New developments for worldline- and worldsheet formulations of lattice field theories

Christof Gattringer, University of Graz, Austria

Maria Anosova, Falk Bruckmann, Philippe de Forcrand, Ydalia Delgado Mercado, Mario Giuliani, Daniel Göschl, Joshua Hoffer, Thomas Kloiber, Alexander Lehmann, Carlotta Marchis, Oliver Orasch, Michael Müller Preussker, Alexander Schmidt, Felix Springer, Tin Sulejmanpasic

Why worldlines and worldsheets? And what are they?

Quantum field theory on the lattice

- In the lattice formulation we give the Feynman path integral a mathematically precise meaning by introducing a space time lattice Λ.

$$\langle O \rangle \;=\; \frac{1}{Z} \int D[\phi]\; e^{-S[\phi]}\; O[\phi]$$

$D[\phi] = \prod_{x \in \Lambda} d\phi(x)$

$S[\phi]$ .... discretized action

- In a Monte Carlo simulation one generates a finite number of field configurations $\phi^{(n)}, n = 1, 2 \dots N$ with probability

$$P[\phi^{(n)}] \quad = \quad \frac{e^{-S[\phi^{(n)}]}}{Z}$$

Observables assume the form of mean values

$$\langle O \rangle \quad = \quad \frac{1}{N} \sum_{n=1}^{N} O[\phi^{(n)}] \;+\; \mathcal{O}(1/\sqrt{N})$$

Complex action problem

- In general, lattice field theories with finite chemical potential μ or a topological term have actions $S[\phi]$ with an imaginary part.

- The Boltzmann factor

$$e^{-S[\phi]} \quad \in \quad \mathbb{C}$$

  thus has a complex phase and cannot be used as a probability weight.

- Standard Monte Carlo simulation techniques are not available for a non-perturbative analysis.

  "Complex action problem" or "Sign problem"

- Generic feature of finite density field theories both, on the lattice and in the continuum, for bosonic and fermionic theories.

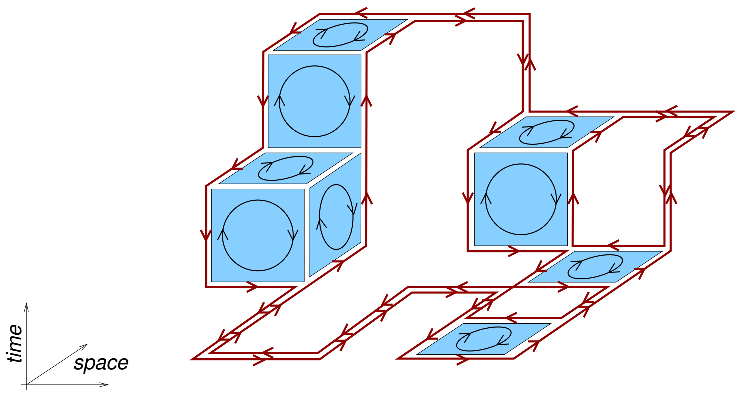
Solving the complex action problem with worldlines and worldsheets

- For some systems one can use strong coupling expansion techniques to rewrite the partition function in terms of new variables ("dual variables") such that it is a sum over real and positive terms.

- The dual variables are worldlines for matter fields and worldsheets for gauge fields.

- The Monte Carlo simulation then is done in terms of the worldlines and worldsheets.

- New Monte Carlo techniques for systems with constraints.



How does the mapping to worldlines and worldsheets work?

... example of a charged scalar field with chemical potential

C. Gattringer, T. Kloiber, Nucl. Phys. B 869 (2013) 56

Worldline representation for the charged ϕ^4 field

- Lattice action: $\quad (\, \phi_n \in \mathbb{C} \, , \quad M^2 = m^2 + 8 \,)$

$$S = \sum_n \Big[\, M^2 \, |\phi_n|^2 \; + \lambda \, |\phi_n|^4 \, \Big] - \sum_{n,\nu} \Big[e^{-\mu \, \delta_{\nu,4}} \, \phi_n^\star \, \phi_{n+\widehat{\nu}} \; + \; e^{\mu \, \delta_{\nu,4}} \, \phi_x \, \phi_{n+\widehat{\nu}}^\star \Big]$$

- Expand the nearest neighbor terms of e^{-S}:

$$\prod_{n,\nu} \exp\left(e^{-\mu \, \delta_{\nu 4}} \, \phi_n^\star \, \phi_{n+\widehat{\nu}} \right) \; = \; \prod_{n,\nu} \sum_{j_{n,\nu}=0}^{\infty} \frac{\left(e^{-\mu \, \delta_{\nu 4}}\right)^{j_{n,\nu}}}{j_{n,\nu} \, !} \left(\phi_n^\star \; \phi_{n+\widehat{\nu}}\right)^{j_{n,\nu}}$$

$$= \sum_{\{j\}} e^{-\mu \sum_n j_{n,4}} \prod_{n,\nu} \frac{1}{j_{n,\nu}!} \prod_n \phi_n^{\sum_\nu j_{n-\hat{\nu},\nu}} \phi_n^{\star \, \sum_\nu j_{n,\nu}}$$

- $j_{n,\nu}$ (and $\overline{j}_{n,\nu}$ for other NN-term) turn into the new worldline degrees of freedom.

Worldline representation - integrating out the fields

- Integral over $\phi_n \sim r\, e^{i\theta}$: ($F, \overline{F}$ are sums of $j_{.,\nu}, \overline{j}_{.,\nu}$ connected to n)

$$\int_{\mathbb{C}} \frac{d\,\phi_n}{2\pi}\; e^{-M^2|\phi_n|^2 - \lambda|\phi_n|^4}\; \phi_n^{\,F}\; \phi_n^{\star\,\overline{F}} \;=$$

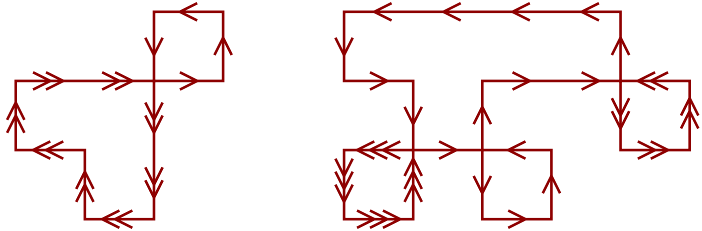
$$= \int_0^\infty dr\; r^{\,1+F+\overline{F}}\; e^{-M^2 r^2 - \lambda r^4} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\; e^{\,i\theta\,(F-\overline{F})} \;=\; \mathcal{R}(F+\overline{F})\;\delta(F-\overline{F})$$

- At every site n there is a weight factor $\mathcal{R}(F+\overline{F})$ and a constraint $\delta(F-\overline{F})$.

- Explicitly the product over the constraints at all sites reads $(d_{n,\nu} = j_{n,\nu} - \overline{j}_{n,\nu})$:

$$\prod_n \delta\Big(\sum_\nu \left[\, d_{n,\nu} - d_{n-\widehat{\nu},\nu}\,\right]\Big) \quad\Leftrightarrow\quad \sum_\nu \left[\, d_{n,\nu} - d_{n-\widehat{\nu},\nu}\,\right] \;=\; \vec{\nabla}\vec{d}_n \;=\; 0 \quad \forall n$$

- Admissible configurations of worldline variables are oriented loops of flux:



Worldline representation - final form

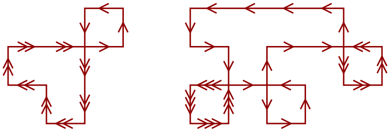
- The original partition function is mapped exactly to a sum over configurations of the worldline variables $j_{n,\nu}, \overline{j}_{n,\nu} \in \mathbb{N}_0$ with $d_{n,\nu} = j_{n,\nu} - \overline{j}_{n,\nu}$.

$$Z \;=\; \sum_{\{j,\overline{j}\}} \mathcal{W}[\,j,\overline{j}\,]\,\mathcal{C}[\,d\,]$$

- $\mathcal{W}[\,j,\overline{j}\,]$: Real and positive weight from radial d.o.f. and combinatorics.

- Constraints from integrating over the symmetry group ($d_{n,\nu} = j_{n,\nu} - \overline{j}_{n,\nu}$):

$$\mathcal{C}[\,d\,] \;=\; \prod_n \delta\Big(\vec{\nabla}\vec{d}_n\Big)$$



- Particle number N $\Leftrightarrow$ temporal winding number $\omega[d]$ of $d_{n,\nu}$-flux:

$$e^{-\mu\sum_n d_{n,4}} \;=\; e^{-\mu\, N_t\,\omega[d]} \;=\; e^{-\mu\,\beta\,\omega[d]} \;\equiv\; e^{-\mu\,\beta\,N}$$

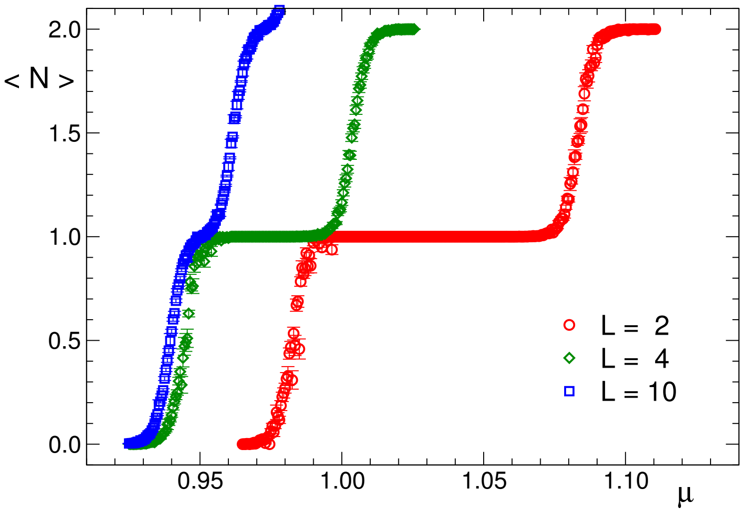
New representation is very powerful ....

Example: Finite density condensation and scattering data

F. Bruckmann, C. Gattringer, T. Kloiber, T. Sulejmanpasic, PRL 115, 231601 (2015)
C. Gattringer, M. Giuliani and O. Orasch, PRL 120, 241601 (2018)

Condensation thresholds

Expectation value $\langle N \rangle$ of the particle number as a function of the chemical potential μ at very low temperature (charged scalar field):



- At critical values $\mu_n(L)$ one observes jumps from $\langle N \rangle = n-1$ to $\langle N \rangle = n$.
- The condensation thresholds $\mu_n(L)$ depend on the spatial extent L.

Connection of condensation thresholds and n-particle energies

- Grand canonical partition sum and grand potential:

$$Z \;=\; \mathrm{Tr}\, e^{-\beta(\hat{H} - \mu \hat{N})} \;=\; e^{-\beta\,\Omega(\mu)}$$

- Low T: In each particle sector Z is governed by the minimal grand potential $\Omega(\mu)$

$$\Omega(\mu) \quad \xrightarrow{T\to 0} \quad \begin{cases} \Omega_{min}^{N=0} \;=\; 0\,, & \mu \;\in\; [0, \mu_1] \\[2ex] \Omega_{min}^{N=1} \;=\; m \;-\; 1\mu\,, & \mu \;\in\; [\mu_1, \mu_2] \\[2ex] \Omega_{min}^{N=2} \;=\; W_2 - 2\mu\,, & \mu \;\in\; [\mu_2, \mu_3] \\[2ex] \Omega_{min}^{N=3} \;=\; W_3 - 3\mu\,, & \mu \;\in\; [\mu_3, \mu_4] \\ \ldots \end{cases}$$

- m: physical mass, W_2: minimal 2-particle energy, W_3: minimal 3-particle energy ...
- Use continuity of $\Omega(\mu)$ to relate the critical μ_n to m and the W_n.

$$m(L) \;=\; \mu_1(L)\,, \quad W_2(L) \;=\; \mu_1(L) + \mu_2(L)\,, \quad \ldots \quad W_n(L) \;=\; \sum_{k=1}^{n} \mu_k(L)$$

Connection of condensation thresholds and n-particle energies

- The multi-particle energies are governed by low energy parameters.
- In particular their finite volume dependence can be related to scattering data.

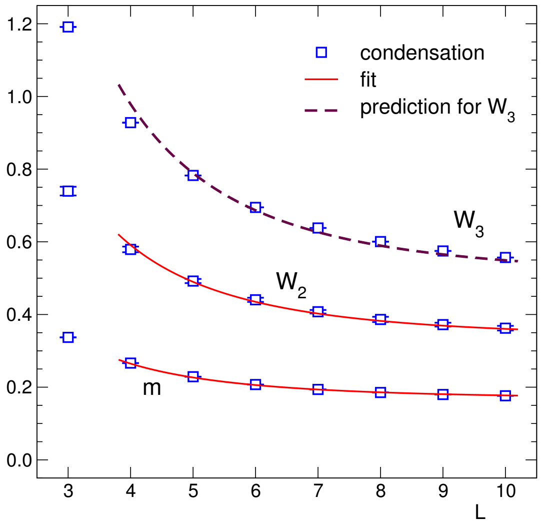
  (K. Huang, C.N. Yang, M. Lüscher, S.R. Beane, W. Detmold, M.J. Savage, S.R. Sharpe, M.T. Hansen)

$\mathcal{I} = -8.914, \mathcal{J} = 16.532$

$$
\begin{aligned}
m(L) &= m_\infty + \frac{A}{L^{\frac{3}{2}}}\, e^{-L\, m_\infty} \\
W_2(L) &= 2m + \frac{4\pi a}{mL^3}\left[1 - \frac{a}{L}\frac{\mathcal{I}}{\pi} + \left(\frac{a}{L}\right)^2 \frac{\mathcal{I}^2 - \mathcal{J}}{\pi^2} + \mathcal{O}\left(\frac{a}{L}\right)^3\right] \\
W_3(L) &= 3m + \frac{12\pi a}{mL^3}\left[1 - \frac{a}{L}\frac{\mathcal{I}}{\pi} + \left(\frac{a}{L}\right)^2 \frac{\mathcal{I}^2 + \mathcal{J}}{\pi^2} + \mathcal{O}\left(\frac{a}{L}\right)^3\right] \\
m(L) &= \mu_1(L)\,, \quad W_2(L) = \mu_1(L) + \mu_2(L)\,, \quad W_3(L) = \mu_1(L) + \mu_2(L) + \mu_3(L)
\end{aligned}
$$

- We thus expect that one can describe the thresholds $\mu_n(L)$ with scattering data.

Comparison of threshold data with the finite volume relations



- Good agreement: Condensation can indeed be described with scattering data.
- Key technical ingredient: Worldline representation

Summary

- All lattice field theories can be exactly rewritten in terms of worldlines and worldsheets.
- In several examples it was seen that this overcomes complex action problems.
- Interesting physics can be explored. Examples: finite density, topological terms ...

Developments:

- Fermions: Resummation of contributions in bag determinants.
- To address the re-ordering problem of non-abelian gauge fields we use abelian color cycles, which are paths through color space along plaquettes.

  (poster by Joshua Hoffer)
- The ACC construction can be generalized by including matter fields.
- Weights for all terms of strong coupling expansion known in closed form.

A glimpse at fermions: Baryon bags

C. Gattringer, PRD 97 (2018) 074506

Framework, separation of Baryon terms

Lattice QCD with one flavor of staggered fermions:

$$Z \;=\; \int D[U]\, e^{S_G[U]}\, Z_F[U] \quad , \qquad Z_F[U] \;=\; \int D[\,\overline{\psi},\psi]\, e^{S_F[\overline{\psi},\psi,U]}$$

Action:

$$S_F[\,\overline{\psi},\psi,U] \;=\; \sum_x \Big[2m\,\overline{\psi}_x \psi_x + \sum_\nu \gamma_{x,\nu} \Big[\overline{\psi}_x U_{x,\nu} \psi_{x+\hat{\nu}} - \overline{\psi}_{x+\hat{\nu}} U^\dagger_{x,\nu} \psi_x \Big]\Big]$$

Expansion of nearest neighbor Boltzmann factor:

$$\begin{aligned} e^{\gamma\,\overline{\psi}U\psi} &= 1 \,+\, \gamma\,(\overline{\psi}U\psi) \,+\, \frac{1}{2!}(\overline{\psi}U\psi)^2 \,+\, \frac{\gamma}{3!}(\overline{\psi}U\psi)^3 \\ &= \Big[1 \,+\, \frac{\gamma}{3!}(\overline{\psi}U\psi)^3\Big]\Big[1 \,+\, \gamma\,(\overline{\psi}U\psi) \,+\, \frac{1}{2!}(\overline{\psi}U\psi)^2\Big] \\ &= e^{\frac{\gamma}{3!}(\overline{\psi}U\psi)^3} \sum_{k=0}^{2} \frac{(\gamma\,\overline{\psi}U\psi)^k}{k!} \end{aligned}$$

Baryon action

Cubic term is independent of gauge fields:

$$
\begin{aligned}
(\overline{\psi} U \psi)^3 \;&=\; (\overline{\psi}_a U_{ab} \psi_b)^3 \;=\; \overline{\psi}_a\,\psi_b\,\overline{\psi}_{a'}\,\psi_{b'}\,\overline{\psi}_{a''}\,\psi_{b''}\,U_{ab}\,U_{a'b'}\,U_{a''b''} \\
&=\; \overline{\psi}_3\,\overline{\psi}_2\,\overline{\psi}_1\,\psi_1\,\psi_2\,\psi_3\,\epsilon_{a\,a'a''}\,U_{ab}\,U_{a'b'}\,U_{a''b''}\,\epsilon_{b\,b'b''} \\
&=\; \overline{\psi}_3\,\overline{\psi}_2\,\overline{\psi}_1\,\psi_1\,\psi_2\,\psi_3\,\epsilon_{a\,a'a''}\,\epsilon_{a\,a'a''}\,\det U \;=\; 3!\,\overline{\psi}_3\,\overline{\psi}_2\,\overline{\psi}_1\,\psi_1\,\psi_2\,\psi_3
\end{aligned}
$$

Definition of baryon fields:

$$
\overline{B}_x \;=\; \overline{\psi}_{x,3}\,\overline{\psi}_{x,2}\,\overline{\psi}_{x,1} \quad , \qquad B_x \;=\; \psi_{x,1}\,\psi_{x,2}\,\psi_{x,3} \qquad \left(\;=\; \epsilon_{abc}\,\psi_{x,a}\,\psi_{x,b}\,\psi_{x,c}\,/\,3!\;\right)
$$

Boltzmann factor:

$$
e^{S_F[\overline{\psi},\psi,U]} \;=\; e^{\,S_B[\overline{B},B]} \;\times\; W_{QD}[\,\overline{\psi},\psi,U]
$$

$$
S_B[\overline{B},B] \;=\; \sum_x \Big[2M\,\overline{B}_x\,B_x + \sum_\nu \gamma_{x,\nu} \Big[\,\overline{B}_x\,B_{x+\hat{\nu}} - \overline{B}_{x+\hat{\nu}}\,B_x \Big] \Big]
$$

Partition sum with factorized baryon contributions

Partition sum:

$$Z = \int D[\overline{\psi}, \psi] \, e^{\,S_B[\overline{B},B]} \int D[U] \, e^{\,S_G[U]} \, W_{QD}[\overline{\psi}, \psi, U] \; = \; \int D[\overline{\psi}, \psi] \, e^{\,S_B[\overline{B},B]} \, W_{QD}^{\,int}[\overline{\psi}, \psi]$$

- Strong coupling: quark and diquark term $W_{QD}^{\,int}$ can be calculated in closed form.
- Grassmann integral can be saturated either with terms from $e^{\,S_B[\overline{B},B]}$ or from $W_{QD}^{\,int}$.
- Decompose space time into baryon bags $\mathcal{B}_i$ and a complementary domain $\overline{\mathcal{B}}$:

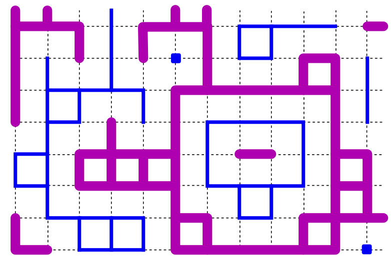
  $\mathcal{B}_i$: baryon terms are used for saturating the Grassmann integral.
  $\overline{\mathcal{B}}$: quark and diquark terms are used for the Grassmann integral.

- $\overline{B}_x = \overline{\psi}_{x,3} \, \overline{\psi}_{x,2} \, \overline{\psi}_{x,1}$ and $B_x = \psi_{x,1} \, \psi_{x,2} \, \psi_{x,3}$ inherit Grassmann properties.
- Contribution from a baryon bag $\mathcal{B}_i$ is the bag determinant $\det \, D^{(i)}$.

Final form of partition sum with baryon bags

Bag-factorized partition sum

$$Z = \sum_{\{\mathcal{B}\}} \prod_i \det D^{(i)} \times Z_{\overline{\mathcal{B}}}$$



- The partition function is a sum over configurations of baryon bags and the path integral is decomposed into baryon bag contributions and terms in the complementary domain.
- Inside the baryon bags $\mathcal{B}_i$ the system chooses a description with freely propagating baryons as degrees of freedom.
- The bag determinants $\det D^{(i)}$ are real and positive. They sum contributions of many worldlines inside $\mathcal{B}_i$.
- In the complementary domain $\overline{\mathcal{B}}$ the relevant degrees of freedom are monomers and dimers for quarks and diquarks. $Z_{\overline{\mathcal{B}}}$ is real and positive.
- The dynamics and scale of the fermion bags depends on the couplings. $\Rightarrow$ MC update

Towards non-abelian gauge fields

C. Gattringer, C. Marchis, Nucl. Phys. B916 (2017) 627
C. Marchis, C. Gattringer, Phys. Rev. D (2018) 034508

Where is the problem?

- Before integrating the scalar fields with $\int D[\phi] = \int \prod_{x} d\phi_{x}$ we had to reorder them:

$$\prod_{x,\nu} (\phi_x^\star \, \phi_{x+\widehat{\nu}})^{j_{x,\nu}} \, (\phi_x \, \phi_{x+\widehat{\nu}}^\star)^{\overline{j}_{x,\nu}} \; = \; \prod_{x} \phi_x^{\sum_\nu (\overline{j}_{x,\nu} + j_{x-\hat{\nu},\nu})} \phi_x^{\star \, \sum_\nu (j_{x,\nu} + \overline{j}_{x-\hat{\nu},\nu})}$$

(works also for abelian gauge fields)

- When reordering fermionic d.o.f. one picks up minus signs (Grassmann numbers).

- Reordering non-abelian gauge fields?

.... one does not even know how to do it!

- We currently explore decomposing the action into smaller building blocks:

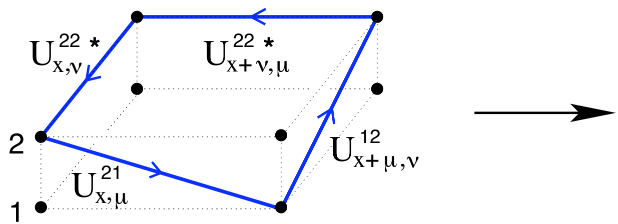
Abelian color cycles (ACC)

Decomposition of the non-abelian action into abelian color cycles:

- Action for SU(2) lattice gauge theory ($U_{x,\mu} \in$ SU(2)) :

$$S \;=\; -\frac{\beta}{2}\sum_{x,\mu<\nu} \text{Tr}\; U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^{\dagger}_{x+\hat{\nu},\mu}\, U^{\dagger}_{x,\nu} \;=\; -\frac{\beta}{2}\sum_{x,\mu<\nu}\;\sum_{a,b,c,d=1}^{2} U^{ab}_{x,\mu}\, U^{bc}_{x+\hat{\mu},\nu}\, U^{dc\,\star}_{x+\hat{\nu},\mu}\, U^{ad\,\star}_{x,\nu}$$

- The products $U^{ab}_{x,\mu}\, U^{bc}_{x+\hat{\mu},\nu}\, U^{dc\,\star}_{x+\hat{\nu},\mu}\, U^{ad\,\star}_{x,\nu}$ are the abelian color cycles (ACC) (= paths through color space along plaquettes) we use for expanding the Boltzmann factor. Example:



$p^{2122}_{x,\mu} \in \boldsymbol{N_0}$

- Suitable parameterization:

$$U_{x,\mu} \;=\; \begin{bmatrix} \cos\theta_{x,\mu}\; e^{i\alpha_{x,\mu}} & \sin\theta_{x,\mu}\; e^{i\beta_{x,\mu}} \\ -\sin\theta_{x,\mu}\; e^{-i\beta_{x,\mu}} & \cos\theta_{x,\mu}\; e^{-i\alpha_{x,\mu}} \end{bmatrix} \qquad \theta_{x,\mu} \in [0,\pi/2]\;,\quad \alpha_{x,\mu}, \beta_{x,\mu} \in [-\pi,\pi]$$

Expansion in ACCs

- Partition sum:

$$Z = \int D[U] \; \exp\left(\frac{\beta}{2} \sum_{x,\mu<\nu} \sum_{a,b,c,d} U^{ab}_{x,\mu} \, U^{bc}_{x+\hat{\mu},\nu} \, U^{dc\,\star}_{x+\hat{\nu},\mu} \, U^{ad\,\star}_{x,\nu} \right) \;, \quad \int D[U] = \prod_{x,\mu} dU_{x,\nu}$$

- Expansion of the Boltzmann factor:

$$\begin{aligned} Z &= \int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d} e^{\frac{\beta}{2} U^{ab}_{x,\mu} \, U^{bc}_{x+\hat{\mu},\nu} \, U^{dc\,\star}_{x+\hat{\nu},\mu} \, U^{ad\,\star}_{x,\nu}} \\ &= \int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d} \sum_{p^{abcd}_{x,\mu\nu}=0}^{\infty} \frac{\left(\frac{\beta}{2}\right)^{p^{abcd}_{x,\mu\nu}}}{p^{abcd}_{x,\mu\nu}\,!} \left(U^{ab}_{x,\mu} \, U^{bc}_{x+\hat{\mu},\nu} \, U^{dc\,\star}_{x+\hat{\nu},\mu} \, U^{ad\,\star}_{x,\nu} \right)^{p^{abcd}_{x,\mu\nu}} \end{aligned}$$

- Reordering the terms:

$$Z = \sum_{\{p\}} \prod_{x,\mu<\nu} \prod_{a,b,c,d} \frac{\left(\frac{\beta}{2}\right)^{p^{abcd}_{x,\mu\nu}}}{p^{abcd}_{x,\mu\nu}\,!} \prod_{x,\mu} \int d_H[\theta_{x,\mu}, \alpha_{x,\mu}, \beta_{x,\mu}] \prod_{ab} \left(U^{ab}_{x,\mu} \right)^{N^{ab}_{x,\mu}[p]} \left(U^{ab\,\star}_{x,\mu} \right)^{\overline{N}^{ab}_{x,\mu}[p]}$$

Remaining link integrals can be solved and give constraints and weights for the configurations $\{p\}$ of the cycle occupation numbers $p^{abcd}_{x,\mu\nu} \in \mathbb{N}_0$.

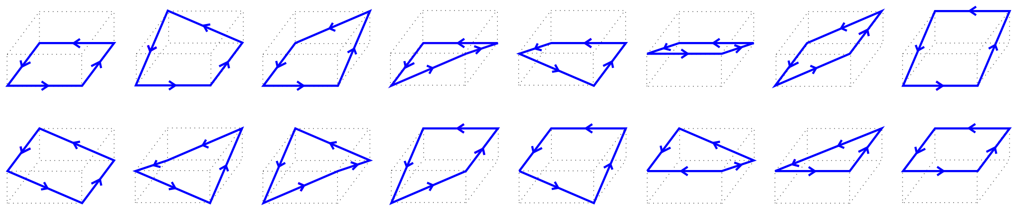
Partition function as sum over occupation numbers of ACCs

- Dual partition sum:

$$Z = \sum_{\{p\}} W_\beta[p]\,(-1)^{\sum_{x,\mu} J^{21}_{x,\mu}} \prod_{x,\mu<\nu} \delta\left(J^{11}_{x,\mu} - J^{22}_{x,\mu}\right)\,\delta\left(J^{12}_{x,\mu} - J^{21}_{x,\mu}\right)$$

$$J^{ab}_{x,\mu} = \text{ total flux from } a \text{ to } b \text{ along the link } x,\mu$$

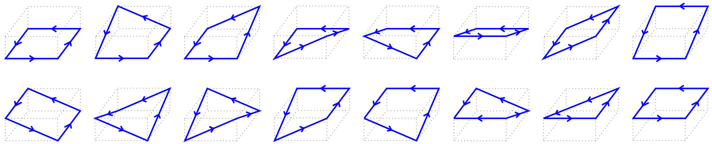
- 16 possible ACCs that can be occupied (i.e., $p^{abcd}_{x,\mu\nu} > 0$):



- Constraints at each link:

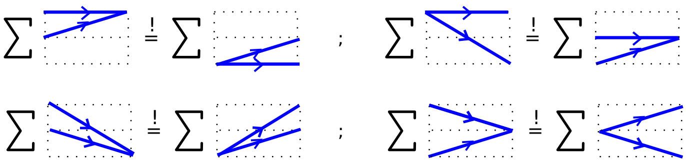
ACC representation - summary

- The partition sum is a sum over configurations of cycle occupation numbers $p_{x,\mu\nu}^{abcd} \in \mathbb{N}_0$.



- At every link the fluxes must obey constraints:

- SU(3) has three color choices with $3^4 = 81$ ACCs and constraints:



- We are working on a resummation strategy to overcome signs.

Summary

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- In several examples it was seen that this overcomes complex action problems.
- Interesting physics can be explored. Examples: finite density, topological terms ...
- Fermions are a challenge. Resummation of contributions in bag determinants.
- To address the re-ordering problem of non-abelian gauge fields we use abelian color cycles, which are paths through color space along plaquettes.
- The ACC construction can be generalized by including matter fields.
- Weights for all terms of the strong coupling expansion are known in closed form.