Emerging magnetic monopoles

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Outline

Descriptions of magnetic monopole within different physical paradigms:

Classical physics $\longrightarrow$ Quantum mechanics $\longrightarrow$ Quantum field theory

(Maxwell, Dirac, 't Hooft, Polyakov, Cho, Maison)

Optionality/emergence of basic *physical properties* of magnetic monopole within different paradigms:

- Existence [sic]
- Mass
- Magnetic charge

Magnetic monopole in classical physics

Maxwell's equations

Maxwell's equations:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= \rho_{\mathrm{e}} & \qquad \boldsymbol{\nabla} \times \boldsymbol{B} - \dot{\boldsymbol{E}} &= \boldsymbol{j}_{\mathrm{e}} \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 & \qquad -\boldsymbol{\nabla} \times \boldsymbol{E} - \dot{\boldsymbol{B}} &= \boldsymbol{0} \end{aligned}$$

Lorentz force:

$$\boldsymbol{F} = q_{\mathrm{e}}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

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Lorentz force:

$$\boldsymbol{F} = q_\mathrm{e}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + q_\mathrm{m}(\boldsymbol{B} - \boldsymbol{v} \times \boldsymbol{E})$$

$\rightarrow$ Introducing the **magnetic charge** to the Maxwell's equations is nothing but trivial...

Point-like magnetic charge (a.k.a. the magnetic monopole)

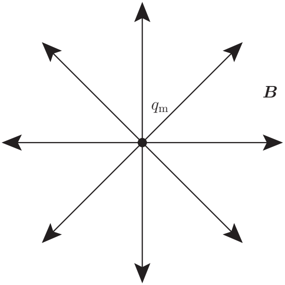
Magnetic monopole (charge q_{m}) is just a point-like magnetic charge density:

- The charge density

$$\rho_{\mathrm{m}} = q_{\mathrm{m}} \delta^3(\boldsymbol{r})$$

- From the Maxwell's equations follows the "Coulomb's law" for magnetic monopole:

$$\boldsymbol{B} = q_{\mathrm{m}} \frac{1}{4\pi} \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$$



- Easy!

Classical monopole: Summary

Introducing magnetic monopoles into the Maxwell's equations is trivial:

- It is even so trivial that all of its properties are completely arbitrary:
  - Its existence is not forbidden, nor required either
  - Mass and charge are arbitrary

In fact, by allowing magnetic charges the Maxwell's equations obtain a new symmetry, a $U(1)$ duality transformation between electric and magnetic fields/charges/currents:

$$\boldsymbol{E}' + \mathrm{i}\boldsymbol{B}' = \mathrm{e}^{\mathrm{i}\alpha}\big(\boldsymbol{E} + \mathrm{i}\boldsymbol{B}\big)$$

- From classical point of view merely an esthetical curiosity
- But perhaps a hint that there is something deeper about the magnetic monopoles...

$\Rightarrow$ Classically, it is a mystery why there are no magnetic monopoles in the Nature
(At least for esthetically minded people...)

Magnetic monopole in quantum mechanics: Dirac monopole

Electromagnetic potentials in QM are indispensable

But the Nature is not classical:

- Non-relativistic Schrödinger equation for an electrically charged (q_e) particle (ψ) in an external electromagnetic field:

$$\left[-\frac{1}{2m}(\boldsymbol{\nabla}-\mathrm{i}q_\mathrm{e}\boldsymbol{A})^2+q_\mathrm{e}\varphi\right]\psi \;=\; \mathrm{i}\frac{\partial}{\partial t}\psi$$

The wave function doesn't couple directly to the elmag fields $\boldsymbol{E}$, $\boldsymbol{B}$, but to the elmag potentials φ, $\boldsymbol{A}$

- We are specificaly interested in behavior of an electrically charged particle in the field of a static magnetic monopole.

$\Rightarrow$ What $\boldsymbol{A}$ corresponds to magnetic monopole? Answer is not trivial...

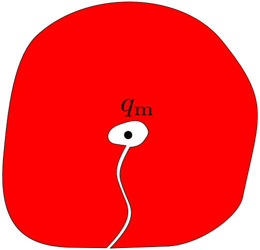
The Dirac string

Turns out that:

- The potential $\boldsymbol{A}$ (such that $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$) can be defined only
  - on a *simply connected* region
  - where magnetic charge density *vanishes*.
- Proof by contradiction:
  - $0 \neq \rho_{\mathrm{m}} = \boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$
  - $0 \neq q_{\mathrm{m}} = \int \mathrm{d}^3 x \, \rho_{\mathrm{m}} = \int \mathrm{d}^3 x \, \boldsymbol{\nabla} \cdot \boldsymbol{B} = \oiint \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{B} = \oiint \mathrm{d}\boldsymbol{S} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$

$\Rightarrow$ For a single magnetic monopole q_{m}:

- $\boldsymbol{A}$ can be defined everywhere *except* on a line stretching from the monopole position to infinity
  $\rightarrow$ the **Dirac string**
- The precise position and shape of the Dirac string is arbitrary and can be moved around by a gauge transformation
  $\Rightarrow$ the Dirac string is non-physical



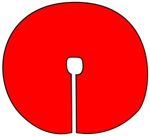
Need for two potentials

$\Rightarrow$ At least two $\boldsymbol{A}$'s are necessary to fully describe field of a magnetic monopole.

(Analogue of the situation in differential geometry where a single map is typically not sufficient to cover the whole manifold.)

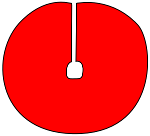
For instance, the following two potentials, $\boldsymbol{A}_+$ and $\boldsymbol{A}_-$, can be used:

- $\boldsymbol{A}_+$ is defined everywhere but on the negative z-axis:



$$\boldsymbol{A}_+ = \frac{q_\mathrm{m}}{4\pi|\boldsymbol{r}|}\frac{1-\cos\theta}{\sin\theta}\boldsymbol{e}_\phi$$

- $\boldsymbol{A}_-$ is defined everywhere but on the positive z-axis:



$$\boldsymbol{A}_- = \frac{q_\mathrm{m}}{4\pi|\boldsymbol{r}|}\frac{-1-\cos\theta}{\sin\theta}\boldsymbol{e}_\phi$$

By construction, both yield the same magnetic field ($\boldsymbol{\nabla}\times\boldsymbol{A}_+ = \boldsymbol{\nabla}\times\boldsymbol{A}_-$), so that on their overlap they are necessarily related by a gauge transformation.

Dirac quantisation condition

Back to an electrically charged particle ($q_{\rm e}$) in the field of a static magnetic monopole ($q_{\rm m}$):

- Each of the two potentials $\boldsymbol{A}_{\pm}$ gives rise to a distinct Schrödinger equation:

$$\begin{aligned} -\frac{1}{2m}(\boldsymbol{\nabla} - {\rm i}q_{\rm e}\boldsymbol{A}_{+})^2\psi_{+} &= {\rm i}\partial_0\psi_{+} \\ -\frac{1}{2m}(\boldsymbol{\nabla} - {\rm i}q_{\rm e}\boldsymbol{A}_{-})^2\psi_{-} &= {\rm i}\partial_0\psi_{-} \end{aligned}$$

- Both equations should lead to the same physics:
  $\Rightarrow$ The wave functions ψ_{+} and ψ_{-} must be the same up to a phase.
- However, turns out that this is in general not possible, *unless* the charges are related as

$$\boxed{q_{\rm e}\, q_{\rm m} \;=\; 2\pi n} \qquad\qquad (n \in \mathbb{Z})$$

  $\Rightarrow$ The **Dirac quantisation condition**

Comments on the Dirac quantisation condition

$$\boxed{q_{\mathrm{e}}\, q_{\mathrm{m}} \;=\; 2\pi n}$$

- Robust: can be derived also by different methods and holds also for the soliton-type magnetic monopoles (to be discussed in a moment...)
- Can be generalized to *dyons* (particles carrying both the electric and magnetic charge).
- The basic implication: If there exists at least a single magnetic monopole (with q_{m}) in the Universe, all electric charges q_{e} must be quantized:

$$q_{\mathrm{e}} \;=\; \frac{2\pi}{q_{\mathrm{m}}} n$$

  - Indeed: all electric charges in nature are quantized
  - In fact, explanation of the electric charge quantization was one of the original Dirac's motivations for introducing the magnetic monopoles
  - Today we know also other explanations for electric charge quantization (anomaly freedom, grand unified theories, ...)

Dirac monopole: Summary

The type of magnetic monopole we have just described is known as the **Dirac (magnetic) monopole**.

Like in classical physics, its existence again optional:

- It is put into the theory by hand (It is not predicted by the theory, nor required either)
- Also its mass is again arbitrary

On the other hand, QM predicts something non-trivial about the magnetic charge:

- Dirac quantization condition: $q_{\mathrm{e}}\, q_{\mathrm{m}} \;=\; 2\pi n$

Its implications (electric charge quantization) make apparent non-existence even more mysterious...

Magnetic monopole in QFT

No magnetic monopoles in QFT. . . ?

But our world is described by QFT:

- We want a QFT description of magnetic monopoles as well
- For "electric monopoles" (all particles known to date) this can be done very *easily* and *naturally*
  $\rightarrow$ the notoriously successful QED or SM

Why not to simply incorporate magnetic monopoles into QED as electrons?

- But as it turns out, the magnetic monopole cannot be (easily and naturally) incorporated into the framework of QFT ✘
- At least as long as one insists on having both electrically and magnetically charged particles in the theory and treating them symmetrically, on the same footing.
- Technically, the problems are due to singular character of $\boldsymbol{A}$ (the Dirac string).
- (In fact, it *is* possible to incorporate monopoles into QED, but for the prize of doubling number of elmag potentials, introducing constraints, having issues with renormalizability, locality... The resulting highly non-elegant Two-Potential QED is sometimes used, but only as an effective description, not as a fundamental theory.)

$\longrightarrow$ There's a way out, but one has to give up the idea that magnetic monopoles are described in the same way as all other particles . . .

A particle in QFT: plane waves

In QFT particles can be treated in two rather different ways:

1. By quantizing plane waves solutions of equations of motion
   - The standard machinery taught in introductory QTF courses: Feynman diagrams, perturbation theory, annihilation/creation operators...
   - $\rightarrow$ "elementary" particles: quarks, leptons, gauge bosons, Higgs, gravitons, ...
   - Perturbative description
   - Genuinely quantum description

Alas, for magnetic monopoles this description doesn't work ✗

A particle in QFT: topological soliton

But there's another way how to describe a particle:

2. As a topological soliton
    - Solution of the classical EOMs with particle-like properties
    - In a way "bound states" of the elementary particles (plane waves)
        - "Emerging": Not visible in the Lagrangian
        - Inherently non-perturbative: Difficult to calculate scattering amplitudes with them
    - Despite the computational difficulties, the "particle-ness" is perhaps more intuitive

(Remarkably, the two seemingly different descriptions (plane waves and solitons) are in fact deeply connected, via S-duality)

$\rightarrow$ The magnetic monopoles! ✓

There are two different solitonic descriptions of monopole
- by 't Hooft and Polyakov (1974)
- by Cho and Maison (1996)

Let's see...

't Hooft–Polyakov magnetic monopole

Georgi–Glashow model

The prototypical example:

- $SU(2)$ gauge theory with Higgs field $\boldsymbol{\phi}$ in triplet representation:

$$\mathcal{L} = \frac{1}{2}(\mathrm{D}_\mu \boldsymbol{\phi})^2 - \frac{1}{4}(\boldsymbol{F}^{\mu\nu})^2 - \frac{\lambda}{4}(\boldsymbol{\phi}^2 - v^2)^2$$

- If $v^2 > 0$, the vacuum $\boldsymbol{\phi}^2 = v^2$ breaks spontaneously the symmetry $SU(2)$ down to the "electromagnetic" $U(1)$.
- The normal perturbative spectrum:
  - Two massive vector bosons $W^\pm$
  - Massless photon
  - Higgs boson

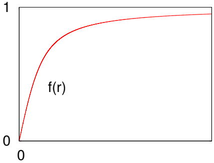
  (The Georgi–Glashow model was once considered as a model of electroweak interactions.)

However, 't Hooft and Polyakov showed that the spectrum contains also the magnetic monopole, described as a topological soliton!

Topological soliton: The main idea

EOMs have *non-trivial static* solution that is *finite-energy*, *regular* and *stable*.

$$\phi = v\frac{\boldsymbol{r}}{r}f(r)$$ where $f(r)$ is like:



How come such a non-trivial field configuration doesn't dissipate?

- Notice that $\phi^2 \xrightarrow[r\to\infty]{} v^2$ (As it must in order that the total energy be finite.)
- We have thus a mapping from one sphere (of spatial infinity $\boldsymbol{r}^2 = \infty$) to another sphere (of vacuum configurations $\phi^2 = v^2$)
- All such mappings can be classified into classes of equivalence ("homotopy" classes) that cannot be continuously deformed one into another
- In mathematical parlance: $\pi_2(SU(2)/U(1)) = \pi_2(S^2) = \mathbb{Z}$
- Our non-trivial solution ϕ is in different homotopy class than a trivial solution

$\Rightarrow \phi$ cannot evolve into a trivial solution and must be stable!

ϕ is topologically stabilized by its boundary condition $\longrightarrow$ a *topological soliton*

Magnetic field

But where is the promised magnetic field? And where is $\boldsymbol{A}_\mu$?

- Finiteness of the total energy requires that the gauge field must be asymptotically *related to* $\boldsymbol{\phi}$ as

$$\boldsymbol{A}_i \quad \underset{r\to\infty}{\longrightarrow} \quad -\frac{1}{gv^2}\boldsymbol{\phi}\times\partial_i\boldsymbol{\phi}$$

Recall that, asymptotically, $SU(2)$ is broken down to electromagnetic $U(1)$
(Since $\boldsymbol{\phi}^2 \longrightarrow v^2$ for $r \longrightarrow \infty$)

- The electromagnetic field-strength tensor can be projected out as

$$F^{\mu\nu} \equiv \frac{1}{v}\boldsymbol{\phi}\cdot\boldsymbol{F}^{\mu\nu} = -\frac{1}{gv^3}\boldsymbol{\phi}\cdot(\partial^\mu\boldsymbol{\phi}\times\partial^\nu\boldsymbol{\phi})$$

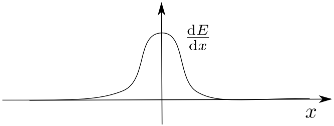
- Magnetic field: $B_k = -\frac{1}{2}\epsilon_{kij}F^{ij}$
- Magnetic charge: $\boxed{q_{\mathrm{m}} = \int_{S^2}\mathrm{d}S_k B_k = \ldots = \frac{4\pi}{g}}$

Having included also the gauge fields, we can call the topological soliton (made by both $\boldsymbol{\phi}$ and $\boldsymbol{A}_\mu$) the *magnetic monopole*

Magnetic monopole as a particle

Basic properties of our soliton:

- Everywhere except in an *extended, but finite* region around its center (the "core") it corresponds to vacuum
- (While in the core the symmetry is in fact unbroken)
- The energy density is localized in the core:



- The total energy (mass) is *finite* and *non-zero*
- It is stable – cannot transform to a vacuum everywhere

⇒ We have every reason to interpret the magnetic monopole as a particle, which is

- massive
- non-point-like
- stable

Magnetic monopole as a non-perturbative object

Physical properties (g is the gauge coupling constant):

- Magnetic charge: $q_{\mathrm{m}} = \dfrac{4\pi}{g}$
- Mass: $M \geq \dfrac{4\pi v}{g}$
- Size: $R \approx \dfrac{1}{vg}$

$\Rightarrow$ Magnetic monopole is a non-perturbative object ($\sim 1/g$)

- Can be interpreted as a very heavy bound state of elementary excitations: the Higgs boson and gauge bosons
  - In the weakly-coupled theory ($g \ll 1$) the monopoles is very heavy and large $\Rightarrow$ can be treated classically
  - In strongly-coupled regime ($g \gg 1$) the description via plane waves and solitons (and correspondingly what is elementary and composite particle) would switch (S-duality)

't Hooft–Polyakov monopole: Summary

The **'t Hooft–Polyakov monopole** has exactly the features we wanted:

- Its existence is not postulated, but predicted within a (suitable) QFT
  (Althought it is not present in the Lagrangian, but emerges as a bound state)
- Its mass is not arbitrary, but calculable in terms of parameters of the model
- Its magnetic charge is predicted (satisfying, of course, the Dirac quantization condition)

Moreover:

- Smooth solution: no singularity in the center of monopole, no Dirac string

This sounds perfect, perhaps *too* perfect...

't Hooft–Polyakov magnetic monopole in SM?

So far we investigated only a toy $SU(2)$ model.

Does the 't Hooft–Polyakov monopole exist in the Standard Model?

✗

No.

Reason is the SSB pattern of SM...

Generalizations

Conditions to have a monopole solution à la 't Hooft–Polyakov:

- Gauge theory with a group G spontaneously broken down to a subgroup H
- The group G must not contain a $U(1)$ factor
- $\Rightarrow$ Only then the maps of the spatial infinity (S^2) to the vacuum manifold (G/H) contain inequivalent classes
  ("Second homotopy group $\pi_2(G/H)$ is nontrivial")

Examples:

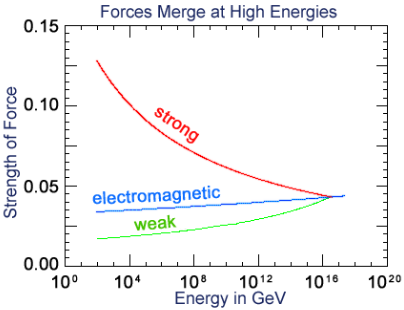
- The Standard Model:
  - $$\left.\begin{array}{rcl} G &=& SU(3)_{\mathrm{C}} \times SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}} \\ H &=& U(1)_{\mathrm{em}} \end{array}\right\} \pi_2(G/H) = \{1\}$$
  - There are no magnetic monopoles ✗
- Grand Unified Theories:
  - $$\left.\begin{array}{rcl} G &=& SU(5),\ SO(10),\ \ldots \\ H &=& SU(3)_{\mathrm{C}} \times SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}} \end{array}\right\} \pi_2(G/H) = \mathbb{Z}$$
  - There are magnetic monopoles ✓

Grand Unified Theories

GUTs:

- Based typically on Lie groups like: $SU(5)$, $SO(10)$, ...
- Therefore allow (actually *predict*) the existence of magnetic monopoles
- GUTs are well motivated theories $\Rightarrow$ the corresponding magnetic monopoles almost certainly exist

"*The existence of magnetic monopoles seems like one of the safest bets that one can make about physics not yet seen.*" (Joe Polchinski)



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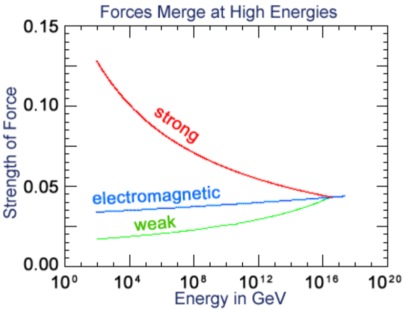

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But:

- Mass is $\sim 10^{16}\,\mathrm{GeV}$
  $\Rightarrow$ Way too heavy to be produced in any human-made particle accelerator
- During the Big Bang they were produced in large numbers, but later diluted by Inflation: Now ~ 1 magnetic monopole within one Hubble horizon
  $\Rightarrow$ No chance of observing them in cosmic radiation

Cho–Maison magnetic monopole

Another source of the desired topology

We argued that SM doesn't have the 't Hooft–Polyakov monopole

- But not so fast
- Perhaps it has some different type (description) of magnetic monopole
- Perhaps the desired topology has to be sought for somewhere else than in the pattern of SSB

Indeed, Cho and Maison (1996) noticed that

- Higgs (complex doublet field) H itself has the desired topology
- We can write it as

$$H \;=\; \frac{1}{\sqrt{2}}(v+\sigma)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \qquad \text{where} \quad |\xi_1|^2 + |\xi_2|^2 = 1$$

- Therefore ξ_1 and ξ_2 form a "circle" in $\mathbb{C}^2$ plane (or real S^3)
- Due to $U(1)_Y$ invariance the points (ξ_1, ξ_2) and $(\mathrm{e}^{\mathrm{i}\alpha}\xi_1, \mathrm{e}^{\mathrm{i}\alpha}\xi_2)$ are physically equivalent and have to be identified
- Set of these identified points forms the complex projective space $\mathbb{C}\mathrm{P}^1 \sim S^2$
- But $\pi_2(\mathbb{C}\mathrm{P}^1) = \pi_2(S^2) = \mathbb{Z}$

$\Rightarrow$ There are topological sectors and, potentially, magnetic monopoles!

Infinite monopole mass and the solution

Indeed, magnetic monopole solution can be found!

- But there's a problem! It has infinite mass:

$$M = \frac{2\pi}{g'^2}\int_0^\infty \frac{\mathrm{d}r}{r^2} + \text{finite terms} = \infty$$

- So indeed, no (finite-mass) monopole in the SM...

However, this can be cured by going <u>beyond SM</u> and modifying the $U(1)_Y$ kinetic term:

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \longrightarrow \quad -\frac{1}{4}f\big(|H|^2\big)F_{\mu\nu}F^{\mu\nu}$$

where f is some positive function

- If $f(0) = 0$, the mass of the monopole comes out finite!

However, since f is in principle arbitrary, also the monopole mass is arbitrary...

Lower mass bound

However, there is a lower bound on mass of the Cho–Maison magnetic monopole:

$$\boxed{M \;\geq\; \frac{2\pi v}{g} \;\approx\; 2.37\,\mathrm{TeV}}$$

(P. B., F. Blaschke, [PTEP 2018 (2018) no.7, 073B03, `arXiv:1711.04842`])

- Regardless of choice of f, the magnetic monopole cannot be lighter
- Technically, the proof relies on separating topological contributions to the mass (easily calculated using the judiciously constructed BPS limit) from the non-topological contributions (volume integrals of energy density) and showing that the latter are positive.
- Magnetic monopoles could be potentialy pair-produced and *observed* at the LHC

  (At least kinematically; another issue is the cross section…)

Summary

Magnetic monopole

- is not just “another hypothetical particle”
- has a long, rich history of rather different, yet related mathematical descriptions
- occurs in QFT in a rather special and unexpected way: as a soliton
- almost certainly exists with mass $\sim 10^{16}\,\mathrm{TeV}$ (’t Hooft–Polyakov)
- can be perhaps as light as several TeV (Cho–Maison) in modest BSMs
- not detected yet, but stay tuned!