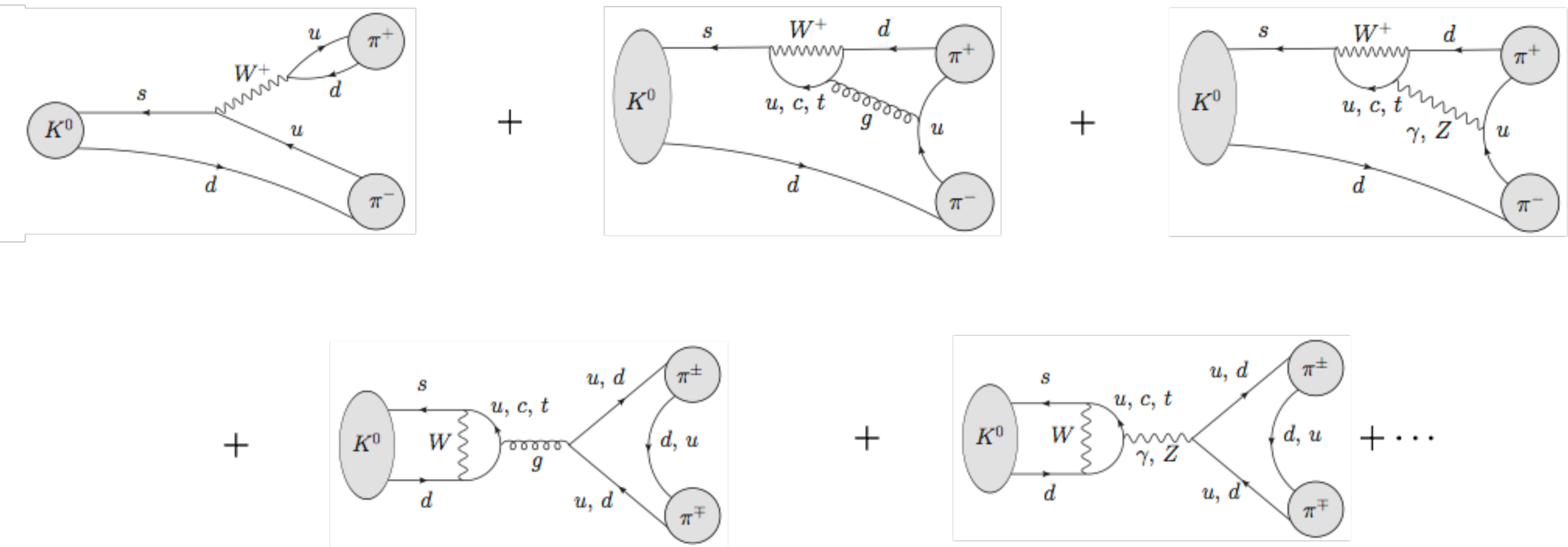
- exploring and mapping the flavour sector remains as important a problem as any other in particle physics
  - why the generation structure? why 3 families?
  - is there a structure in the values of quark masses and CKMs?
  - is new physics lingering out there?

- strong support to a synergic exp/th flavour programme crucial; **what can future colliders offer?**

- **eagerly waiting for Belle II, LHCb Upgrade II, kaon expts.**

backup slides

resonance/multihadron final states



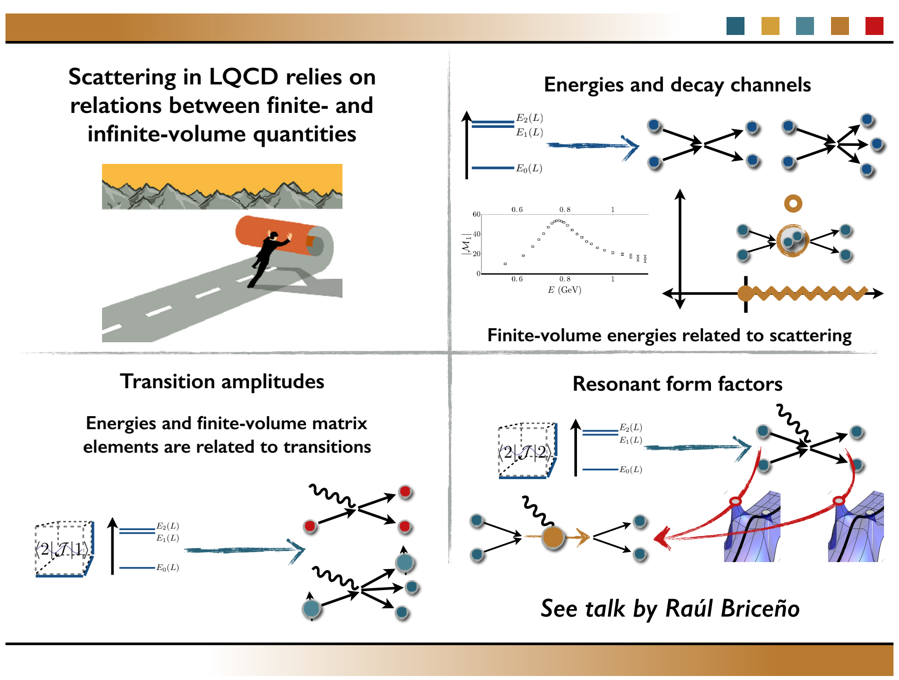
$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right)_{\mathrm{SM}} = 1.38(5.15)(4.43) \times 10^{-4}$$

cf. $$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right)_{\mathrm{exp}} = 16.6(2.3) \times 10^{-4}$$

[RBC/UKQCD Collaboration, PRL 115 (2015) 212001]

resonance/multihadron final states

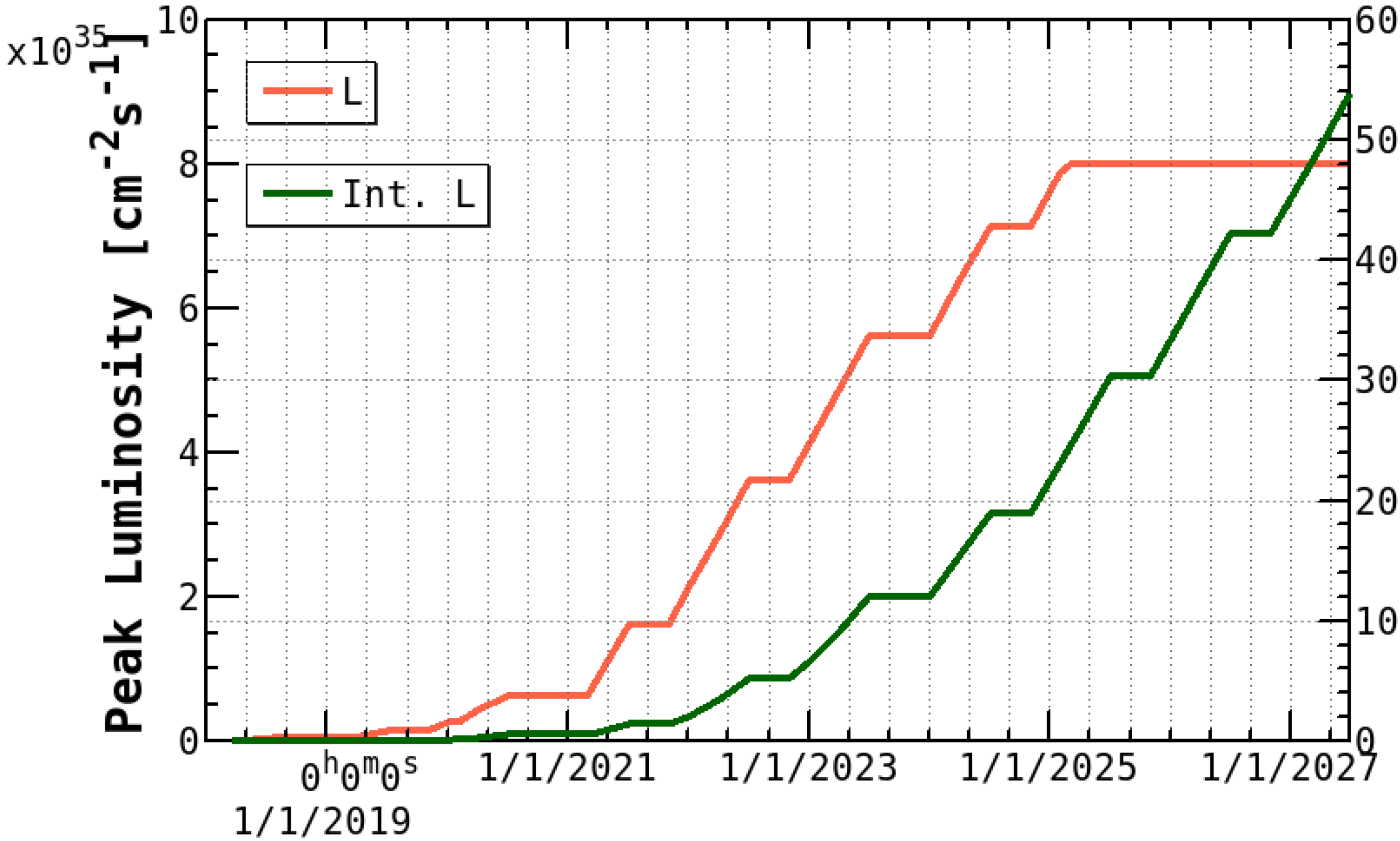
- QFT aspects well understood in simplest 1→2 transitions (e.g., $K \to \pi\pi$) — large errors down to algorithmic/computational issues.
- huge recent QFT developments in the wider picture
  - up to 2→3 processes worked out in detail
  - detailed characterisation of resonances, including their coupling to currents



[see, e.g., MT Hansen & R Briceño @ Confinement XIII]

- non-trivial QFT tools in place, good prospects for resonances in final state (e.g., $B \to K^*$); non-leptonic decay, couplings to 4-quark operators still very demanding numerically.

Belle II timeline



[K Hara @ 6th KEK Flavor Factory Workshop, 2019/02]

meeting the challenge from experiment

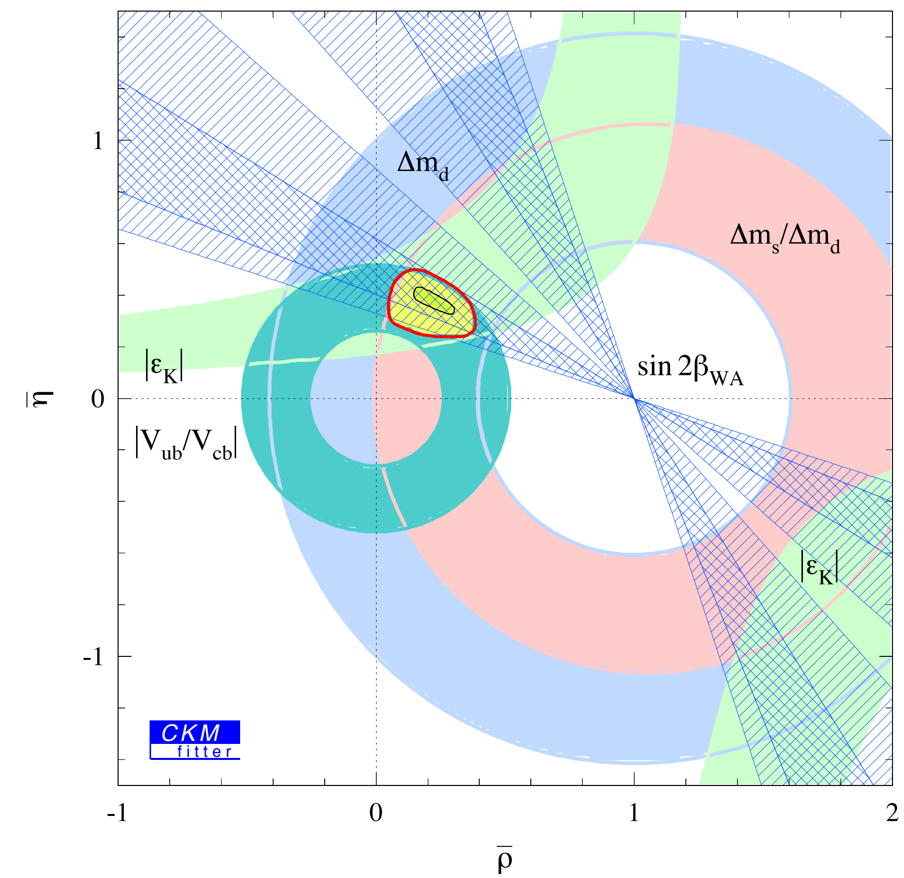
extremely active experimental programme in coming decade(s):

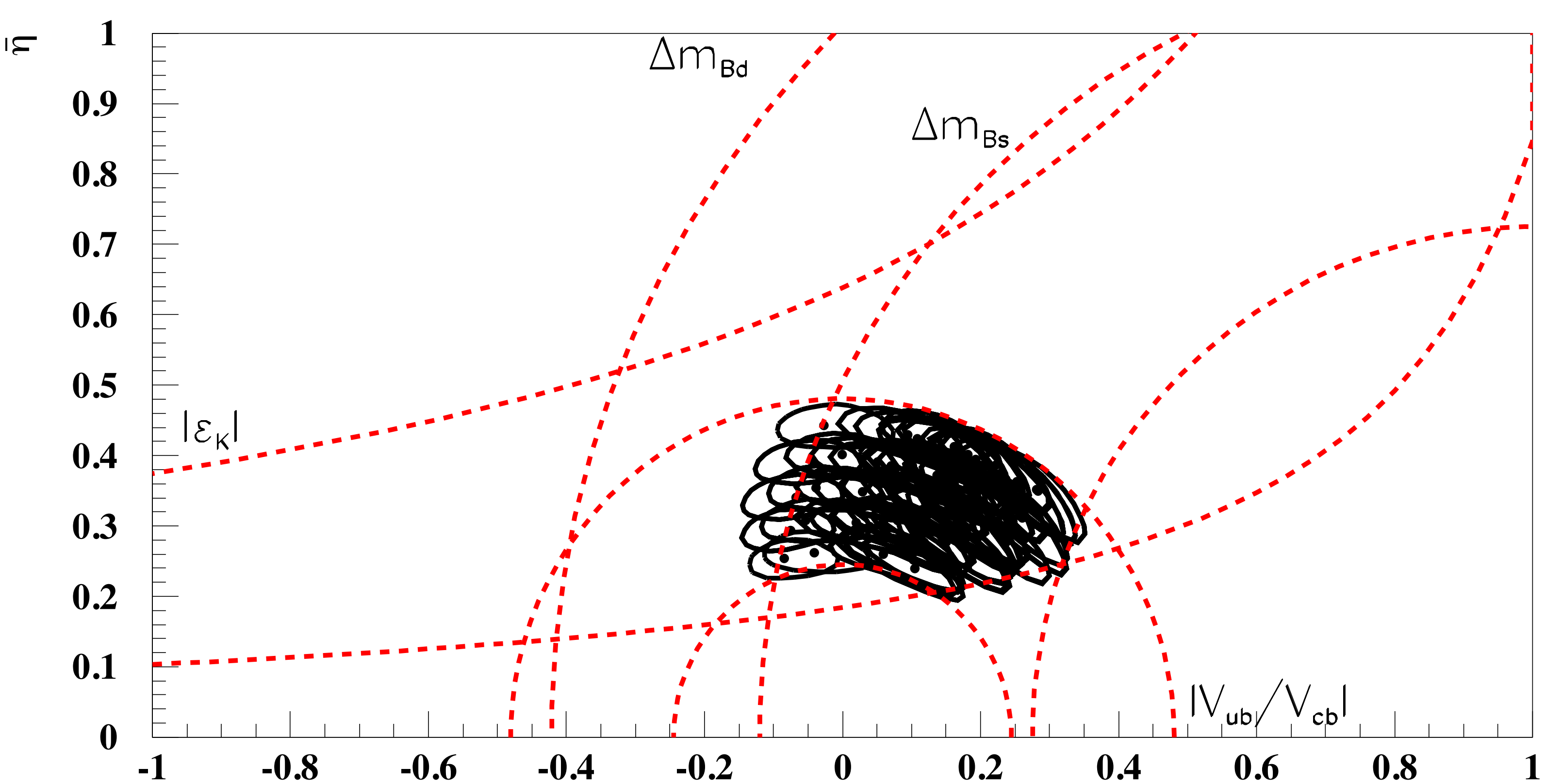
- heavy quark physics: LHCb, Belle II, BESIII (charm), ...
- kaon physics: NA62, KOTO, ...

Observable	Current LHCb	LHCb 2025	Belle II	Upgrade II	ATLAS & CMS
CKM tests					
γ, with $B_s^0 \to D_s^+ K^-$	$(^{+17}_{-22})^\circ$ [136]	4°	–	1°	–
γ, all modes	$(^{+5.0}_{-5.8})^\circ$ [167]	1.5°	1.5°	0.35°	–
$\sin 2\beta$, with $B^0 \to J/\psi K_S^0$	0.04 [609]	0.011	0.005	0.003	–
ϕ_s, with $B_s^0 \to J/\psi \phi$	49 mrad [44]	14 mrad	–	4 mrad	22 mrad [610]
ϕ_s, with $B_s^0 \to D_s^+ D_s^-$	170 mrad [49]	35 mrad	–	9 mrad	–
$\phi_s^{s\bar{s}s}$, with $B_s^0 \to \phi\phi$	154 mrad [94]	39 mrad	–	11 mrad	Under study [611]
$a_{\rm sl}^s$	33×10^{-4} [211]	10×10^{-4}	–	3×10^{-4}	–
$\lvert V_{ub}\rvert/\lvert V_{cb}\rvert$	6% [201]	3%	1%	1%	–
$\boldsymbol{B_s^0, B^0 \to \mu^+\mu^-}$					
$\mathcal{B}(B^0 \to \mu^+\mu^-)/\mathcal{B}(B_s^0 \to \mu^+\mu^-)$	90% [264]	34%	–	10%	21% [612]
$\tau_{B_s^0 \to \mu^+\mu^-}$	22% [264]	8%	–	2%	–
$S_{\mu\mu}$	–	–	–	0.2	–
$\boldsymbol{b \to c\ell^-\bar{\nu}_l}$ **LUV studies**					
$R(D^*)$	0.026 [215, 217]	0.0072	0.005	0.002	–
$R(J/\psi)$	0.24 [220]	0.071	–	0.02	–

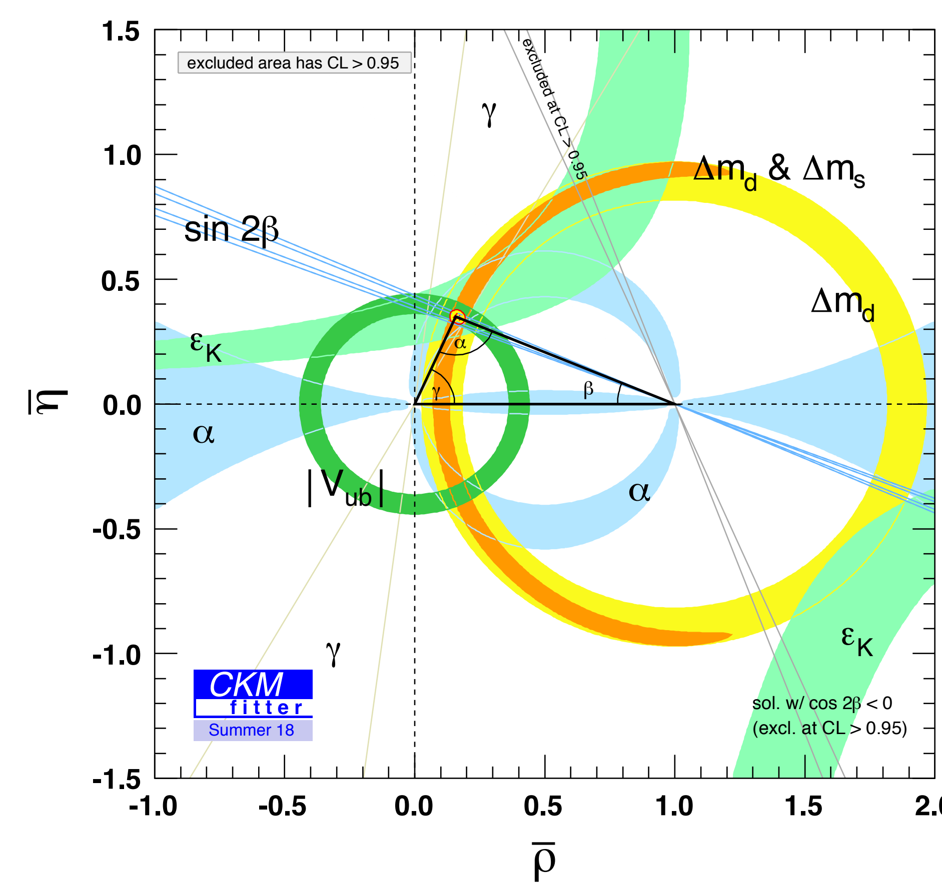
[C Langenbruch @ *Implications of LHC measurements and future prospects*, 2018/10]

evolution

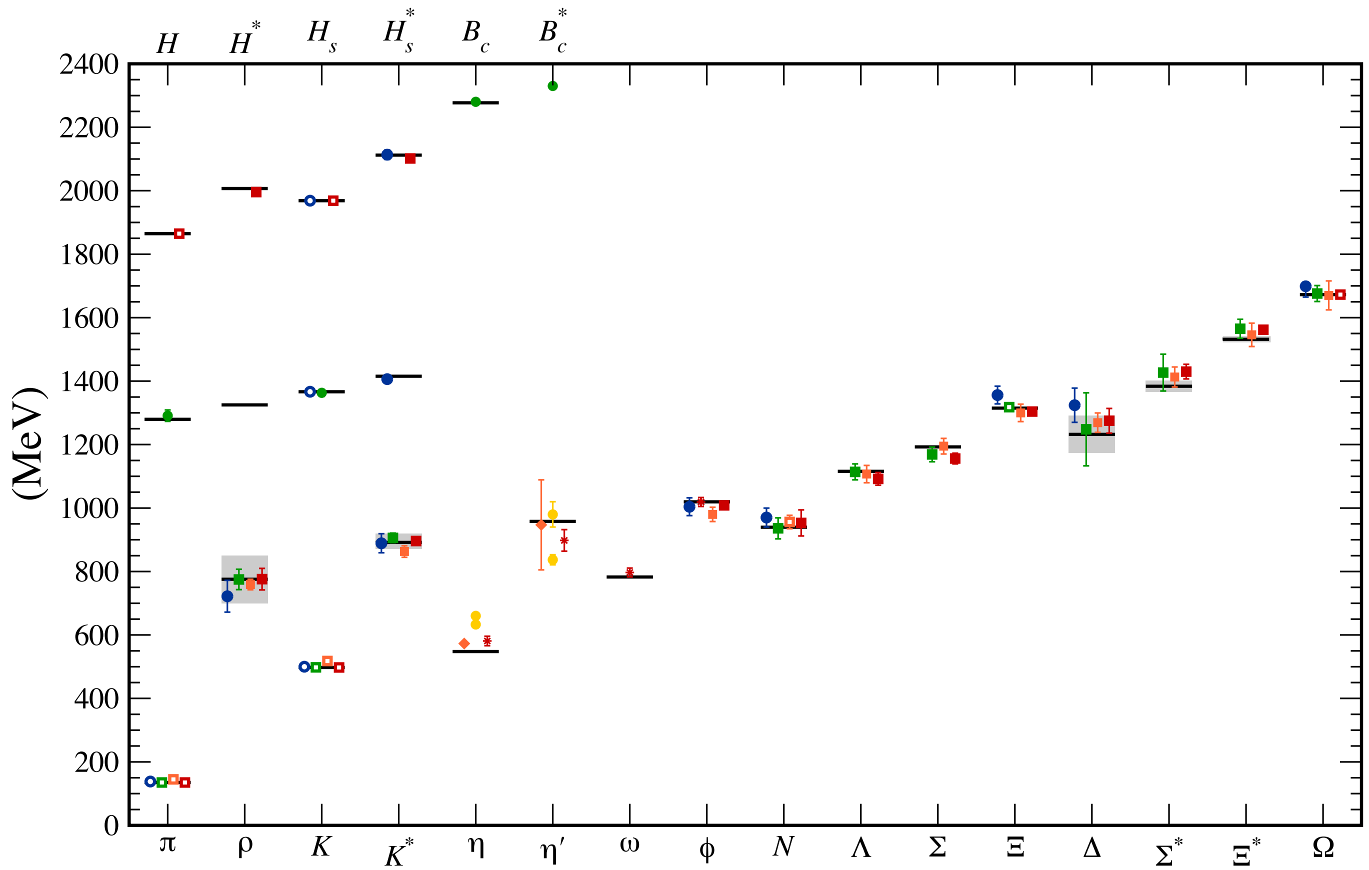




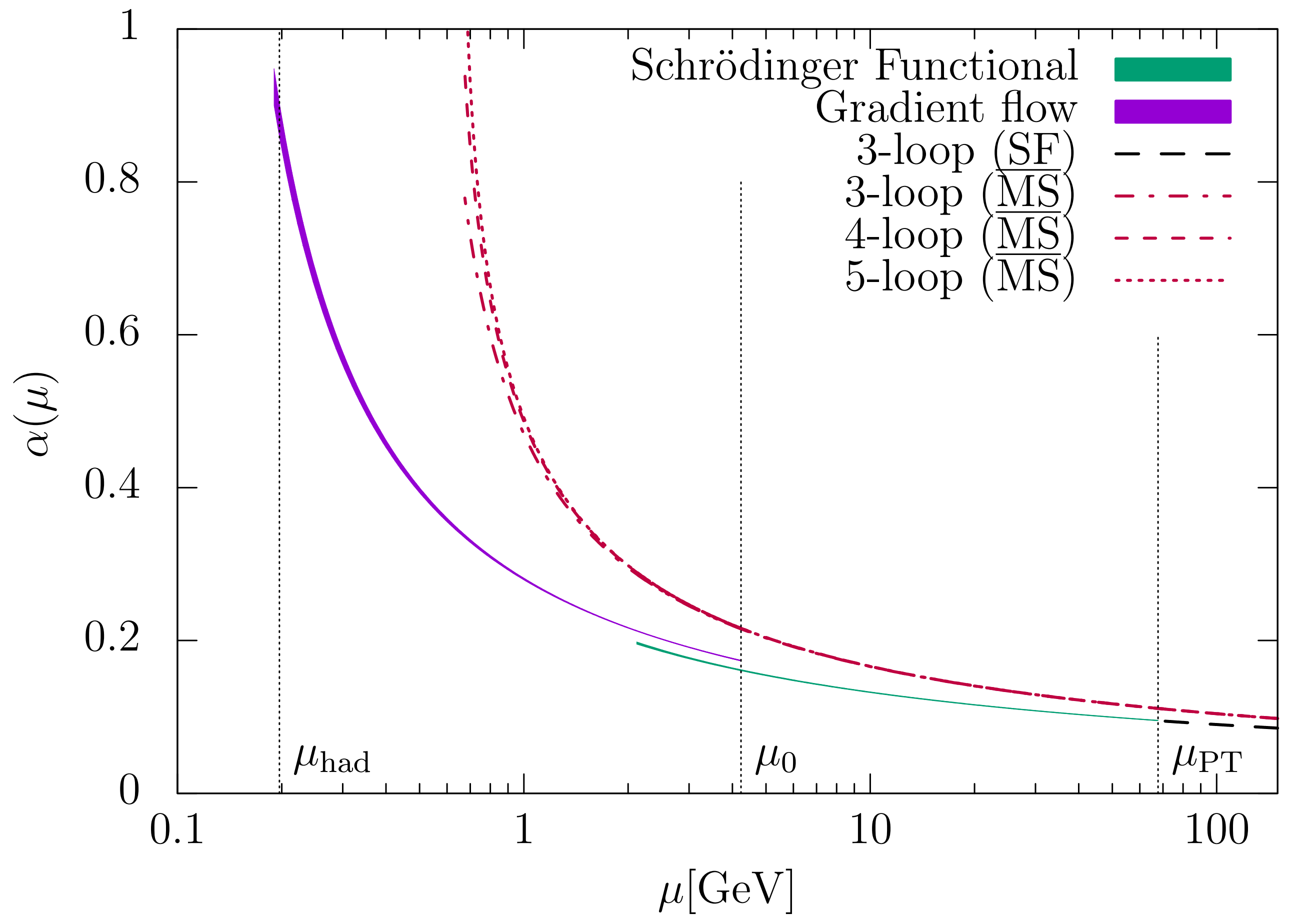
[BaBar Physics Book, 1999]



lattice QCD: state-of-the-art



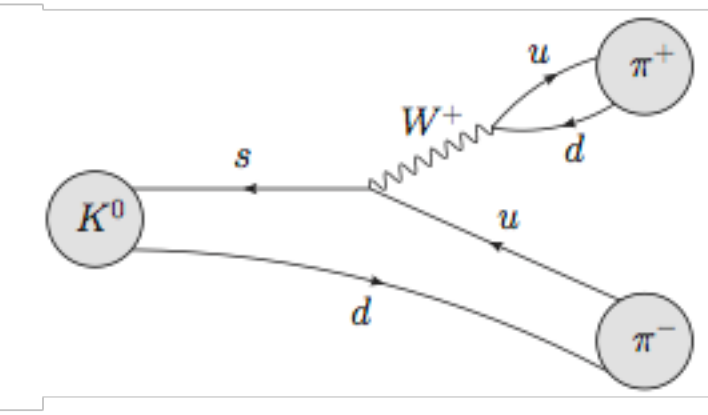
[Kronfeld, Annu. Rev. Nucl. Part. Sci. 62 (2012)]



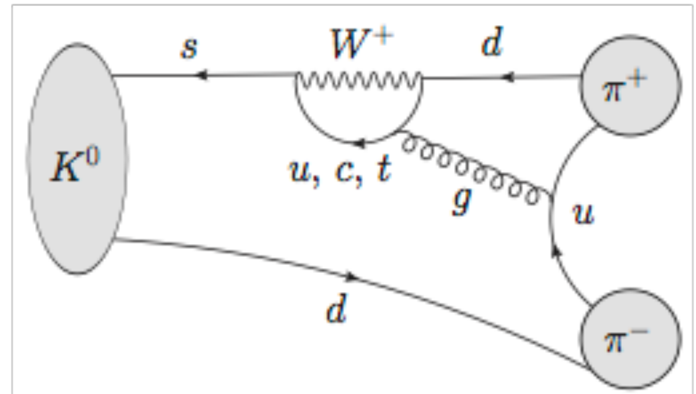
$$\Rightarrow \alpha_s^{\overline{\mathrm{MS}}}(M_Z) = 0.11852(84)$$

[ALPHA Collaboration, PRL 119 (2017) 102001]

lattice QCD: state-of-the-art



$+$



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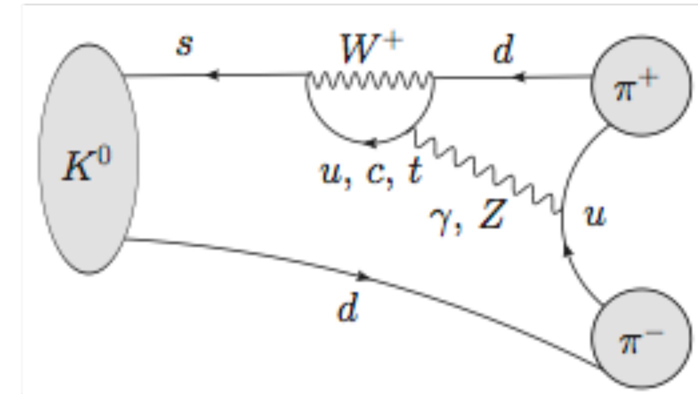

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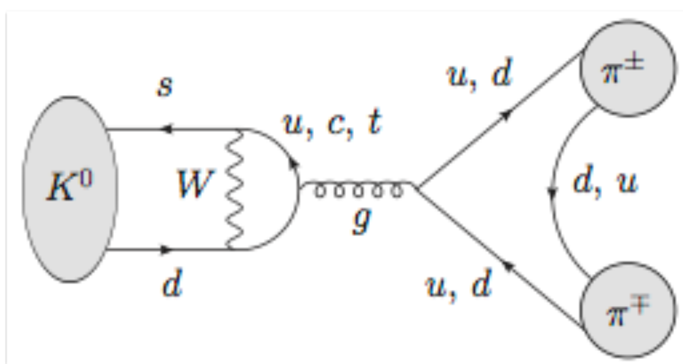

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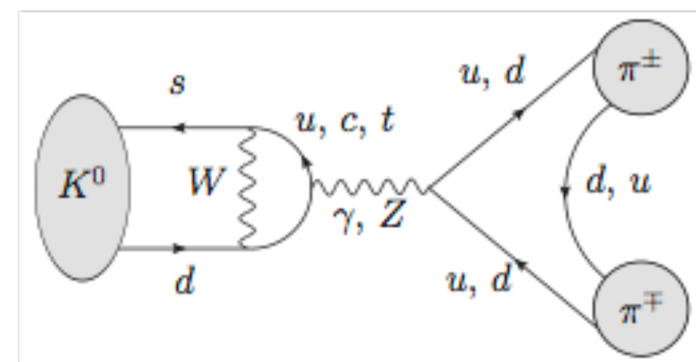


$+\cdots$

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right)_{\mathrm{SM}} = 1.38(5.15)(4.43) \times 10^{-4}$$

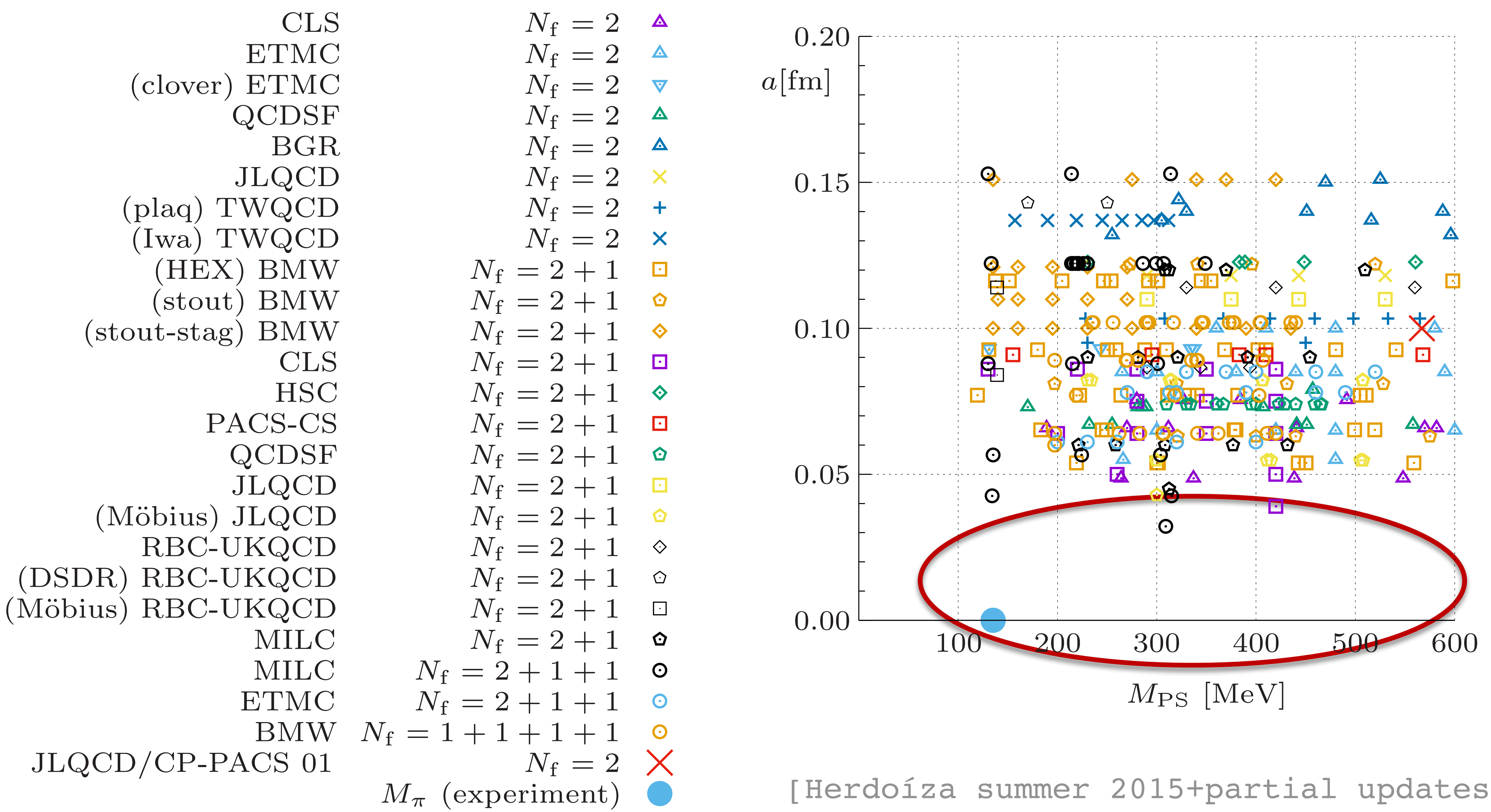
cf. $$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right)_{\mathrm{exp}} = 16.6(2.3) \times 10^{-4}$$

[RBC/UKQCD Collaboration, PRL 115 (2015) 212001]

fully relativistic *b* quarks

$$(am_b)^2 \lesssim \frac{1}{3} \leftrightarrow a \lesssim 0.03 \text{ fm}$$

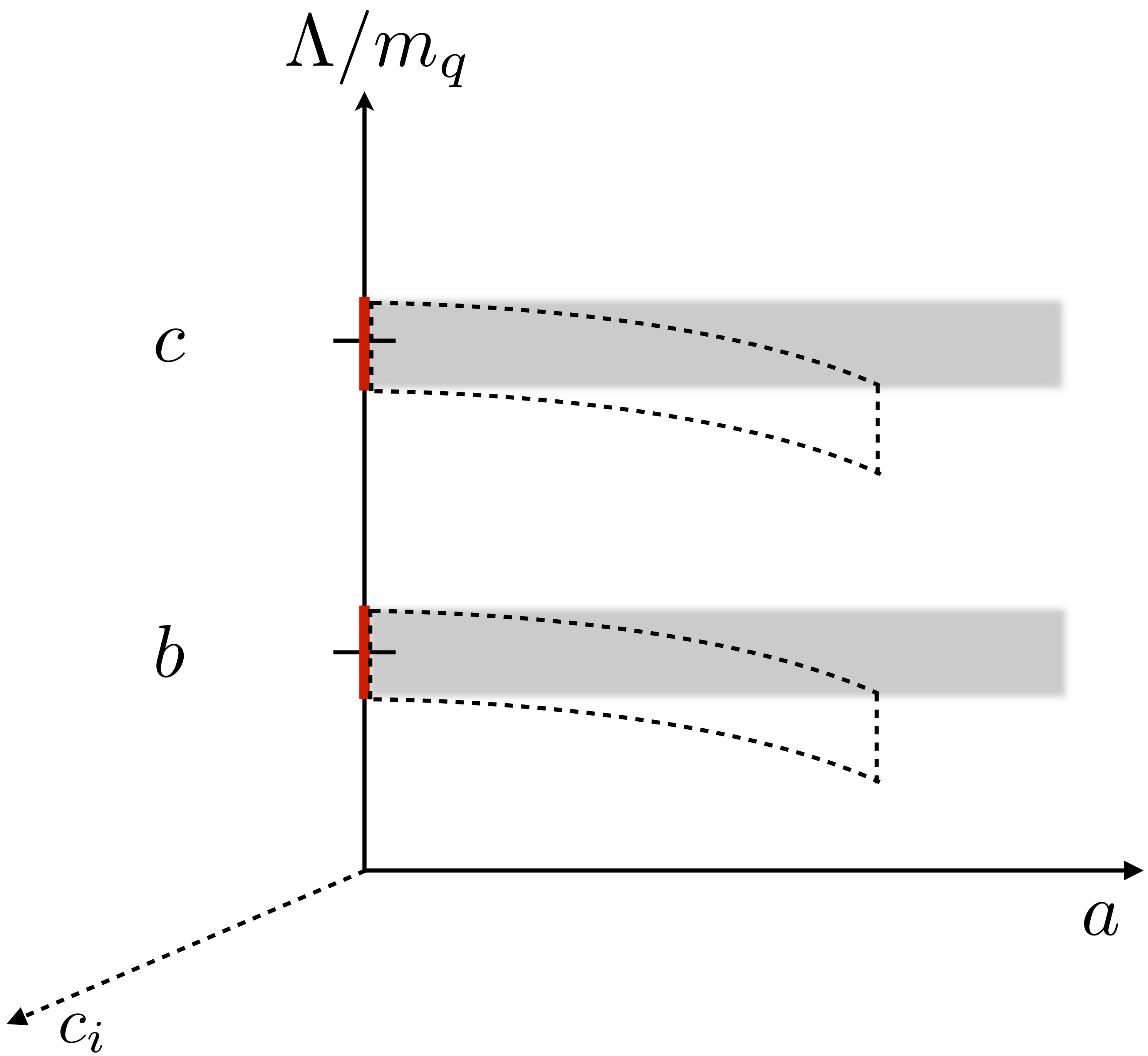
$\Rightarrow$ populate lower lattice spacings in simulation landscape



[Herdoíza summer 2015+partial updates]

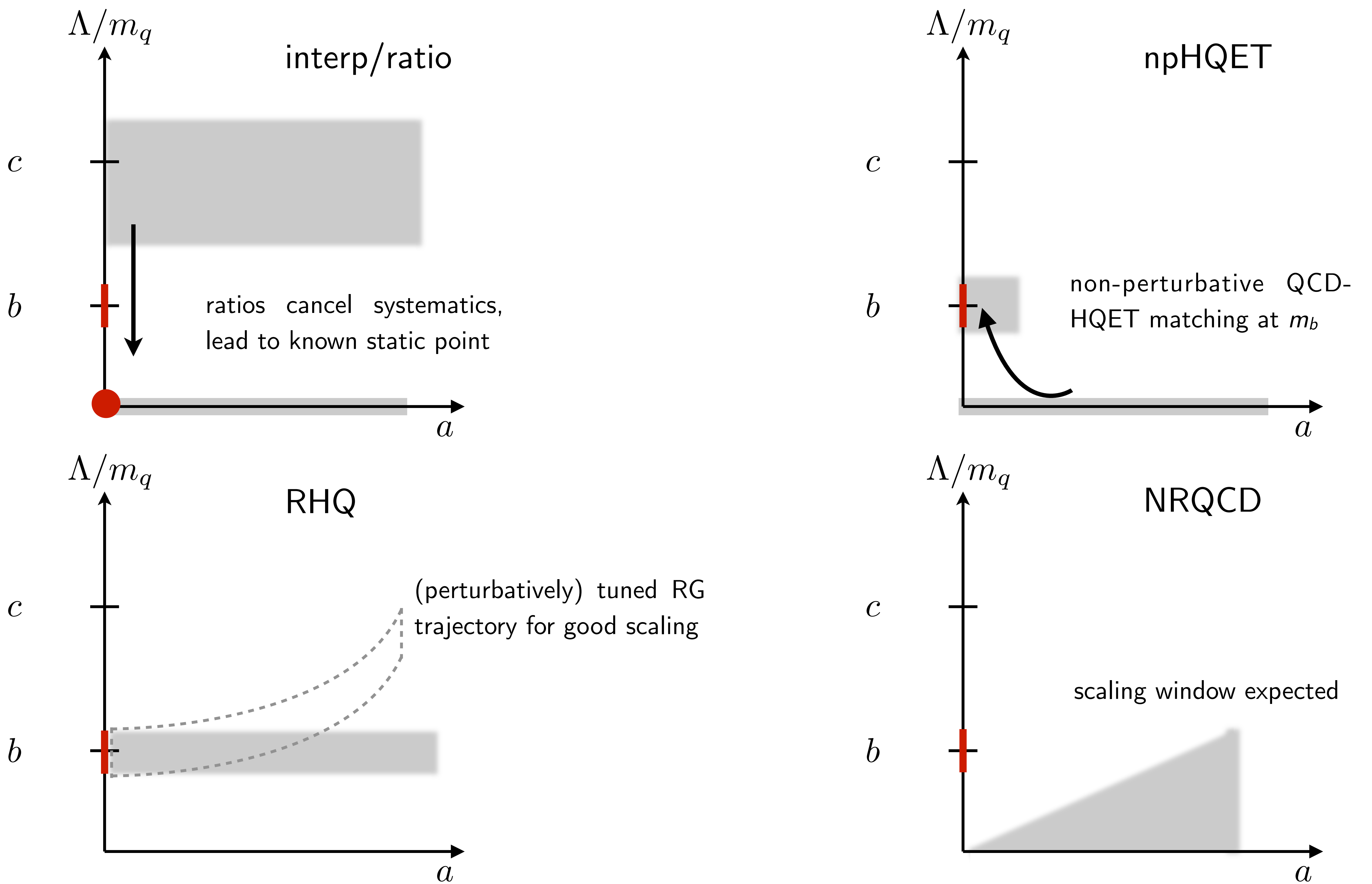
approaches to B physics

what one would like to do [cf. MILC's finest lattices]



approaches to B physics

effective theory used differently, different pros/cons balance: **crosschecks crucial**



lattice QCD for phenomenology: FLAG

Flavour Lattice Averaging Group: your one-stop repository of lattice results, world averages / estimates

FLAG2019 4th edition: results up to 2018/09/30 [Aoki et al., arXiv:1902.08191]

advisory board: S Aoki, M Golterman, R Van de Water, A Vladikas

editorial board: G Colangelo, S Hashimoto, A Jüttner, S Sharpe, U Wenger

working groups:

quark masses	T Blum, A Portelli, A Ramos
V_{ud}, V_{us}	S Simula, T Kaneko, JN Simone
LECs	S Dürr, H Fukaya, UM Heller
kaon mixing	P Dimopoulos, G Herdoíza B Mawhinney
heavy leptonic + mixing	D Lin, Y Aoki, M Della Morte
heavy semileptonic	E Lunghi, D Bečirević, S Gottlieb, CP
α_s	R Sommer, R Horsley, T Onogi
nuclear matrix elements	R Gupta, S Collins, A Nicholson, H Wittig

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what FLAG provides for each quantity:

- complete list of references
- summary of relevant formulae and notation
- quick-look summary tables
- quality assessment of computation setup: colour-coded tables
- averages/estimates (if sensible)
- a “lattice dictionary” for non-experts
- thorough appendix tables with details of all computations for experts
- between-editions updates at http://itpwiki.unibe.ch/flag

cite the original works!

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tables:

Collaboration	Ref.	N_f	publication status	continuum extrapolation	chiral extrapolation	finite volume	renormalization/matching	heavy-quark treatment	f_{B^+}	f_{B^0}	f_B	f_{B_s}
FNAL/MILC 17	[5]	2+1+1	A	★	★	★	★	✓	189.4(1.4)	190.5(1.3)	189.9(1.4)	230.7(1.2)
HPQCD 17A	[72]	2+1+1	A	★	★	★	○	✓	–	–	196(6)	236(7)
ETM 16B	[27]	2+1+1	A	★	○	○	○	✓	–	–	193(6)	229(5)
ETM 13E	[551]	2+1+1	C	★	○	○	○	✓	–	–	196(9)	235(9)
HPQCD 13	[71]	2+1+1	A	★	★	★	○	✓	184(4)	188(4)	186(4)	224(5)
RBC/UKQCD 14	[76]	2+1	A	○	○	○	○	✓	195.6(14.9)	199.5(12.6)	–	235.4(12.2)
RBC/UKQCD 14A	[75]	2+1	A	○	○	○	○	✓	–	–	219(31)	264(37)
RBC/UKQCD 13A	[552]	2+1	C	○	○	○	○	✓	–	–	$191(6)^{\diamond}_{\rm stat}$	$233(5)^{\diamond}_{\rm stat}$
HPQCD 12	[74]	2+1	A	○	○	○	○	✓	–	–	191(9)	228(10)
HPQCD 12	[74]	2+1	A	○	○	○	○	✓	–	–	$189(4)^{\triangle}$	–
HPQCD 11A	[73]	2+1	A	★	○	★	★	✓	–	–	–	$225(4)^{\nabla}$
FNAL/MILC 11	[63]	2+1	A	○	○	★	○	✓	197(9)	–	–	242(10)
HPQCD 09	[78]	2+1	A	○	○	○	○	✓	–	–	$190(13)^{\bullet}$	$231(15)^{\bullet}$
ALPHA 14	[77]	2	A	★	★	★	★	✓	–	–	186(13)	224(14)
ALPHA 13	[553]	2	C	★	★	★	★	✓	–	–	187(12)(2)	224(13)
ETM 13B, 13C†	[65, 554]	2	A	★	○	★	○	✓	–	–	189(8)	228(8)
ALPHA 12A	[555]	2	C	★	★	★	★	✓	–	–	193(9)(4)	219(12)
ETM 12B	[556]	2	C	★	○	★	○	✓	–	–	197(10)	234(6)
ALPHA 11	[557]	2	C	★	○	★	★	✓	–	–	174(11)(2)	–
ETM 11A	[197]	2	A	○	○	★	○	✓	–	–	195(12)	232(10)
ETM 09D	[558]	2	A	○	○	○	○	✓	–	–	194(16)	235(12)

$^{\diamond}$Statistical errors only.
$^{\triangle}$Obtained by combining f_{B_s} from HPQCD 11A with f_{B_s}/f_B calculated in this work.
$^{\nabla}$This result uses one ensemble per lattice spacing with light to strange sea-quark mass ratio $m_\ell/m_s \approx 0.2$.
$^{\bullet}$This result uses an old determination of $r_1 = 0.321(5)$ fm from Ref. [559] that has since been superseded.
†Update of ETM 11A and 12B.

★/✓ allows for satisfactory control of systematics

○ allows for reasonable (but improvable) estimate of systematics

■ unlikely to allow for reasonable control of systematics

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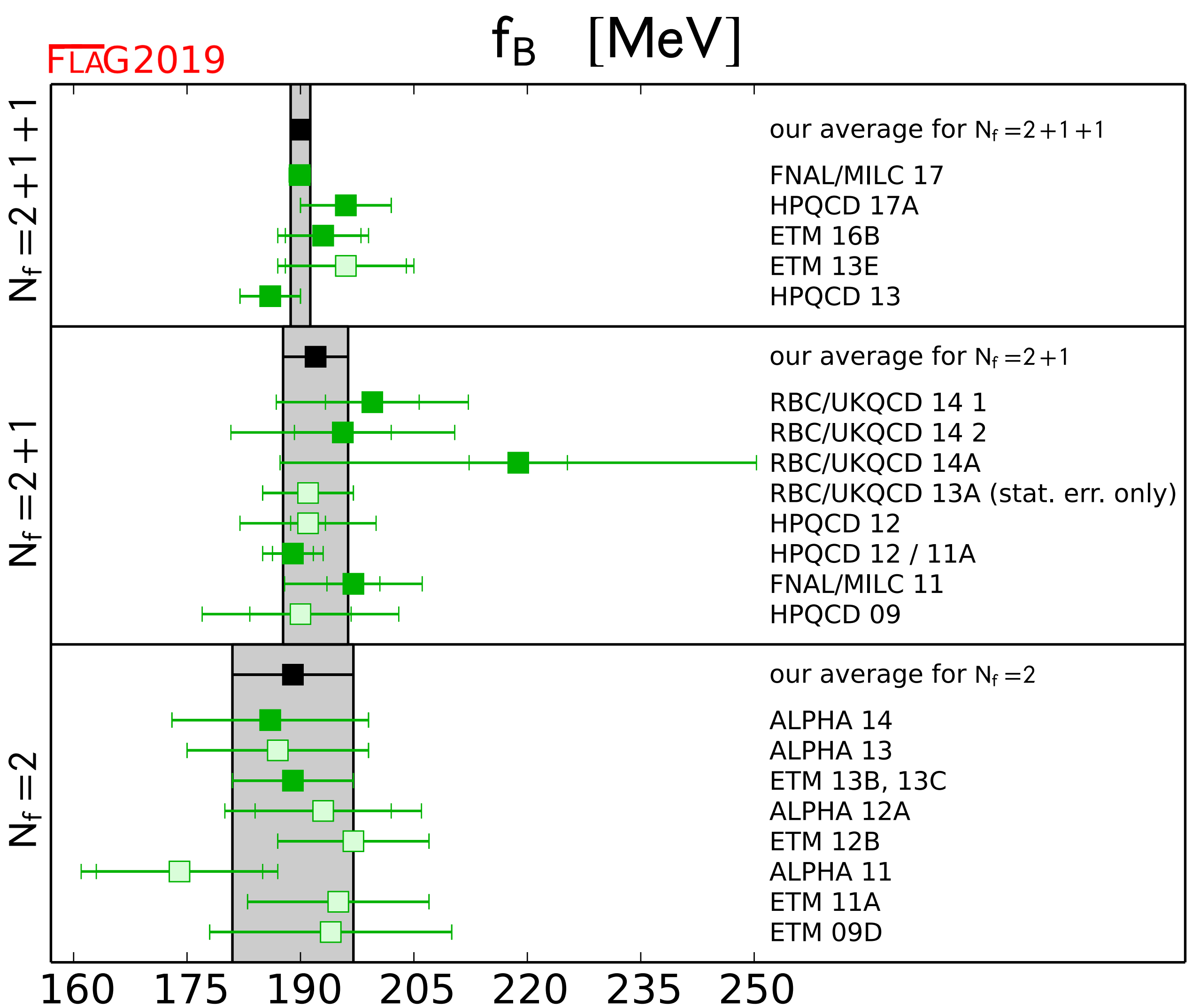
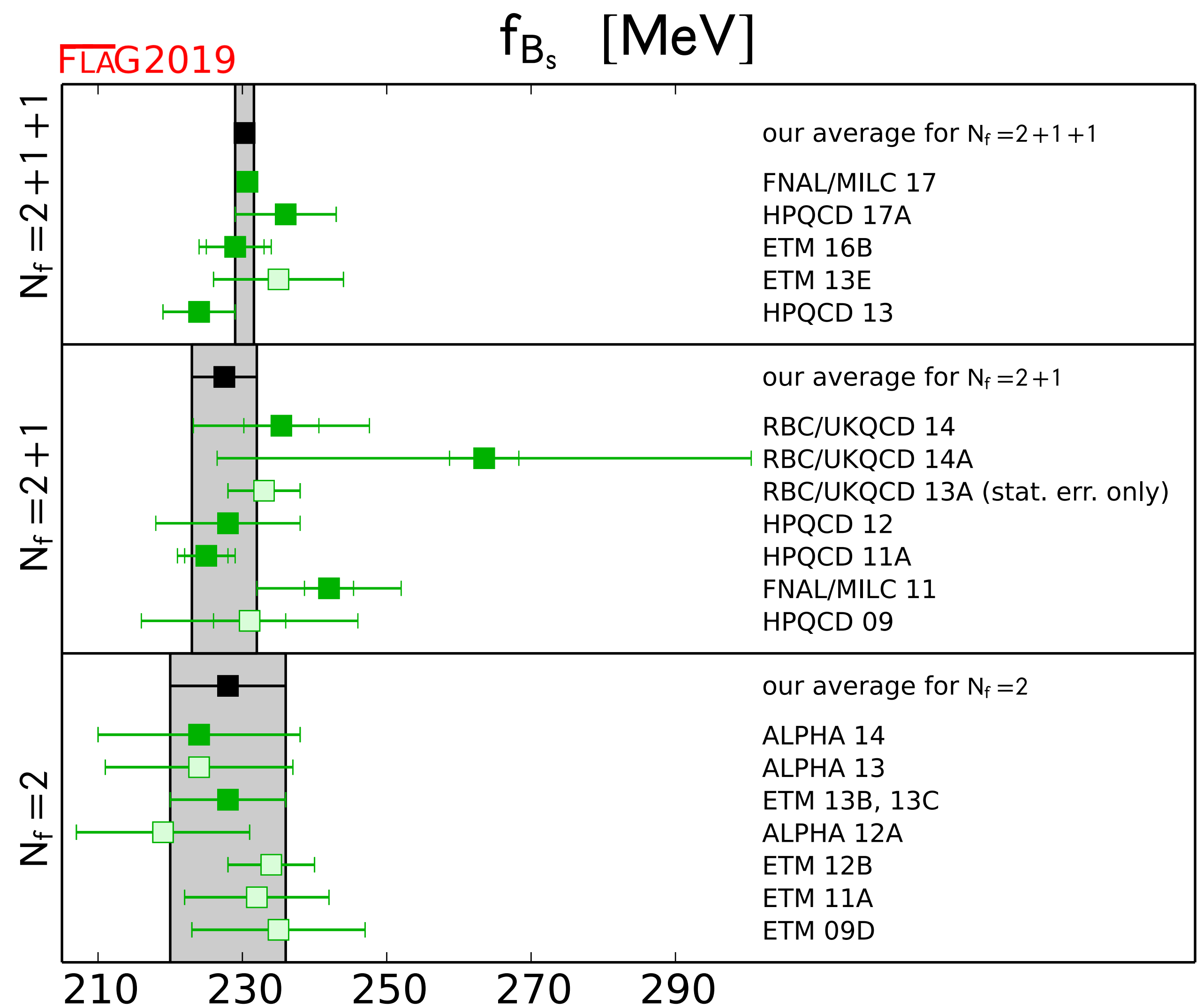
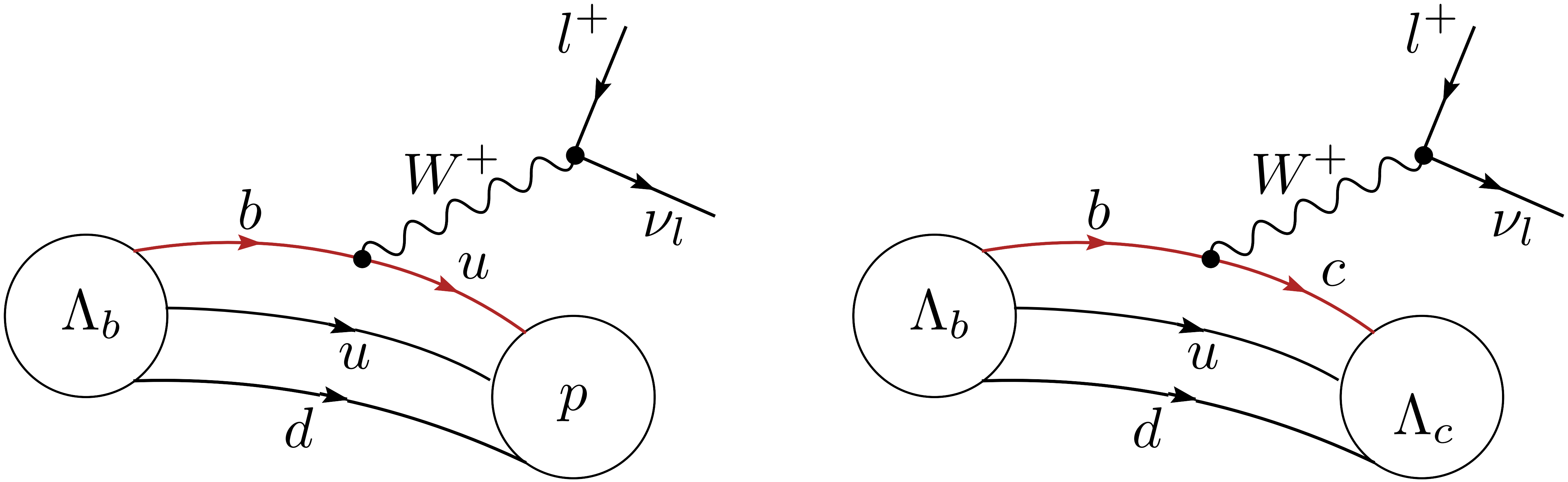
plots:

- ■ result included in average or estimate
- □ result OK but not included (superseded, unpublished, ...)
- □ all other results





baryon SL decay



new exclusive determination of $|V_{cb}|/|V_{ub}|$ from LHCb measurement + LQCD computation of form factors

[Detmold, Lehner, Meinel PRD 92 (2015) 034503]
[LHCb, Nature Phys 11 (2015) 743]

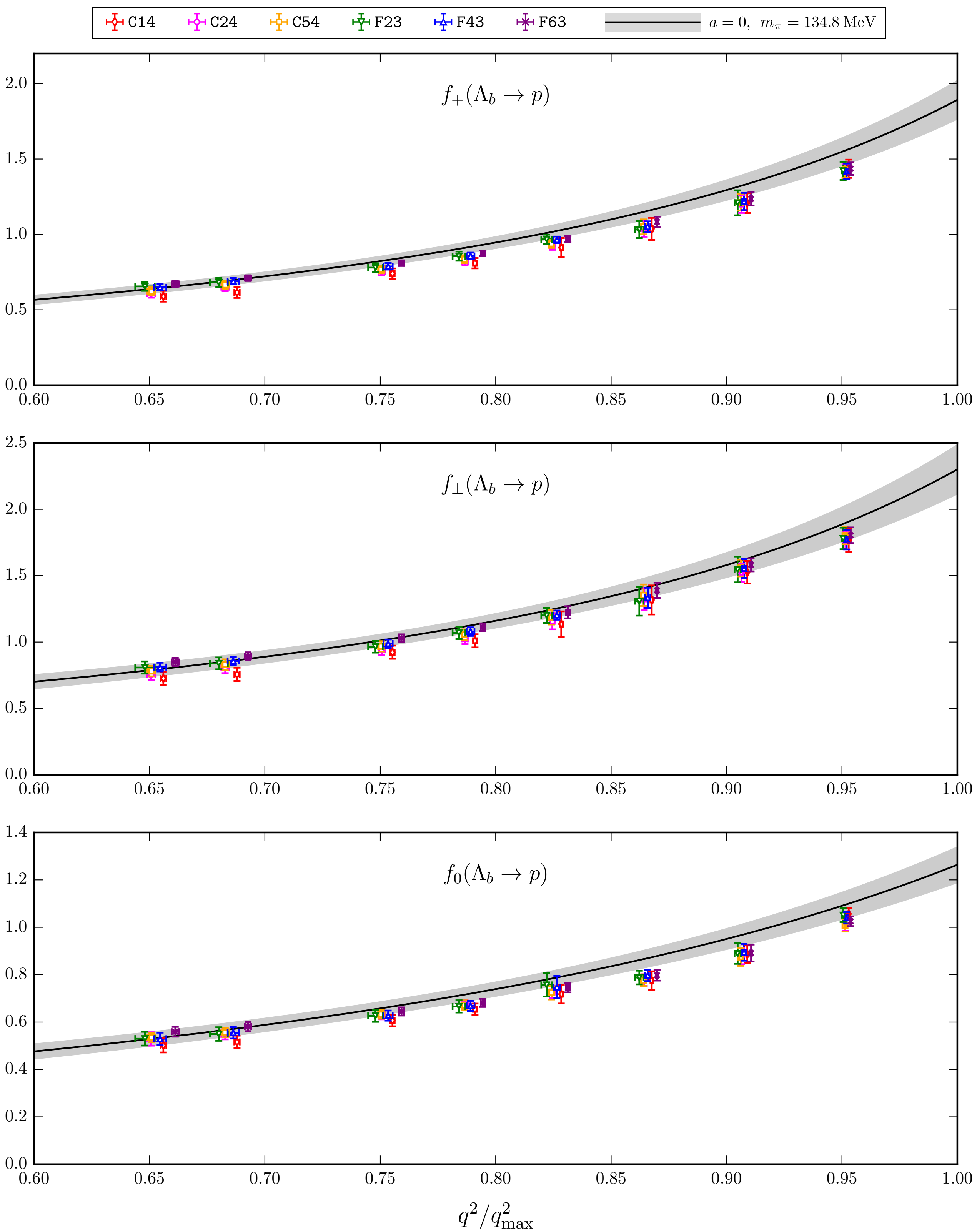
work since extended to charm channels, radiative decays, ...

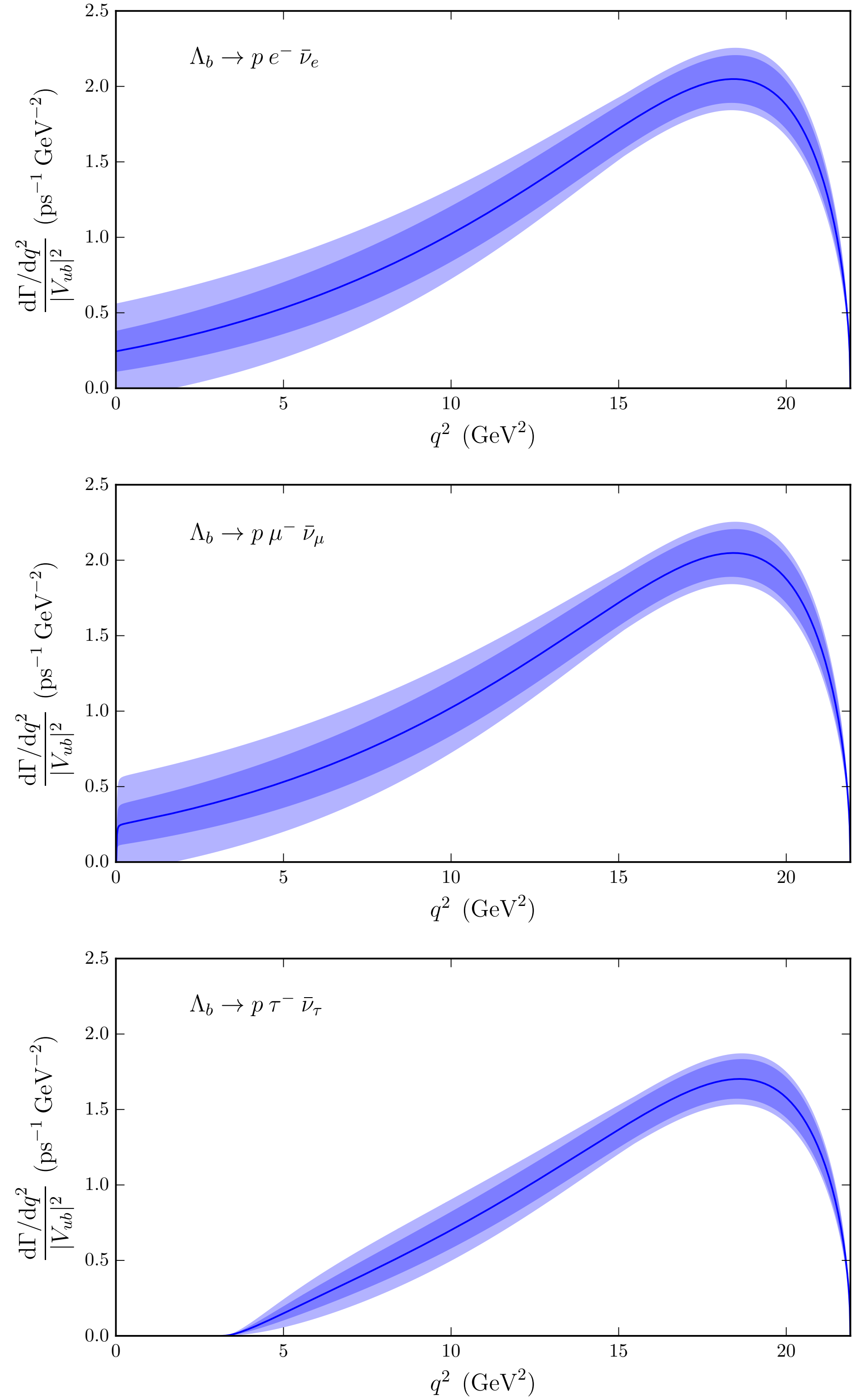
[Detmold, Meinel PRD 93 (2016) 074501]
[Meinel, Rendon PoS Lattice2016 (2016) 299]
[Meinel PRL 118 (2017) 082001]
[Meinel PRD 97 (2018) 034511]

baryonic decays

[Detmold, Lehner, Meinel PRD 92 (2015) 034503]

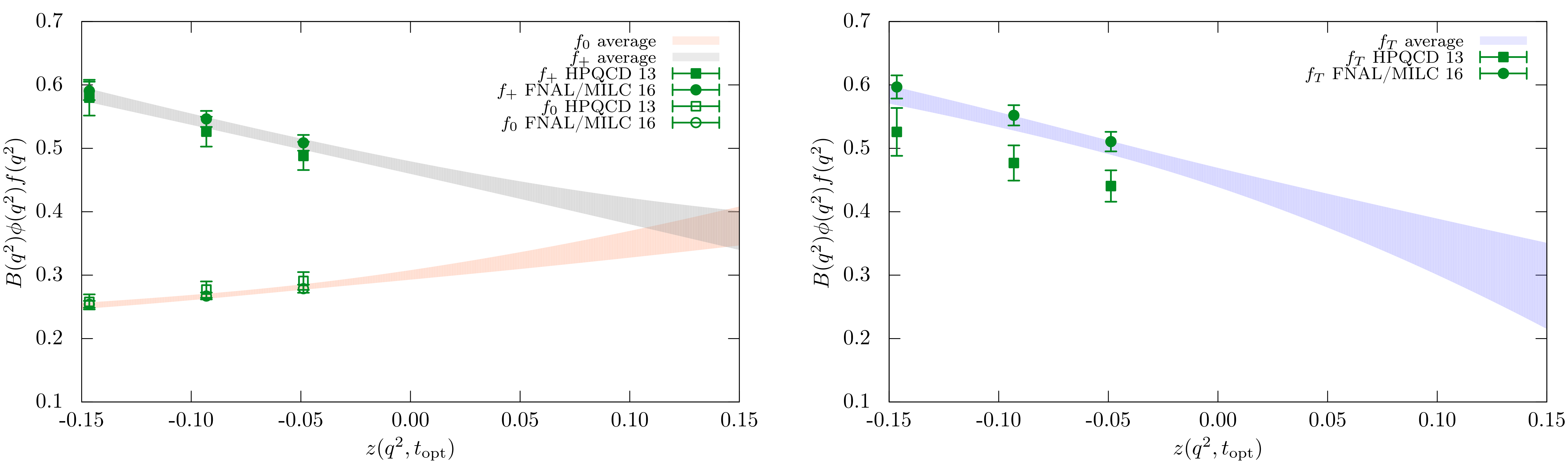
[cf. Detmold, Lin, Meinel, Wingate PRD 88 (2013) 014512]





radiative decays/BSM

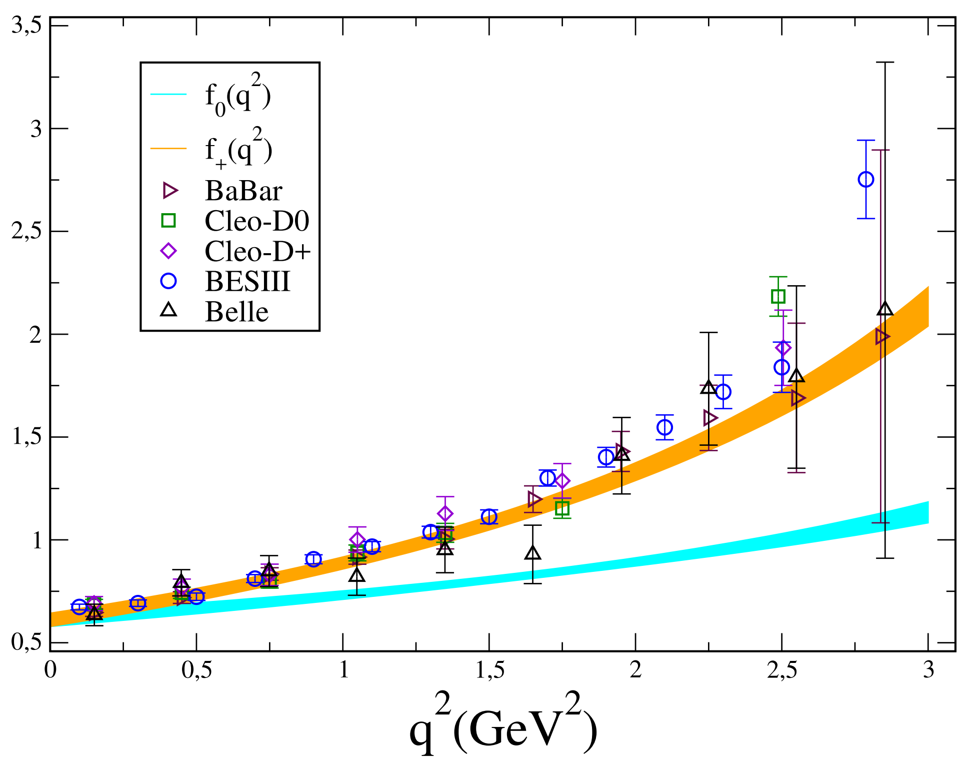
- lattice results at similar level of maturity as for SM tree-level decays
- channels with vectors in final state (e.g. K^*) much more complicated: treatment of resonances in Euclidean amplitudes quite non-trivial
- matrix elements of charmed penguins in H_w involve similar difficulties as n non-leptonic K and B decay — difficult nut to crack. (bounds?)

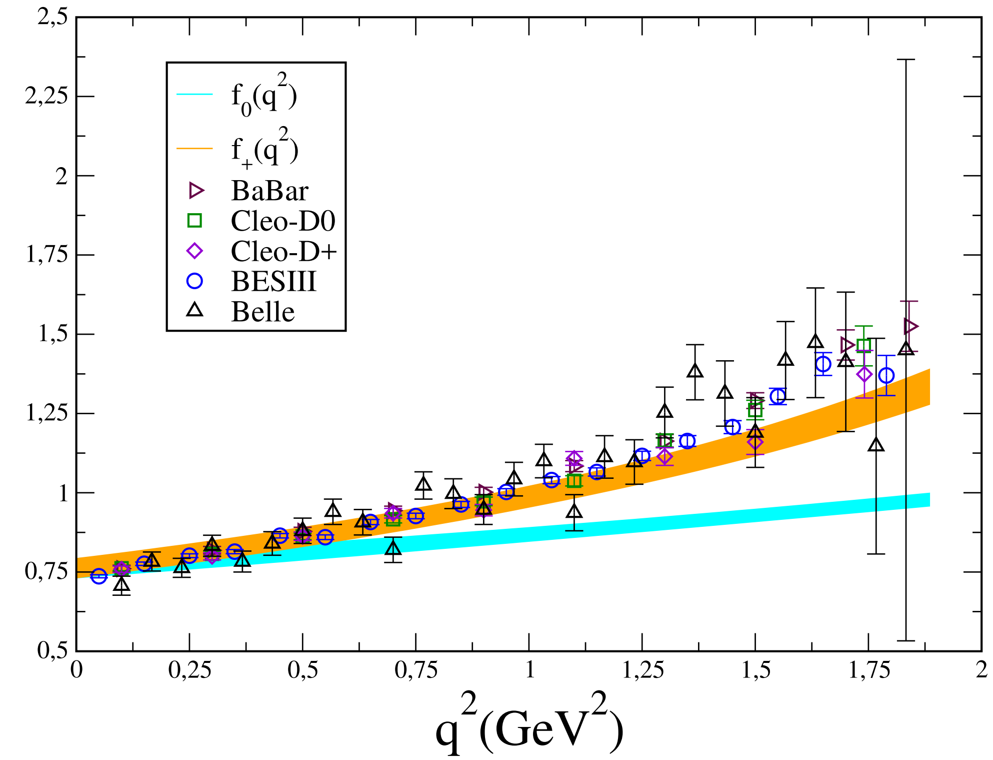


[FLAG4, Aoki et al., arXiv:1902.08191]

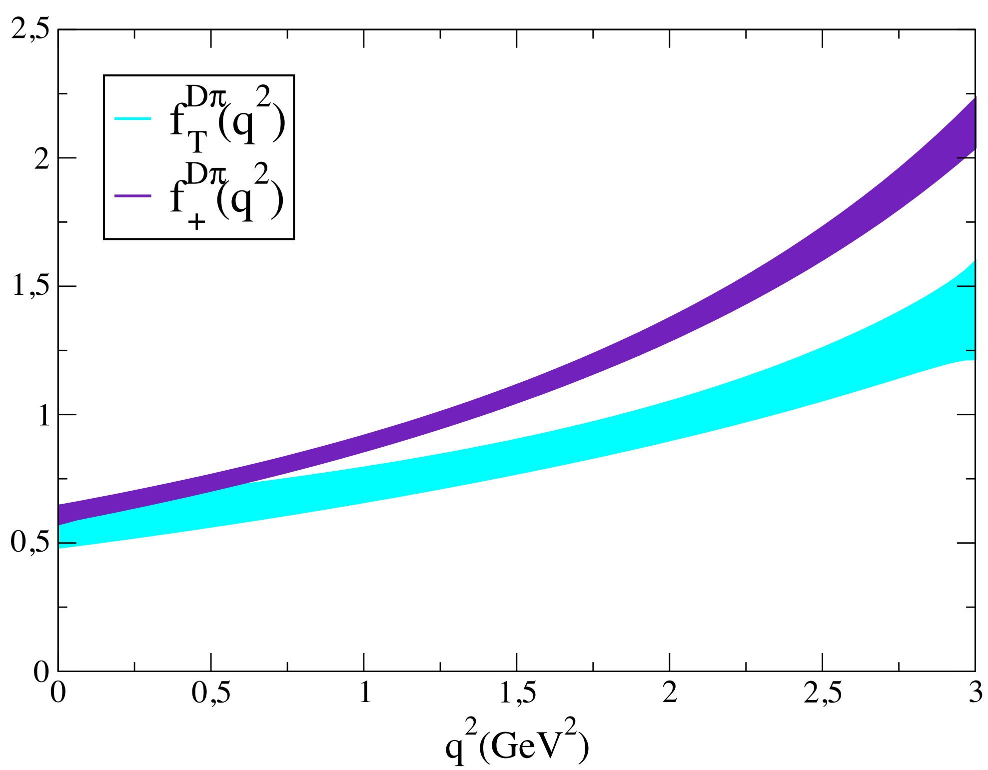
⇒ O_7, O_9, O_{10} (similar for $B \to \pi$ by FNAL/MILC, id. charm ETM)

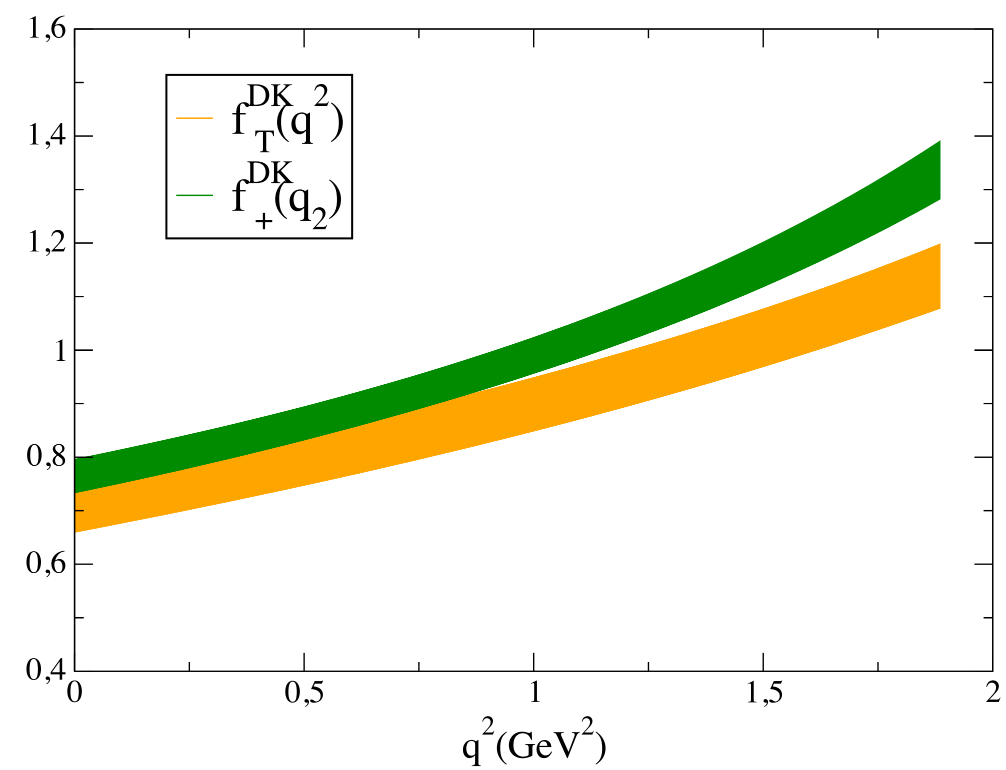
radiative decays/BSM





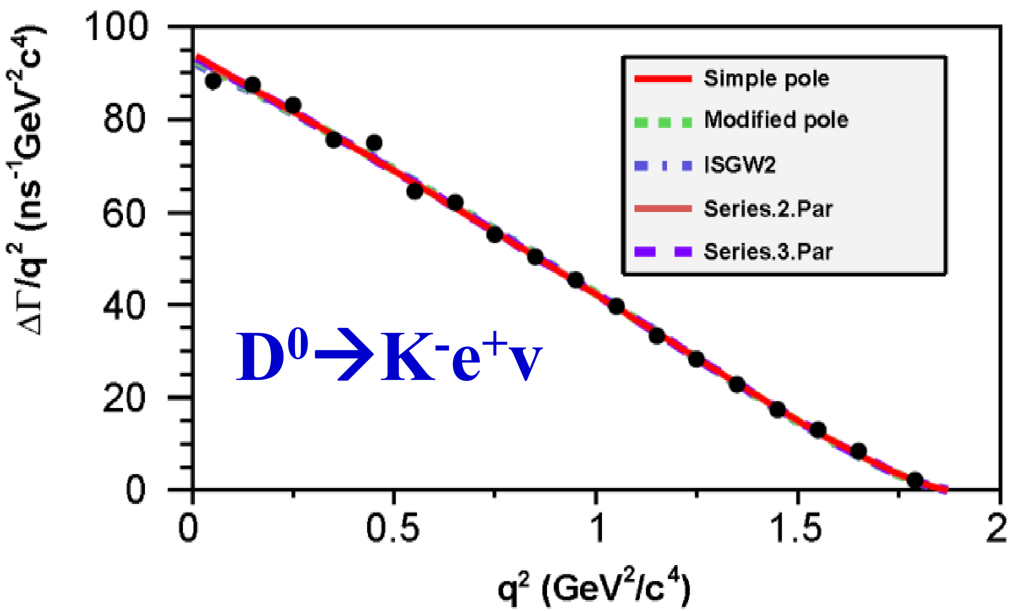
[ETMC PRD 96 (2017) 054514]

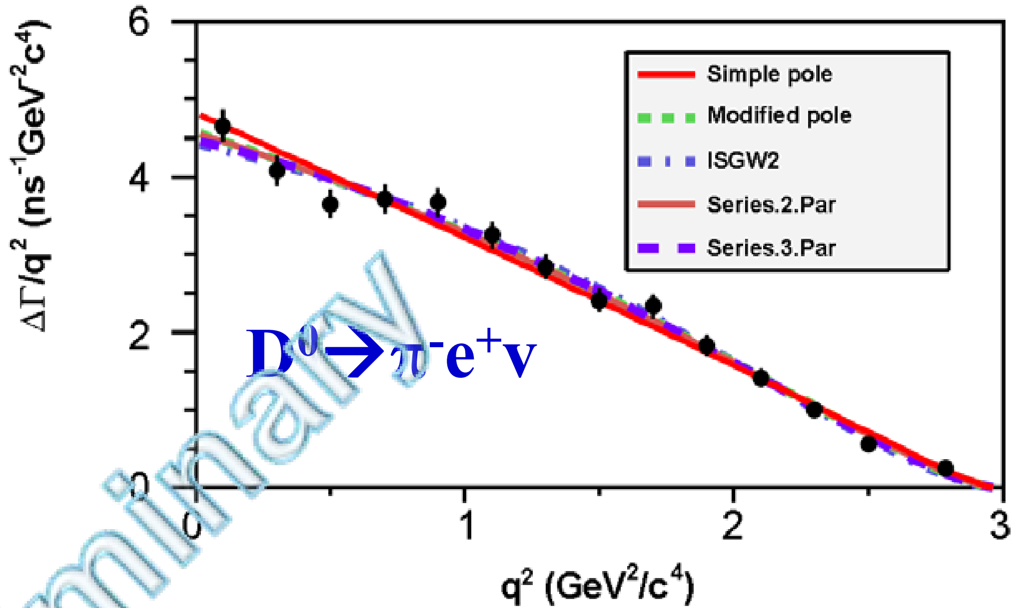




[ETMC arXiv:1710.07121]

q^2 dependence of form factors





		$D^0 \to K^- e^+ \nu$		$D^0 \to \pi^- e^+ \nu$
Simple Pole	$f_K^+(0)\|V_{cs}\|$	0.7209±0.0022±0.0033	$f_\pi^+(0)\|V_{cd}\|$	0.1475±0.0014±0.0005
	M_{pole}	1.9207±0.0103±0.0069	M_{pole}	1.9114±0.0118±0.0038
Mod. Pole	$f_K^+(0)\|V_{cs}\|$	0.7163±0.0024±0.0034	$f_\pi^+(0)\|V_{cd}\|$	0.1437±0.0017±0.0008
	α	0.3088±0.0195±0.0129	α	0.2794±0.0345±0.0113
ISGW2	$f_K^+(0)\|V_{cs}\|$	0.7139±0.0023±0.0034	$f_\pi^+(0)\|V_{cd}\|$	0.1415±0.0016±0.0006
	r_{ISGW2}	1.6000±0.0141±0.0091	r_{ISGW2}	2.0688±0.0394±0.0124
Series.2.Par	$f_K^+(0)\|V_{cs}\|$	0.7172±0.0025±0.0035	$f_\pi^+(0)\|V_{cd}\|$	0.1435±0.0018±0.0009
	r_1	-2.2278±0.0864±0.0575	r_1	-2.0365±0.0807±0.0260
Series.3.Par	$f_K^+(0)\|V_{cs}\|$	0.7196±0.0035±0.0041	$f_\pi^+(0)\|V_{cd}\|$	0.1420±0.0024±0.0010
	r_1	-2.3331±0.1587±0.0804	r_1	-1.8434±0.2212±0.0690
	r_2	3.4223±3.9090±2.4092	r_2	-1.3871±1.4615±0.4677

[from H Ma's talk on behalf of BESIII at CHARM 2015]

a benchmark case: $f_+(B \to \pi l\nu)$

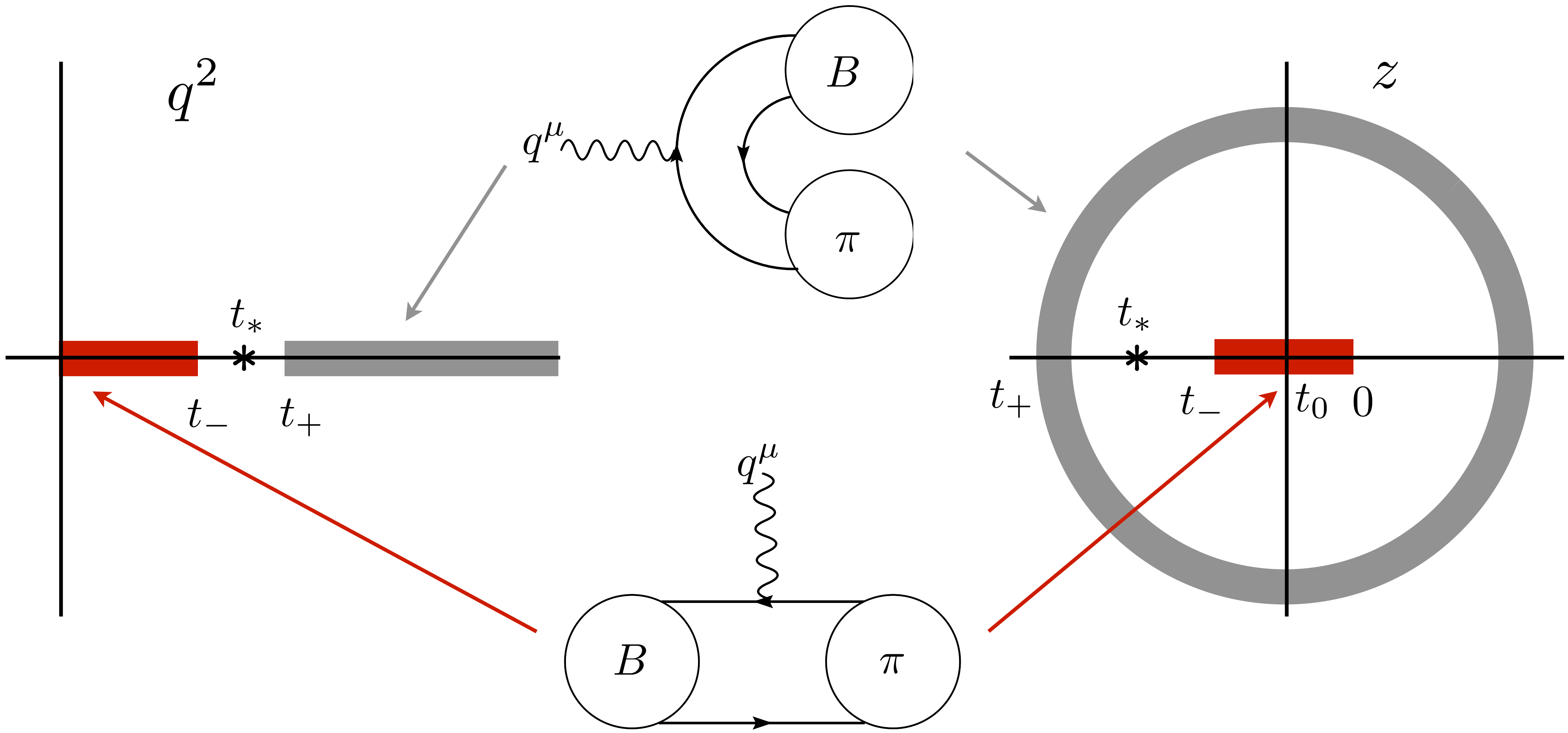
various parametrisations based on pole dominance: Bećirević-Kaidalov, Ball-Zwicky, Hill, ... difficult to systematically improve precision

[Bečirević, Kaidalov PLB 478 (2000) 417]
[Ball, Zwicky PRD 71 (2005) 014015]
[Hill PRD 73 (2006) 014012]

z-parametrisations proposed to solve this issue (almost) rigourously by exploiting unitarity and crossing symmetry

[Okubo PRD 3 (1971) 2807, 4 (1971) 725]
[Bourrely, Machet, de Rafael NPB 189 (1981) 157]
[Boyd, Grinstein, Lebed PRL 74 (1995) 4603]
[Lellouch NPB 479 (1996) 353]
[Bourrely, Caprini, Micu EJPC 27 (2003) 439]
[Arnesen, Grinstein, Rothstein, Stewart PRL 95 (2005) 071802]
[Becher, Hill PLB 633 (2006) 61]
[Flynn, Nieves PRD 75 (2007) 013008]
[Bourrely, Caprini, Lellouch PRD 79 (2009) 013008]

a benchmark case: $f_+(B \to \pi l \nu)$



$$z = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}$$

$$t_+ = (m_B + m_\pi)^2 \,, \qquad t_0 < t_+$$

$$\Rightarrow$$

$$f_+(q^2) = \frac{1}{B(q^2)\phi(q^2, t_0)} \sum_{n \ge 0} a_n z(q^2, t_0)^n$$

unitarity bound: $\sum_{m,n} B^{(\phi)}_{mn} a_m a_n \le 1$

a benchmark case: $f_+(B \to \pi l \nu)$

$$f_+(q^2) = \frac{1}{B(q^2)\phi(q^2,t_0)} \sum_{n\geq 0} a_n z(q^2,t_0)^n \qquad B(q^2) = z(q^2, m_{B^*}^2)$$

BGL: complicated outer function $\phi \quad \longrightarrow \quad \sum_{n\geq 0} |a_n|^2 \lesssim 1$

[Boyd, Grinstein, Lebed PRL 74 (1995) 4603]

BCL: $f_+(q^2) = \frac{1}{1-q^2/m_{B^*}^2} \sum_{n\geq 0} a_n z^n \quad \longrightarrow \quad \sum_{m,n\geq 0} B_{mn} a_m a_n \lesssim 1$

(recommended by FLAG)

[Bourrely, Caprini, Lellouch PRD 79 (2009) 013008]

crucial for optimal use:

- *all* sub-threshold poles included in Blaschke factor
- fixed kinematics (coefficients implicitly depend on quark masses)

does the unitarity bound apply?

- using a z-parametrisation as part of a global fit including a, m_q, ... (modified z-expansion) tricky
  - poles can cross threshold as quark masses change
  - complicated entanglement of (m_q,a) dependence (complete form factor vs. z-parametrisation coefficient)
- pole structure not always well-known (scalar channels, D decay), or complicated (Λ_b decay)
- missing sub-threshold poles may imply convergence breakdown (proton charge radius analysis by Hill, Paz et al, D semileptonic decay data by Bećirević et al)

[Hill, Paz PRD 82 (2010) 113005]
[Bhattacharya, Hill, Paz PRD 84 (2011) 073006]
[Epstein, Paz, Roy PRD 90 (2014) 074027]
[Bečirević et al arXiv:1407.1019]