lattice QCD for precision flavour physics
⇒ precision prospects for CKM determination

Carlos Pena

CERN Council Open Symposium on the Update of the European Strategy for Particle Physics - Granada, May 2019
why we care

- NP: energy frontier has revealed the/a Higgs boson+barren (?) land
  - exquisite control of SM predictions needed to dig up possible new Physics
  - hadronic sector: $\alpha_s$, quark masses, ...

---

**Standard Model Production Cross Section Measurements**

**Status:**

March 2019

**ATLAS Preliminary Run 1,2 $\sqrt{s} = 5,7,8,13$ TeV**

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[ATLAS 2019]
why we care

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● intensity frontier
  - land of opportunity (LHCb, Belle II, BESIII; NA62, KOTO; $(g-2)_\mu$ programme; nEDM; …)
  - strong interaction effects key
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- intensity frontier
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  - strong interaction effects key

- is the SM’s CKM mechanism the only source of flavour-changing interactions, CP violation? [and: is LFU preserved?]

\[ V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \]

[ Cabibbo PRL 10 (1963) 531 ]
[ Kobayashi, Maskawa Prog. Theor. Phys. 49 (1973) 652 ]
meeting the challenge from experiment

extremely active experimental programme in coming decade(s):

- heavy quark physics: LHCb, Belle II, BESIII (charm), ...
- kaon physics: NA62, KOTO, ...

lattice QCD needs to keep up with experimental precision — and make an effort to deliver PREdictions (including new physics).
meeting the challenge from experiment

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- heavy quark physics: LHCb, Belle II, BESIII (charm), ...
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projections — including reduction in theory (lattice) uncertainty:

<table>
<thead>
<tr>
<th>Observables</th>
<th>Belle (2017)</th>
<th>Belle II</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>5 ab⁻¹</td>
<td>50 ab⁻¹</td>
</tr>
<tr>
<td>$</td>
<td>V_{cb}</td>
<td>$ incl.</td>
</tr>
<tr>
<td>$</td>
<td>V_{cb}</td>
<td>$ excl.</td>
</tr>
<tr>
<td>$</td>
<td>V_{ub}</td>
<td>$ incl.</td>
</tr>
<tr>
<td>$</td>
<td>V_{ub}</td>
<td>$ excl. (WA)</td>
</tr>
<tr>
<td>$\mathcal{B}(B \to \tau\nu)$ [10⁻⁶]</td>
<td>91 \cdot (1 \pm 24%)</td>
<td>9%</td>
</tr>
<tr>
<td>$\mathcal{B}(B \to \mu\nu)$ [10⁻⁶]</td>
<td>&lt; 1.7</td>
<td>20%</td>
</tr>
<tr>
<td>$R(B \to D\tau\nu)$ (Had. tag)</td>
<td>0.374 \cdot (1 \pm 16.5%)</td>
<td>6%</td>
</tr>
<tr>
<td>$R(B \to D^*\tau\nu)$ (Had. tag)</td>
<td>0.296 \cdot (1 \pm 7.4%)</td>
<td>3%</td>
</tr>
</tbody>
</table>

meeting the challenge from experiment

extremely active experimental programme in coming decade(s):

- heavy quark physics: LHCb, Belle II, BESIII (charm), ...
- kaon physics: NA62, KOTO, ...

to do list:

- bring precision standards of lattice B-physics to (or below) 1% for (semi)leptonic meson decay, as already achieved in kaon sector.
- ditto, few % in baryon channels, neutral meson mixing.
- make inroads in multihadron/(broad) resonance final states.
- long-distance OPE: rare decays, charm CP violation, ...
where we stand

global fit:

$$|V_{\text{CKM}}| = \begin{pmatrix}
0.97446(10) & 0.22452(44) & 0.00365(12) \\
0.22438(44) & 0.97359^{(10)}_{(11)} & 0.04214(76) \\
0.00896^{(24)}_{(23)} & 0.04133(74) & 0.999105(32)
\end{pmatrix}$$

[PDG 2018]
where we stand

global fit:

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\end{pmatrix}$$

exclusive determination with lattice input: errors between few permille (light, strange) and few percent (charm, bottom)

$$|V_{\text{CKM}}| = \begin{pmatrix}
0.97437(16) & 0.2249(7) & 0.00373(14) \\
0.2166(7)(50) & 1.004(2)(16) & 0.0401(10)
\end{pmatrix}$$

[PDG 2018]

[FLAG 2018]
Thus, the processes typically considered for determining the absolute values of the CKM matrix elements are the following:

\[
|V_{\text{CKM}}| = \begin{pmatrix}
0.97437(16) & 0.2249(7) & 0.00373(14) \\
0.2166(7)(50) & 1.004(2)(16) & 0.0401(10)
\end{pmatrix}
\]

[FLAG 2018]

exclusive determination with lattice input:

1st row:
where we stand

exclusive determination with lattice input:

\[
|V_{\text{CKM}}| = \begin{pmatrix}
0.97437(16) & 0.2249(7) & 0.00373(14) \\
0.2166(7)(50) & 1.004(2)(16) & 0.0401(10)
\end{pmatrix}
\]

[FLAG 2018]
where we stand

exclusive determination with lattice input:

\[
|V_{\text{CKM}}| = \begin{pmatrix}
0.97437(16) & 0.2249(7) & 0.00373(14) \\
0.2166(7)(50) & 1.0042(16) & 0.0401(10)
\end{pmatrix}
\]

\[f_{B(s)}^2 B_{B(s)}\]

\[B_K, |V_{cb}|^4\]
OPE for weak decays of hadrons

electromagnetic corrections to hadronic weak matrix elements traditionally neglected in lattice studies.

\[
\frac{g_w^2}{p^2 - M_W^2} = -2\sqrt{2}G_F \left[ 1 + \mathcal{O} \left( \frac{p^2}{M_W^2} \right) \right]
\]

as precision has started to approach percent levels, estimation of e.m. effects has become an issue.
lattice QCD

\[ \mathcal{L}_{\text{QCD}} = -\frac{1}{2g^2} \text{tr} \left[ F_{\mu\nu} F^{\mu\nu} \right] + \sum_{q=1}^{N_f} \bar{\psi}_q \left[ i \not{D} - m_q \right] \psi_q + \frac{i\theta}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} \left[ F_{\mu\nu} F_{\rho\sigma} \right] \]

first-principles approach = control all systematic uncertainties

- spacetime = Euclidean lattice
- allows to define path integral rigorously and compute it via Monte Carlo methods
- QCD recovered by removing cutoffs at physical kinematics
- values of Lagrangian parameters fixed by \( N_f + 1 \) hadron masses/decay constants — everything else are predictions

[Wilson 1974]
It would allow to study QCD in different conditions such as high density or temperature as took place in the early universe or in very dense systems such as neutron stars. QCD is in some sense a model field theory for many extensions of the SM as well as for the lattice approach. In QCD we know where the UV fixed point lies so we know where the continuum limit is and how to approach it. The lattice method might be necessary to study other field theories such as those in models of technicolor or dynamical gauge symmetry breaking where things might not be so easy. Clearly having solved QCD is a benchmark to guide future investigations.

Giving the spread of quark masses that span six orders of magnitude, dealing with all quarks in a lattice simulation is very difficult since approaching the continuum limit in controlled conditions would require extremely fine lattices. This brute force approach is not practical. Fortunately, when we try to describe the low energy regime, the effect of the heavy quarks can be accurately described by an effective theory that results from integrating them out. It is a consequence of the decoupling theorem that the effects of the heavy quarks in the low energy dynamics are well represented by local operators of the light fields only (gluons and the lighter quarks) where the effect of the heavy scales is reabsorbed in the couplings.

We are also interested however in processes involving heavy hadrons. An efficient way to do this is to consider them as static sources as is done in the heavy quark effective theory. I refer to Roman Sommer's lectures for a detailed discussion of this effective theory as an efficient tool to study heavy flavours on the lattice.
lattice QCD for phenomenology: FLAG

Flavour Lattice Averaging Group: your one-stop repository of lattice results, world averages / estimates


advisory board: S Aoki, M Golterman, R Van de Water, A Vladikas

editorial board: G Colangelo, S Hashimoto, A Jüttner, S Sharpe, U Wenger

working groups:

quark masses

\( V_{ud}, V_{us} \)

LECs

kaon mixing

heavy leptonic + mixing

heavy semileptonic

\( \alpha_s \)

nuclear matrix elements

T Blum, A Portelli, A Ramos

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CITE THE ORIGINAL WORKS!
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[SWME 15A] PRD 93 (2016) 01451
[TWQCD 14] PLB 736 (2014) 231
### leptonic decay

\[
\mathcal{B}(B^+ \to l^+ \nu_l) = \frac{G_F^2}{8\pi} \frac{m_l^2 m_{B^+}^2}{m_{B^+}^2} \left(1 - \frac{m_l^2}{m_{B^+}^2}\right)^2 |V_{ub}|^2 f_{B^+}^2
\]

\[
\langle 0 | \bar{b} \gamma^\mu \gamma^5 q | B^+(p) \rangle = f_{B^+} p^\mu
\]

\[
\mathcal{B}(B_q \to l^+ l^-) = \frac{G_F^2}{\pi} Y \left(\frac{\alpha}{4\pi \sin^2 \theta_W}\right)^2 m_{B_q} m_l^2 \sqrt{1 - 4 \frac{m_l^2}{m_{B_q}^2}} |V_{tb}^* V_{tq}|^2 f_{B_q}^2
\]
leptonic decay

\[ \mathcal{B}(B^+ \to l^+ \nu_l) = \frac{G_F^2 m_l^2 m_{B^+}^2}{8\pi} \left( 1 - \frac{m_l^2}{m_{B^+}^2} \right)^2 |V_{ub}|^2 f_{B^+}^2 \]

\[ \langle 0 | \bar{b} \gamma^\mu \gamma^5 q | B^+(p) \rangle = f_{B^+} p^\mu \]

precision at few-per-mille, QED+IB corrections crucial for next stage
leptonic decay

\[ B(B^+ \to l^+\nu_l) = \frac{G_F^2 m_l^2 m_{B^+}^2}{8\pi} \left(1 - \frac{m_l^2}{m_{B^+}^2}\right)^2 |V_{ub}|^2 f_{B^+}^2 \]

\[
\langle 0|\bar{b}\gamma^\mu\gamma^5 q|B^+(p)\rangle = f_{B^+}p^\mu
\]

precision at \textbf{few-per-mille}, QED+IB corrections crucial for next stage
It is clear that the decay constants for charged and neutral $B$ mesons play different roles in flavour-physics phenomenology. As already mentioned above, the knowledge of the $B^+$ meson decay constant $f_{B^+}$ is essential for extracting $|V_{ub}|$ from leptonic $B^+$ decays. The neutral $B^0$ meson decay constants $f_{B^0}$ and $f_{B_s}$ are inputs for the search of new physics in rare leptonic $B^0$ decays. In view of this, it is desirable to include isospin-breaking effects in lattice computations for these quantities, and have results for $f_{B^+}$ and $f_{B^0}$. With the increasing precision of recent lattice calculations, isospin splittings for $B$-meson decay constants are significant, and will play an important role in the foreseeable future. A few collaborations reported $f_{B^+}$ and $f_{B^0}$ separately by taking into account strong isospin effects in the valence sector, and estimated the corrections from electromagnetism. To properly use these results for extracting phenomenologically relevant information, one would have to take into account QED effects in the $B$-meson leptonic decay rates. Currently, errors on the experimental measurements on these decay rates are still very large. In this review, we will then concentrate on the isospin-averaged result $f_B$ and the $B_s$-meson decay constant, as well as the $SU(3)$-breaking ratio $f_{B_s}/f_{B^+}$. For the determination of $f_{B^+}$ and $f_{B_s}/f_{B^+}$, we refer the reader to the latest work from the Particle Data Group [132]. Notice that the $N_f=2+1$ lattice result used in Ref. [132] and the current review are identical. We will discuss this in further detail at the end of this subsection.

The status of lattice-QCD computations for $B$-meson decay constants and the $SU(3)$-breaking ratio, using gauge-field ensembles with light dynamical fermions, is summarized in Tabs. 34 and 35, while Figs. 23 and 24 contain the graphical presentation of the collected results and our averages. Many results in these tables and plots were already reviewed in detail in the previous FLAG report. Below we will describe the new results that appeared after January 2016.

Figure 23: Decay constants of the $B$ and $B_s$ mesons. The values are taken from Tab. 34 (the $f_{B^+}$ entry for FNAL/MILC 11 represents $f_{B^+}$). The significance of the colours is explained in Sec. 2. The black squares and grey bands indicate our averages in Eqs. (196), (199), (202), (197), (200) and (203). No new $N_f=2$ and $N_f=2+1$ projects for computing $f_B$, $f_{B_s}$ and $f_{B_s}/f_{B^+}$ were completed after the publication of the previous FLAG review [3]. Therefore, our averages for these cases were calculated using the results from previous reviews.

See Ref. [204] for a strategy that has been proposed to account for QED effects.
semileptonic decay

\[
\frac{d\Gamma(P_i \to P_f l\nu)}{dq^2} = \frac{G_F^2 |V_{jk}|^2}{24\pi^3} \frac{(q^2 - m_i^2)^2 \sqrt{E_f^2 - m_f^2}}{q^4 m_i^2} \\
\times \left[ \left(1 + \frac{m_i^2}{2q^2}\right) m_i^2 (E_f^2 - m_f^2) |f_+(q^2)|^2 + \frac{3m_i^2}{8q^2} (m_i^2 - m_f^2)^2 |f_0(q^2)|^2 \right]
\]

\[
\langle P_f(p')|\bar{D}_k\gamma_\mu U_j|P_i(p)\rangle = f_+(q^2) \left( p_\mu + p'_\mu - \frac{m_i^2 - m_f^2}{q^2} q_\mu \right) + f_0(q^2) \frac{m_i^2 - m_f^2}{q^2} q_\mu, \quad q = p - p'
\]
semileptonic decay: $K \rightarrow \pi$

$$f_+(0) = 1 + f_2 + f_4 + \ldots$$

While the first sum on the right extends over nonstrange intermediate states, the second runs over exotic states with strangeness $\pm 2a$.

The expansion of $f_+(0)$ in $SU(3)$ chiral perturbation theory in powers of $m_u$, $m_d$, and $m_s$ starts with

$$f_+(0) = 1 + f_2$$

Since all of the low-energy constants occurring in $f_2$ can be expressed in terms of $M_\pi$, $M_K$, and $f_\pi$, the NLO correction is known. In the language of the sum rule (73), $f_2$ stems from nonstrange intermediate states with three mesons.

Like all other nonexotic intermediate states, it lowers the value of $f_+(0)$:

$$f_2 = -0.023$$

when using the experimental value of $f_\pi$ as input. The corresponding expressions have also been derived in quenched or partially quenched (staggered) chiral perturbation theory.

At the same order in the $SU(2)$ expansion, $f_+(0)$ is parameterized in terms of $M_\pi$ and two a priori unknown parameters. The latter can be determined from the dependence of the lattice results on the masses of the quarks. Note that any calculation that relies on the $\chi$PT formula for $f_2$ is subject to the uncertainties inherent in NLO results: instead of using the physical value of the pion decay constant $f_\pi$, one may use the constant $f_0$ that occurs in the effective Lagrangian and represents the value of $f_\pi$ in the chiral limit.

Although trading $f_\pi$ for $f_0$ in the expression for the NLO term affects the result only at NNLO, it may make a significant numerical difference in calculations where the latter are not explicitly accounted for. (Lattice results concerning the value of the ratio $f_\pi/f_0$ are reviewed in Sec. 5.3.)

**Figure 8:** Comparison of lattice results (squares) for $f_+(0)$ and $f_K/\pi$ with various model estimates based on $\chi$PT (blue circles). The ratio $f_K/\pi$ is obtained in pure QCD including the $SU(2)$ isospin-breaking correction (see Sec. 4.3). The black squares and grey bands indicate our estimates. The significance of the colours is explained in Sec. 2.

The lattice results shown in the left panel of Fig. 8 indicate that the higher order contributions $\Delta f_+ \equiv f_+(0) - 1 - f_2$ are negative and thus amplify the effect generated by $f_2$. This confirms the expectation that the exotic contributions are small. The entries in the lower part of the left panel represent various model estimates for $f_4$. In Ref. [249], the symmetry-breaking precision at few-per-mille, QED+IB corrections crucial for next stage
precision for CKMs still theory-dominated (exp results for $K$ decay much more precise)

negligible dependence on charm mass, good agreement among various determinations.
very few results, although ETMC (+ ongoing FNAL/MILC) has the first computation of the $q^2$ dependence of all form factors

(relevant: extrapolation of exp rates to $q^2=0$ already sensitive to parametrisation)
semileptonic decay: $B \to D$, $B \to \pi$

parametrisation of $q^2$ dependence plays a key role
semileptonic decay: \( B \to D, \ B \to \pi \)

CKMs: few % errors
neutral meson mixing

\[\epsilon_K = \frac{A[K_L \to (\pi\pi)_{I=0}]}{A[K_S \to (\pi\pi)_{I=0}]} = \exp(i\phi_\epsilon) \sin(\phi_\epsilon) \left[ \frac{\text{Im}(M_{12}^{SD})}{\Delta M_K} + \frac{\text{Im}(M_{12}^{LD})}{\Delta M_K} + \frac{\text{Im}(A_0)}{\text{Re}(A_0)} \right] \]

\[\text{Im}(M_{12}^{SD}) = \frac{G_F^2 M_W^2}{12\pi^2} \left[ \lambda_c^2 S_0(x_c) \eta_1 + \lambda_t^2 S_0(x_t) \eta_2 + 2 \lambda_c \lambda_t S_0(x_c, x_t) \eta_3 \right] f_K^2 m_K \hat{B}_K \]

\[= \frac{G_F^2 M_W^2}{12\pi^2} |V_{cb}|^2 \lambda^2 \eta \left[ S_0(x_c) \eta_1 + |V_{cb}|^2 (1 - \bar{\rho}) S_0(x_t) \eta_2 + S_0(x_c, x_t) \eta_3 \right] f_K^2 m_K \hat{B}_K \]

\[\lambda_q = V_{qs}^* V_{qd}, \quad x_q = \frac{m_q^2}{M_W^2} \]

\[B_K = \frac{\langle \bar{K}^0 | (\bar{s}_L \gamma^\mu d_L) (\bar{s}_L \gamma_\mu d_L) | K^0 \rangle}{\frac{8}{3} f_K^2 m_K^2} \]

long-distance contribution relevant at \(\sim 1\%\) precision
neutral meson mixing

\[ \Delta m_q = 2 |M_{12}| \]

\[ M_{12} = \frac{G_F^2 M_W^2}{12 \pi^2} (V_{tq}^* V_{tb})^2 S_0 \left( \frac{m_t^2}{M_W^2} \right) \eta_B m_{B_q} f_{B_q}^2 \hat{B}_{B_q} \]

\[ B_{B_q} = \frac{\langle \bar{B}_q^0 | (\bar{b}_L \gamma^\mu q_L) (\bar{b}_L \gamma^\mu q_L) | B_q^0 \rangle}{\frac{8}{3} f_{B_q}^2 m_{B_q}^2} \]
neutral meson mixing

FLAG average sports 1.3% error — work out long-distance contribution, QED corrections

paucity of results wrt kaon sector glaring, 5-8% errors: largest room for improvement among basic CKM quantities
the present

- CKMs from pion/kaon physics receive \textit{permille} uncertainties from the lattice; \textbf{few \%} in charm, bottom CKMs. Kaon mixing at \%.

- several \textbf{exclusive} channels allow for crosschecks
  - pion, kaon, charm: leptonic+semileptonic (including $\Lambda_c$).
  - bottom: baryon decay ($\Lambda_b$, $p$, ...); $B \rightarrow D^* l \nu$; \textbf{predictions} for $B_s \rightarrow K l \nu$, $B_s \rightarrow D_s(*) l \nu$, $B_c \rightarrow (M) l \nu$, ...; first information on channels with other vector resonances.
    - \textbf{bonus}: same techniques provide equally-precise \textit{BSM} input.

- largest room for bread-and-butter improvement: charm SL, B mixing

- developing: multihadron/resonances in final state
the (short-term) future

- fully tame the B sector: fully relativistic $b$ quarks

- systematically add electromagnetic + strong isospin breaking
  - QCD+QED
  - working examples

- work out long-distance OPE contributions
  - bonus: open new channels (rare $K$ decays, charm CP violation, ...)

- improve channels with resonances / $>1$ hadron in final state

- ............
fully relativistic $b$ quarks

$$(a m_b)^2 \lesssim \frac{1}{3} \quad \Leftrightarrow \quad a \lesssim 0.03 \text{ fm}$$

$\Rightarrow$ populate lower lattice spacings in simulation landscape
fully relativistic $b$ quarks

algorithmic issue: strong lattice space dependence of autocorrelations

[Del Debbio, Panagopoulos, Vicari 2002]
[Schaefer, Sommer, Virotta 2010]

[Lüscher, Schaefer 2011; CLS $N_f=2+1$ obs programme]
[Mages et al. 2015; Laio et al. 2015; Brower et al. 2015; Detmold, Endres 2016]
fully relativistic $b$ quarks

algorithmic issue: strong lattice space dependence of autocorrelations

- improve algorithmic performance by simulating with non-trivial boundary conditions.

- estimate finite-volume corrections stemming from long autocorrelations (MILC’s quark masses, decay constants).

<table>
<thead>
<tr>
<th>$Q^2_{\text{sample}}/Q^2_{\text{XPT}}$</th>
<th>$m_f = \text{physical}$</th>
</tr>
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<tbody>
<tr>
<td>$f_K/f_\pi$</td>
<td>1.19680(0.00014)[0.00015]</td>
</tr>
<tr>
<td>$am_\pi$</td>
<td>0.028964(0.000020)[0.000008]</td>
</tr>
<tr>
<td>$af_D$</td>
<td>0.045389(0.000245)[0.000006]</td>
</tr>
<tr>
<td>$am_D$</td>
<td>0.400678(0.000258)[0.000001]</td>
</tr>
<tr>
<td>$af_{D_s}$</td>
<td>0.053582(0.000025)[0.000000]</td>
</tr>
<tr>
<td>$am_{D_s}$</td>
<td>0.422041(0.000037)[0.000000]</td>
</tr>
</tbody>
</table>

reliance on effective theory being rapidly eroded
\[ f_{\pi^\pm} = 130.2(0.8) \text{ MeV} \quad (0.6\%) \]
\[ f_{K^\pm} = 155.7(0.3) \text{ MeV} \quad (0.2\%) \]
\[ f_+(0) = 0.9706(27) \quad (0.3\%) \]
\[ \delta \chi_{\text{e.m.}}^{\text{PT}}(\pi^- \rightarrow l^- \bar{\nu}) = 1.8\% \]
\[ \delta \chi_{\text{e.m.}}^{\text{PT}}(K^- \rightarrow l^- \bar{\nu}) = 1.1\% \]
\[ \delta \chi_{\text{e.m.}}^{\text{PT}}(K \rightarrow \pi l \bar{\nu}) = 0.5\% - 3.0\% \]


precision of standalone QCD computation in isospin limit well below the size of e.m.+IB corrections
QED (+ isospin breaking)

no mass gap in QED ⇒ massless photons in physical spectrum ⇒ not easy to work in finite volume; two ways out:

• expand observables in $\alpha_{\text{em}}$ and $m_u - m_d$, compute coefficients of expansion non-perturbatively in QCD
  
  [de Divitiis et al. (RM123) PRD 87 (2013) 114505]

• simulate QCD+QED directly, including isolated charges — possibly at unphysically large values of $\alpha_{\text{em}}$ and $m_u - m_d$ + extrapolation.

  - formulate QED in finite volume, treat zero modes by hand
    
    [Hayakawa, Uno Prog. Theor. Phys. 120 (2008) 413]

  - introduce photon mass (fixed gauge), extrapolate to massless photon limit
    
    [Endres et al. PRL 117 (2016)]

  - introduce non-trivial $C^*$ boundary conditions
    
ab-initio computation of baryon mass splittings

light-meson leptonic rates

meson masses and HVP
[RBC/UKQCD JHEP 1709 (2017) 153]

strong IB in \((g-2)_{\mu}\)
[FNAL/MILC+HPQCD PRL 120 (2018) 152001]
OPE long-distance contributions
(\(+\) rare decays/charm CP)

“long-distance” contributions appear when loops involve exchanges of light d.o.f. in the effective weak theory description.

\[
\Delta m_K \equiv m_{K_L} - m_{K_S} = 2\mathcal{P} \sum_\alpha \frac{\langle \bar{K}^0 | \mathcal{H}_W | \alpha \rangle \langle \alpha | \mathcal{H}_W | K^0 \rangle}{m_K - E_\alpha} = 3.483(6) \times 10^{-12} \text{ MeV.}
\]

practical implementation on the lattice worked out by RBC/UKQCD

[Christ et al. PRD88 (2013) 014508]

preliminary result:  \( \Delta m_K = 5.5(1.7) \times 10^{-12} \text{ MeV} \)

[Bai et al. Lattice 2017]
OPE long-distance contributions
(+ rare decays/charm CP)

with this technique in place, other similar problems can be attacked.

- rare kaon decays: $K \to \pi l^+ l^-$, $K \to \pi \nu \bar{\nu}$


“emerging kaon UT”

[Lehner, Lunghi, Soni PLB 759 (2016) 82]
OPE long-distance contributions
( + rare decays/charm CP)

with this technique in place, other similar problems can be attacked.

- CP-conserving rare kaon decays: $K \rightarrow \pi l^+ l^-, \ K \rightarrow \pi \nu \bar{\nu}$
  

- charm CP violation???
conclusions & outlook

- lattice flavour phenomenology has long reached its age of maturity, keeping apace with/abreast of experiment.

- upcoming era will require sub-percent precision in staple observables. tools are in place.
  - finer lattice spacings for precision B-physics
  - quantitative control of e.m. and strong isospin breaking corrections

- new avenues being open for lattice studies.
  - baryon decay
  - long-distance contributions to OPE
  - multihadron/resonances in final state
  - inclusive rates

- lattice collaborations have become large and resource-intensive, in both human and computational terms; sustained support is needed to keep synergy with experimental efforts.
conclusions & outlook

• exploring and mapping the flavour sector remains as important a problem as any other in particle physics
  - why the generation structure? why 3 families?
  - is there a structure in the values of quark masses and CKMs?
  - is new physics lingering out there?

• strong support to a synergic exp/th flavour programme crucial; what can future colliders offer?

• eagerly waiting for Belle II, LHCb Upgrade II, kaon expts.
backup slides
resonance/multihadron final states

\[ \text{Re} \left( \frac{\epsilon'}{\epsilon} \right)_{\text{SM}} = 1.38(5.15)(4.43) \times 10^{-4} \]

\[ \text{cf. Re} \left( \frac{\epsilon'}{\epsilon} \right)_{\text{exp}} = 16.6(2.3) \times 10^{-4} \]
resonance/multihadron final states

- QFT aspects well understood in simplest 1→2 transitions (e.g., $K \to \pi \pi$) — large errors down to algorithmic/computational issues.

- huge recent QFT developments in the wider picture
  - up to 2→3 processes worked out in detail
  - detailed characterisation of resonances, including their coupling to currents

- non-trivial QFT tools in place, good prospects for resonances in final state (e.g., $B \to K^*$); non-leptonic decay, couplings to 4-quark operators still very demanding numerically.

[see, e.g., MT Hansen & R Briceño @ Confinement XIII]
Belle II timeline

[Accumulate 50 ab$^{-1}$, x50 of Belle/KEKB]

[K Hara @ 6th KEK Flavor Factory Workshop, 2019/02]
meeting the challenge from experiment

extremely active experimental programme in coming decade(s):

- heavy quark physics: LHCb, Belle II, BESIII (charm), …
- kaon physics: NA62, KOTO, …

### Table: Prospects summary

<table>
<thead>
<tr>
<th>Observable</th>
<th>Current LHCb</th>
<th>LHCb 2025</th>
<th>Belle II</th>
<th>Upgrade II</th>
<th>ATLAS &amp; CMS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CKM tests</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma$, with $B_s^0 \rightarrow D_s^+ K^-$</td>
<td>$^{+17}_{-22}$° [136]</td>
<td>$4^\circ$</td>
<td>–</td>
<td>$1^\circ$</td>
<td>–</td>
</tr>
<tr>
<td>$\gamma$, all modes</td>
<td>$^{+5.0}_{-5.8}$° [167]</td>
<td>$1.5^\circ$</td>
<td>$1.5^\circ$</td>
<td>$0.35^\circ$</td>
<td>–</td>
</tr>
<tr>
<td>sin$2\beta$, with $B^0 \rightarrow J/\psi K^0$</td>
<td>0.04 [609]</td>
<td>0.011</td>
<td>0.005</td>
<td>0.003</td>
<td>–</td>
</tr>
<tr>
<td>$\phi_s$, with $B^0 \rightarrow J/\psi \phi$</td>
<td>49 mrad [44]</td>
<td>14 mrad</td>
<td>–</td>
<td>4 mrad</td>
<td>22 mrad [610]</td>
</tr>
<tr>
<td>$\phi_s$, with $B^0 \rightarrow D_s^+ D_s^-$</td>
<td>170 mrad [49]</td>
<td>35 mrad</td>
<td>–</td>
<td>9 mrad</td>
<td>–</td>
</tr>
<tr>
<td>$\phi_{sss}$, with $B^0 \rightarrow \phi \phi$</td>
<td>154 mrad [94]</td>
<td>39 mrad</td>
<td>–</td>
<td>11 mrad</td>
<td>Under study [611]</td>
</tr>
<tr>
<td>$a_{sl}^s$</td>
<td>$33 \times 10^{-4}$ [211]</td>
<td>$10 \times 10^{-4}$</td>
<td>–</td>
<td>$3 \times 10^{-4}$</td>
<td>–</td>
</tr>
<tr>
<td>$</td>
<td>V_{ub}</td>
<td>/</td>
<td>V_{cb}</td>
<td>$</td>
<td>6% [201]</td>
</tr>
<tr>
<td><strong>$B_s^0, B^0 \rightarrow \mu^+\mu^-$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathcal{B}(B^0 \rightarrow \mu^+\mu^-)/\mathcal{B}(B_s^0 \rightarrow \mu^+\mu^-)$</td>
<td>90% [264]</td>
<td>34%</td>
<td>–</td>
<td>10%</td>
<td>21% [612]</td>
</tr>
<tr>
<td>$\tau_{B_s^0 \rightarrow \mu^+\mu^-}$</td>
<td>22% [264]</td>
<td>8%</td>
<td>–</td>
<td>2%</td>
<td>–</td>
</tr>
<tr>
<td>$S_{\mu\mu}$</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.2</td>
<td>–</td>
</tr>
<tr>
<td><strong>$b \rightarrow c l^- \bar{\nu}_l$ LUV studies</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R(D^*)$</td>
<td>0.026 [215, 217]</td>
<td>0.0072</td>
<td>0.005</td>
<td>0.002</td>
<td>–</td>
</tr>
<tr>
<td>$R(J/\psi)$</td>
<td>0.24 [220]</td>
<td>0.071</td>
<td>–</td>
<td>0.02</td>
<td>–</td>
</tr>
</tbody>
</table>

[C Langenbruch @ Implications of LHC measurements and future prospects, 2018/10]
\[ \Delta m_{\text{Bd}} \]

\[ \Delta m_{\text{Bs}} \]

\[ |e_K| \]

\[ |V_{\mu\mu}/V_{CP}| \]

\[ \text{excluded area has } CL > 0.95 \]

\[ \text{Summer 18 CKM fit filter} \]

[BayBar Physics Book, 1999]
As a satisfying feature of Figure 2 is that the result is on top of data from experiments (i.e., lattice spacing and sea quark mass). Horizontal bars respectively. Asterisks represent anisotropic lattices. Circles, squares, and diamonds stand for staggered, Wilson, and naive chiral fermions. The most striking aspect of the spectrum is how well it agrees experimentally measured masses (widths).

\[
\text{Figure 2 includes predictions for mesons with quark content}
\]

Part. Sci. 62 (2012) - 4000 MeV.

\[
\text{The isospin-1 light mesons and the isospin-}
\]

21st Century Lattice QCD has been used to verify the mass spectrum of quarks. In the di-

\[
\eta
\]

The isosinglet scalar glueballs are all 800–900 MeV higher than the lowest s

\[
\text{LL}
\]

await confirmation.

\[
\text{and first radially excited}
\]

[Part. Sci. 62 (2012) - 4000 MeV.

\[
\Rightarrow \alpha^\text{MS}_s(M_Z) = 0.11852(84)
\]

[ALPHA Collaboration, PRL 119 (2017) 102001]
lattice QCD: state-of-the-art

\begin{align*}
\text{Re} \left( \frac{\epsilon'}{\epsilon} \right)_{\text{SM}} &= 1.38(5.15)(4.43) \times 10^{-4} \\
\text{cf.} \quad \text{Re} \left( \frac{\epsilon'}{\epsilon} \right)_{\text{exp}} &= 16.6(2.3) \times 10^{-4}
\end{align*}

fully relativistic $b$ quarks

$$(a m_b)^2 \lesssim \frac{1}{3} \iff a \lesssim 0.03 \text{ fm} \quad \Rightarrow \text{populate lower lattice spacings in simulation landscape}$$
approaches to B physics

what one would like to do [cf. MILC’s finest lattices]
approaches to B physics

effective theory used differently, different pros/cons balance: crosschecks crucial

\[ \frac{\Lambda}{m_q} \]

interp/ratio

\[ \frac{\Lambda}{m_q} \]

ratios cancel systematics, lead to known static point

\[ \frac{\Lambda}{m_q} \]

(perturbatively) tuned RG trajectory for good scaling

\[ \frac{\Lambda}{m_q} \]

non-perturbative QCD-HQET matching at \(m_b\)

\[ \frac{\Lambda}{m_q} \]

scaling window expected

\[ \frac{\Lambda}{m_q} \]
lattice QCD for phenomenology: FLAG

Flavour Lattice Averaging Group: your one-stop repository of lattice results, world averages / estimates


advisory board: S Aoki, M Golterman, R Van de Water, A Vladikas

editorial board: G Colangelo, S Hashimoto, A Jüttner, S Sharpe, U Wenger

working groups:

- quark masses
- $V_{ud}, V_{us}$
- LECs
- kaon mixing
- heavy leptonic + mixing
- heavy semileptonic
- $\alpha_s$
- nuclear matrix elements

T Blum, A Portelli, A Ramos
S Simula, T Kaneko, JN Simone
S Dürr, H Fukaya, UM Heller
P Dimopoulos, G Herdoíza B Mawhinney
D Lin, Y Aoki, M Della Morte
E Lunghi, D Bečirević, S Gottlieb, CP
R Sommer, R Horsley, T Onogi
R Gupta, S Collins, A Nicholson, H Wittig
what FLAG provides for each quantity:

- complete list of references
- summary of relevant formulae and notation
- quick-look summary tables
- quality assessment of computation setup: colour-coded tables
- averages/estimates (if sensible)
- a “lattice dictionary” for non-experts
- thorough appendix tables with details of all computations for experts
- between-editions updates at http://itpwiki.unibe.ch/flag

cite the original works!
### Collaboration Refs.

- ETM 12B
- ETM 13B, 13C
- ALPHA 14
- HPQCD 09
- RBC/UKQCD 14
- RBC/UKQCD 14A
- RBC/UKQCD 13A
- HPQCD 12
- HPQCD 12
- HPQCD 11A
- FNAL/MILC 11
- HPQCD 09
- ALPHA 14
- ALPHA 13
- EMT 13B, 13C\(^1\)
- ALPHA 12A
- EMT 12B
- ALPHA 11
- EMT 11A
- EMT 09D

### Quality Criteria, Averaging and Error Estimation

The essential characteristics of our approach to the problem are presented in the various sections of the present review. The quality of a calculation. We stress, however, the importance to provide some compact information as regards the values and range of the chosen parameters, and its systematic error of a given lattice result, disqualifies it from being used.

### Tables:

<table>
<thead>
<tr>
<th>Collaboration</th>
<th>Ref.</th>
<th>(N_f)</th>
<th>Publication status</th>
<th>Extrapolation</th>
</tr>
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<tbody>
<tr>
<td>FNAL/MILC 17</td>
<td>[5]</td>
<td>2+1+1</td>
<td>A ★★★★★★✓</td>
<td>189.4(1.4)</td>
</tr>
<tr>
<td>HPQCD 17A</td>
<td>[72]</td>
<td>2+1+1</td>
<td>A ★★★★★★✓</td>
<td>190.5(1.3)</td>
</tr>
<tr>
<td>EMT 16B</td>
<td>[27]</td>
<td>2+1+1</td>
<td>A ★★★★★★✓</td>
<td>189.9(1.4)</td>
</tr>
<tr>
<td>EMT 13E</td>
<td>[551]</td>
<td>2+1+1</td>
<td>A ★★★★★★✓</td>
<td>230.7(1.2)</td>
</tr>
<tr>
<td>HPQCD 13</td>
<td>[71]</td>
<td>2+1+1</td>
<td>A ★★★★★★✓</td>
<td>184(4)</td>
</tr>
<tr>
<td>RBC/UKQCD 14</td>
<td>[76]</td>
<td>2+1</td>
<td>A ★★★★★★✓</td>
<td>195.6(14.9)</td>
</tr>
<tr>
<td>RBC/UKQCD 14A</td>
<td>[75]</td>
<td>2+1</td>
<td>A ★★★★★★✓</td>
<td>199.5(12.6)</td>
</tr>
<tr>
<td>RBC/UKQCD 13A</td>
<td>[552]</td>
<td>2+1</td>
<td>C ★★★★★★✓</td>
<td>191(6) stat</td>
</tr>
<tr>
<td>HPQCD 12</td>
<td>[74]</td>
<td>2+1</td>
<td>A ★★★★★★✓</td>
<td>227(7)</td>
</tr>
<tr>
<td>HPQCD 12</td>
<td>[74]</td>
<td>2+1</td>
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<td>227(7)</td>
</tr>
<tr>
<td>HPQCD 11A</td>
<td>[73]</td>
<td>2+1</td>
<td>A ★★★★★★✓</td>
<td>233(5) stat</td>
</tr>
<tr>
<td>FNAL/MILC 11</td>
<td>[63]</td>
<td>2+1</td>
<td>A ★★★★★★✓</td>
<td>229(5)</td>
</tr>
<tr>
<td>HPQCD 09</td>
<td>[78]</td>
<td>2+1</td>
<td>A ★★★★★★✓</td>
<td>224(5)</td>
</tr>
</tbody>
</table>

- ★✓ allows for satisfactory control of systematics
- ○ allows for reasonable (but improvable) estimate of systematics
- ■ unlikely to allow for reasonable control of systematics

\(^a\)Statistical errors only.
\(^\Delta\)Obtained by combining \(f_B\) from HPQCD 11A with \(f_{\bar{B}}/f_B\) calculated in this work.
\(^\triangledown\)This result uses one ensemble per lattice spacing with light to strange sea-quark mass ratio \(m_l/m_s\approx 0.2\).
\(^*\)This result uses an old determination of \(r_1 = 0.321(5)\) fm from Ref. [559] that has since been superseded.
\(^1\)Update of EMT 11A and 12B.
plots:

- result included in average or estimate
- result OK but not included (superseded, unpublished, ...)
- all other results
baryon SL decay

new exclusive determination of $|V_{cb}|/|V_{ub}|$ from LHCb measurement + LQCD computation of form factors

[Detmold, Lehner, Meinel PRD 92 (2015) 034503]

work since extended to charm channels, radiative decays, ...

[Detmold, Meinel PRD 93 (2016) 074501]
[Meinel PRL 118 (2017) 082001]
[Meinel PRD 97 (2018) 034511]
baryonic decays

[Detmold, Lehner, Meinel PRD 92 (2015) 034503]
radiative decays/BSM

- lattice results at similar level of maturity as for SM tree-level decays
- channels with vectors in final state (e.g. $K^*$) much more complicated: treatment of resonances in Euclidean amplitudes quite non-trivial
- matrix elements of charmed penguins in $H_w$ involve similar difficulties as n non-leptonic $K$ and $B$ decay — difficult nut to crack. (bounds?)

$B(q^2)\phi(q^2)\langle q^2 \rangle$

$z(q^2, t_{opt})$

$B(q^2)\phi(q^2)\langle q^2 \rangle$

$z(q^2, t_{opt})$

⇒ $O_7$, $O_9$, $O_{10}$ (similar for $B\to\pi$ by FNAL/MILC, id. charm ETM)
radiative decays/BSM

Figure 5.

Figure 8:
$q^2$ dependence of form factors

<table>
<thead>
<tr>
<th>$D^0 \rightarrow K^- e^+ \nu$</th>
<th>$D^0 \rightarrow \pi^- e^+ \nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Simple Pole</strong></td>
<td><strong>Simple pole</strong></td>
</tr>
<tr>
<td>$f_K^+(0)</td>
<td>V_{cs}</td>
</tr>
<tr>
<td>$0.7209\pm0.0022\pm0.0033$</td>
<td>$0.1475\pm0.0014\pm0.0005$</td>
</tr>
<tr>
<td>$M_{pole}$</td>
<td>$M_{pole}$</td>
</tr>
<tr>
<td>$1.9207\pm0.0013\pm0.0069$</td>
<td>$1.9114\pm0.0118\pm0.0038$</td>
</tr>
<tr>
<td><strong>Mod. Pole</strong></td>
<td><strong>ISGW2</strong></td>
</tr>
<tr>
<td>$f_K^+(0)</td>
<td>V_{cs}</td>
</tr>
<tr>
<td>$0.7163\pm0.0024\pm0.0034$</td>
<td>$0.1437\pm0.0017\pm0.0008$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>$0.3088\pm0.0195\pm0.0129$</td>
<td>$0.2794\pm0.0345\pm0.0113$</td>
</tr>
<tr>
<td><strong>ISGW2</strong></td>
<td><strong>ISGW2</strong></td>
</tr>
<tr>
<td>$f_K^+(0)</td>
<td>V_{cs}</td>
</tr>
<tr>
<td>$0.7139\pm0.0023\pm0.0034$</td>
<td>$0.1415\pm0.0016\pm0.0006$</td>
</tr>
<tr>
<td>$r_{ISGW2}$</td>
<td>$r_{ISGW2}$</td>
</tr>
<tr>
<td>$1.6000\pm0.0141\pm0.0091$</td>
<td>$2.0688\pm0.0394\pm0.0124$</td>
</tr>
<tr>
<td><strong>Series.2.Par</strong></td>
<td><strong>Series.3.Par</strong></td>
</tr>
<tr>
<td>$f_K^+(0)</td>
<td>V_{cs}</td>
</tr>
<tr>
<td>$0.7172\pm0.0025\pm0.0035$</td>
<td>$0.1435\pm0.0018\pm0.0009$</td>
</tr>
<tr>
<td>$r_1$</td>
<td>$r_1$</td>
</tr>
<tr>
<td>$-2.2278\pm0.0864\pm0.0575$</td>
<td>$-2.0365\pm0.0807\pm0.0260$</td>
</tr>
<tr>
<td><strong>Series.3.Par</strong></td>
<td><strong>Series.3.Par</strong></td>
</tr>
<tr>
<td>$f_K^+(0)</td>
<td>V_{cs}</td>
</tr>
<tr>
<td>$0.7196\pm0.0035\pm0.0041$</td>
<td>$0.1420\pm0.0024\pm0.0010$</td>
</tr>
<tr>
<td>$r_1$</td>
<td>$r_1$</td>
</tr>
<tr>
<td>$-2.3331\pm0.1587\pm0.0804$</td>
<td>$-1.8434\pm0.2212\pm0.0690$</td>
</tr>
<tr>
<td>$r_2$</td>
<td>$r_2$</td>
</tr>
<tr>
<td>$3.4223\pm3.9090\pm2.4092$</td>
<td>$-1.3871\pm1.4615\pm0.4677$</td>
</tr>
</tbody>
</table>

[from H Ma’s talk on behalf of BESIII at CHARM 2015]
a benchmark case: $f_+ (B \to \pi l \nu)$

various parametrisations based on pole dominance: Bećirević-Kaidalov, Ball-Zwicky, Hill, ... difficult to systematically improve precision

[Ball, Zwicky PRD 71 (2005) 014015]
[Hill PRD 73 (2006) 014012]

z-parametrisations proposed to solve this issue (almost) rigourously by exploiting unitarity and crossing symmetry

[Bourrely, Machet, de Rafael NPB 189 (1981) 157]
[Boyd, Grinstein, Lebed PRL 74 (1995) 4603]
[Lellouch NPB 479 (1996) 353]
[Bourrely, Caprini, Micu EJPC 27 (2003) 439]
[Arnesen, Grinstein, Rothstein, Stewart PRL 95 (2005) 071802]
[Flynn, Nieves PRD 75 (2007) 013008]
[Bourrely, Caprini, Lellouch PRD 79 (2009) 013008]
\[ z = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}} \]

\[ t_+ = (m_B + m_\pi)^2, \quad t_0 < t_+ \]

a benchmark case: \( f_+(B \to \pi l\nu) \)

\[ f_+(q^2) = \frac{1}{B(q^2)\phi(q^2, t_0)} \sum_{n \geq 0} a_n z(q^2, t_0)^n \]

unitarity bound: \( \sum_{m,n} B_{mn}^{(\phi)} a_m a_n \leq 1 \)
**a benchmark case:** $f_+(B \rightarrow \pi l \nu)$

$$f_+(q^2) = \frac{1}{B(q^2)\phi(q^2,t_0)} \sum_{n \geq 0} a_n z(q^2,t_0)^n$$

$$B(q^2) = z(q^2, m_{B*}^2)$$

**BGL:** complicated outer function $\phi \xrightarrow{} \sum_{n \geq 0} |a_n|^2 \lesssim 1$

[Boyd, Grinstein, Lebed PRL 74 (1995) 4603]

**BCL:** $f_+(q^2) = \frac{1}{1 - q^2/m_{B*}^2} \sum_{n \geq 0} a_n z^n \xrightarrow{} \sum_{m,n \geq 0} B_{mn}a_m a_n \lesssim 1$

(recommended by FLAG)

[Bourrely, Caprini, Lellouch PRD 79 (2009) 013008]

**crucial for optimal use:**

- all sub-threshold poles included in Blaschke factor
- fixed kinematics (coefficients implicitly depend on quark masses)
does the unitarity bound apply?

• using a $z$-parametrisation as part of a global fit including $a$, $m_q$, ... 
  (modified $z$-expansion) tricky
  - poles can cross threshold as quark masses change
  - complicated entanglement of $(m_q,a)$ dependence (complete form factor vs. $z$-parametrisation coefficient)

• pole structure not always well-known (scalar channels, $D$ decay),
  or complicated ($\Lambda_b$ decay)

• missing sub-threshold poles may imply convergence breakdown
  (proton charge radius analysis by Hill, Paz et al, $D$ semileptonic decay data by Bećirević et al)

[Hill, Paz PRD 82 (2010) 113005]
[Bhattacharya, Hill, Paz PRD 84 (2011) 073006]
[Epstein, Paz, Roy PRD 90 (2014) 074027]