Symplectic kicks from an electron cloud pinch

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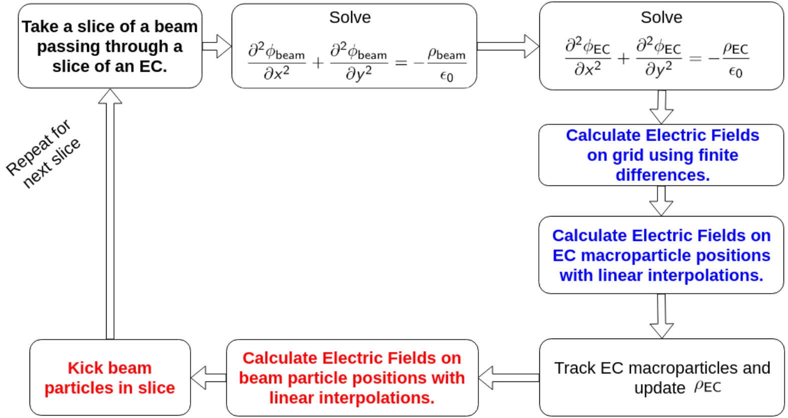
PyECLOUD-PyHEADTAIL simulations



Beams are sliced in $\zeta = s\frac{\beta}{\beta_0} - \beta ct$. From the point of view of the EC, s does not change and ζ is basically **time!**

We propose to modify the kicks to the beam particles to support long-term tracking.

Symplecticity Condition

Suppose we have a map[1] $\mathcal{M}$ that transforms $\vec{x} = (x_1, x_2)$ to $\vec{X} = (X_1, X_2)$:

$$\vec{X} = \mathcal{M}\vec{x}$$

The map $\mathcal{M}$ is symplectic if

$$\mathbf{MSM^T} = \mathbf{S} \tag{1}$$

where $\mathbf{M}$ is the Jacobian with $M_{ij} = \frac{\partial X_i}{\partial x_j}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
If the map can be represented with a matrix multiplication, then the Jacobian is the matrix that multiplies $\vec{x}$.

[1] This example is in 1D but can be trivially generalized to any dimension.

Symplecticity - Simple Map

Simple transverse **thin-lens** electrostatic kick:

$$
\begin{aligned}
x^f &= x^i \\
p_x^f &= p_x^i + A \cdot E_x(x^i, y^i) \\
y^f &= y^i \\
p_y^f &= p_y^i + A \cdot E_y(x^i, y^i), \\
A &= \frac{qL}{\beta^2 \gamma m c^2}
\end{aligned}
\quad \xrightarrow{\textit{Jacobian}} \quad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ A\partial_x E_x & 1 & A\partial_y E_x & 0 \\ 1 & 0 & 1 & 0 \\ A\partial_x E_y & 0 & A\partial_y E_y & 1 \end{pmatrix}
$$

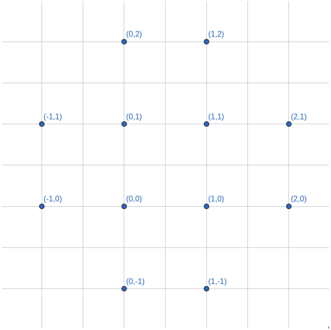
Symplecticity condition $\mathbf{MSM^T} = \mathbf{S}$ boils down to:

$$
\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}
$$

Symplecticity - PyECLOUD

Map in PyECLOUD[2]is:

$$x^f = x^i$$
$$p_x^f = p_x^i + A \cdot E_x(x^i, y^i)$$
$$y^f = y^i$$
$$p_y^f = p_y^i + A \cdot E_y(x^i, y^i),$$
$$A = \frac{qL}{\beta^2 \gamma m c^2}$$



where E_x, E_y are bilinear interpolations on finite differences.

$$E_{x,y}\left(\tilde{x}, \tilde{y}\right) = a_{x,y} + b_{x,y}\tilde{x} + c_{x,y}\tilde{y} + d_{x,y}\tilde{x}\tilde{y},$$

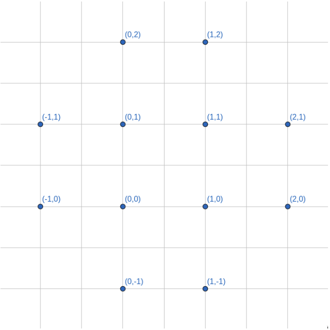
$$\tilde{x} = \frac{x}{\Delta x} \in [0,1]\,,\ \tilde{y} = \frac{y}{\Delta y} \in [0,1]$$

Symplecticity - PyECLOUD

$$E_{x,y}(\tilde{x},\tilde{y}) = a_{x,y} + b_{x,y}\tilde{x} + c_{x,y}\tilde{y} + d_{x,y}\tilde{x}\tilde{y}$$

where the coefficients are defined as:

$$a_{x,y} = E^{(0,0)}_{x,y}$$
$$b_{x,y} = -E^{(0,0)}_{x,y} + E^{(1,0)}_{x,y}$$
$$c_{x,y} = -E^{(0,0)}_{x,y} + E^{(0,1)}_{x,y}$$
$$d_{x,y} = E^{(0,0)}_{x,y} - E^{(0,1)}_{x,y} - E^{(1,0)}_{x,y} + E^{(1,1)}_{x,y},$$



and the derivatives are:

$$\partial_x E_{x,y}(\tilde{x},\tilde{y}) = \frac{b_{x,y} + d_{x,y}\tilde{y}}{\Delta x}, \qquad \partial_y E_{x,y}(\tilde{x},\tilde{y}) = \frac{c_{x,y} + d_{x,y}\tilde{x}}{\Delta y}$$

Symplecticity - PyECLOUD

Symplecticity Condition:

$$\partial_x E_y(\tilde{x}, \tilde{y}) - \partial_y E_x(\tilde{x}, \tilde{y}) = 0 \Rightarrow$$
$$\frac{b_y}{\Delta x} + \frac{d_y}{\Delta x}\tilde{y} - \frac{c_x}{\Delta y} - \frac{d_x}{\Delta y}\tilde{x} = 0$$

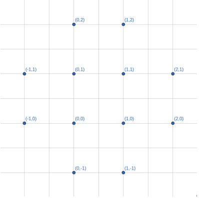
All coefficients of the polynomial must vanish.

$$\Rightarrow \begin{cases} \frac{b_y}{\Delta x} - \frac{c_x}{\Delta y} & = 0 \\ d_x & = 0 \\ d_y & = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{-E_y^{(0,0)} + E_y^{(1,0)}}{\Delta x} - \frac{-E_x^{(0,0)} + E_x^{(0,1)}}{\Delta y} & = 0 \\ E_x^{(0,0)} - E_x^{(1,0)} - E_x^{(0,1)} + E_x^{(1,1)} & = 0 \\ E_y^{(0,0)} - E_y^{(1,0)} - E_y^{(0,1)} + E_y^{(1,1)} & = 0 \end{cases}$$

Symplecticity - PyECLOUD

Fields defined as central differences:



$$E_x^{(i,j)} = -\frac{\phi^{(i+1,j)} - \phi^{(i-1,j)}}{2\Delta x}$$

$$E_y^{(i,j)} = -\frac{\phi^{(i,j+1)} - \phi^{(i,j-1)}}{2\Delta y}$$

$$\begin{cases} \frac{-E_y^{(\mathbf{0},\mathbf{0})}+E_y^{(\mathbf{1},\mathbf{0})}}{\Delta x} - \frac{-E_x^{(\mathbf{0},\mathbf{0})}+E_x^{(\mathbf{0},\mathbf{1})}}{\Delta y} & = 0 \\ E_x^{(0,0)} - E_x^{(1,0)} - E_x^{(0,1)} + E_x^{(1,1)} & = 0 \\ E_y^{(0,0)} - E_y^{(1,0)} - E_y^{(0,1)} + E_y^{(1,1)} & = 0 \end{cases}$$

Substituting the electric fields with the finite differences, we get:

$$\begin{cases} \phi^{(0,1)} - \phi^{(0,-1)} - \phi^{(1,1)} + \phi^{(1,-1)} - \phi^{(1,0)} + \phi^{(-1,0)} + \phi^{(1,1)} - \phi^{(-1,1)} & = 0 \\ -\phi^{(1,0)} + \phi^{(-1,0)} + \phi^{(2,0)} - \phi^{(0,0)} + \phi^{(1,1)} - \phi^{(-1,1)} - \phi^{(2,1)} + \phi^{(0,1)} & = 0 \\ -\phi^{(0,1)} + \phi^{(0,-1)} + \phi^{(1,1)} - \phi^{(1,-1)} + \phi^{(0,2)} - \phi^{(0,0)} - \phi^{(1,2)} + \phi^{(1,0)} & = 0 \end{cases}$$

Symplecticity - PyECLOUD

The present map is symplectic if:

$$\begin{cases} \phi^{(0,1)} - \phi^{(0,-1)} - \phi^{(1,1)} + \phi^{(1,-1)} - \phi^{(1,0)} + \phi^{(-1,0)} + \phi^{(1,1)} - \phi^{(-1,1)} & = 0 \\ -\phi^{(1,0)} + \phi^{(-1,0)} + \phi^{(2,0)} - \phi^{(0,0)} + \phi^{(1,1)} - \phi^{(-1,1)} - \phi^{(2,1)} + \phi^{(0,1)} & = 0 \\ -\phi^{(0,1)} + \phi^{(0,-1)} + \phi^{(1,1)} - \phi^{(1,-1)} + \phi^{(0,2)} - \phi^{(0,0)} - \phi^{(1,2)} + \phi^{(1,0)} & = 0 \end{cases}$$

- These conditions are not linearly independent of the equations consisting the Poisson solver.
- Imposing them leads to an **overdetermined** system of equations which has no solution.
- **Linearly interpolating on finite differences produces non-symplectic kicks.**

Symplectifying in 4D

Step #1

Given a regular grid of a scalar potential $\phi^{(i,j)}$, calculate E_x, E_y such that the kick is symplectic, i.e.:

$$\frac{\partial E_y}{\partial x} = \frac{\partial E_x}{\partial y}$$

By defining $E_x = -\frac{\partial \phi}{\partial x}$ and $E_y = -\frac{\partial \phi}{\partial y}$, we need only find an analytical approximation of ϕ such that

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)$$

The condition holds if we interpolate ϕ with a scheme that guarantees continuous derivatives. (C^1-Continuity)

Extending to a 6D kick

A Hamiltonian that produces this 4D kick is:

$$H(x, y; s) = \frac{qL}{\beta^2 \gamma mc^2} \phi\,(x, y)\,\delta(s)$$

where $\delta(s)$ is Dirac's delta function. Actually, we have $\phi(x, y)$ for each step of ζ. We can use a 3D interpolation method to get a function for $\phi(x, y, \zeta)$.
Finally, our Hamiltonian is[3]:

$$H(x, y, \zeta; s) = \frac{qL}{\beta^2 \gamma mc^2} \phi\,(x, y, \zeta)\,\delta(s)$$

[3] The full family of thin symplectic 6D kicks can be found in the appendix.

6D Symplectic Kick

and it produces the kick:

$$
\begin{aligned}
x &\mapsto x \\
p_x &\mapsto p_x - \frac{qL}{\beta^2\gamma mc^2}\frac{\partial\phi}{\partial x}(x,y,\zeta) \\
y &\mapsto y \\
p_y &\mapsto p_y - \frac{qL}{\beta^2\gamma mc^2}\frac{\partial\phi}{\partial y}(x,y,\zeta) \\
\zeta &\mapsto \zeta \\
\delta &\mapsto \delta - \frac{qL}{\beta^2\gamma mc^2}\frac{\partial\phi}{\partial\zeta}(x,y,\zeta)
\end{aligned}
$$

The problem reduces to approximating ϕ in such a way that it is analytically differentiable once and C^1 continuous.

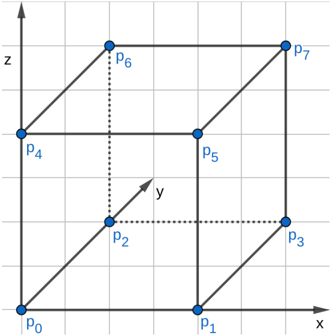
Section 4

Tricubic Interpolation

How to interpolate[4]

Objective

Given a regular 3D grid of any function $f^{i,j,k}$, we interpolate **locally** in a way that the first derivatives are continuous globally.



- To be continuous globally, first we must be continuous at the corners. To do that we fix the values $\left\{ f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\}$ at the 8 corners of our cube $\rightarrow$ 32 constraints.

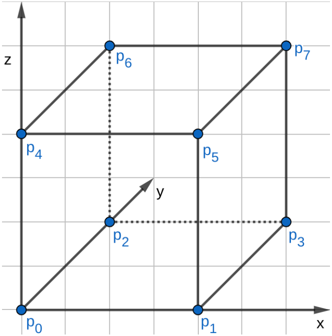
[3] Lekien, F & J. E., Marsden. (2005). Tricubic Interpolation in Three Dimensions. International Journal for Numerical Methods in Engineering. 63. 10.1002/nme.1296.

Interpolating Function

Objective

For simplicity, we want to interpolate with a polynomial function :

$$f(x, y, z) = \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} a_{ijk} x^i y^j z^k$$

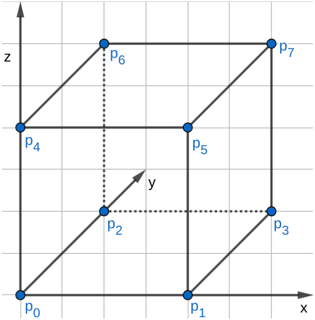


Degrees of Freedom $= (N + 1)^3$

- $N = 3 \Rightarrow 27$ Degrees of freedom. **Not enough!**
- $N = 4 \Rightarrow 64$ Degrees of freedom. **More than enough, but we need 32 additional constraints.**

Violation of Charge Conservation

Can we impose values for $\left\{\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}\right\}$ to conserve charge simultaneously?



$$f(x,y,z) = \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} a_{ijk} x^i y^j z^k$$

$$\frac{\partial f}{\partial x}(x,y,z) = \sum_{i=1}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} i a_{ijk} x^{i-1} y^j z^k$$

$$\xrightarrow{y,z=0}$$

$$\begin{cases} f(x,0,0) = a_{000} + a_{100}x + a_{200}x^2 + a_{300}x^3 \\ \frac{\partial f}{\partial x}(x,0,0) = a_{100} + 2a_{200}x + 3a_{300}x^2 \end{cases}$$

$$\Rightarrow \begin{cases} f|_{p_0} = a_{000} \\ f|_{p_1} = a_{000} + a_{100} + a_{200} + a_{300} \\ \frac{\partial f}{\partial x}|_{p_0} = a_{100} \\ \frac{\partial f}{\partial x}|_{p_1} = a_{100} + 2a_{200} + 3a_{300} \end{cases}$$

$$\frac{\partial^2 f}{\partial x^2}(x,0,0) = 2a_{200} + 6a_{300}x$$

$\frac{\partial^2 f}{\partial x^2}$ is linearly dependent on the existing constraints.

Additional Constraints

We cannot use the derivatives

$$\left\{ \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2} \right\}$$

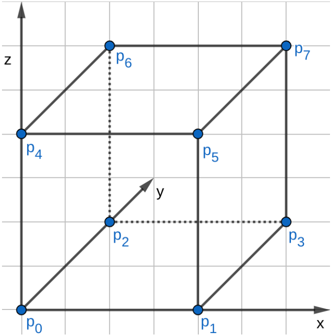
As it turns out, the simplest set of constraints that is linearly independent to our other constraints is

$$\left\{ \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial y \partial z}, \frac{\partial^3 f}{\partial x \partial y \partial z} \right\}$$

Finally, our full set of constraints (input) is

$$\left\{ f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial y \partial z}, \frac{\partial^3 f}{\partial x \partial y \partial z} \right\}$$

Tricubic Interpolation



$$b_i = \begin{cases} f\big|_{p_i} & 0 \le i \le 7 \\ \frac{\partial f}{\partial x}\big|_{p_{i-8}} & 8 \le i \le 15 \\ \frac{\partial f}{\partial y}\big|_{p_{i-16}} & 16 \le i \le 23 \\ \frac{\partial f}{\partial z}\big|_{p_{i-24}} & 24 \le i \le 31 \\ \frac{\partial^2 f}{\partial x \partial y}\big|_{p_{i-32}} & 32 \le i \le 39 \\ \frac{\partial^2 f}{\partial x \partial z}\big|_{p_{i-40}} & 40 \le i \le 47 \\ \frac{\partial^2 f}{\partial y \partial z}\big|_{p_{i-48}} & 48 \le i \le 55 \\ \frac{\partial^3 f}{\partial x \partial y \partial z}\big|_{p_{i-56}} & 56 \le i \le 63 \end{cases}$$

$$f(x, y, z) = \sum_{i=0}^{3} \sum_{j=0}^{3} \sum_{k=0}^{3} a_{ijk} x^i y^j z^k$$

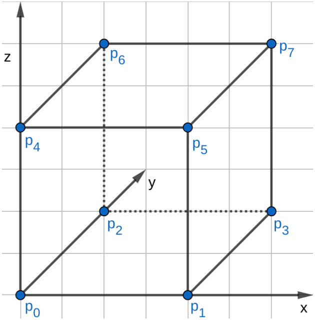
$$\alpha_{i+4j+16k} = a_{ijk}$$

$$\mathbf{B}\boldsymbol{\alpha} = \mathbf{b} \Rightarrow \boxed{\boldsymbol{\alpha} = \mathbf{B}^{-1}\mathbf{b}}$$

where $\mathbf{B}^{-1}$ is an integer 64×64 matrix.

C^1-Continuity

Interpolation is obviously C^1-continuous inside the cube. Is it on the boundaries?



$$f(x,y,z) = \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} a_{ijk}x^i y^j z^k$$

$$\frac{\partial f}{\partial x}(x,y,z) = \sum_{i=1}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} i a_{ijk}x^{i-1} y^j z^k$$

$$\xrightarrow{y=1,z=0}$$

$$\begin{cases} \sum_{j=0}^{3} a_{ijk} = b_{ik} \\ f(x,1,0) = b_{00} + b_{10}x + b_{20}x^2 + a_{30}x^3 \\ \frac{\partial f}{\partial x}(x,1,0) = b_{10} + 2b_{20}x + 3b_{30}x^2 \end{cases}$$

$$\Rightarrow \begin{cases} f|_{p_2} = b_{00} \\ f|_{p_3} = b_{00} + b_{100} + b_{200} + b_{30} \\ \frac{\partial f}{\partial x}|_{p_2} = b_{10} \\ \frac{\partial f}{\partial x}|_{p_3} = b_{10} + 2b_{20} + 3b_{30} \end{cases}$$

Which is the same interpolation.

We can repeat for all faces and all edges to check that we are continuous everywhere.

Tricubic Interpolation

This Tricubic Interpolation has been implemented in a Python package.

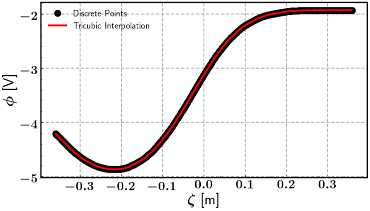
- It can use exact derivatives if provided, or use finite differences to estimate them.
- It has been thoroughly tested to check that it can **exactly** reconstruct any "tricubic" polynomial when using exact derivatives.
- It can be found in https://github.com/kparasch/TricubicInterpolation/.

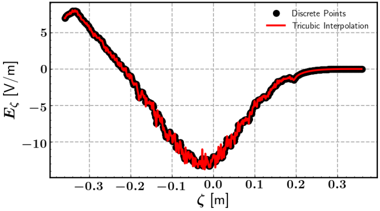
Section 5

Examples

Problem #1 - Simulation can be (very) Noisy

At $x = \Delta x, y = 0$:

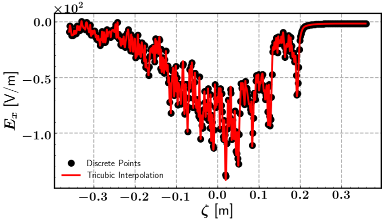


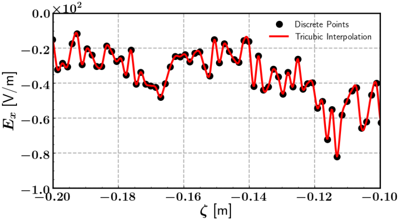


- Interpolation of ϕ is flawless.
- Derivative on the other hand can be very noisy.

Problem #1 - Pinch is Noisy

At $x = \Delta x, y = 0$:



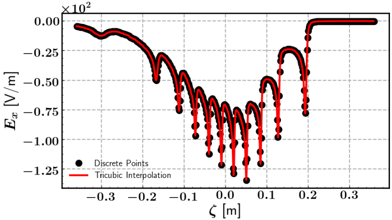


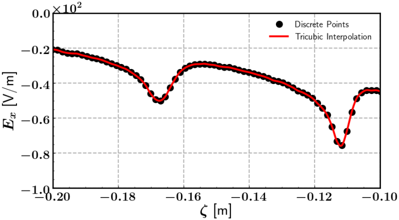
Zoom in $\zeta \in (-0.2, -0.1)$ of left figure.

- Even for noisy simulations of pinches, interpolation scheme does not disappoint.
- Simulation of the pinch still suffers from macroparticle noise.
- Solution: Reduce noise by averaging many pinches.

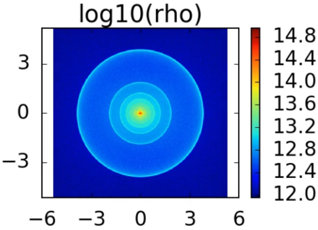
Problem #1 - Pinch is Noisy

At $x = \Delta x, y = 0$:





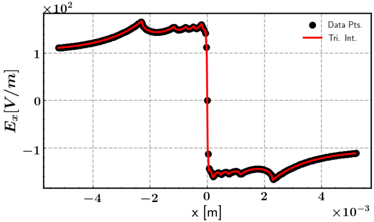
Zoom in $\zeta \in (-0.2, -0.1)$ of left figure.



- Averaging 2000 pinches reveals clear structure.
- Interpolation scheme is looks good.

If step size is not small enough

Worst case is when we look with respect to a transverse direction. The potential **flips** very quickly. (Beam sigma here is $3.66 \cdot 10^{-4}$ m)



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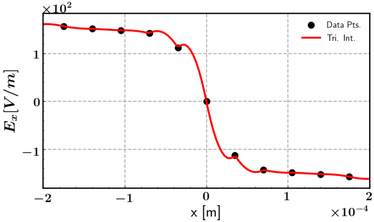

Very sharp changes can lead to unnatural "wiggles" inbetween cells.

If step size is not small enough

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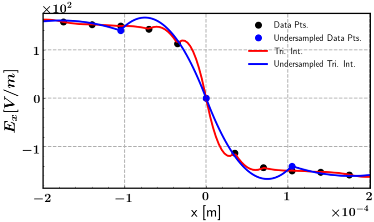


Very sharp changes can lead to unnatural "wiggles" inbetween cells. Through undersampling, the bumps get worse. We need to find a way to quantify and control these artifacts.

Conclusions

- We symplectified our kick by using the Tricubic Interpolation scheme.
- We implemented the Tricubic Interpolation in a tested Python package.
- We studied the behaviour of the interpolation scheme in order to predict possible problems.

Next steps:

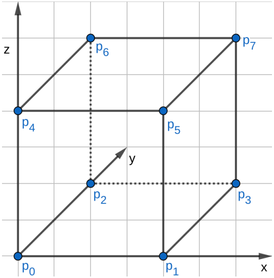
- See if the interpolation can be improved or if the "wiggles" can be quantified.
- Begin some preliminary tracking in PySixtrack.
- Do some serious tracking with SixTrackLib.

I thank you for your attention!

Appendices

Kinks

Can the kinks be the artifact of the other dimensions?



$$f(x,y,z) = \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} \alpha_{ijk} x^i y^j z^k$$

$$\frac{\partial f}{\partial x}(x,y,z) = \sum_{i=1}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} i\alpha_{ijk} x^{i-1} y^j z^k$$

$$\xrightarrow{y,z=0}$$

$$\begin{cases} f(x,0,0) = \alpha_{000} + \alpha_{100}x + \alpha_{200}x^2 + \alpha_{300}x^3 \\ \frac{\partial f}{\partial x}(x,0,0) = \alpha_{100} + 2\alpha_{200}x + 3\alpha_{300}x^2 \end{cases}$$

$$\Rightarrow \begin{cases} f|_{p_0} = \alpha_{000} \\ f|_{p_1} = \alpha_{000} + \alpha_{100} + \alpha_{200} + \alpha_{300} \\ \frac{\partial f}{\partial x}|_{p_0} = \alpha_{100} \\ \frac{\partial f}{\partial x}|_{p_1} = \alpha_{100} + 2\alpha_{200} + 3\alpha_{300} \end{cases}$$

On an edge of the cube, the interpolation depends only on the values of the function and its derivative with respect to the independent variable of the edge. Answer is no!

Symplecticity - Why

Violation of symplecticity implies that integrals of motion are no longer conserved. Long-term tracking simulations **can** lead to wrong conclusions.

Consider a linear one-turn map M and

- a thick quadrupole map:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \begin{pmatrix} \cos(k\Delta s) & \frac{1}{k}\sin(k\Delta s) \\ -k\sin(k\Delta s) & \cos(k\Delta s) \end{pmatrix} \cdot M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

- a 1st order Taylor approximation of thick quadrupole:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \begin{pmatrix} 1 & \Delta s \\ -k^2\Delta s & 1 \end{pmatrix} \cdot M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

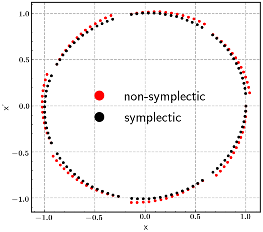
with

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

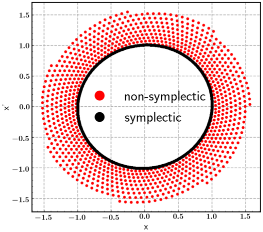
Non-Symplectic Tracking

Tracking with a **large** symplectic error ($k = 0.3, \Delta s = 0.1$):

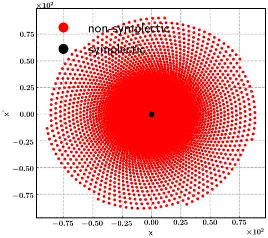
- Turns = 100



- Turns = 1000



- Turns = 10000



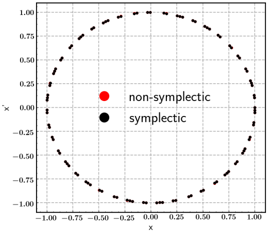
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \begin{pmatrix} \cos(k\Delta s) & \frac{1}{k}\sin(k\Delta s) \\ -k\sin(k\Delta s) & \cos(k\Delta s) \end{pmatrix} \cdot M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{i}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \begin{pmatrix} 1 & \Delta s \\ -k^2\Delta s & 1 \end{pmatrix} \cdot M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{i}$$

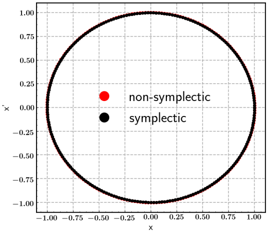
Non-Symplectic Tracking

Tracking with a **small** symplectic error ($k = 0.3, \Delta s = 0.01$):

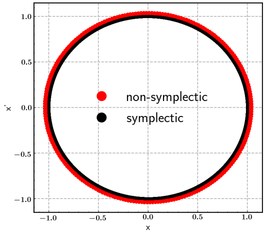
- Turns = 100



- Turns = 1000



- Turns = 10000



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \begin{pmatrix} \cos(k\Delta s) & \frac{1}{k}\sin(k\Delta s) \\ -k\sin(k\Delta s) & \cos(k\Delta s) \end{pmatrix} \cdot M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{i}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \begin{pmatrix} 1 & \Delta s \\ -k^2\Delta s & 1 \end{pmatrix} \cdot M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{i}$$

If the symplectic error is small, is symplecticity actually necessary?

Symplectifying in 6D

Step #2

Given a regular grid of a scalar potential $\phi^{(i,j)}$ at regular steps of ζ, produce a symplectic thin-lens 6D kick.

$$
\begin{aligned}
x &\mapsto x \\
p_x &\mapsto p_x - \frac{qL}{\beta^2\gamma mc^2}\frac{\partial\phi}{\partial x}(x,y,\zeta) \\
y &\mapsto y \\
p_y &\mapsto p_y - \frac{qL}{\beta^2\gamma mc^2}\frac{\partial\phi}{\partial y}(x,y,\zeta) \\
\zeta &\mapsto \zeta \\
\delta &\mapsto \delta + f(x,y,\zeta)
\end{aligned}
$$

where $f(x,y,\zeta)$ is an arbitrary function of x,y,ζ. In addition to the previous condition, $f(x,y,\zeta)$ must satisfy:

$$
\frac{\partial f}{\partial x} = -\frac{qL}{\beta^2\gamma mc^2}\frac{\partial}{\partial\zeta}\left(\frac{\partial\phi}{\partial x}\right)
$$

$$
\frac{\partial f}{\partial y} = -\frac{qL}{\beta^2\gamma mc^2}\frac{\partial}{\partial\zeta}\left(\frac{\partial\phi}{\partial y}\right)
$$

[3] Thin-lens in the sense that x, y, ζ remain unchanged.

Symplectifying in 6D
First condition

- $$\frac{\partial f}{\partial x} = -\frac{qL}{\beta^2 \gamma m c^2} \frac{\partial}{\partial \zeta}\left(\frac{\partial \phi}{\partial x}\right)$$
  Integration gives:

  $$f(x, y, \zeta) = \int \frac{\partial f}{\partial x} \mathrm{d}x = -\int \frac{qL}{\beta^2 \gamma m c^2} \frac{\partial}{\partial \zeta}\left(\frac{\partial \phi}{\partial x}\right) \mathrm{d}x$$

  Because we approximate ϕ such that it has globally continuous derivatives,

  $$f(x, y, \zeta) = -\frac{qL}{\beta^2 \gamma m c^2} \frac{\partial \phi}{\partial \zeta}(x, y, \zeta) + g(y, \zeta)$$

  where $g(y, \zeta)$ is again an arbitrary function.

Symplectifying in 6D

$$f(x, y, \zeta) = -\frac{qL}{\beta^2 \gamma m c^2} \frac{\partial \phi}{\partial \zeta}(x, y, \zeta) + g(y, \zeta)$$

Second condition:

- Replacing f into the other condition:
  $\frac{\partial f}{\partial y} = -\frac{qL}{\beta^2 \gamma m c^2} \frac{\partial}{\partial \zeta}\left(\frac{\partial \phi}{\partial y}\right)$
  We arrive to

$$\frac{\partial g}{\partial y} = 0$$

which means that

$$g(y, \zeta) = g(\zeta)$$

Symplectic Kick

Summary

The 6D map will be symplectic for all momentum deviation kicks of the form

$$\delta \mapsto \delta - \frac{qL}{\beta^2 \gamma m c^2} \frac{\partial \phi}{\partial \zeta}(x, y, \zeta) + g(\zeta)$$

with an arbitrary $g(\zeta)$ function.

The simplest choice is to set $g(\zeta) = 0$. Analytical calculations on the **physical** thin-lens approximation[5] of the electron cloud interaction on the beam particles arrive on the same map with $g(\zeta) = 0$.

[3]See future presentation.