Short Treatment of (Special) Relativity

How can we deal with moving charged particles ?

Given the subject of the school, with very strong emphasis on electrodynamics and accelerators

CERN Accelerator School, 8-21 September 2019, Vysoke-Tatry, Slovakia

Reading Material

[1]R.P. Feynman, Feynman lectures on Physics, Vol. 1 + 2.

[2]A. Einstein, *Zur Elektrodynamik bewegter Körper*, Ann. Phys. 17, (1905).

[3]A. Michelson, E. Morley, *On the Relative Motion of the Earth and the Luminiferous Ether*, American Journal of Science. 34, (203), 333-345.

[4]J. Freund, *Special Relativity*, (World Scientific, 2008).

[5]J.D. Jackson, *Classical Electrodynamics* (Wiley, 1998 ..)

[6]J. Hafele and R. Keating, Science 177, (1972) 166.

[7]W. Herr, *Short Theory of Special Relativity*, Proc. CAS Accelerator School on FEL and ERL, (Hamburg, Germany, 31 May - 10 June 2016), ed. R. Bailey, CERN-2018-001-SP,

[8]M. Ferrario, *Lecture on Special Relativity*, CAS Accelerator School Introductory Course, (Constanta, Romania, 16 - 29 September 2018)

[9]R.P. Feynman, *Six Not-So-Easy Pieces*, Basic Books, New York

[10] S. Sheehy, *Particle motion in Hamiltonian formalism* (This school)

Variables and units used in this lecture

Formulae use SI units throughout.

$\vec{E}(\vec{r},t)$	=	**electric field [V/m]**
$\vec{H}(\vec{r},t)$	=	**magnetic field [A/m]**
$\vec{D}(\vec{r},t)$	=	**electric displacement [C/m^2]**
$\vec{B}(\vec{r},t)$	=	**magnetic flux density [T]**
q	=	**electric charge [C]**
e	=	**elementary charge 1.60218$\cdot 10^{-19} [C]$**
$\rho(\vec{r},t)$	=	**electric charge density [C/m^3]**
$\vec{I}, \ \vec{j}(\vec{r},t)$	=	**current [A], current density [A/m^2]**
μ_0	=	**permeability of vacuum, 4 $\pi \cdot 10^{-7}$ [H/m or N/A^2]**
ϵ_0	=	**permittivity of vacuum, 8.854 $\cdot 10^{-12}$ [F/m]**
c	=	**speed of light in free space, 299792458.0 m/s**
h	=	**Planck constant, 6.62607 $\cdot 10^{-34}$ Js (4.13570 $\cdot 10^{-15}$ eVs)**

OUTLINE and Learning Goals

- Principles of Relativity (Newton, Galilei, Poincare)
  - Motivation, Ideas and Terminology
  - Formalism, Examples
- Principles of Special Relativity (Einstein)
  - Postulates, Formalism and Consequences
  - Four-vectors → allow calculations with a minimum amount of mathematics and/or without hand-waving arguments
- If time permits: relativistically correct formulation of Maxwell's equations
- Ambition (satisfy different "learning expectations"):
  1. Provide "ready-to-use" formulae for daily work
  2. Spend some time on fundamentals for those interested
  3. For your pleasure: added a few exercises you may or may not want to try, solutions can (soon) be found on Indico ..

For lack of time, cannot discuss experiments and "so-called" paradoxes, please consult the literature

Setting the scene (terminology) ..

- **An observer (lets call it O) assigns/describes an "event" E, e.g. an explosion in my office (lets call the office a "frame" S) with Space coordinates: $\vec{x} = (x, y, z)$ and Time: t**

- **Another observer (lets call it O') assigns/describes the same "event" in its own coordinate system (lets call it a "frame" S') with**
  - **Space coordinates: $\vec{x'} = (x', y', z')$ and Time: t**

→ **Physics laws cannot depend on <u>where</u> you are**

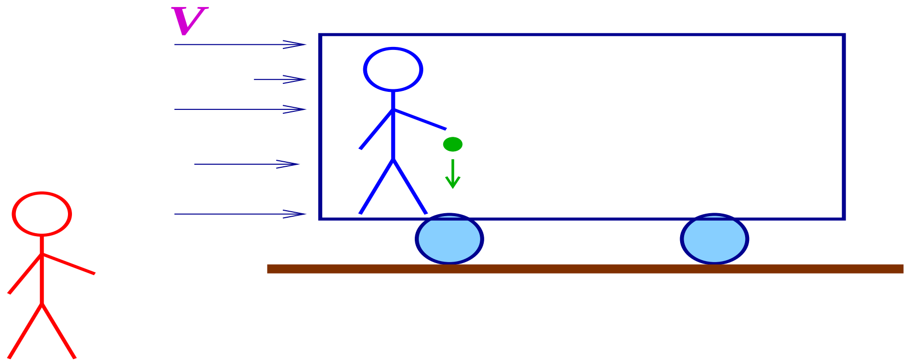
→ **Relativity: teaches us the connection and relationship between different observations**

Easy enough, if a coordinate system is displaced in x-direction by d:

$$(x,\ y,\ z\ ) \implies (x + d,\ y,\ z\ )$$

What if the frames are moving relative to each other (it means that d is now a function of time $d(t)$) ?

Assume a frame at rest (S) and another frame (S') moving in x-direction with velocity $\vec{V} = (V, 0, 0) \quad \longrightarrow \quad d(t) = V \cdot t$



- **Observer O'/Passenger** observes an event (e.g. a falling pear) <u>within</u> moving frame
- **Observer O** observes the same event*) <u>from</u> resting frame

Physics laws must not depend on whether or not one moves with constant velocity: there cannot be two different sets of physics laws

*) This is all important !

Formulated by Newton and Galilei: Principles of Relativity

Definition:

A frame moving at constant velocity is an (Inertial System)

Physics laws are the same in all inertial systems

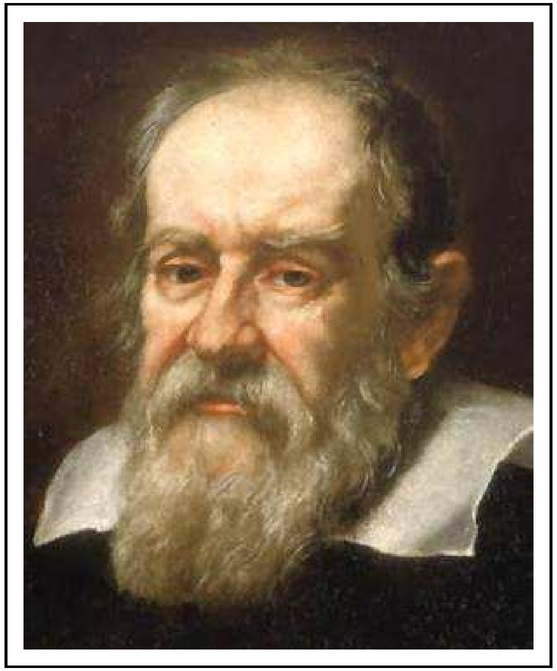
Example: we would like to have:

$Force \; = \; m \cdot a$ **and** $Force' \; = \; m \cdot a'$

Now we need a transformation for:

(x, y, z) **and** t → (x', y', z') **and** t' **(and anything that is derived from it).**

Galilei transformation



$$x' = x - V_x t$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Galilei transformations relate observations in two frames moving relative to each other (here with constant velocity V_x in x-direction).

Only the position (in direction of V_x) is changing with time

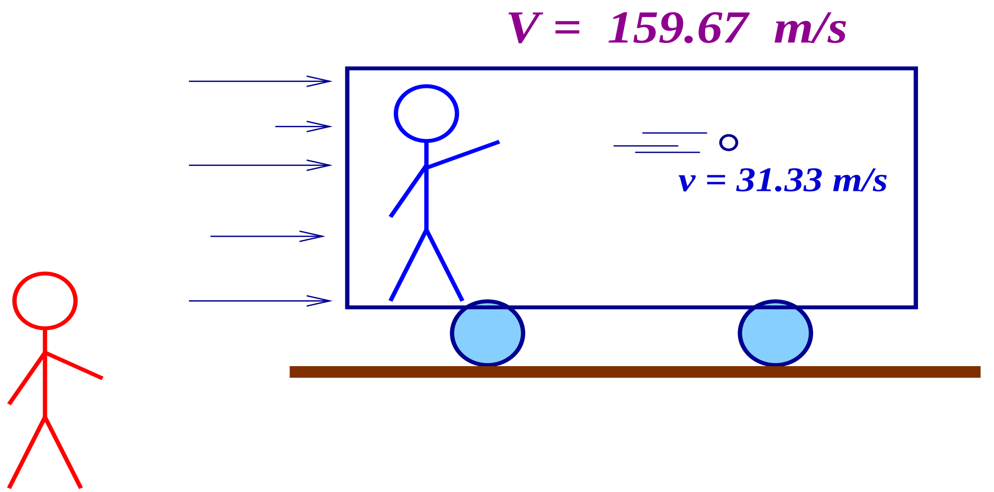
Frame moves in x-direction with velocity V_x:

- **Space coordinates are changed, time is not changed !**
- **Space, mass and time are <u>independent</u> quantities**
  - **Absolute space where physics laws are the same**
  - **Absolute time where physics laws are the same**
- **Some examples, plug it in:**

$$m \cdot a \quad = \quad m \cdot \ddot{x} \quad = \quad m \cdot \ddot{x}' \quad = \quad m \cdot a'$$

$$v_{x'} \quad = \quad \frac{dx'}{dt} \quad = \quad \frac{dx}{dt} \; - \; v_x$$ **(velocities can be added)**

Newton: space and time are absolute → independent of objects and perception



Fling a ball with 31.33 m/s in a frame moving with 159.67 m/s:

Observed from a non-moving frame: $v_{tot} = V + v$

Speed of ping-pong ball seen from outside: v_{tot} **= 191 m/s**

Watch out ! ➡ One is the velocity of the reference frame (V) and the other is the velocity of an object (v), relevant throughout the lecture ..

Given enough power, v_{tot} can reach any speed ...

Looks good, but there are problems if we speed up !

Galilei transformations are incompatible with experiments and observations:

(1) Postulate: Laws of physics must be the same in all inertial systems

(2) Measurement (e.g. [3, 9]): The constant C (Speed of light in free space) is finite and independent of the motion of the source (299792458.0 m/s) and cannot be exceeded by any object

- If (1) is right $\Longrightarrow$ (2) is wrong
- If (2) is right $\Longrightarrow$ (1) is wrong
- If both are right: $\mathbf{v}_{tot} = \mathbf{v} + \mathbf{V}$ is wrong *)

(3) There is no ether, i.e. no absolute reference frame, light is not a (conventional) "wave"

One has to expect some issues with electromagnetism !

*) Otherwise the earth would not move at all !! (see e.g. [1])

Problems with Galilei transformation → Maxwell's equations

Maxwell describes light as waves, wave equation reads:

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)\Psi = 0$$

With Galilei transformation $x = x' - vt,\ y' = y,\ z' = z,\ t' = t$ ***:***

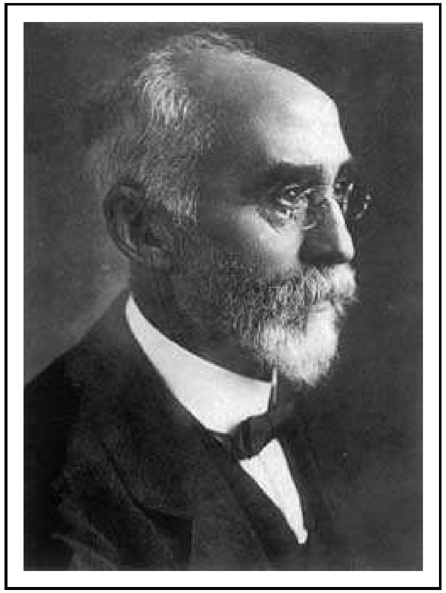
$$\left(\left[1 - \frac{v^2}{c^2}\right]\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x \partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = 0$$

... not quite the same shape !

Reason: Conventional waves move in a medium (ether !) observed from another frame the speed is different ...

Side note: according to Newton (!!), light is made of particles and he predicted that it is bend by gravitational fields ... (like predicted by Special Relativity) !

Mixed derivatives are involved, can only be "solved" if time is transformed as well. By trial and error Lorentz showed such a transformation (a derivation becomes totally trivial later, no waste of time here)



$$x' = \frac{x - Vt}{\sqrt{(1 - \frac{V^2}{c^2})}} = \gamma \cdot (x - Vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{V \cdot x}{c^2}}{\sqrt{(1 - \frac{V^2}{c^2})}} = \gamma \cdot (t - \frac{V \cdot x}{c^2})$$

Transformation for constant velocity V along x-axis
Time is now also transformed

Note: for $V \ll c$ (or $c = \infty$) it reduces to a Galilei transformation !

Transformation of velocities

Frame S' moves with constant velocity of V relative to frame S

Object inside moving frame moves with $\vec{v}' = (v'_x, v'_y, v'_z)$

What is the velocity $\vec{v} = (v_x, v_y, v_z)$ of the object in the frame S ?

$$v_x = \frac{v'_x + V}{1 + \frac{v'_x V}{c^2}} \qquad v_y = \frac{v'_y}{\gamma(1 + \frac{v'_x V}{c^2})} \qquad v_z = \frac{v'_z}{\gamma(1 + \frac{v'_x V}{c^2})}$$

addition of velocities : $v = v_1 + v_2$ → $v = \dfrac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$

$\mathbf{v}_{tot} = \mathbf{v} + \mathbf{V}$ is wrong → **C + v = C**

The speed C can never be exceeded by adding velocities !

Maybe surprising: in Special Relativity the fact that C is the maximum possible speed is the main issue, not light:

in matter particles (e.g. electrons, pions) can go faster than light, but cannot exceed C !

Lorentz transformations, solve one problem but got many others, possible solutions:

1. Galilean relativity applies to classical mechanics, but not to electromagnetic effects and light has a reference frame (some sort of ether, but not in the original sense). Was defended by many people - e.g. Lorentz, sometimes with obscure concepts *) - just for the purpose of explaining away the difficulties.
2. Maxwell's equations are wrong and should be modified to be consistent with Galilei's relativity
3. <u>Enter Einstein:</u> A relativity principle different from Galilei but valid for <u>both</u> classical mechanics and electrodynamics (requires modification of the laws of classical mechanics an a different formulation of electrodynamics) Maxwell is right, but the formulation is wrong !

Sounds very simple, but has mind-bending implications, most irritating:

- With Lorentz transformations: Newton's equation of motions are wrong ! (but see later)
- The key is that C is the maximum speed for any object !
  - The "Speed of Light" C is not about Light !
  - Light moves at the "Speed of Light" because it cannot go faster !

*) see later ...

Postulates of Special Relativity (Einstein)

All physical laws (and results of experiments) in inertial frames must have equivalent forms

Speed of light in free space C must be the same in all frames - <u>and finite</u>

→ No object can move faster than C

Step 1: On the way to a reformulation of "Classical Mechanics"

➞ Apply the postulates to classical objects

The motion of (classical) objects is described by the change of coordinates as a function of time. This is only meaningful using a sensible definition of time

A basic assumption is that time is always measured using some sort of a "clock".

At the place of the clock it is easy to define the time (just look at the clock), but how about linking the sequence/timing of events at different places ? In all cases where time is important, a judgement on the Simultaneity of events is automatically involved. Otherwise: an absolute time exists.

The constancy of C requires more thorough considerations

➞ Observation of Simultaneity of events at different places, each place has its own clock ..

Simultaneity between moving frames

Assume two events in frame S at (different) positions x_1 and x_2 happen simultaneously at times $t_1 = t_2$, what happens in S' ?

- **Classical case: $t'_1 = t'_2$ ➡ requires an absolute time (troubles !)**
- **Relativistic case: no absolute time, the times t'_1 and t'_2 in S' we get from:**

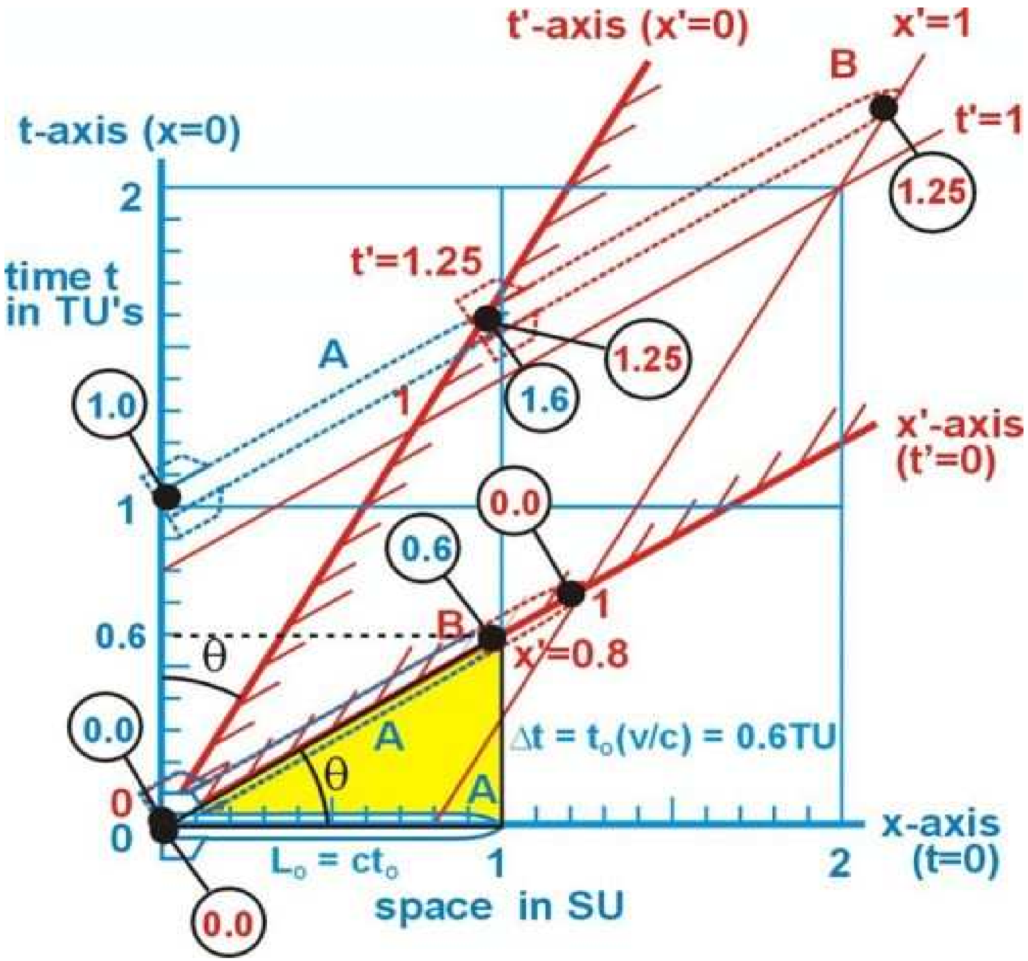
$$t'_1 = \frac{t_1 - \frac{V \cdot x_1}{c^2}}{\sqrt{(1 - \frac{V^2}{c^2})}} \quad \text{and} \quad t'_2 = \frac{t_2 - \frac{V \cdot x_2}{c^2}}{\sqrt{(1 - \frac{V^2}{c^2})}}$$

$x_1 \neq x_2$ **in S implies that $t'_1 \neq t'_2$ in frame S' !!**

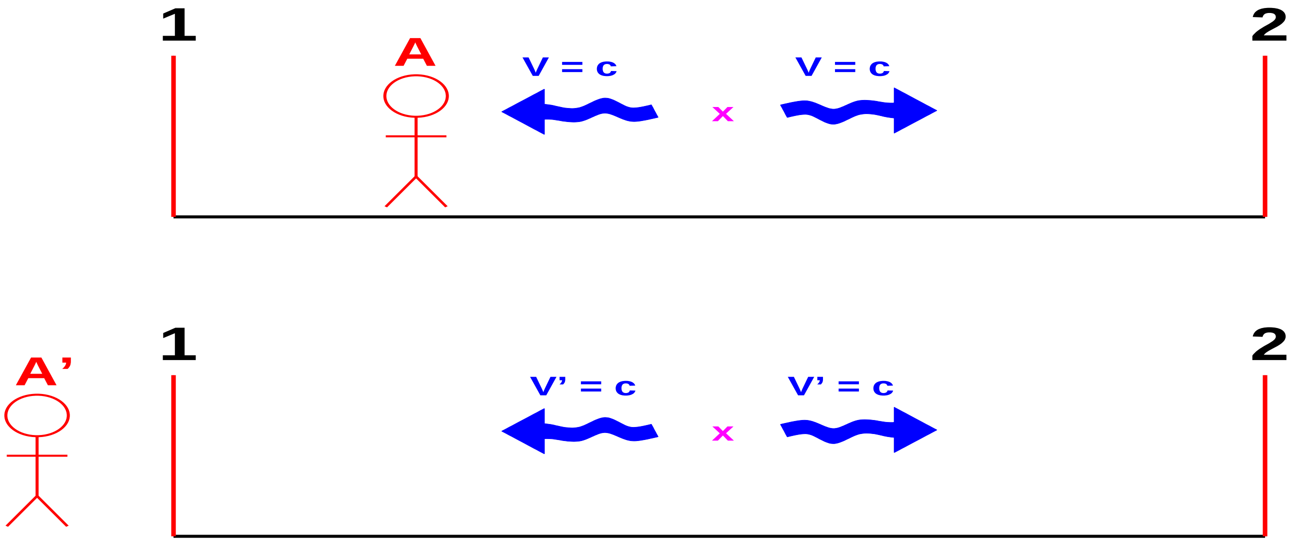
➤ **Two events simultaneous at (different !) positions $x_1 \neq x_2$ in S are not simultaneous in S'**

Rather formal, maybe easier to understand: some sketches and illustrations ➡

Lack of Simultaneity - explanation:



Simultaneity in resting frames



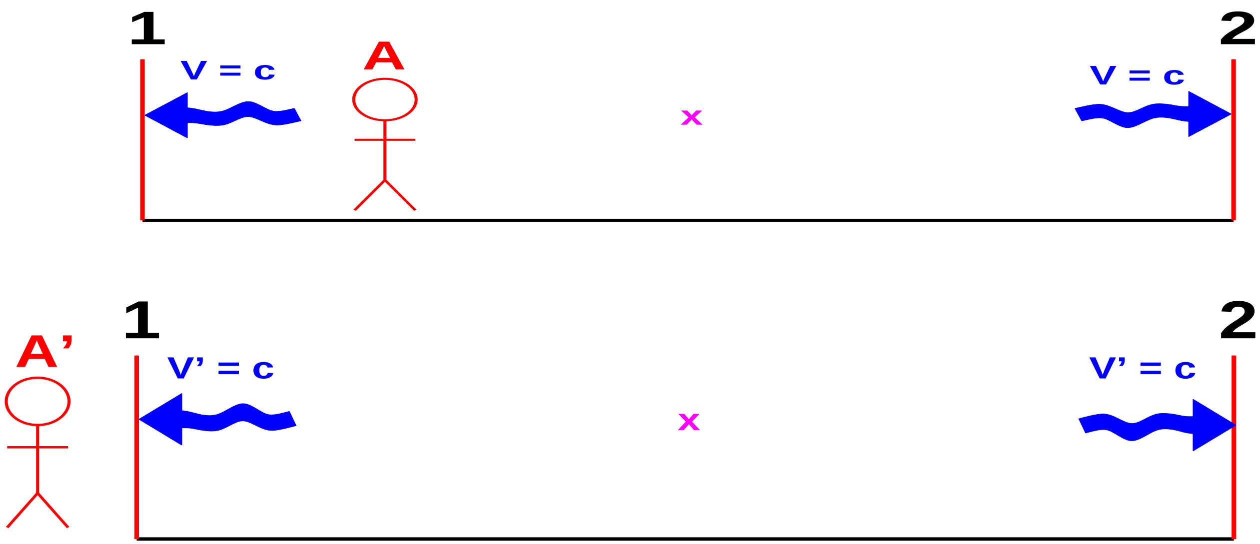
- System with a light source (x) and detectors (1, 2)

  Two flashes of light emitted simultaneously towards the detectors

  Observer (A) inside this frame

  Observer (A') outside

After some time:

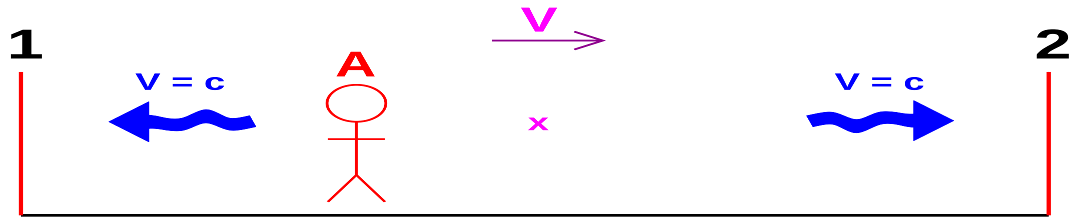


A: both flashes arrive simultaneously at 1 and 2

A': both flashes arrive simultaneously at 1 and 2

What if the frame A is moving relative to observer A' with a velocity V?

Now one frame is moving with speed V:



<u>Classical case:</u> V' = c - V and V' = c + V, flashes <u>arrive simultaneously</u> for A'

<u>Relativistic case:</u> V' = c and V' = c , flashes arrive at <u>different times</u> for A'

A simultaneous event in S is not simultaneous in S': if $x_1 \neq x_2$

Why care about simultaneity ?

- Simultaneity is not frame independent
- Vital for measurement processes and the concept of "time"
- Almost all paradoxes are explained by that (see previous comment) !
- Different observers see a different reality, in particular the sequence of events can change !
  - For $t_1 < t_2$ we may find (not always$^{*)}$!) a frame where $t_1 > t_2$ (concept of before and after depends on the observer)

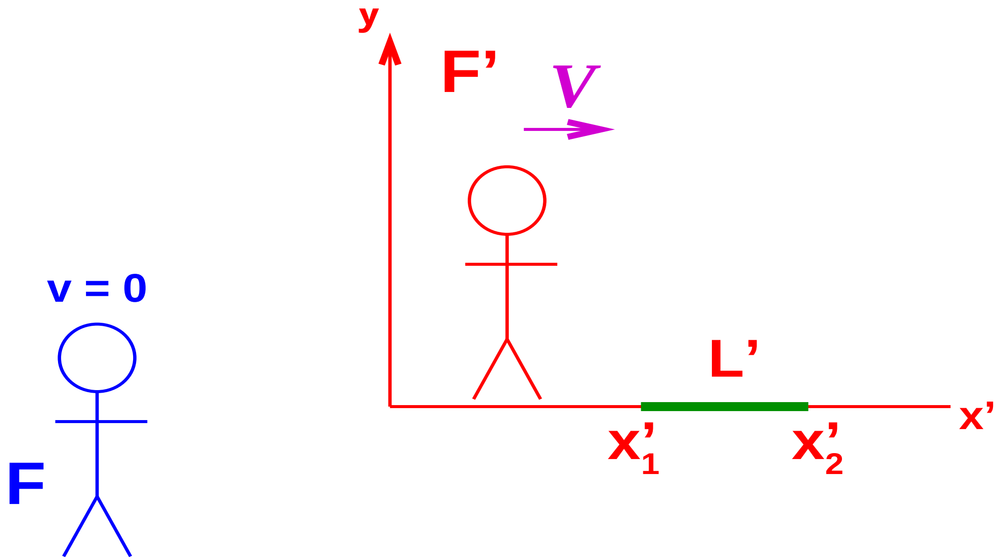
"Hurst made the 1st and the 3rd goal, whichever came first"

(R. Michel, German sport journalist, 1966)

$^{*)}$ A key to anti-matter, (why anti-particles must exist) - if you are interested: ask a lecturer (or read [7] or [10]) ...

But careful: this must not be confused with a loss of causality, it is never violated (an extremely important topic, but not relevant for accelerators, see e.g. [7] for details)

Step 2: How to measure the length of an object ?



Have to measure position of both ends simultaneously*)

Length of a rod in S' is $L' = x'_2 - x'_1$, measured simultaneously at a fixed time t' in frame S',

What is the length L measured from S ??

*) ..sounds already like troubles

We have to measure simultaneously (again !) the ends of the rod at a fixed time t' in frame F', i.e.: $L' = x'_2 - x'_1$

Lorentz transformation of "rod coordinates" into rest frame:

$$x'_1 = \gamma \cdot (x_1 - V t) \quad \text{and} \quad x'_2 = \gamma \cdot (x_2 - V t)$$

$$L' = x'_2 - x'_1 = \gamma \cdot (x_2 - x_1) = \gamma \cdot L$$

$$\rightarrow \quad \boxed{L = L'/\gamma}$$

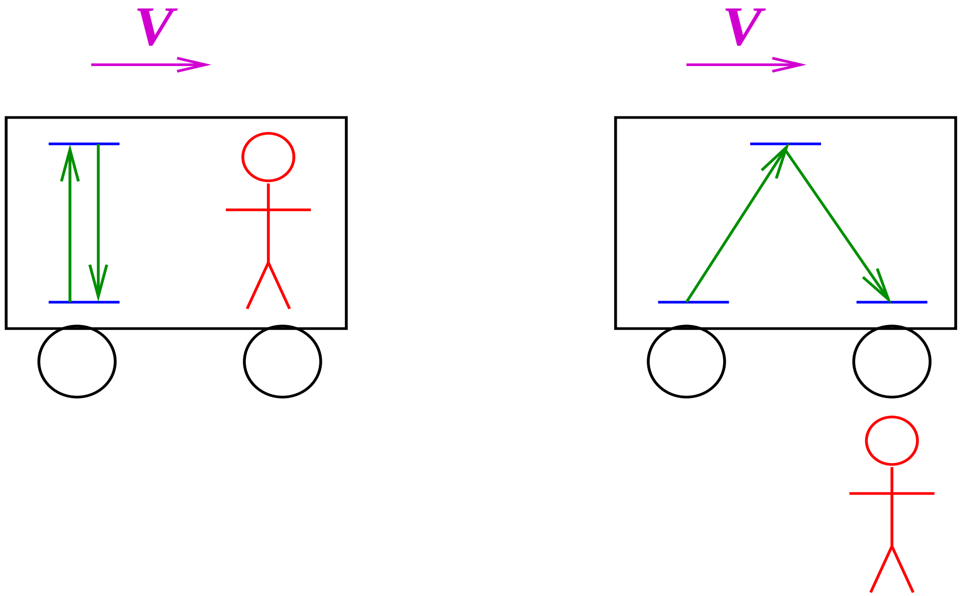
In <u>normal</u> life (low speed) not important, but in accelerators (high speed) : bunch length, electromagnetic fields, magnets, ...

Explanations:

- **Lorentz (defend the ether !) : <u>objects are "deformed"</u> (mechanically) by the ether, both atoms and macroscopic objects**
- **Einstein (dismissed the ether !): <u>space itself is deformed</u>**

Step 3: Time dilation - schematic

Reflection of light (it does not have to be light !) between 2 mirrors seen inside moving frame and from outside

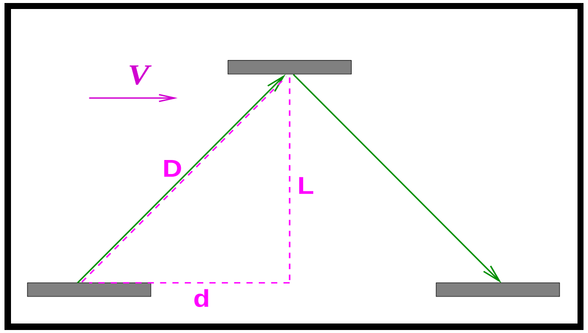


Frame moving with velocity V

Seen from outside the path is longer, but C must be the same ..

➡ **Something wrong with** $v = \frac{\Delta L}{\Delta t}$

No Lorentz transformation needed in this case, a simple geometry will do. Just the postulate than C is independent of motion of a source



In frame S′ : light travels L in time $\Delta t'$

In frame S : light travels D in time Δt

$L = c \cdot \Delta t' \quad D = c \cdot \Delta t \quad d = V \cdot \Delta t$

$(c \cdot \Delta t)^2 = (c \cdot \Delta t')^2 + (V \cdot \Delta t)^2$

$$\boxed{\Delta t = \gamma \cdot \Delta t'}$$

"If an event takes a certain amount of time as measured by an observer at rest with respect to this event, the time for that event to occur is longer for an observer moving with respect to this event" (Einstein)

Is time dilation a headache ?

You can interprete this two ways:

1. The car is moving: $\Delta t = \gamma \cdot \Delta t'$
2. The observer is moving: $\Delta t' = \gamma \cdot \Delta t$

Seems like a contradiction (but no frame can be privileged)...

No, fixed by the concept of proper time τ:

The time measured by the observer at rest relative to the process

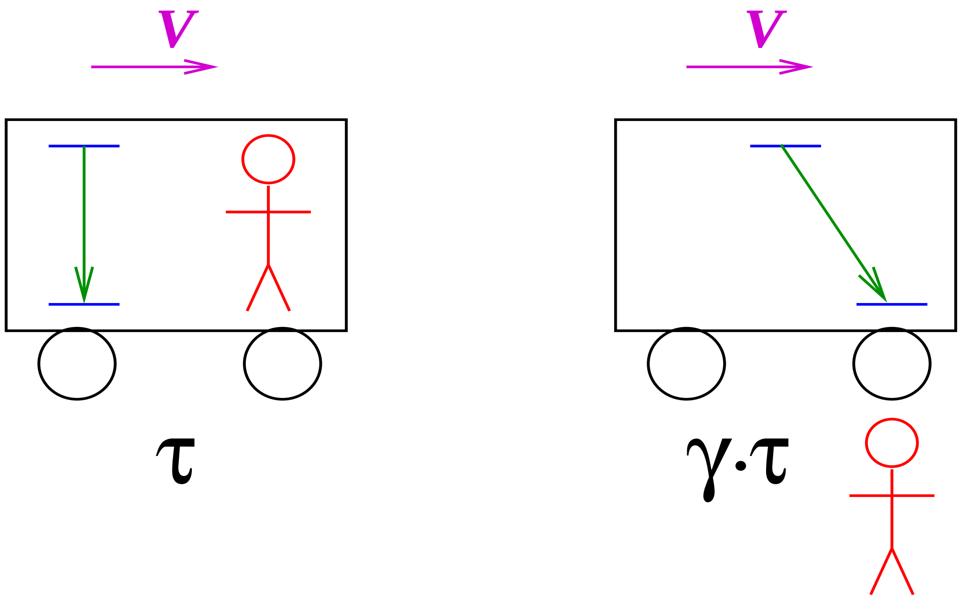
Or: The proper time for a given observer is measured by the clock that travels with the observer (and always the same):

$$c^2 \Delta\tau^2 \quad = \quad c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \quad = \quad c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2$$

Ditto for Lorentz contraction ...

An interesting consequence: $c^2\Delta\tau^2$ can be positive or negative with strong implications (e.g. for the concept of before and after, a detailed discussion in [10])

Falling object in a moving car:



Observer within the moving car measures the proper time τ, no matter how fast the car is moving

Observer outside measures the time $\gamma \cdot \tau$

Proper Length and Proper Time

Time and distances are relative :

- τ is a fundamental time: proper time τ
- The time measured by an observer in its own frame
- Your lifetime in your own frame
- From frames moving relative to it, time appears longer

- $\mathcal{L}$ is a fundamental length: proper length $\mathcal{L}$
- The length measured by an observer in its own frame
- Your proportions in your own frame
- From frames moving relative to it, it appears shorter

Sometimes called "things-as-they-are" (in contrast to "things-as-they-appear")

Standard Example: muon μ decay ..

Lifetime of the muon:

In lab frame: $\gamma \cdot \ \tau$ **In frame of muon:** $\tau \ \approx \ 2 \cdot 10^{-6}$ **s**

A clock in the muon frame shows the proper time and the muon decays in $\approx \ 2 \cdot 10^{-6}$ s, independent of the muon's speed.

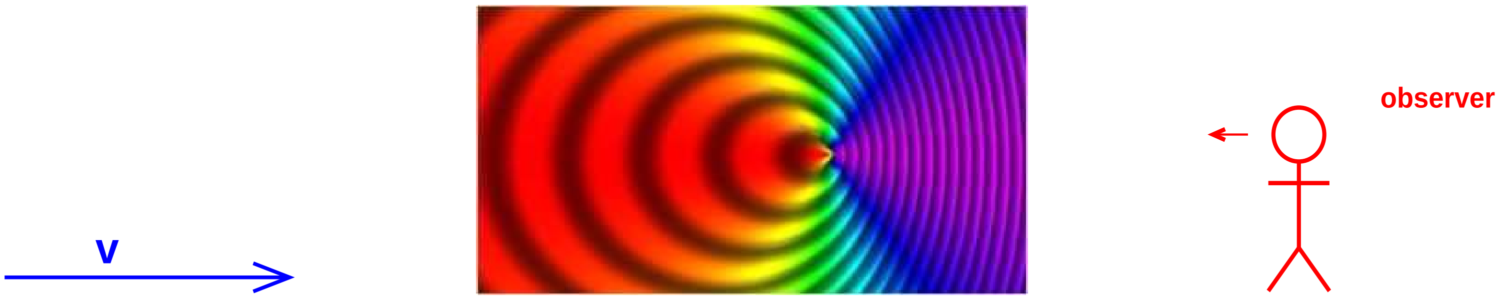
Seen from the lab frame the muon lives γ times longer, many seconds ...

Time is relative not absolute !

1 minute can be perceived very differently, depending on who you are (muon) or where you are !

(... for example on which side of the bathroom door you are on)
(based on saying by Einstein)

Example: moving light source with speed $\mathbf{v} \approx \mathbf{c}$



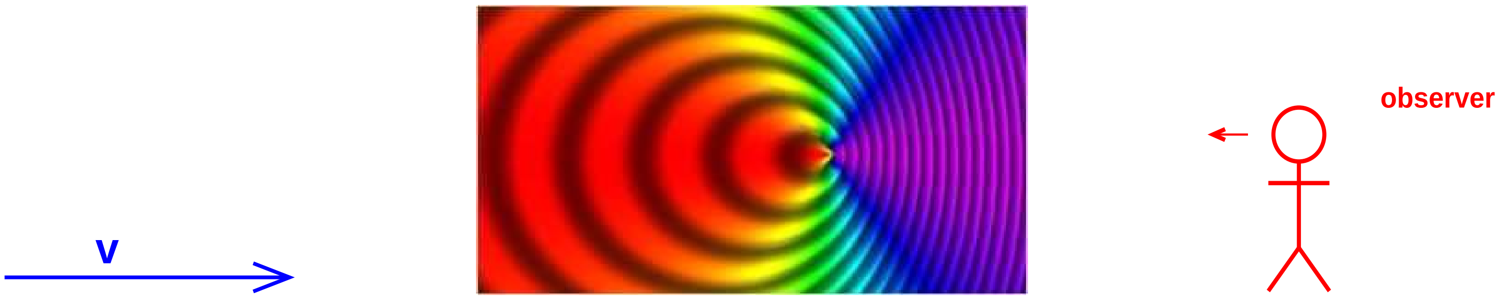
Relativistic Doppler effect (important for FEL):

Unlike sound: no medium of propagation

Observed frequency depends on observation angle θ

→ **frequency is changed:** $\nu = \nu_0 \cdot \gamma \cdot (1 - \beta_r \cos(\theta))$

Example: moving light source with speed **v ≈ c**



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Travelling at $v \approx c$ through space can damage your health !

Another every day example (GPS satellite):

- 20000 km above ground, (unlike popular believe: not on geostationary orbits, this would not work)
- Orbital speed 14000 km/h (i.e. relative to observer on earth)
- On-board clock accuracy 1 ns
- Relative longitudinal precision of satellite orbit $\leq 10^{-8}$
- At GPS receiver, for 5 m need clock accuracy $\approx$ 10 ns

Exercise 1: Do we have to correct for (this) relativistic effect ?

To make it clear:

Key to understand relativity

- **Lorentz contraction:**
  - It is not the matter that is compressed (was believed before Einstein, e.g. Lorentz)
  - It is the space that is modified
- **Time dilation:**
  - It is not the clock that is changed
  - It is the time that is modified

What about the mass m and momentum $\vec{p}$?

Assume an object inside moving frame S' moves with $\vec{v}' = (0, v'_y, 0)$
Transverse momentum conservation in both frames requires:

$$p_y = p'_y \;\rightarrow\; m\ v_y = m'\ v'_y \;\rightarrow\; m\ v'_y/\gamma = m'\ v'_y \;\rightarrow\; \boxed{m = \gamma m'}$$

For momentum conservation: <u>mass must also be transformed</u> !

Using : $m' = m_0 \longrightarrow m = \gamma \cdot m_0$

for small velocities : $m \cong m_0 + \frac{1}{2} m_0 v^2 \left(\frac{1}{c^2}\right) + \ldots.$

and multiplied by c^2 : $mc^2 \cong m_0 c^2 + \frac{1}{2} m_0 v^2 = m_0 c^2 + T$

Interpretation:

- **Total energy E is $E = mc^2$ for <u>any</u> object**
- m **is the mass (energy) of any object "in motion"**
  → **Any form of Energy possesses inertia, including light**
- m_0 **is the mass (energy) of the object "at rest"**
- **The mass m is not the same in all inertial systems, the rest mass m_0 is !**

Practical and impractical units

Standard (SI) units are not very convenient, easier to use:

[E] = eV [p] = eV/c [m] = eV/c^2 ("energy equivalent units")

then: $E^2 = m_0^2 + p^2$

If rest mass of objects can be totally converted into energy:

Mass of a proton: $m_p = 1.672 \cdot 10^{-27}$ **Kg**
Energy(at rest): $m_p c^2 = 938$ **MeV** $= 0.15$ **nJ**

Ping-pong ball: $m_{pp} = 2.7 \cdot 10^{-3}$ **Kg** **(** $\approx 1.6\ 10^{24}$ **protons)**
Energy(at rest): $m_{pp} c^2 = 1.5 \cdot 10^{27}$ **MeV** $= 2.4 \cdot 10^{14}$ **J**

$\approx$ **750000 times the full LHC beam**

$\approx$ **60 kilotons of TNT**

(12 kilotons of TNT correspond to about 0.5 g)

What about Newton's equations ?

The (most) practical solution:
Replace m' in all classical formulae by $m = \gamma m'$ and everything is saved ...

What about masses in accelerators ?

The mass of a fast moving particle is increasing like (in the frame of the control room):

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

When we accelerate:

- **For v $\ll$ c: E, m, p, v increase ...**
- **For v $\approx$ c: E, m, p increase, but v does (almost) not !**

$$\beta = \frac{v}{c} \approx \sqrt{1 - \frac{m_0^2 c^4}{T^2}}$$

→ **Concept of transition (synchrotrons)**

Kinematic relations - very useful for daily work

	cp	**T**	**E**	γ
$\beta =$	$\frac{1}{\sqrt{(\frac{E_0}{cp})^2+1}}$	$\sqrt{1-\frac{1}{(1+\frac{T}{E_0})^2}}$	$\sqrt{1-(\frac{E_0}{E})^2}$	$\sqrt{1-\gamma^{-2}}$
cp =	cp	$\sqrt{T(2E_0+T)}$	$\sqrt{E^2-E_0^2}$	$E_0\sqrt{\gamma^2-1}$
$E_0 =$	$\frac{cp}{\sqrt{\gamma^2-1}}$	$T/(\gamma-1)$	$\sqrt{E^2-c^2p^2}$	E/γ
T =	$cp\sqrt{\frac{\gamma-1}{\gamma+1}}$	**T**	$E-E_0$	$E_0(\gamma-1)$
$\gamma =$	$cp/E_0\beta$	$1+T/E_0$	E/E_0	γ

Kinematic relations - logarithmic derivatives (even more useful)

	$\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{dE}{E} = \frac{d\gamma}{\gamma}$
$\frac{d\beta}{\beta} =$	$\frac{d\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{dp}{p}$	$\frac{1}{\gamma(\gamma+1)}\frac{dT}{T}$	$\frac{1}{(\beta\gamma)^2}\frac{d\gamma}{\gamma}$
$\frac{dp}{p} =$	$\gamma^2\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$[\gamma/(\gamma+1)]\frac{dT}{T}$	$\frac{1}{\beta^2}\frac{d\gamma}{\gamma}$
$\frac{dT}{T} =$	$\gamma(\gamma+1)\frac{d\beta}{\beta}$	$(1+\frac{1}{\gamma})\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{\gamma}{(\gamma-1)}\frac{d\gamma}{\gamma}$
$\frac{dE}{E} =$	$(\beta\gamma)^2\frac{d\beta}{\beta}$	$\beta^2\frac{dp}{p}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$
$\frac{d\gamma}{\gamma} =$	$(\gamma^2-1)\frac{d\beta}{\beta}$	$\frac{dp}{p} - \frac{d\beta}{\beta}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$

Example LHC (7 TeV): $\frac{\Delta p}{p} \approx 10^{-4}$ **implies:** $\frac{\Delta v}{v} = \frac{\Delta\beta}{\beta} \approx 2 \cdot 10^{-12}$

Exercise 2: What is the speed of a particle when its kinetic energy is the same as its rest mass ?

Is $E = mc^2$ a general concept ? Consider a macroscopic object:

Spinning a boiled egg :

Total Energy of object <u>increases</u>

More Energy – Larger Mass ?

According to Einstein: yes

- **Try it (very easy, e.g. use 10 Hz)**
- **Compute it (easy, e.g. use: R = 2 cm, h = 6 cm, m = 50 g)**
- **Measure it (not so easy)**

Relativity was controversial at the beginning - a lot of argumentation ..

1 It is not based on a "real" theoretical concept

→ Only experimental evidence and "Gedanken experiments"

2 Inconceivable effects (easy to compute - difficult to believe):

→ Simultaneity, Time dilation, Length contraction ...

→ General application of $E = m c^2$ (e.g. eggs, flashlights)

→ ...

3 Incompatible with Quantum mechanics

4 Active reasearch/controversy: does a battery change its mass when charged ?

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➡ ...

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First summary

➡ **Physics laws the same in all inertial frames ...**

➡ **Speed of light in free space C is the same in all frames and requires Lorentz transformation**

➡ **No object or signal (having physical consequences) can move faster than C**

➡ **The most important formula: V + C = C**

- **Moving objects appear shorter for an observer in another frame**
- **Moving clocks appear to go slower for an observer in another frame**
- **Mass is not independent of motion ($m = \gamma \cdot m_0$) and total energy is $E = m \cdot c^2$ (second most important formula)**
- **No absolute space or time: where it happens and when it happens is not independent ➡ Space-Time**

➡ **Next: how to calculate something and applications ...**

Introducing four-vectors

Since space and time are not independent, must reformulate physics taking both into account:

$t, \quad \vec{a} = (x,\ y,\ z)$ ➡ Replace by one vector including the time

We need two types of four-vectors[a] (here position four-vector):

$$X^{\mu} = (ct,\ x,\ y,\ z) \quad \text{and} \quad X_{\mu} = (ct,\ -x,\ -y,\ -z)$$

We have a temporal and a spatial part
(time t multiplied by c to get the same units, other conventions exist)

A comment: Sometimes $X^{\mu} = (ict,\ x,\ y,\ z)$ is used, but the i is merely a (completely unnecessary) mathematical trick to fake the "appearance" of the scalar product. It has no physical meaning whatsoever (and can mess up and hide important physics concepts, e.g. in EM theory and in Particle and Accelerator Physics) ...

[a] Due to "skewed" reference system, for details ask one of the lecturers ..

Life becomes really simple ➡

Lorentz transformation can be written in matrix form:

$$X'^{\mu} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Transformation Matrix } \Lambda} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = X^{\mu}$$

$$X'^{\mu} = \Lambda \circ X^{\mu} \qquad (\Lambda \quad \text{for "Lorentz"})$$

Here for motion in x-direction, but can always rotate into direction of motion

but note:

$$X'_\mu = \begin{pmatrix} ct' \\ -x' \\ -y' \\ -z' \end{pmatrix} = \begin{pmatrix} \gamma & +\gamma\beta & 0 & 0 \\ +\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ -x \\ -y \\ -z \end{pmatrix} = X_\mu$$

This matrix is the inverse of the previous matrix

F.A.Q: Why bother about this $^\mu$ or $_\mu$ stuff ??
Is it useful or just abracadabra ??

Necessary for a formulation of electrodynamics consistent with Special Relativity, and makes the life very easy to calculate various effects applied to accelerators

For many calculations, just blindly follow a few simple rules ➡

Most important concept: Scalar products

Cartesian Scalar Product (Euclidean metric in 3D):

$$\vec{x} \cdot \vec{y} \;=\; (x_a, y_a, z_a) \;\cdot\; (x_b, y_b, z_b) = (x_a \cdot x_b \;+ y_a \cdot y_b \;+\; z_a \cdot z_b)$$

Space-time four-vectors like:

$$A^\mu \;=\; (ct_a, x_a, y_a, z_a) \qquad B_\mu \;=\; (ct_b, -x_b, -y_b, -z_b)$$

Four-vector Scalar Product (for more rigorous treatment see [7]):

$$A^\mu B_\mu = \underbrace{\sum_{\mu=0}^{3} A^\mu B_\mu}_{\text{Einstein convention}} = (ct_a \cdot ct_b \;-\; x_a \cdot x_b \;-\; y_a \cdot y_b \;-\; z_a \cdot z_b)$$

For many applications you can use this simplified rule:

$$AB = (ct_a \cdot ct_b \;-\; x_a \cdot x_b \;-\; y_a \cdot y_b \;-\; z_a \cdot z_b)$$

Why bother about four-vectors ?

- **We want the same laws of all physics in different frames**
- **The solution: write the laws of physics in terms of four vectors and use Lorentz transformation**
- **Without proof*): any four-vector (scalar) product $Z^{\mu}Z_{\mu}$ has the same value in all inertial frames:**

$$Z^{\mu}Z_{\mu} \quad = \quad Z'^{\mu}Z'_{\mu}$$ (whatever Z is, do not have to be the same type...)

All scalar products of any four-vectors are invariant !

The value and shape are invariant !

but : $Z^{\mu}Z^{\mu}$ and $Z'_{\mu}Z'_{\mu}$ are obviously not !!*)

*) **The proofs are extremely simple !**

The most important four-vectors:

Coordinates : $X^\mu = (ct, x, y, z) = (ct, \vec{x})$

Velocities : $U^\mu = \dfrac{dX^\mu}{d\tau} \; = \; \gamma(c, \dot{\vec{x}}) = \gamma(c, \vec{u})$

Momenta : $P^\mu = mU^\mu = m\gamma(c, \vec{u}) = \gamma(mc, \vec{p})$

Force : $F^\mu = \dfrac{dP^\mu}{d\tau} = \gamma \dfrac{d}{d\tau}(mc, \vec{p})$

Wave propagation vector : $K^\mu = (\dfrac{\omega}{c}, \vec{k})$

Also the Gradient : $\partial^\mu = (\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$

ALL four-vectors A^μ transform like:

$$A'^\mu = \Lambda \circ A^\mu \qquad \text{(and} \quad A'_\mu = \Lambda^{-1} \circ A_\mu)$$

A special invariant

From the velocity four-vectors:

$$U^{\mu} \; = \; \gamma(C, \vec{u}) \qquad \text{and} \qquad U_{\mu} \; = \; \gamma(C, -\vec{u})$$

we get the scalar product:

$$U^{\mu}U_{\mu} = \gamma^{2}(C^{2} - \vec{u}^{2}) = C^{2} \; !!$$

➡ C **is an invariant, has the same value in all inertial frames**

$$U^{\mu}U_{\mu} = U'^{\mu}U'_{\mu} = C^{2}$$

➡ **The invariant of the velocity four-vector U is C and it is the same in ALL frames, i.e. independent of relative motion (good news !)**

Note: the naive choice $U_{\mu} \; = \; (C, -\vec{u})$ is <u>not</u> a four-vector, $U_{\mu} \; = \; \gamma(C, -\vec{u})$ is

Another important invariant

Momentum four-vector P of a particle with mass m and energy E:

$$P^\mu = m_0 U^\mu = m_0 \gamma(c, \vec{u}) = (mc, \vec{p}) \;=\; (\frac{E}{c}, \vec{p})$$

$$P'^\mu = m_0 U'^\mu = m_0 \gamma(c, \vec{u'}) = (mc, \vec{p'}) \;=\; (\frac{E'}{c}, \vec{p'})$$

We can get another invariant:

$$P^\mu P_\mu \;=\; P'^\mu P'_\mu \;=\; m_0^2 c^2$$

Invariant of the four-momentum vector is the mass m_0

➡ **The rest mass m_0 is the same in all frames !**
(otherwise we could tell whether we are moving or not !!)

Exercise 3a: **what is the four-momentum of a photon (a.k.a. $\gamma^{*)}$) ?**

Exercise 3b: **The rest mass of a photon is 0, what is (real) mass of a photon ?**

Bonus: **how many photons are needed for half a pound of green light ?**

$^{*)}$ **(an extremly unfortunate coincidence ...)**

- **Use of four-vectors simplify calculations significantly, follow the rules and look for invariants, in particular kinematic relationships, e.g.**

- **The momentum four-vector of a system of particles (collision or decay) is the sum of the individual four-vectors:** $P^{\mu} = P_1^{\mu} + P_2^{\mu}$

  - **Particle decay: important for secondary beams ! (find mass of parent particle or decay products, details in [8])**

  - **Particle collisions: not only colliders**

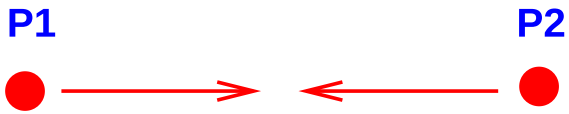
Exercise 4: Consider two mechanisms for proton-antiproton pair production, colliding with a proton at rest p

1. $p + p \longrightarrow p + p + (p + \bar{p})$
2. $\gamma + p \longrightarrow p + (p + \bar{p})$

Using four-momenta, compute the minimum energies and speed of the incoming p **and** γ **to produce the proton-antiproton pair**

Handy formulae - What is the available centre of mass energy E_{cm} ?

Collider



$$P_1^\mu = (E, \vec{p}) \qquad P_2^\mu = (E, -\vec{p})$$

$$P^\mu = P_1^\mu + P_2^\mu = (2E, 0)$$

$$E_{cm} = \sqrt{P^\mu P_\mu} = 2 \cdot E$$

Stationary Target

$$P_1^\mu = (E, \vec{p}) \qquad P_2^\mu = (m_0, 0)$$

$$P^\mu = P_1^\mu + P_2^\mu = (E + m_0, \vec{p})$$

$$E_{cm} = \sqrt{P^\mu P_\mu} = \sqrt{2m_0 E \ + \ 2m_0^2}$$

Works for as many beams/particles as you like :

$P^\mu = P_1^\mu + P_2^\mu + P_3^\mu + \ldots$ **(difficult in practice ..)**

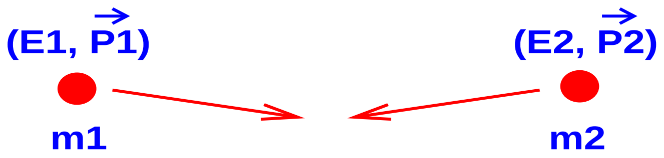
Examples:

collision	E beam energy	E_{cm} (collider)	E_{cm} (fixed target)
pp	315 (GeV)	630 (GeV)	24.3 (GeV)
pp	6500 (GeV)	13000 (GeV)	110.4 (GeV)
pp	90 (PeV)$^{*)}$	180 (PeV)	13000 (GeV)
e+e−	100 (GeV)	200 (GeV)	0.320 (GeV)

$^{*)}$ for B$_{NC}$ $\approx$ 3 T → $C \approx$ 480 000 km (Jupiter $\approx$ 450 000 km)
(although cosmic ray particles can have MUCH higher energies, more than 10^{20} eV, $\gamma \approx 10^{11}$ ..)

"Looks" more complicated when the beams cross at an angle Θ

Collider with crossing angle



Yet the calculation becomes equally trivial[a] using the four-momenta:

$P_1^\mu = (E_1, \vec{p_1})$ and $P_2^\mu = (E_2, \vec{p_2})$ **(m_1 may or may not be m_2)**

$$P^\mu = P_1^\mu + P_2^\mu = (E_1 + E_2, \; \vec{p_1} + \vec{p_1})$$

$$P^\mu P_\mu = E_1^2 + E_1^2 + 2E_1E_2 - \vec{p_1}^{\,2} - \vec{p_2}^{\,2} - 2\vec{p_1}\vec{p_2}$$

$$\Longrightarrow \quad E_{cm} = \sqrt{m_1^2 + m_2^2 + 2E_1E_2 - 2\vec{p_1}\vec{p_2}}$$

$$\boxed{\Longrightarrow \quad E_{cm} = \sqrt{m_1^2 + m_2^2 + 2E_1E_2(1 - \beta_1\beta_2 \cos\Theta)}}$$

---

[a] Trivial: $\leq$ 5 lines .. (try it without four-vectors, wish you all the best)

Relativity and electrodynamics

- Back to the original problems: Electrodynamics and Maxwell equations
- Why not try again four-vectors

Write potentials and currents as four-vectors:

$$\Phi,\ \vec{A} \quad \Rightarrow \quad A^{\mu} \ = \ (\frac{\Phi}{c},\ \vec{A})$$

$$\rho,\ \vec{j} \quad \Rightarrow \quad J^{\mu} \ = \ (\rho \cdot c,\ \vec{j})$$

What about the transformation of current and potentials ?

Also note: many textbooks on Special Relativity start from here, all other effects (time dilation, length contraction, proper time, etc. ...) follow almost automatically.

Transform the four-current like:

$$\begin{pmatrix} \rho' c \\ j'_x \\ j'_y \\ j'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho c \\ j_x \\ j_y \\ j_z \end{pmatrix}$$

Surprise, it transforms via: $J'^{\mu} = \Lambda \ J^{\mu}$ **(always the same Λ)**

Ditto for: $A'^{\mu} = \Lambda \ A^{\mu}$ **(always the same Λ)**

Another invariant: $\partial_{\mu} J^{\mu}$ $^{*)}$ $= \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\vec{j} = 0$ **(charge conservation)**

$^{*)}$ **remember: <u>any</u> product of four-vectors is invariant**

Electromagnetic fields using these potentials: $A^\mu = (\frac{\Phi}{c}, \vec{A})$

conventional : $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$

Written as e.g. x-components:

$$E_x = -\frac{\partial A_0}{\partial x} - \frac{\partial A_1}{\partial t} = -\frac{\partial A_t}{\partial x} - \frac{\partial A_x}{\partial t}$$

$$B_x = +\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = +\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

→ **using all components: fields are described by a "field-matrix"** $F^{\mu\nu}$**:**

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & \frac{-E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

Electromagnetic fields described by field-matrix $F^{\mu\nu}$:

$$F'^{\nu\mu} = \begin{pmatrix} 0 & \frac{-E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \frac{+E_x}{c} & \frac{+E_y}{c} & \frac{+E_z}{c} \\ \frac{-E_x}{c} & 0 & -B_z & B_y \\ \frac{-E_y}{c} & B_z & 0 & -B_x \\ \frac{-E_z}{c} & -B_y & B_x & 0 \end{pmatrix} = F^{\nu\mu}$$

It transforms via: $F'^{\mu\nu} = \Lambda \; F^{\mu\nu} \; \Lambda^T$ **(same Λ as before)**

Interesting: electric fields change sign, magnetic fields <u>do not</u>, why ? (the reason is a bit tricky and maybe surprising, ask a lecturer ...) !!!

Transformation of fields into a moving frame (x-direction):

Using $F'^{\mu\nu} \;=\; \Lambda\; F^{\mu\nu}\; \Lambda^{T}$ **it becomes almost trivial**
(you will hardly find the "classical" derivation (not using four-vectors) in lectures ... !)

One obtains the handy formulae:

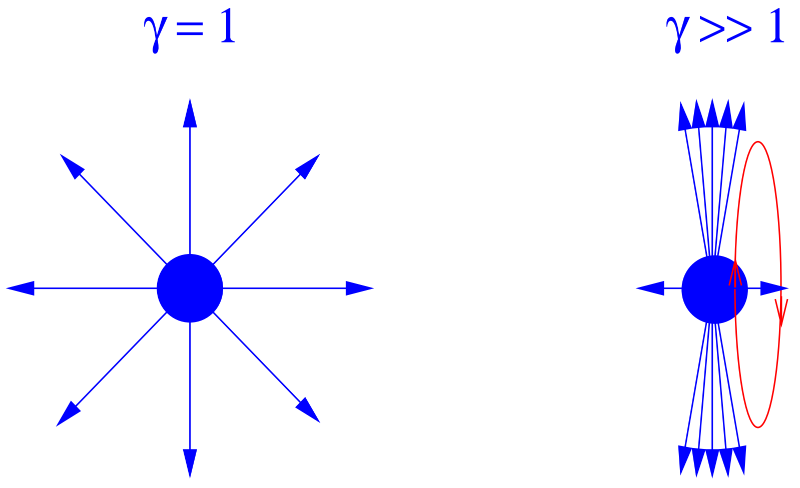
$$
\begin{aligned}
E'_x &= E_x & B'_x &= B_x \\
E'_y &= \gamma(E_y - v \cdot B_z) & B'_y &= \gamma(B_y + \frac{v}{c^2} \cdot E_z) \\
E'_z &= \gamma(E_z + v \cdot B_y) & B'_z &= \gamma(B_z - \frac{v}{c^2} \cdot E_y)
\end{aligned}
$$

Fields perpendicular to movement are transformed

Strong dependence on γ

How do the field components look like as a function of γ **?**

Coulomb field of a charge moving at constant velocity



$$\vec{E}_{\parallel} = \frac{q}{4\pi\epsilon_0} \frac{1}{\gamma^2 r^2} \frac{\vec{r}}{r}$$

$$\vec{E}_{\perp} = \frac{q}{4\pi\epsilon_0} \frac{\gamma}{r^2} \frac{\vec{r}}{r}$$

r radial distance

$\vec{r}$ radial direction

- **For large γ the longitudinal field component disappears**
- **Only a transverse (radial) component is left**
- **All important for collective effects (e.g. space charge, beam-beam, wake fields, etc.)**

(here only the results, a nice derivation in [8])

What about forces ??

Start with the (four-)force as the time derivative of the four-momentum:

$$\mathcal{F}_L^{\mu} \quad = \quad \frac{\partial P^{\mu}}{\partial \tau}$$

Without any effort one gets the four-vector for the Lorentz force, with the well known expression in the second part:

$$\mathcal{F}_L^{\mu} \;=\; \gamma q(\frac{\vec{E} \cdot \vec{u}}{c},\; \vec{E} \;+\; \vec{u} \times \vec{B}) \quad = \quad q \cdot F^{\mu\nu} U_{\nu}$$

Quote Einstein (1905):

"For a charge moving in an electromagnetic field, the force experienced by the charge is equal to the electric force, transformed into the rest frame of the charge"

There is no mystic, velocity dependent coupling between a charge and a magnetic field ! (because it does not exist)

It is just a consequence of an electric field in two reference frames

An important consequence - remember:

$$
\begin{aligned}
E'_x &= E_x & B'_x &= B_x \\
E'_y &= \gamma(E_y - v \cdot B_z) & B'_y &= \gamma(B_y + \tfrac{v}{c^2} \cdot E_z) \\
E'_z &= \gamma(E_z + v \cdot B_y) & B'_z &= \gamma(B_z - \tfrac{v}{c^2} \cdot E_y)
\end{aligned}
$$

Assuming that $\vec{B}' = 0$, we get for the <u>transverse</u> forces:

$$\vec{F}_{mag} = -\beta^2 \cdot \vec{F}_{el}$$

For $\beta = 1$, Electric and Magnetic <u>forces</u> cancel, plenty of consequences, e.g. Space Charge, beam-beam, ...

Most important for stability of beams (so watch out for $\beta \ll 1$) !

Hamiltonians and all that ..

A particularly useful and powerful method to decribe the motion of particle in accelerator elements is the Hamiltonian formalism [7]. No details on the formalism, just writing down the Hamiltonian function as it comes out of our treatment.

What is needed are the potentials and $E^2 \;=\; p^2c^2 \;\;+\;\; m^2c^4$

(Sorry for those of you who have been cheated into believing that this is "too complicated for students", the opposite is true)

Without proof, the Hamiltonian is: $H = T + V$ **= kinetic energy + potential energy**

Hamiltonian for a (ultra relativistic, i.e. $\gamma \gg 1,\ \beta \approx\ 1$**) particle in an electro-magnetic field is given by (any textbook on Electrodynamics, in particular [5]):**

$$H(\vec{x},\vec{p},t) = c\sqrt{(\vec{p} - e\vec{A}(\vec{x},t))^2 + m_0^2c^2} + e\Phi(\vec{x},t) \qquad \text{(ugly so far ...)}$$

where $\vec{A}(\vec{x},t)$**,** $\Phi(\vec{x},t)$ **are the vector and scalar potentials (i.e. the** V**)**

Maybe not yet completely obvious, this allows a straightforward formalism to derive the motion of particle in any accelerator magnet [10] and beyond (e.g. fields due to other sorces, for example beam-beam, space charge forces etc.)

Next step*) : Maxwell's equations using four-vectors and $F^{\mu\nu}$:

$$\nabla \vec{E} = \frac{\rho}{\epsilon_0} \quad \textbf{and} \quad \nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \quad \xrightarrow{1+3}$$

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \quad \text{(Inhomogeneous Maxwell equation)}$$

$$\nabla \vec{B} = 0 \quad \textbf{and} \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \xrightarrow{1+3}$$

$$\partial_\gamma F^{\mu\nu} + \partial_\mu F^{\nu\lambda} + \partial_\nu F^{\lambda\mu} = 0 \quad \text{(Homogeneous Maxwell equation)}$$

We have Maxwell's equation in a very compact form, transformation between moving systems very easy:

just transform gradient four-vector ∂_ν and field four-vector $F^{\nu\lambda}$

*) **Next 4 slides are more abstract: they show the final and most important result of the theory but less suited for immediate applications, will whiz through them more quickly**

How to use all that stuff ???   Look at first equation:

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

Written explicitly (Einstein convention, sum over μ):

$$\partial_\mu F^{\mu\nu} = \sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} = \partial_0 F^{0\nu} + \partial_1 F^{1\nu} + \partial_2 F^{2\nu} + \partial_3 F^{3\nu} = \mu_0 J^\nu$$

Choose e.g. $\nu = 0$ and replace $F^{\mu\nu}$ by corresponding matrix elements:

$$\partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \mu_0 J^0 \quad \Rightarrow$$

$$0 + \partial_x \frac{E_x}{c} + \partial_y \frac{E_y}{c} + \partial_z \frac{E_z}{c} = \mu_0 J^0 = \mu_0 \, c\rho$$

This corresponds exactly to (Gauss' law):

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad (c^2 = \epsilon_0 \mu_0)$$

For $\nu = 1, 2, 3$ one gets Ampere's law

For example in the x-plane ($\nu = 1$) and the S frame:

$$\partial_y B_z \; - \; \partial_z B_y \; - \; \partial_t \frac{E_x}{c} \; = \; \mu_0 J^x$$

after transforming ∂^γ and $F^{\mu\nu}$ to the S' frame:

$$\partial'_y B'_z \; - \; \partial'_z B'_y \; - \; \partial'_t \frac{E'_x}{c} \; = \; \mu_0 J'^x$$

Now Maxwell's equation have the identical form in S and S'

(In matter: can be re-written with $\vec{D}$ and $\vec{H}$ using "magnetization tensor")

Finally: since $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

$$\partial_\gamma F^{\mu\nu} + \partial_\mu F^{\nu\lambda} + \partial_\nu F^{\lambda\mu} = 0$$

We can re-write them two-in-one in a new form:

$$\boxed{\frac{\partial^2 A^\mu}{\partial x_\nu \partial x^\nu} = \mu_0 J^\mu}$$

This contains all four Maxwell's equations, it is not only the beauty of the equation but shows that Maxwell's equations are now the same in all frames, i.e. it shows their invariance !!

There are no separate electric and magnetic fields, just a frame dependent manifestation of a single electromagnetic field

Quite obvious dealing with Quantum Electrodynamics !

Where are we ?

- Can deal with moving charges in accelerators
- Electromagnetism and fundamental laws of classical mechanics are consistent in the framework of Special Relativity
- Ad hoc introduction of Lorentz force unnecessary, it is a consequence of Special Relativity
- Observations of electromagnetic phenomena are explained
- Classical EM-theory was not consistent with Quantum theory, Special Relativity is the key

Summary I (things to understand)

- Special Relativity is relatively simple and facinating because of the enormous return out of a small investment of facts:
  - Physics laws are the same in all inertial systems
  - The universal constant C (a.k.a. Speed of light in free space) is the same in all inertial systems and the maximum speed for any object

- Everyday phenomena lose their meaning, do not ask what is "real": (leave that to the philosophers)
  - Only union of space and time preserve an independent reality: Space and Time are relative, Spacetime is absolute (a detailed discussion can be found in [7])
  - Electric and magnetic fields do not exist as separate "objects", just different aspects of a single electromagnetic field
  - Its manifestation, i.e. division into electric $\vec{E}$ and magnetic $\vec{B}$ components, depends on the chosen reference frame

- Experimental evidence can only be explained infering that electromagnetic radiation (light, RF, laser, ...) is not a conventional wave (better: not a wave at all !)

Summary II (things for daily and practical work)

- Definition and use of β and γ
- Time dilation and Lorentz contraction: $L = L'/\gamma$ and $\Delta t = \gamma \cdot \Delta t'$
- Formulae for collider performance (E, Luminosity, etc.)
- Kinematic formulae (tables) and practical units: eV, eV/c, eV/c^2
- $E^2 = p^2 + m^2$ and $E = mc^2$
- Mass "in motion" depends on speed $m = \gamma \cdot m_0$ (important for transition energy)
- Transformations of electric and magnetic fields and consequences (e.g. space charge, beam-beam, etc.)
- Concept of four-vectors (qualitatively for daily work)

Maybe interesting, but not treated here:

- Principles of Special Relativity apply to inertial (non-accelerated) systems
- Is it conceivable that the principle applies to accelerated systems ?
- Yes, Introduces General Relativity, with more (hard to believe) consequences:
  - Space and time are dynamical entities:
    - → space and time change in the presence of matter
  - Explanation of gravity (sort of ..)
  - Correction factor for the deflection/bending of light
  - Time Warp again, Black holes, Gravitational Waves, ...
  - Time depends on gravitational potential, different at different heights (RF frequency, Airplanes, GPS !)
- For some effects the Quantum concepts become important (synchrotron radiation, some for beam instrumentation, ..)
- Relativity and philosophy/philosophers

A last word ...

If you do not yet have enough or are bored, look up some of the popular paradoxes (entertaining but mostly irrelevant for accelerators):

- Ladder-garage paradox (*)
- Twin paradox (**)
- Bug - Rivet paradox (**)
- J. Bell's rocket-rope paradox (***)
- ...

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Thanks for spending some time on this event

Solutions of exercise 1:

Orbital speed 14000 km/h $\approx$ 3.9 km/s

➡ $\beta \approx 1.3 \cdot 10^{-5}$, $\gamma \approx 1.000000000084$

Small, but accumulates 7 μs during one day compared to reference time on earth !

After one day: your position wrong by $\approx$ 2 km !!

(including general relativity error is 10 km per day, for the interested: backup slide, coffee break or after dinner discussions)

➡ Countermeasures:

(1) Minimum 4 satellites (avoid reference time on earth)

(2) Detune data transmission frequency from 1.023 MHz to 1.022999999543 MHz prior to launch

Solutions of exercise 2:

$$T = E - m_0c^2 \longrightarrow T = m_0c^2 \; [\gamma - 1]$$

for : $T = m_0c^2$

$$\longrightarrow \gamma\, m_0c^2 - m_0c^2 = m_0c^2$$

$$\longrightarrow \gamma\, m_0c^2 = 2\, m_0c^2$$

$$\longrightarrow \gamma = 2 = \sqrt{1 - \beta^2}$$

$$\longrightarrow \beta = \frac{\sqrt{3}}{2}$$

$$\longrightarrow v = \frac{\sqrt{3}}{2}c$$

Solutions of exercise 3:

3a. What is the four-momentum vector of a photon ?

$E^2 \;=\; p^2 \;+\; m_0^2$ **is also valid for photons**

With proper c: $E^2 \;=\; c^2\, p^2 \;+\; c^4\, m_0^2$
With the general definition of the four-momentum $(\frac{E}{c},\; \vec{p})$

Since the (rest !) mass m_0 of a photon is zero:

$$P_\gamma^\mu \;=\; (\frac{E}{c},\; \frac{E}{c},\; 0,\; 0)$$

3b. What is the (real !) mass of a photon ?

$$m \;=\; \frac{E}{c^2} \;=\; \frac{h\nu}{c^2}$$

Solutions of exercise 4:

3a: $p \quad + \quad p \quad \longrightarrow \quad p \quad + \quad p \quad + \quad (p \quad + \quad \bar{p})$

$$p_0 = (M,0,0,0) \quad p_1 = (E,\vec{p}) \quad \rightarrow \quad p_\mu = (E+M,\vec{p}) \quad p^\mu = (E+M,-\vec{p})$$

$$p_\mu \; p^\mu = M^2 + 2EM + M^2 = 2EM + 2M^2 \quad (= 16M^2)$$

$$2EM = 14M^2$$

$$\underline{E = 7M}$$

3b: $\gamma \quad + \quad p \quad \longrightarrow \quad p \quad + \quad (p \quad + \quad \bar{p})$

$$p_0 = (M,0,0,0) \quad \gamma = (E,E,0,0) \quad \rightarrow \quad p_\mu = (E+M,E,0,0) \quad p^\mu = (E+M,-E,0,0)$$

$$p_\mu \; p^\mu = 2EM + M^2 = 2EM + M^2 \quad (= 9M^2)$$

$$2EM = 8M^2$$

$$\underline{E = 4M}$$

Note: M = proton mass, c = 1

Solutions of exercise 5:

Gravitational time dilation

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{Rc^2}}$$

$$\frac{d\tau}{dt} \approx 1 - \frac{GM}{Rc^2}$$

$$\Delta\tau = \frac{GM}{c^2} \cdot \left(\frac{1}{R_{earth}} - \frac{1}{R_{gps}}\right)$$

With:

$R_{earth} = 6357000$ **m,** $R_{gps} = 26541000$ **m**

$G = 6.674 \cdot 10^{-11} \ \frac{N \cdot m^2}{kg^2}$ $M = 5.974 \cdot 10^{24}$ **kg**

We have:

$$\Delta\tau \approx 5.3 \cdot 10^{-10}$$

Do the math:

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