Sum rules and asymptotic behaviors of neutrino mixing in matter

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Humboldt postdoctoral fellow at LMU Munich 1993—1996

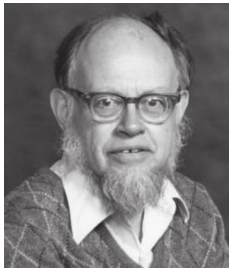
- ★ **Introduction:** matter matters for neutrino oscillations
- ★ **Formulas:** sum rules for neutrino mixing in a medium
- ★ **Application (1):** extremes of the Jarlskog parameter
- ★ **Application (2):** understanding asymptotic behaviors

@ Kitzbuehel Humboldt Kolleg, Hotel Kaiserhof, 23 — 28 June 2019

Part A

1978: Wolfenstein

PHYSICAL REVIEW D VOLUME 17, NUMBER 9 1 MAY 1978



Neutrino oscillations in matter

L. Wolfenstein
Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213

(Received 6 October 1977; revised manuscript received 5 December 1977)

at the age of 55

The effect of coherent forward scattering must be taken into account when considering the oscillations of neutrinos traveling through matter. In particular, for the case of massless neutrinos for which vacuum oscillations cannot occur, oscillations can occur in matter if the neutral current has an off-diagonal piece connecting different neutrino types. Applications discussed are solar neutrinos and a proposed experiment involving transmission of neutrinos through 1000 km of rock.

Ref. [8]

[8] I am indebted to Dr. Daniel Wyler for pointing out the importance of the charged-current terms.

Who was studying double refraction

ACKNOWLEDGMENTS

I wish to thank E. Zavattini for asking the right question, and J. Ashkin, J. Russ, J. F. Donoghue, L. F. Li, S. Adler, and D. Wyler for discussions. This research was supported in part by the U. S Energy Research and Development Administration.

Lincoln Wolfenstein (2004): I think I have learnt as much from all my students as they have learnt from me.

Effective Hamiltonian

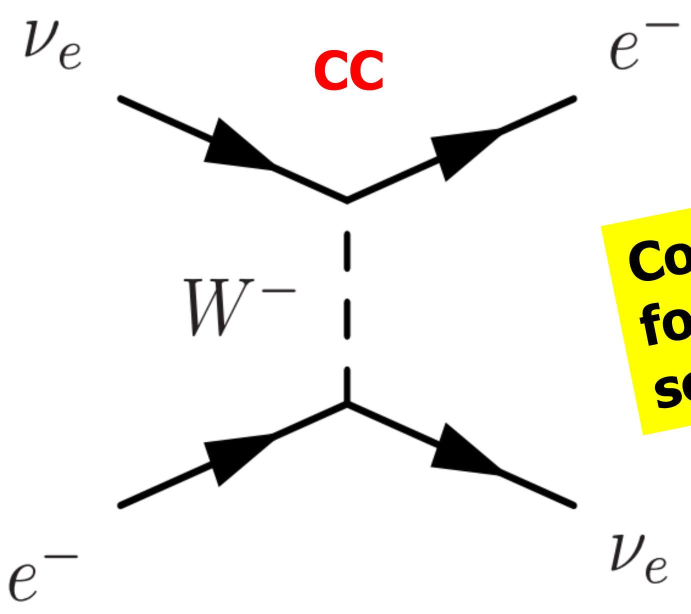
In vacuum the evolution of three neutrino mass eigenstates with time

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} = H_0\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}, \quad H_0 = \frac{1}{2E}\begin{pmatrix}m_1^2 & & \\ & m_2^2 & \\ & & m_3^2\end{pmatrix}, \quad \begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = U\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}$$

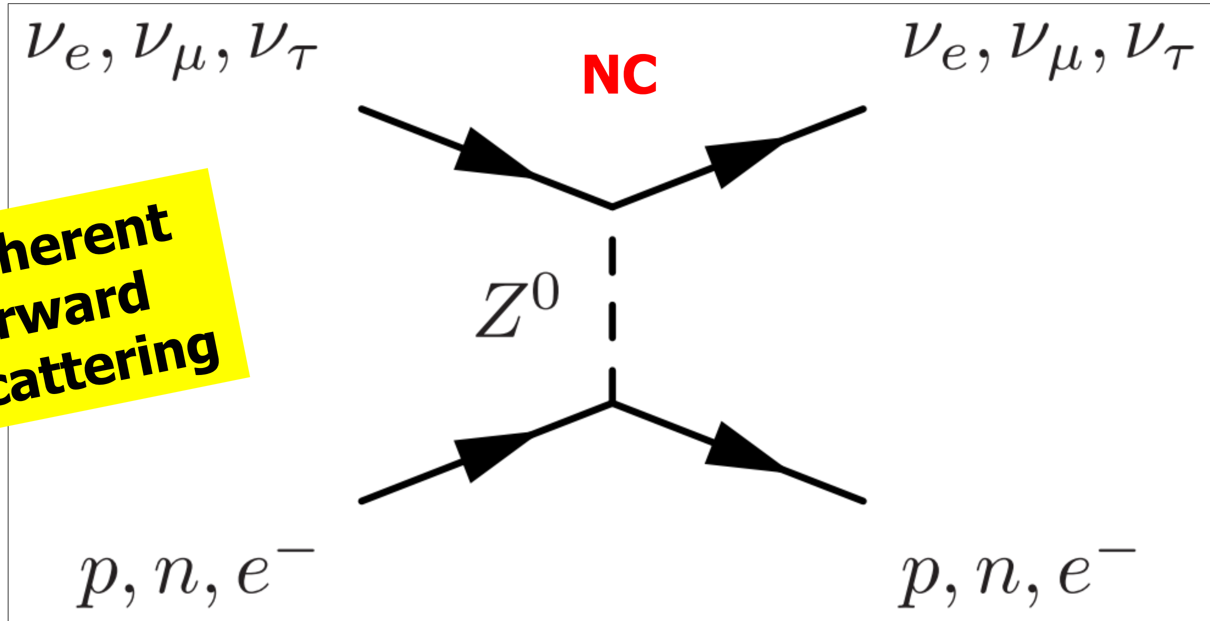
In the flavor basis the evolution of three neutrino flavors is described by the Schroedinger-like equation:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = UH_0U^\dagger\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}$$

Propagating in a medium, neutrinos may have CC and NC interactions



Coherent forward scattering

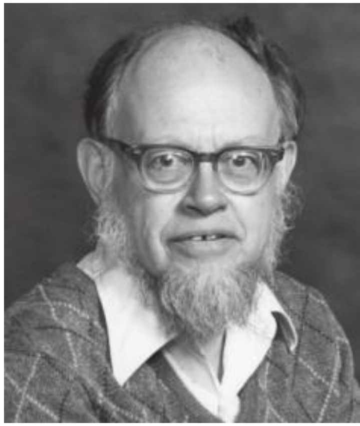


Matter potential

In this case the effective Hamiltonian with a matter potential is

$$H_{\rm m} = U H_0 U^\dagger + \begin{pmatrix} V_{\rm CC} & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} V_{\rm NC} & & \\ & V_{\rm NC} & \\ & & V_{\rm NC} \end{pmatrix}$$

from electron **from neutron**



The NC contributions from electrons and protons cancel each other, since we stay with normal matter:

$$\boxed{N_e = N_p}$$

$$V_{\rm CC} = +\sqrt{2}\, G_{\rm F} N_e$$

$$V^n_{\rm NC} = -\frac{1}{\sqrt{2}} G_{\rm F} N_n$$

$$V^p_{\rm NC} = +\frac{1}{\sqrt{2}} G_{\rm F} N_p \left(1 - 4\sin^2\theta_{\rm w}\right)$$

$$V^e_{\rm NC} = -\frac{1}{\sqrt{2}} G_{\rm F} N_e \left(1 - 4\sin^2\theta_{\rm w}\right)$$

- ♣ **The NC term is universal for three neutrino flavors, and hence it can be neglected in the standard case.**
- ♣ **When an antineutrino beam is taken into consideration, the CC and NC terms flip their signs, and simultaneously the flavor mixing matrix *U* needs to be complex conjugated.**
- ♣ **The NC term should not be ignored if sterile neutrinos are included.**

Part A

1986: MSW effects

IL NUOVO CIMENTO VOL. 9 C, N. 1 Gennaio-Febbraio 1986

Resonant Amplification of ν Oscillations in Matter and Solar-Neutrino Spectroscopy.

S. P. MIKHEYEV and A. YU. SMIRNOV

Institute for Nuclear Research of Academy of Sciences
60th October Anniversary prosp. 7a, Moscow 117 342, USSR

(ricevuto il 3 Maggio 1985)

Wolfenstein's formula:

$$\tan 2\tilde{\theta} = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - 2\sqrt{2}\, G_F N_e E}$$

(1) MSW resonance
$\tilde{\theta} = \pi/4$

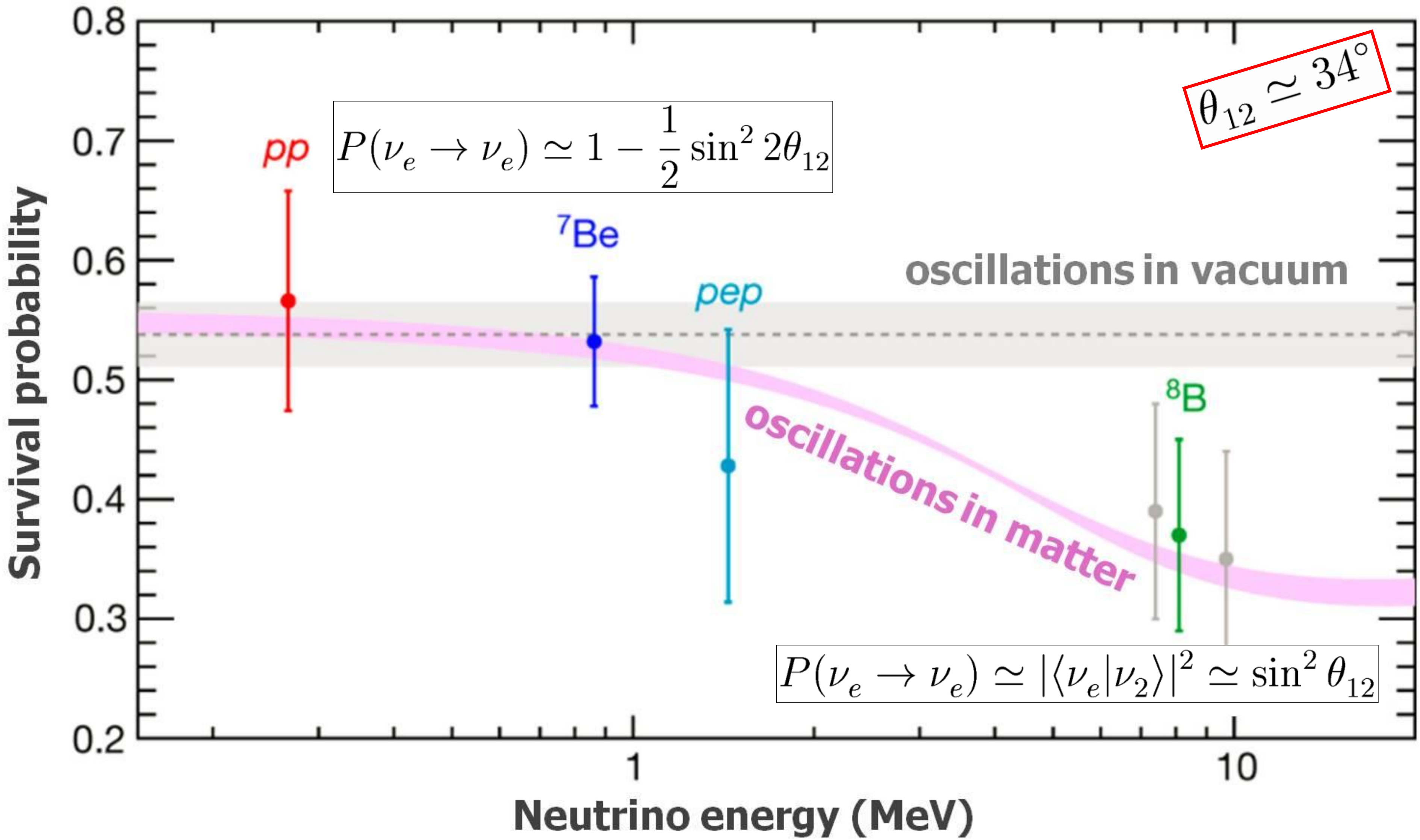
(2) $N_e \to \infty$
$\tilde{\theta} = \pi/2$

G.T. Zatsepin helped publish their paper (rejected by PLB) in Nuovo Cimento.

Summary. — For small mixing angles θ the amplification of ν oscillations in matter has the resonance form (resonance in neutrino energy or matter density). In the Sun resonance effect results in nontrivial changing (suppression) of ν-flux for a wide range of neutrino parameters $\Delta m^2 = (3 \cdot 10^{-4} \div 10^{-8})\,(\text{eV})^2$, $\sin^2 2\theta > 10^{-4}$.

Solar neutrino deficits

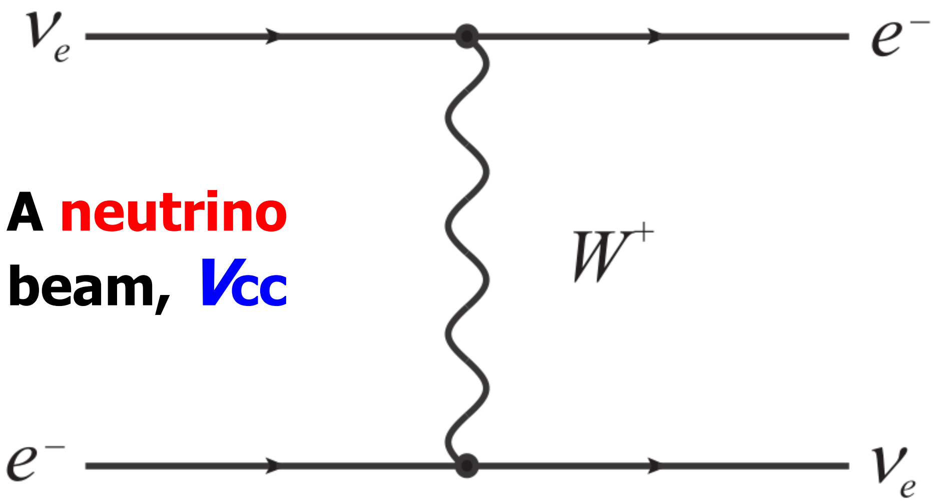
Low-energy solar neutrinos: dominated by vacuum oscillations;
High-energy solar neutrinos: dominated by matter effects.



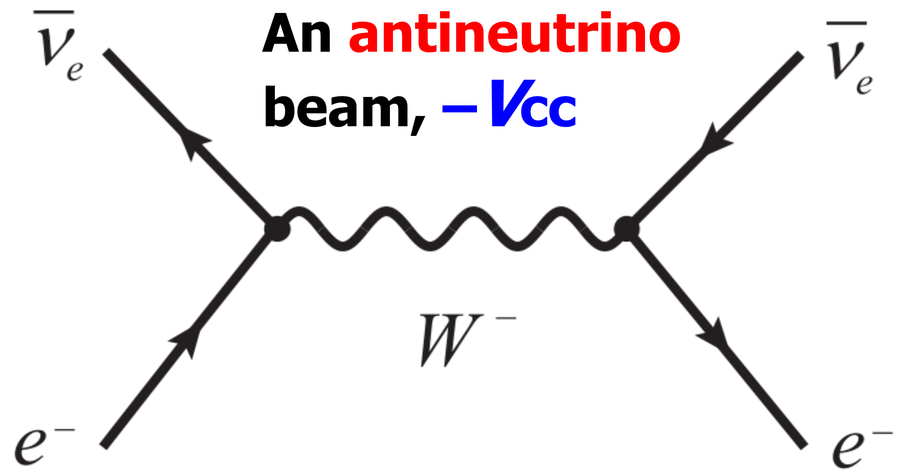
Effective quantities

Matter effects on neutrinos and antineutrinos are ***NOT*** **CP-symmetric** in a normal medium, leading to fake CP violation.

A **neutrino** beam, $V_{\rm cc}$



An **antineutrino** beam, $-V_{\rm cc}$



The effective Hamiltonian in matter can be expressed using **effective quantities** in the same way as in vacuum —— **form invariance**:

$$\mathcal{H}_{\rm m} = \frac{1}{2E} U \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} U^\dagger + \begin{pmatrix} V_{\rm cc} + V_{\rm nc} & 0 & 0 \\ 0 & V_{\rm nc} & 0 \\ 0 & 0 & V_{\rm nc} \end{pmatrix} \equiv \frac{1}{2E} \widetilde{U} \begin{pmatrix} \widetilde{m}_1^2 & 0 & 0 \\ 0 & \widetilde{m}_2^2 & 0 \\ 0 & 0 & \widetilde{m}_3^2 \end{pmatrix} \widetilde{U}^\dagger$$

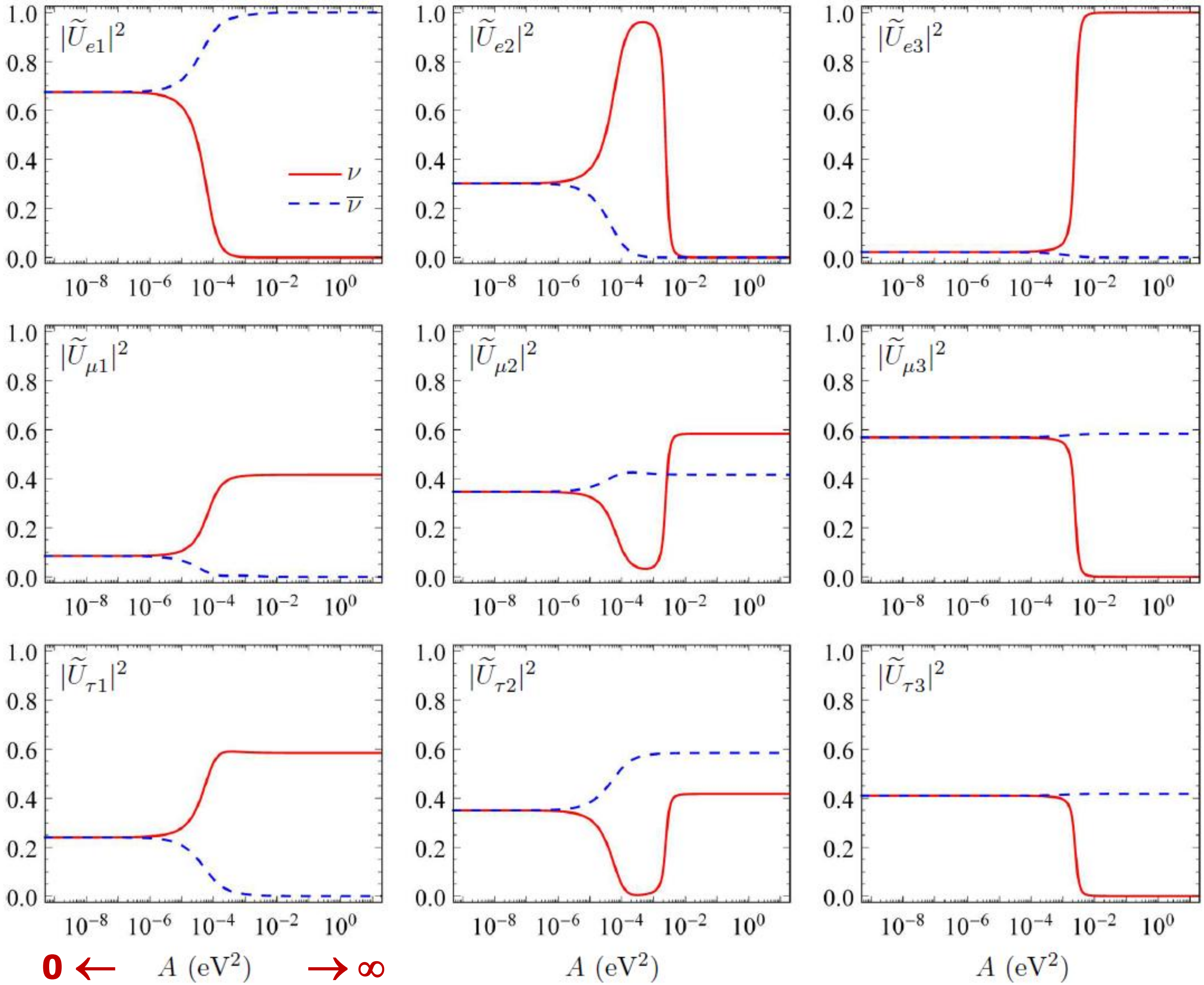
in vacuum — **matter potential** — **in matter**

The **renormalization-group-equation**-like relations between effective & fundamental parameters (Chiu, Kuo, 2017; ZZX, Zhou, Zhou, 2018)

Part A

Asymptotic behaviors (1)

Normal Mass Ordering



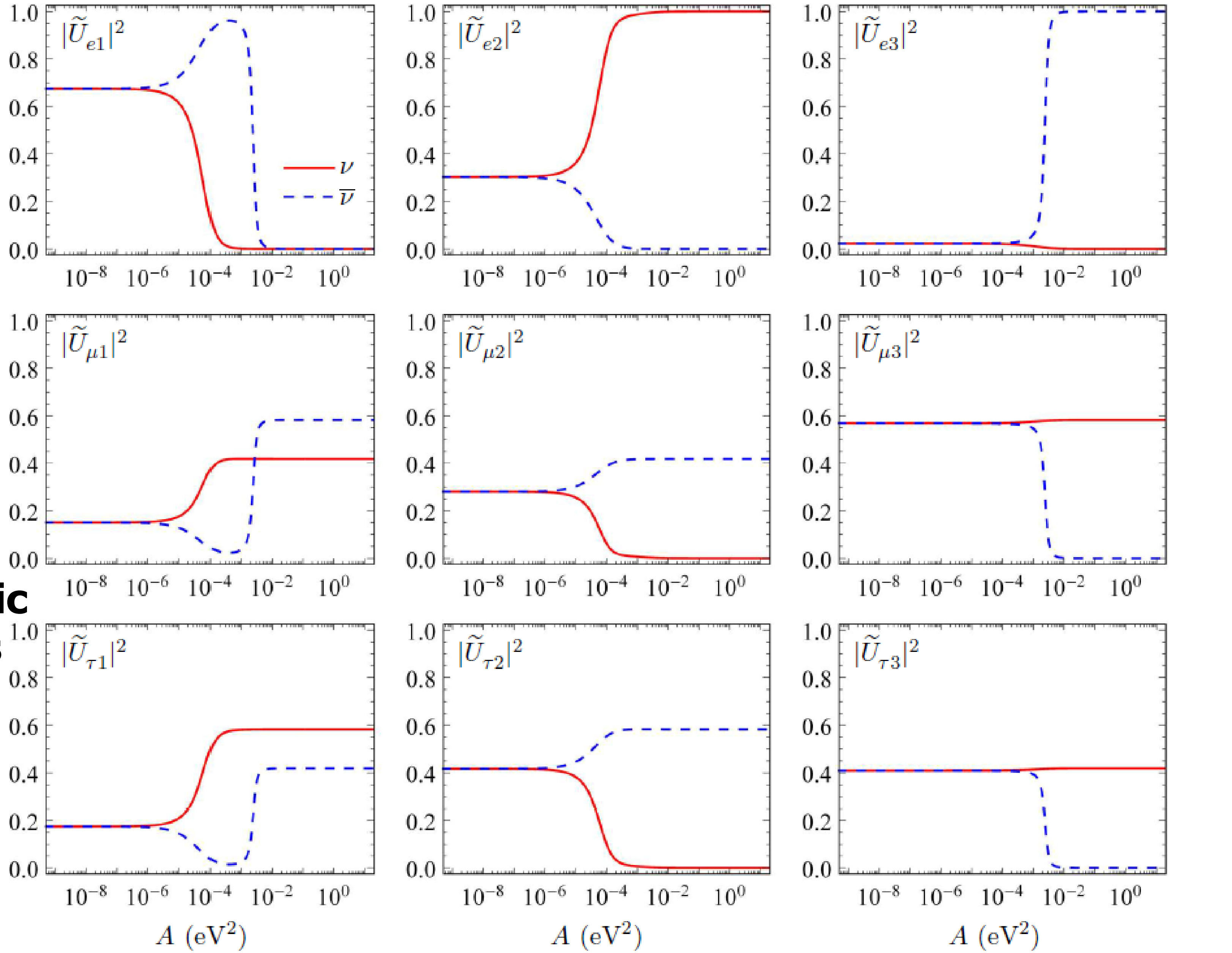
$0 \leftarrow$ $\rightarrow \infty$

$A \rightarrow 0$ vacuum

$A \rightarrow \infty$ dense matter

Asymptotic behaviors (2)

Inverted Mass Ordering



Unitarity assures asymptotic behaviors

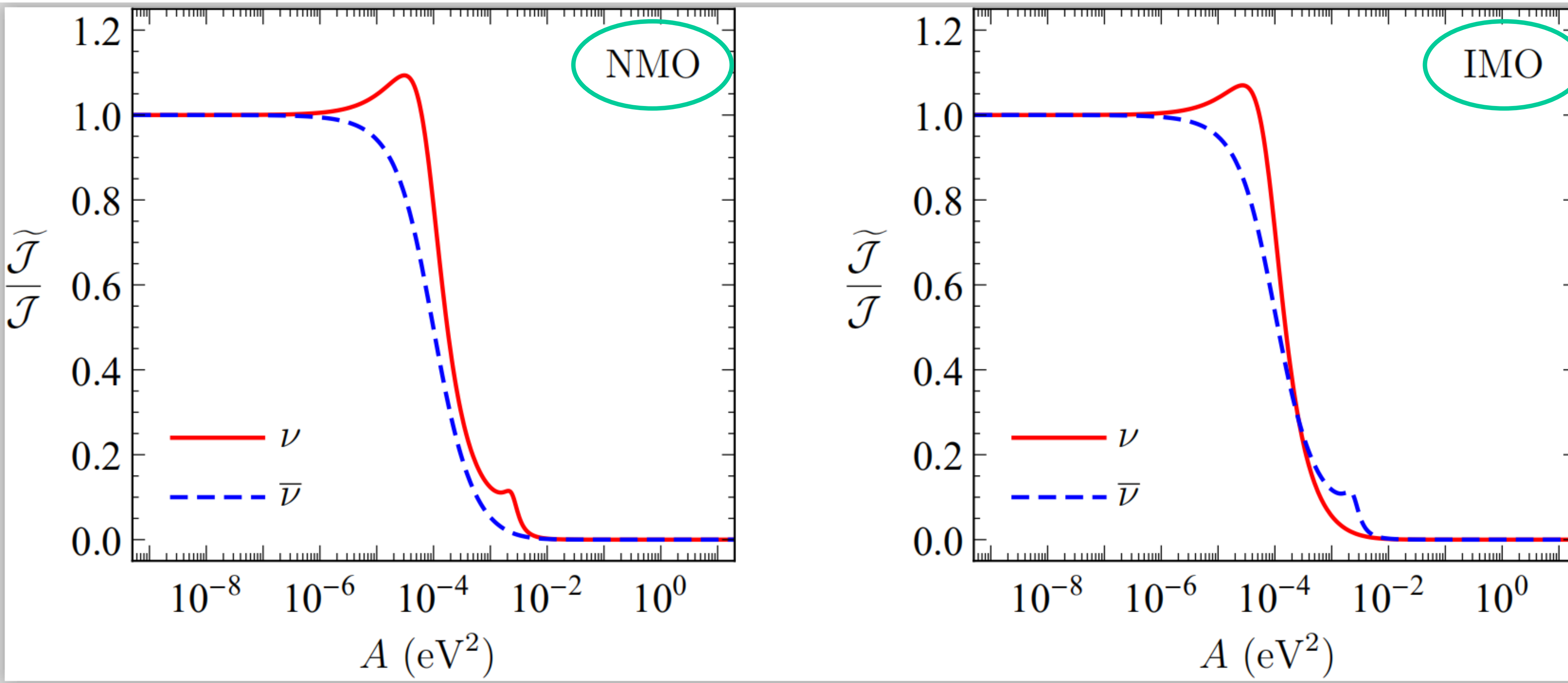
Part A

Jarlskog invariant

CP violation **in ν-oscillations is measured by a J-invariant. It has peaks in matter, but it is not significantly enhanced.**

$$\mathrm{Im}(U_{\alpha i}U_{\beta j}U^*_{\alpha j}U^*_{\beta i}) = \mathcal{J}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\sum_{k}\varepsilon_{ijk}$$

$$\mathrm{Im}(\widetilde{U}_{\alpha i}\widetilde{U}_{\beta j}\widetilde{U}^*_{\alpha j}\widetilde{U}^*_{\beta i}) = \widetilde{\mathcal{J}}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\sum_{k}\varepsilon_{ijk}$$



Targets: **understand the asymptotic behaviors in the $A \to \infty$ limit and Jarlskog's peaks. Based on ZZX + J.Y. Zhu, 1905.08644; 1603.02002 (JHEP)**

Preliminary formulas

Rewrite the effective Hamiltonian in the following way:

$$\mathcal{H}'_{\rm m} = \frac{1}{2E}\left[U\begin{pmatrix}0 & 0 & 0\\ 0 & \Delta_{21} & 0\\ 0 & 0 & \Delta_{31}\end{pmatrix}U^\dagger + \begin{pmatrix}A & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right] \equiv \frac{1}{2E}\left[\widetilde{U}\begin{pmatrix}0 & 0 & 0\\ 0 & \widetilde{\Delta}_{21} & 0\\ 0 & 0 & \widetilde{\Delta}_{31}\end{pmatrix}\widetilde{U}^\dagger + BI\right]$$

with matter parameters + mass-squared differences

$$A = 2EV_{\rm cc}$$

$$B = \widetilde{m}_1^2 - m_1^2 - 2EV_{\rm nc}$$

Normal mass ordering (NMO)

$$\widetilde{\Delta}_{21} = \frac{2}{3}\sqrt{x^2-3y}\sqrt{3\left(1-z^2\right)}$$

$$\widetilde{\Delta}_{31} = \frac{1}{3}\sqrt{x^2-3y}\left[3z+\sqrt{3\left(1-z^2\right)}\right]$$

$$B = \frac{1}{3}x - \frac{1}{3}\sqrt{x^2-3y}\left[z+\sqrt{3\left(1-z^2\right)}\right]$$

Inverted mass ordering (IMO)

$$\widetilde{\Delta}_{21} = \frac{1}{3}\sqrt{x^2-3y}\left[3z-\sqrt{3\left(1-z^2\right)}\right]$$

$$\widetilde{\Delta}_{31} = -\frac{2}{3}\sqrt{x^2-3y}\sqrt{3\left(1-z^2\right)}$$

$$B = \frac{1}{3}x - \frac{1}{3}\sqrt{x^2-3y}\left[z-\sqrt{3\left(1-z^2\right)}\right]$$

$$x = \Delta_{21} + \Delta_{31} + A$$

$$y = \Delta_{21}\Delta_{31} + A\left[\Delta_{21}\left(1-|U_{e2}|^2\right) + \Delta_{31}\left(1-|U_{e3}|^2\right)\right]$$

$$z = \cos\left[\frac{1}{3}\arccos\frac{2x^3-9xy+27A\Delta_{21}\Delta_{31}|U_{e1}|^2}{2\sqrt{\left(x^2-3y\right)^3}}\right]$$

$$B = \frac{1}{3}\left(\Delta_{21} + \Delta_{31} + A - \widetilde{\Delta}_{21} - \widetilde{\Delta}_{31}\right)$$

Part B

Sum rules

A full set of linear equations of 3 unknown variables: they're solvable!

$$\sum_{i=1}^{3} \widetilde{U}_{\alpha i} \widetilde{U}^*_{\beta i} \widetilde{\Delta}_{i1} = \sum_{i=1}^{3} U_{\alpha i} U^*_{\beta i} \Delta_{i1} + A\delta_{e\alpha}\delta_{e\beta} - B\delta_{\alpha\beta}$$

$$\sum_{i=1}^{3} \widetilde{U}_{\alpha i} \widetilde{U}^*_{\beta i} \widetilde{\Delta}_{i1} (\widetilde{\Delta}_{i1} + 2B) = \sum_{i=1}^{3} U_{\alpha i} U^*_{\beta i} \Delta_{i1} \left[\Delta_{i1} + A(\delta_{e\alpha} + \delta_{e\beta})\right] + A^2\delta_{e\alpha}\delta_{e\beta} - B^2\delta_{\alpha\beta}$$

$$\sum_{i=1}^{3} \widetilde{U}_{\alpha i} \widetilde{U}^*_{\beta i} = \sum_{i=1}^{3} U_{\alpha i} U^*_{\beta i} = \delta_{\alpha\beta}$$

Setting $\alpha = \beta$, we obtain 9 moduli of the PMNS matrix elements in matter.

Analytical results

$$|\widetilde{U}_{\alpha 1}|^2 = \frac{\zeta - 2\xi B - \xi\widetilde{\Delta}_{21} - \xi\widetilde{\Delta}_{31} + \widetilde{\Delta}_{21}\widetilde{\Delta}_{31}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}}$$

$$|\widetilde{U}_{\alpha 2}|^2 = \frac{\xi\widetilde{\Delta}_{31} + 2\xi B - \zeta}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{32}}$$

$$|\widetilde{U}_{\alpha 3}|^2 = \frac{\zeta - 2\xi B - \xi\widetilde{\Delta}_{21}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}$$

$$\xi = \Delta_{21}|U_{\alpha 2}|^2 + \Delta_{31}|U_{\alpha 3}|^2 + A\delta_{e\alpha} - B\ ,$$

$$\zeta = \Delta_{21}(\Delta_{21} + 2A\delta_{e\alpha})|U_{\alpha 2}|^2 + \Delta_{31}(\Delta_{31} + 2A\delta_{e\alpha})|U_{\alpha 3}|^2 + A^2\delta_{e\alpha} - B^2$$

For example

The expressions of the moduli squared can be explicitly written as:

$$|\widetilde{U}_{e1}|^2 = \frac{1}{9}\left[\frac{\widetilde{\Delta}_{21}+\widetilde{\Delta}_{31}+\Delta_{31}+\Delta_{32}-A}{\widetilde{\Delta}_{31}}\cdot\frac{\widetilde{\Delta}_{21}+\widetilde{\Delta}_{31}+\Delta_{21}-\Delta_{32}-A}{\widetilde{\Delta}_{21}}|U_{e1}|^2\right.$$
$$+\frac{\widetilde{\Delta}_{21}+\widetilde{\Delta}_{31}+\Delta_{31}+\Delta_{32}-A}{\widetilde{\Delta}_{31}}\cdot\frac{\widetilde{\Delta}_{21}+\widetilde{\Delta}_{31}-\Delta_{21}-\Delta_{31}-A}{\widetilde{\Delta}_{21}}|U_{e2}|^2$$
$$\left.+\frac{\widetilde{\Delta}_{21}+\widetilde{\Delta}_{31}+\Delta_{21}-\Delta_{32}-A}{\widetilde{\Delta}_{31}}\cdot\frac{\widetilde{\Delta}_{21}+\widetilde{\Delta}_{31}-\Delta_{21}-\Delta_{31}-A}{\widetilde{\Delta}_{21}}|U_{e3}|^2\right]$$

$$|\widetilde{U}_{e2}|^2 = \frac{1}{9}\left[\frac{\widetilde{\Delta}_{32}-\widetilde{\Delta}_{21}+\Delta_{31}+\Delta_{32}-A}{\widetilde{\Delta}_{32}}\cdot\frac{\widetilde{\Delta}_{21}-\widetilde{\Delta}_{32}+\Delta_{32}-\Delta_{21}+A}{\widetilde{\Delta}_{21}}|U_{e1}|^2\right.$$
$$+\frac{\widetilde{\Delta}_{32}-\widetilde{\Delta}_{21}+\Delta_{31}+\Delta_{32}-A}{\widetilde{\Delta}_{32}}\cdot\frac{\widetilde{\Delta}_{21}-\widetilde{\Delta}_{32}+\Delta_{21}+\Delta_{31}+A}{\widetilde{\Delta}_{21}}|U_{e2}|^2$$
$$\left.+\frac{\widetilde{\Delta}_{32}-\widetilde{\Delta}_{21}-\Delta_{32}+\Delta_{21}-A}{\widetilde{\Delta}_{32}}\cdot\frac{\widetilde{\Delta}_{21}-\widetilde{\Delta}_{32}+\Delta_{21}+\Delta_{31}+A}{\widetilde{\Delta}_{21}}|U_{e3}|^2\right]$$

In the $A \to \infty$ limit, these quantities will approach finite values.

Part B

Asymptotic results (1)

The effective (matter-corrected) neutrino mass-squared differences + PMNS matrix elements in the $A \to \infty$ limit:

	NMO (neutrinos)	IMO (neutrino)
$\widetilde{\Delta}_{21}$	$\Delta_{31}(1-\lvert U_{e3}\rvert^2) - \Delta_{21}\lvert U_{e1}\rvert^2$	A
$\widetilde{\Delta}_{31}$	A	$\Delta_{31}(1-\lvert U_{e3}\rvert^2) - \Delta_{21}\lvert U_{e1}\rvert^2$
$\widetilde{\Delta}_{32}$	A	$-A$
$\widetilde{U}$	$\begin{pmatrix} 0 & 0 & 1 \\ \sqrt{1-\lvert U_{\mu 3}\rvert^2} & \lvert U_{\mu 3}\rvert & 0 \\ -\lvert U_{\mu 3}\rvert & \sqrt{1-\lvert U_{\mu 3}\rvert^2} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ \sqrt{1-\lvert U_{\mu 3}\rvert^2} & 0 & \lvert U_{\mu 3}\rvert \\ -\lvert U_{\mu 3}\rvert & 0 & \sqrt{1-\lvert U_{\mu 3}\rvert^2} \end{pmatrix}$

Illustration:

$$\left(\lvert \widetilde{U}_{\alpha i}\rvert^2\right)\Big|_{A\to\infty}^{(\mathrm{NMO},\ \nu)} = \begin{pmatrix} 0 & 0 & 1 \\ 0.417 & 0.583 & 0 \\ 0.583 & 0.417 & 0 \end{pmatrix}$$

$$\left(\lvert \widetilde{U}_{\alpha i}\rvert^2\right)\Big|_{A\to\infty}^{(\mathrm{IMO},\ \nu)} = \begin{pmatrix} 0 & 1 & 0 \\ 0.418 & 0 & 0.582 \\ 0.582 & 0 & 0.418 \end{pmatrix}$$

Part B

Asymptotic results (2)

The effective (matter-corrected) neutrino mass-squared differences + PMNS matrix elements in the $A \to \infty$ limit:

	NMO (antineutrino)	IMO (antineutrino)
$\widetilde{\Delta}_{21}$	A	$-\Delta_{31}\left(1-\lvert U_{e3}\rvert^2\right)+\Delta_{21}\lvert U_{e1}\rvert^2$
$\widetilde{\Delta}_{31}$	A	$-A$
$\widetilde{\Delta}_{32}$	$\Delta_{31}\left(1-\lvert U_{e3}\rvert^2\right)-\Delta_{21}\lvert U_{e1}\rvert^2$	$-A$
$\widetilde{U}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & \sqrt{1-\lvert U_{\mu3}\rvert^2} & \lvert U_{\mu3}\rvert\\ 0 & -\lvert U_{\mu3}\rvert & \sqrt{1-\lvert U_{\mu3}\rvert^2}\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 1\\ \lvert U_{\mu3}\rvert & \sqrt{1-\lvert U_{\mu3}\rvert^2} & 0\\ -\sqrt{1-\lvert U_{\mu3}\rvert^2} & \lvert U_{\mu3}\rvert & 0\end{pmatrix}$

Illustration:

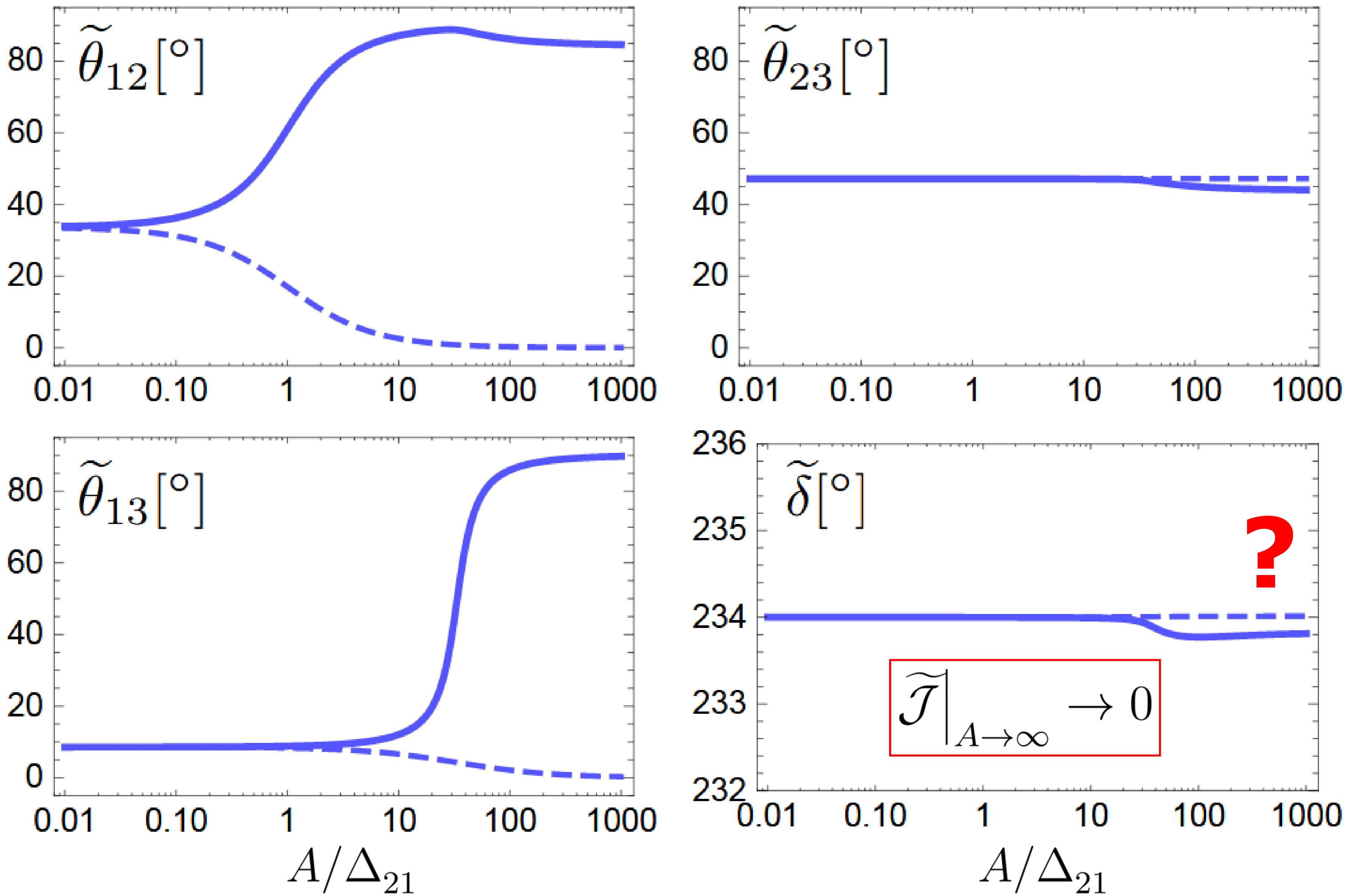
$$\left(|\widetilde{U}_{\alpha i}|^2\right)\Big|^{(\mathrm{NMO},\ \overline{\nu})}_{A\to\infty} = \begin{pmatrix}1 & 0 & 0\\ 0 & 0.417 & 0.583\\ 0 & 0.583 & 0.417\end{pmatrix}$$

$$\left(|\widetilde{U}_{\alpha i}|^2\right)\Big|^{(\mathrm{IMO},\ \overline{\nu})}_{A\to\infty} = \begin{pmatrix}0 & 0 & 1\\ 0.582 & 0.418 & 0\\ 0.418 & 0.582 & 0\end{pmatrix}$$

Part B

Something misleading?

Why the CP-violating phase remains finite in the $A \to \infty$ limit?



The key point is …

Planck scale $\Lambda \sim 10^{19}\,\mathrm{GeV}$

GUT scale? $\Lambda \sim 10^{16}\,\mathrm{GeV}$

Seesaw scale? $\Lambda \sim 10^{12}\,\mathrm{GeV}$

TeV / SUSY? $\Lambda \sim 10^{3}\,\mathrm{GeV}$

Fermi scale $\Lambda \sim 10^{2}\,\mathrm{GeV}$

QCD scale $\Lambda \sim 10^{2}\,\mathrm{MeV}$

μ

heaven

a parametrization of the effective PMNS matrix is basis-dependent!

Analytical continuation allows the phase to approach a finite value.

Example: generating finite θ_{13} via RGE evolution from seesaw scale down to Fermi scale, by inputting a finite initial value of δ.

To every man is given the key to the gates of heaven, the same key opens the gates of hell.

hell

RGE-like evolution

vacuum ***A*** **matter**

A = 0 ***A = ∞***

Part C

Low matter density

Small parameters for expansion:

$$\alpha \equiv \Delta_{21}/\Delta_{31}$$

$$\beta \equiv A/\Delta_{31}$$

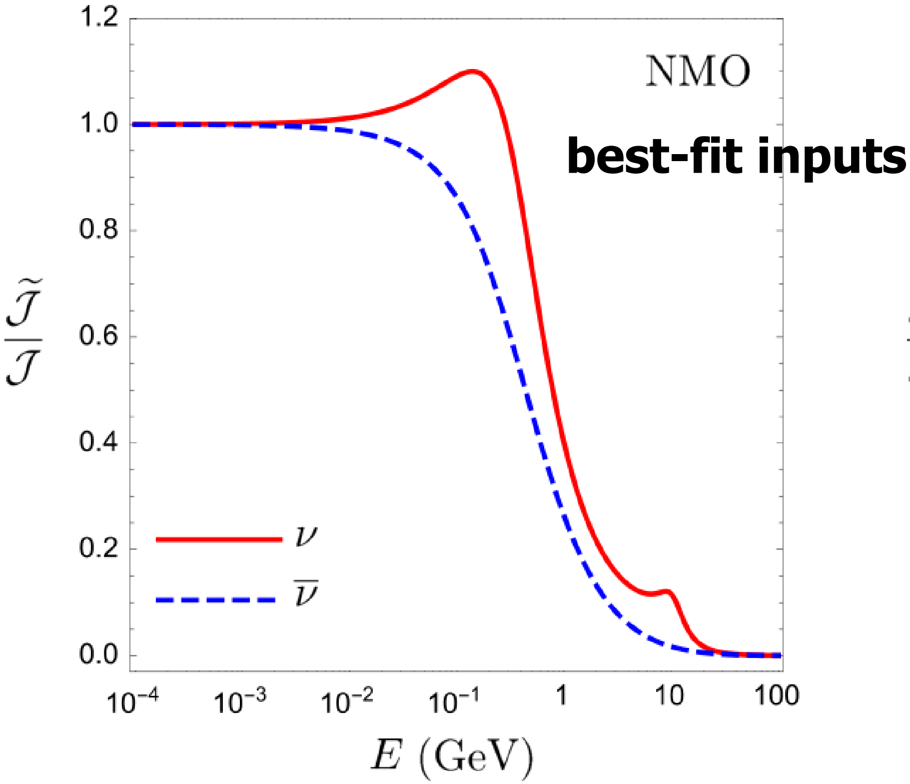
Matter parameter:

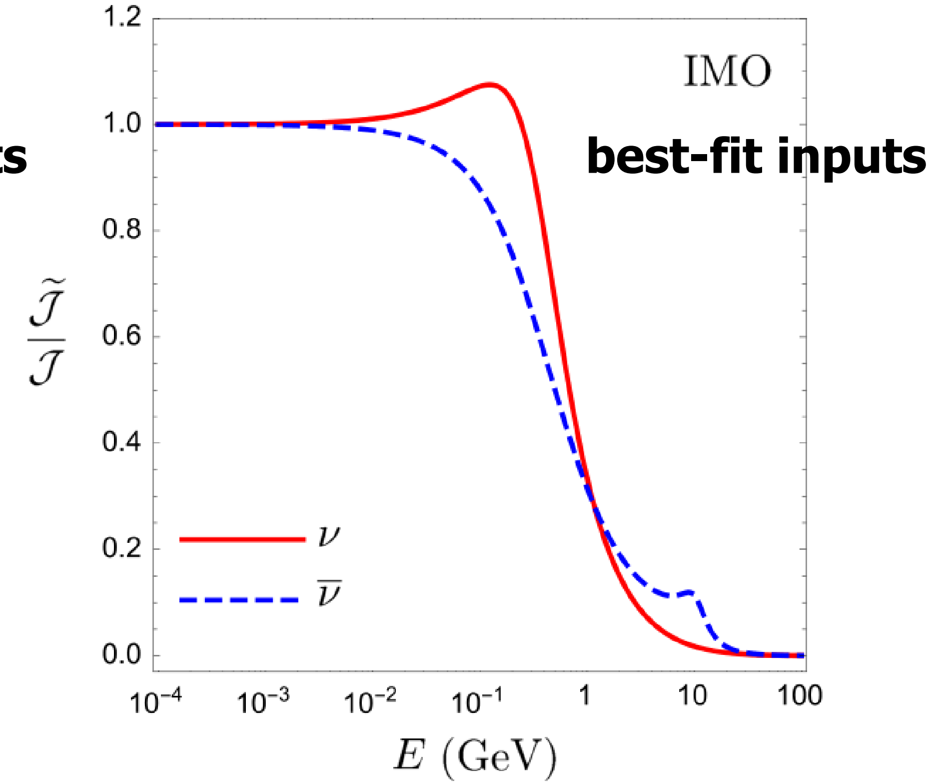
$$A \simeq 2.28 \times 10^{-4}\ \mathrm{eV}^2\ (E/\mathrm{GeV})$$

$$x = \Delta_{31}\left(1+\alpha+\beta\right) ,$$

$$y = \Delta_{31}^2\left[\alpha + \beta\left(|U_{e1}|^2 + |U_{e2}|^2\right) + \alpha\beta\left(1-|U_{e2}|^2\right)\right]$$

$$z = \cos\left[\frac{1}{3}\arccos\frac{2x^3 - 9xy + 27\Delta_{31}^3\alpha\beta|U_{e1}|^2}{2\sqrt{\left(x^2-3y\right)^3}}\right]$$





Part C

Analytical approximations

In the region of *E* > 0.5 GeV, there is a good analytical approximation by M. Freund (2001). Here we focus on the low-energy region.

$$\widetilde{\Delta}^{(\mathrm{N})}_{21} \simeq \Delta_{31}\left(1-\frac{1}{2}\alpha-\frac{1}{2}\beta\right)\epsilon\ ,$$

$$\epsilon \equiv \sqrt{\alpha^2 - 2\left(|U_{e1}|^2-|U_{e2}|^2\right)\alpha\beta+\left(1-2|U_{e3}|^2\right)\beta^2}$$

$$\widetilde{\Delta}^{(\mathrm{N})}_{31} \simeq \Delta_{31}\left[1-\frac{1}{2}\alpha-\frac{1}{2}\left(1-3|U_{e3}|^2\right)\beta+\frac{1}{2}\epsilon-\frac{1}{4}(\alpha+\beta)\epsilon\right]$$

$$\widetilde{\Delta}^{(\mathrm{N})}_{32} \simeq \Delta_{31}\left[1-\frac{1}{2}\alpha-\frac{1}{2}\left(1-3|U_{e3}|^2\right)\beta-\frac{1}{2}\epsilon+\frac{1}{4}(\alpha+\beta)\epsilon\right]$$

normal hierarchy

$$\alpha \equiv \Delta_{21}/\Delta_{31}$$

$$\beta \equiv A/\Delta_{31}$$

inverted hierarchy

$$\widetilde{\Delta}^{(\mathrm{I})}_{21} \simeq -\Delta_{31}\left[1-\frac{1}{2}\alpha-\frac{1}{2}\left(1-3|U_{e3}|^2\right)\beta-\frac{1}{2}\epsilon+\frac{1}{4}(\alpha+\beta)\epsilon\right]$$

$$\widetilde{\Delta}^{(\mathrm{I})}_{31} \simeq \Delta_{31}\left(1-\frac{1}{2}\alpha-\frac{1}{2}\beta\right)\epsilon\ ,$$

$$\widetilde{\Delta}^{(\mathrm{I})}_{32} \simeq \Delta_{31}\left[1-\frac{1}{2}\alpha-\frac{1}{2}\left(1-3|U_{e3}|^2\right)\beta+\frac{1}{2}\epsilon-\frac{1}{4}(\alpha+\beta)\epsilon\right]\ ,$$

A neutrino beam may develop a resonance

$$\frac{\widetilde{\mathcal{J}}}{\mathcal{J}} \simeq \frac{\alpha}{\sqrt{\alpha^2-2\cos 2\theta_{12}\cos^2\theta_{13}\ \alpha\beta+\cos 2\theta_{13}\ \beta^2}}\left(1+\frac{1}{2}\alpha+\frac{3}{2}\beta\right)$$

Resonances

We find a unique energy upper limit for $\widetilde{\mathcal{J}}/\mathcal{J} \gtrsim 1$: $E_0 \simeq \Delta_{21} \cos 2\theta_{12}/(\sqrt{2}\ G_{\rm F} N_e) \simeq 0.25\ {\rm GeV}$

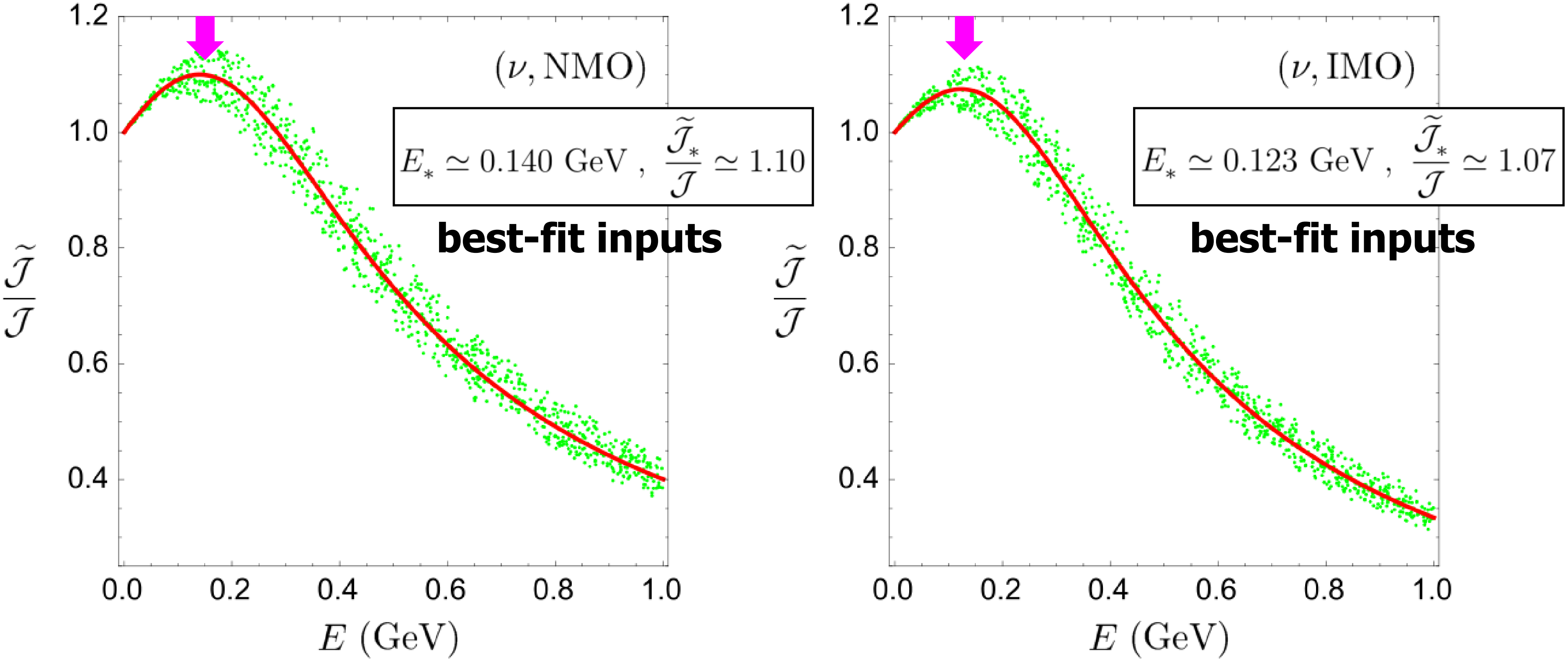
best-fit inputs

$$E_* \simeq \frac{\Delta_{21}}{2\sqrt{2}\ G_{\rm F} N_e} \left[\cos 2\theta_{12} \left(1 + \sin^2\theta_{13}\right) + \frac{3}{2} \sin^2 2\theta_{12} \frac{\Delta_{21}}{\Delta_{31}}\right]$$

resonant energy

$$\frac{\widetilde{\mathcal{J}}_*}{\mathcal{J}} \simeq \frac{1}{\sin 2\theta_{12}} \left[1 + \frac{1}{2}\left(1 + 3\cos 2\theta_{12}\right)\frac{\Delta_{21}}{\Delta_{31}} + {\rm smaller\ terms}\right]$$

enhancive peak

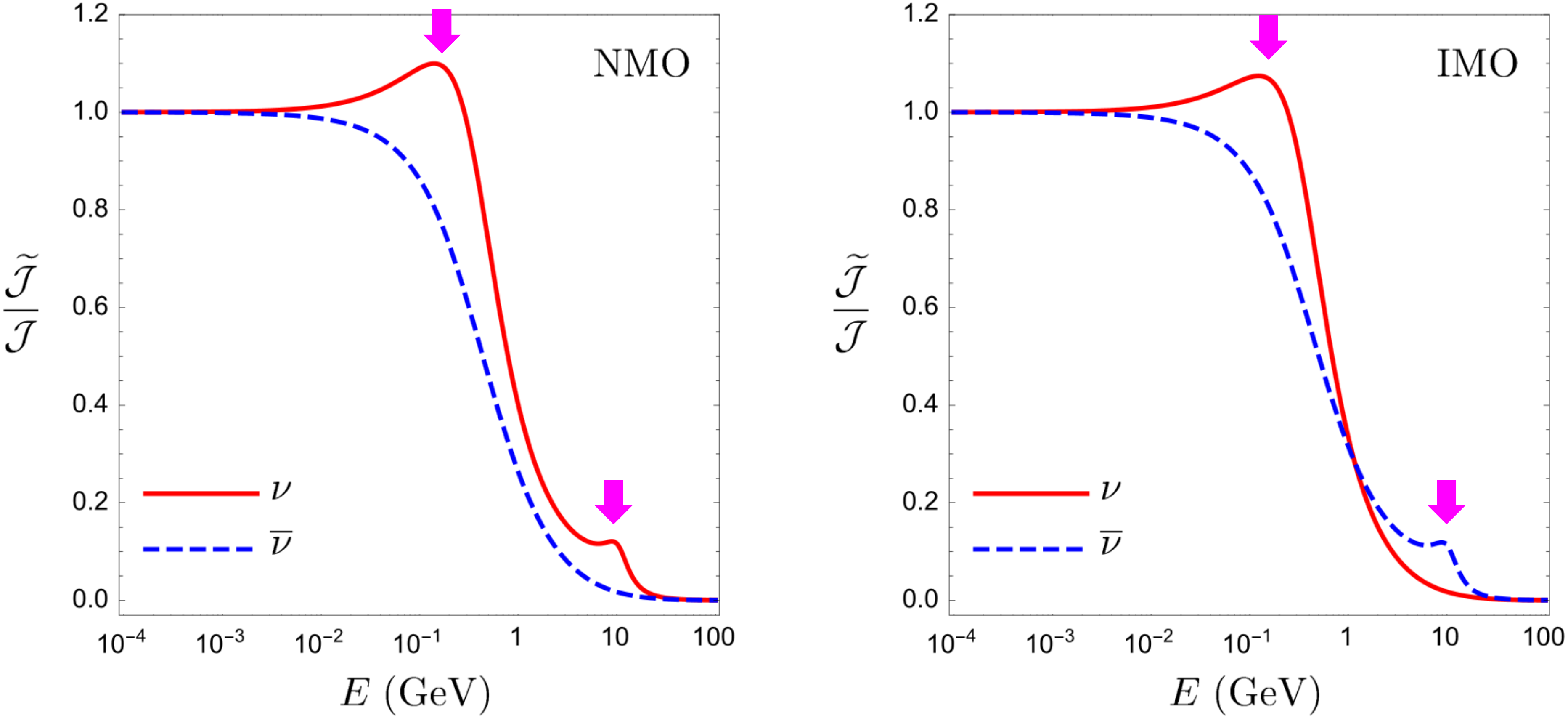


$$\nu \text{ beam } (\Delta_{31} > 0): \quad E_* \simeq 0.140 \text{ GeV}, \quad \frac{\widetilde{\mathcal{J}}_*}{\mathcal{J}} \simeq 1.10; \quad E'_* \simeq 8.906 \text{ GeV}, \quad \frac{\widetilde{\mathcal{J}}'_*}{\mathcal{J}} \simeq 0.12$$

$$\nu \text{ beam } (\Delta_{31} < 0): \quad E_* \simeq 0.123 \text{ GeV}, \quad \frac{\widetilde{\mathcal{J}}_*}{\mathcal{J}} \simeq 1.07;$$

$$\overline{\nu} \text{ beam } (\Delta_{31} < 0): \quad E'_* \simeq 8.828 \text{ GeV}, \quad \frac{\widetilde{\mathcal{J}}'_*}{\mathcal{J}} \simeq 0.12.$$

$$\beta'_* = \frac{\pm A'_*}{\Delta_{31}} \simeq \frac{3\cos 2\theta_{13} + \sqrt{1 - 9\sin^2 2\theta_{13}}}{4}$$



PMNS unitarity triangles

There are 6 unitarity triangles in the complex plane. Only the 3 so-called *Dirac* triangles are related to neutrino oscillations

$$\triangle_e: \quad U_{\mu 1}U^*_{\tau 1} + U_{\mu 2}U^*_{\tau 2} + U_{\mu 3}U^*_{\tau 3} = 0$$
$$\triangle_\mu: \quad U_{\tau 1}U^*_{e 1} + U_{\tau 2}U^*_{e 2} + U_{\tau 3}U^*_{e 3} = 0$$
$$\triangle_\tau: \quad U_{e 1}U^*_{\mu 1} + U_{e 2}U^*_{\mu 2} + U_{e 3}U^*_{\mu 3} = 0$$

In matter, the effective *MNS* unitarity triangles must depart from those in vacuum.

In a low-energy region, we find matter-induced corrections to the Dirac unitarity triangles. The area of every triangle is 0.5 × Jarlskog invariant.

The triangles will shrink quickly when *E* is larger.

$$\widetilde{\triangle}_e: \begin{cases} \widetilde{U}_{\mu 1}\widetilde{U}^*_{\tau 1} \simeq \dfrac{\alpha}{\epsilon}\, U_{\mu 1}U^*_{\tau 1} - \dfrac{1}{2}\left(1 - \dfrac{\alpha-\beta}{\epsilon}\right) U_{\mu 3}U^*_{\tau 3} \\ \widetilde{U}_{\mu 2}\widetilde{U}^*_{\tau 2} \simeq \dfrac{\alpha}{\epsilon}\, U_{\mu 2}U^*_{\tau 2} - \dfrac{1}{2}\left(1 - \dfrac{\alpha+\beta}{\epsilon}\right) U_{\mu 3}U^*_{\tau 3} \\ \widetilde{U}_{\mu 3}\widetilde{U}^*_{\tau 3} \simeq U_{\mu 3}U^*_{\tau 3} \end{cases}$$

$$\widetilde{\triangle}_\mu: \begin{cases} \widetilde{U}_{\tau 1}\widetilde{U}^*_{e 1} \simeq \dfrac{\alpha}{\epsilon}\, U_{\tau 1}U^*_{e 1} - \dfrac{1}{2}\left(1 + \beta - \dfrac{\alpha+\beta}{\epsilon}\right) U_{\tau 3}U^*_{e 3} \\ \widetilde{U}_{\tau 2}\widetilde{U}^*_{e 2} \simeq \dfrac{\alpha}{\epsilon}\, U_{\tau 2}U^*_{e 2} - \dfrac{1}{2}\left(1 + \beta - \dfrac{\alpha-\beta}{\epsilon}\right) U_{\tau 3}U^*_{e 3} \\ \widetilde{U}_{\tau 3}\widetilde{U}^*_{e 3} \simeq (1+\beta)\, U_{\tau 3}U^*_{e 3} \end{cases}$$

$$\widetilde{\triangle}_\tau: \begin{cases} \widetilde{U}_{e 1}\widetilde{U}^*_{\mu 1} \simeq \dfrac{\alpha}{\epsilon}\, U_{e 1}U^*_{\mu 1} - \dfrac{1}{2}\left(1 + \beta - \dfrac{\alpha+\beta}{\epsilon}\right) U_{e 3}U^*_{\mu 3} \\ \widetilde{U}_{e 2}\widetilde{U}^*_{\mu 2} \simeq \dfrac{\alpha}{\epsilon}\, U_{e 2}U^*_{\mu 2} - \dfrac{1}{2}\left(1 + \beta - \dfrac{\alpha-\beta}{\epsilon}\right) U_{e 3}U^*_{\mu 3} \\ \widetilde{U}_{e 3}\widetilde{U}^*_{\mu 3} \simeq (1+\beta)\, U_{e 3}U^*_{\mu 3} \end{cases}$$

On the resonance

The matter-induced corrections to two sides of each triangle are large

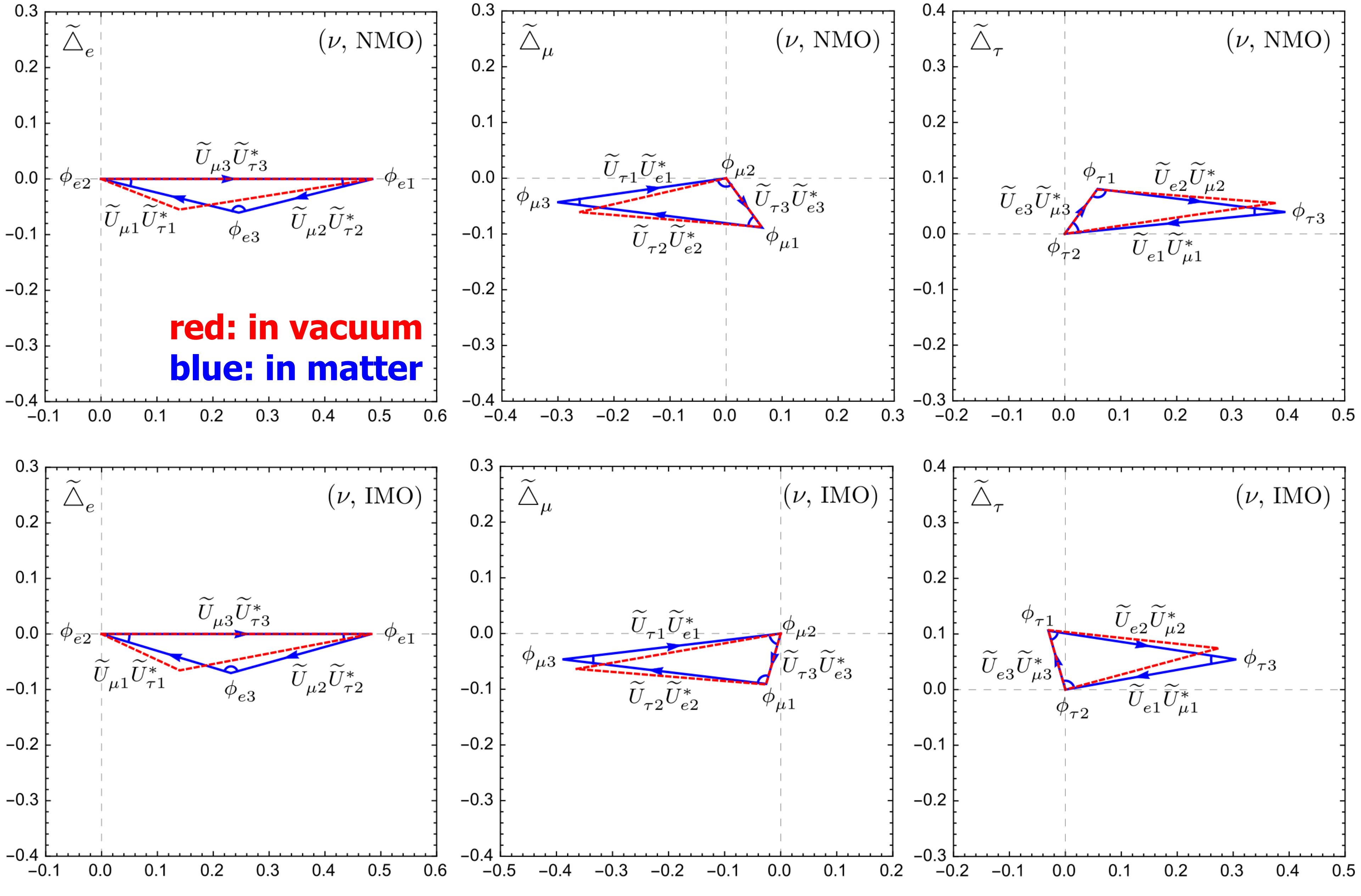
$$\widetilde{\triangle}_e : \begin{cases} \widetilde{U}_{\mu 1}\widetilde{U}^*_{\tau 1} \simeq \dfrac{1}{\sin 2\theta_{12}} U_{\mu 1}U^*_{\tau 1} - \dfrac{1-\tan\theta_{12}}{2} U_{\mu 3}U^*_{\tau 3} \simeq 1.09\ U_{\mu 1}U^*_{\tau 1} - 0.17\ U_{\mu 3}U^*_{\tau 3} \\ \widetilde{U}_{\mu 2}\widetilde{U}^*_{\tau 2} \simeq \dfrac{1}{\sin 2\theta_{12}} U_{\mu 1}U^*_{\tau 1} - \dfrac{1-\cot\theta_{12}}{2} U_{\mu 3}U^*_{\tau 3} \simeq 1.09\ U_{\mu 1}U^*_{\tau 1} + 0.26\ U_{\mu 3}U^*_{\tau 3} \\ \widetilde{U}_{\mu 3}\widetilde{U}^*_{\tau 3} \simeq U_{\mu 3}U^*_{\tau 3} \end{cases}$$

$$\widetilde{\triangle}_\mu : \begin{cases} \widetilde{U}_{\tau 1}\widetilde{U}^*_{e 1} \simeq \dfrac{1}{\sin 2\theta_{12}} U_{\tau 1}U^*_{e 1} - \dfrac{1-\cot\theta_{12}}{2} U_{\tau 3}U^*_{e 3} \simeq 1.09\ U_{\tau 1}U^*_{e 1} + 0.26\ U_{\tau 3}U^*_{e 3} \\ \widetilde{U}_{\tau 2}\widetilde{U}^*_{e 2} \simeq \dfrac{1}{\sin 2\theta_{12}} U_{\tau 2}U^*_{e 2} - \dfrac{1-\tan\theta_{12}}{2} U_{\tau 3}U^*_{e 3} \simeq 1.09\ U_{\tau 2}U^*_{e 2} - 0.17\ U_{\tau 3}U^*_{e 3} \\ \widetilde{U}_{\tau 3}\widetilde{U}^*_{e 3} \simeq U_{\tau 3}U^*_{e 3} \end{cases}$$

$$\widetilde{\triangle}_\tau : \begin{cases} \widetilde{U}_{e 1}\widetilde{U}^*_{\mu 1} \simeq \dfrac{1}{\sin 2\theta_{12}} U_{e 1}U^*_{\mu 1} - \dfrac{1-\cot\theta_{12}}{2} U_{e 3}U^*_{\mu 3} \simeq 1.09\ U_{e 1}U^*_{\mu 1} + 0.26\ U_{e 3}U^*_{\mu 3} \\ \widetilde{U}_{e 2}\widetilde{U}^*_{\mu 2} \simeq \dfrac{1}{\sin 2\theta_{12}} U_{e 2}U^*_{\mu 2} - \dfrac{1-\tan\theta_{12}}{2} U_{e 3}U^*_{\mu 3} \simeq 1.09\ U_{e 2}U^*_{\mu 2} - 0.17\ U_{e 3}U^*_{\mu 3} \\ \widetilde{U}_{e 3}\widetilde{U}^*_{\mu 3} \simeq U_{e 3}U^*_{\mu 3} \end{cases}$$

Part C

Real shapes on resonance

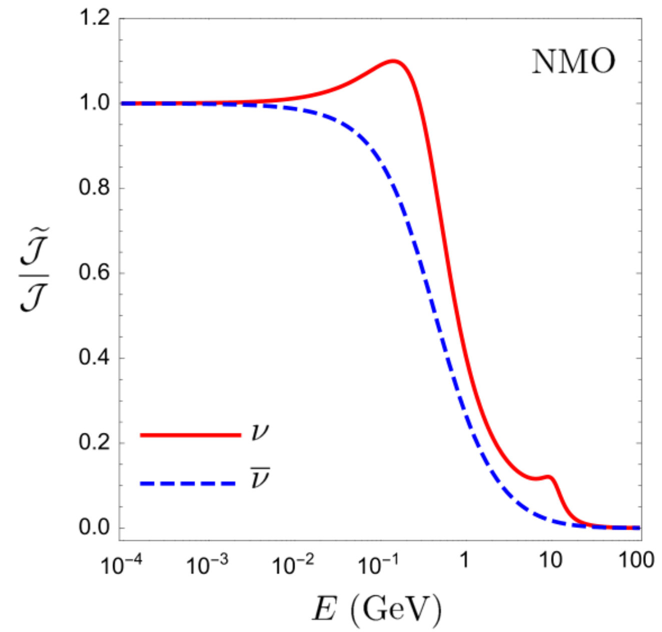


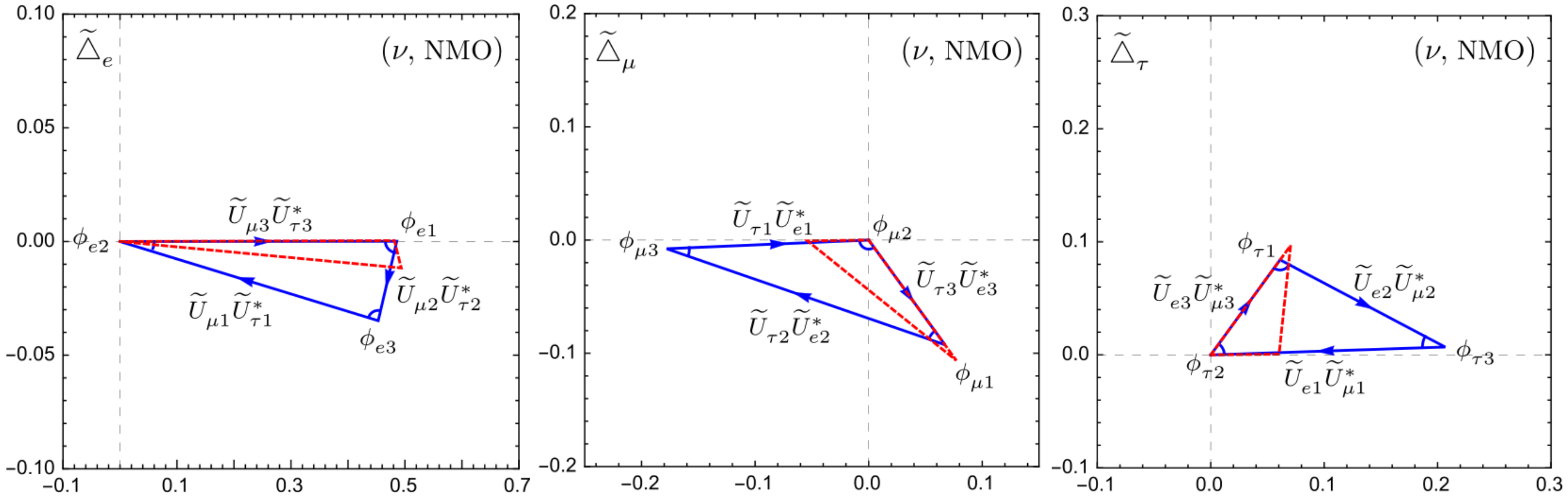
T2K vs NOνA

The effective MNS triangles are very small.

$$\text{T2K } (E \simeq 0.6\ \text{GeV}): \quad \frac{\widetilde{\mathcal{J}}}{\mathcal{J}} \simeq \begin{cases} 0.633 & (\nu\ \text{beam}, \Delta_{31} > 0) \\ 0.568 & (\nu\ \text{beam}, \Delta_{31} < 0) \\ 0.402 & (\overline{\nu}\ \text{beam}, \Delta_{31} > 0) \\ 0.448 & (\overline{\nu}\ \text{beam}, \Delta_{31} < 0) \end{cases}$$

$$\text{NO}\nu\text{A } (E \simeq 2\ \text{GeV}): \quad \frac{\widetilde{\mathcal{J}}}{\mathcal{J}} \simeq \begin{cases} 0.216 & (\nu\ \text{beam}, \Delta_{31} > 0) \\ 0.150 & (\nu\ \text{beam}, \Delta_{31} < 0) \\ 0.132 & (\overline{\nu}\ \text{beam}, \Delta_{31} > 0) \\ 0.190 & (\overline{\nu}\ \text{beam}, \Delta_{31} < 0) \end{cases}$$





A possible experiment?

The possibility of measuring **CP/T** violation in **low** energy ν-oscillation experiments was discussed by **Minakata** and **Nunokawa** in 2000. Now the question is **how low** is **low**.

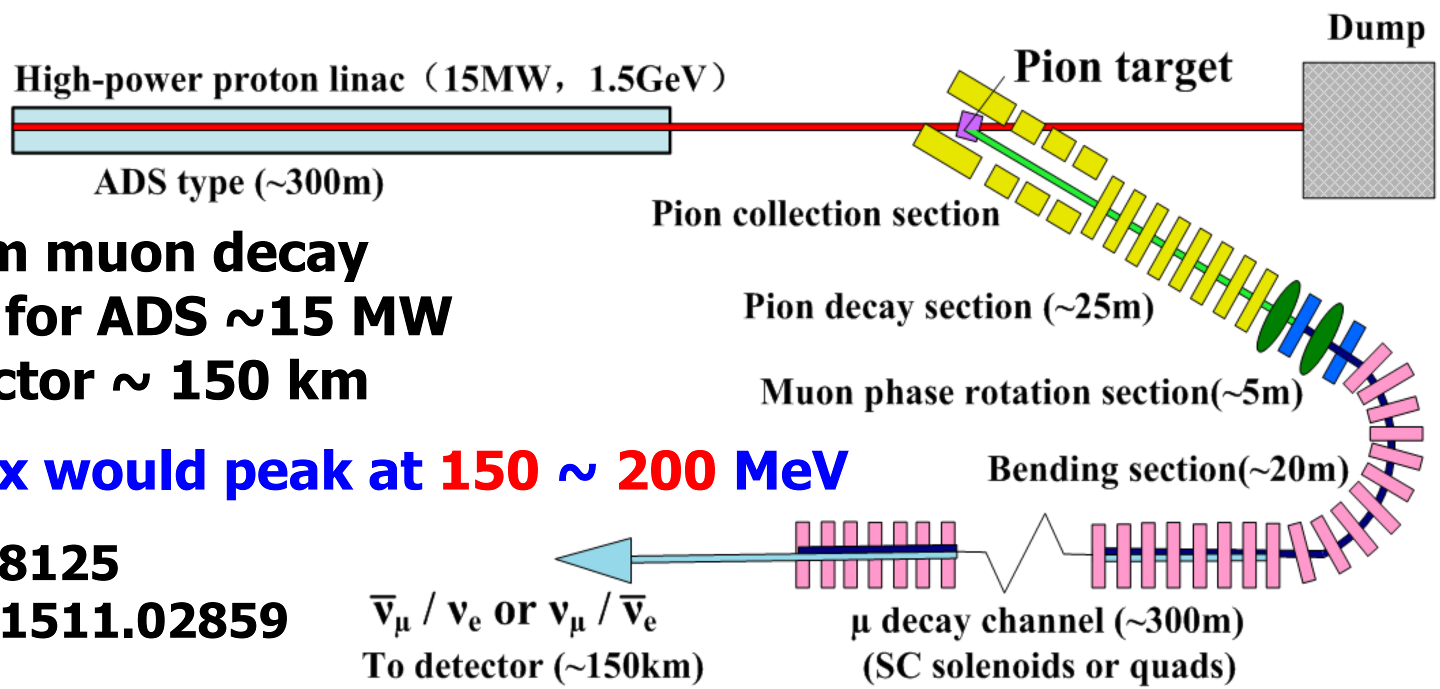
Good things: 1) much smaller terrestrial matter effects and thus more transparent links between effective and intrinsic quantities; 2) much shorter baseline length; ...

Bad things: 1) much smaller cross sections; 2) much lower flux due to the larger beam opening angle; ...

Example: MOMENT

- ♣ Neutrinos from muon decay
- ♣ Proton LINAC for ADS ~15 MW
- ♣ To JUNO detector ~ 150 km

The neutrino flux would peak at 150 ~ 200 MeV

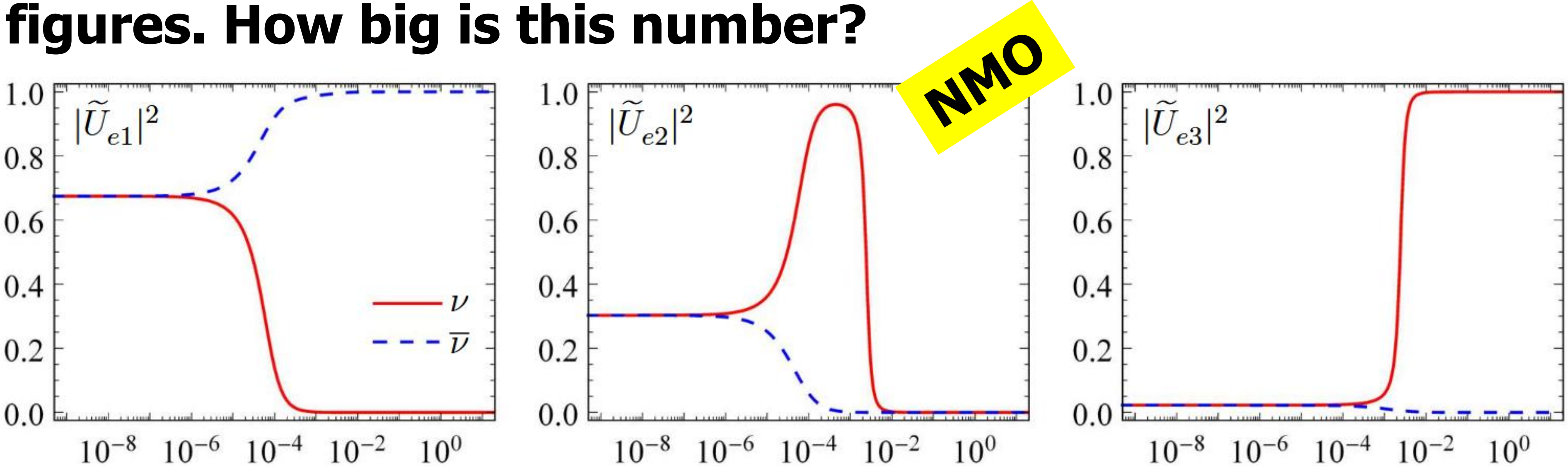


J. Cao et al, 1401.8125
M. Blennow et al, 1511.02859

Final remarks

- The study of matter effects is a kind of **material science**.

- **The $A \to \infty$ limit** is at least **conceptually interesting**, and in practice it is equivalent to $A \gtrsim 10^{-2}\ \mathrm{eV}^2$, as shown in the figures. How big is this number?



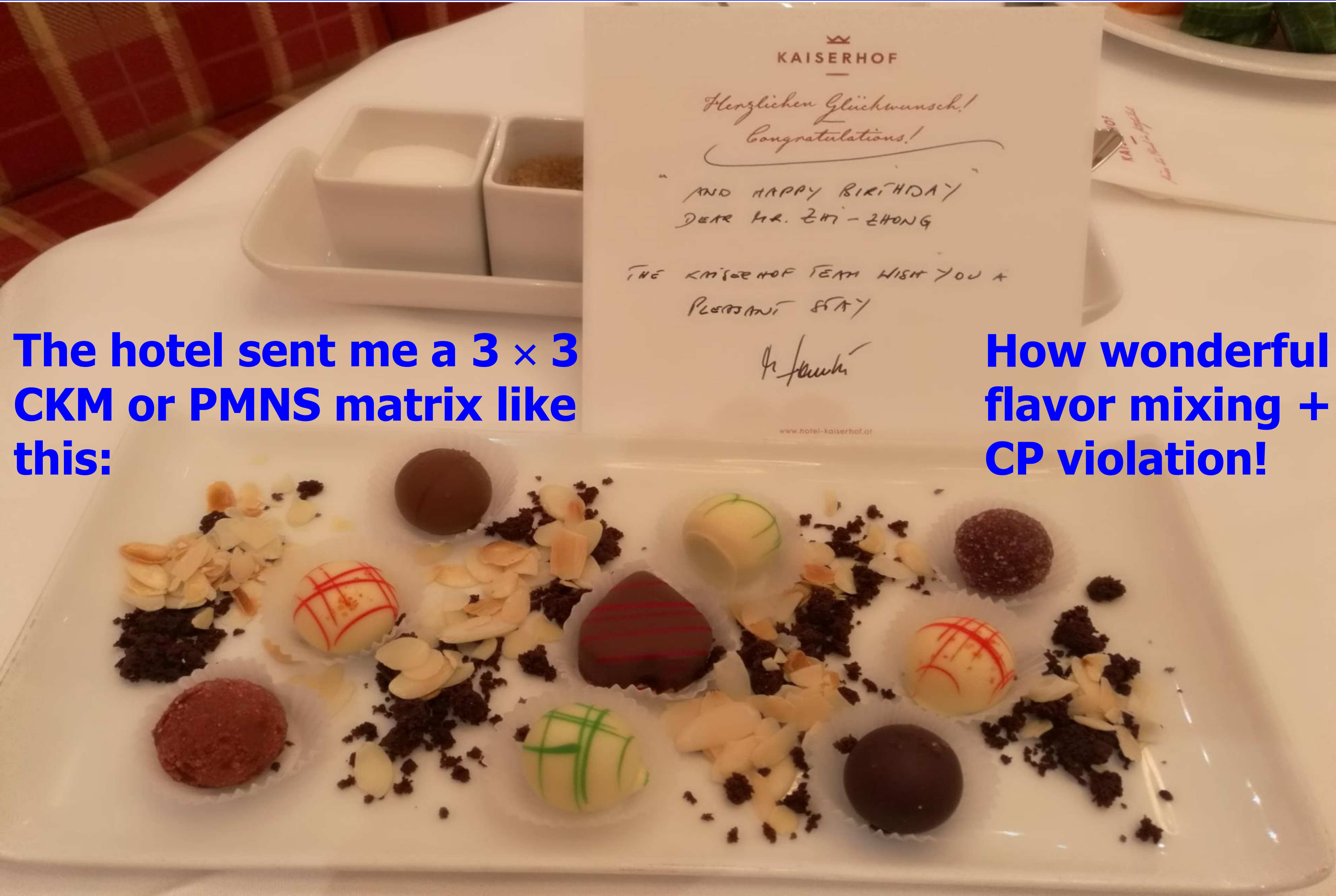
In the very center of the **Sun**, the matter density is not that big at all!

$$N_e(0) \approx 6 \times 10^{25}\ \mathrm{cm}^{-3}$$

$$A \simeq 7.5 \times 10^{-6}\,(E/\mathrm{MeV})\,\mathrm{eV}^2$$

- **A homework** to be done: analytical understanding of the peaks and turning points in the figures by calculating the derivatives of $|\widetilde{U}_{\alpha i}|^2$.

A surprise in the morning



The hotel sent me a 3 × 3 CKM or PMNS matrix like this:

How wonderful flavor mixing + CP violation!