Introducing the Helicity-Flow Method

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Work in Progress

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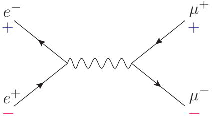
Outline

1. Introducing the Spinor-Helicity Method
2. Building a (Massless) QED Helicity-Flow + Examples
3. (Massless) QCD Helicity Flow + Example
4. Summary & Outlook

Calculating an Amplitude: the Spinor-Helicity Idea

- Standard Feynman diagrams have matrix structure
  - Takes much effort to square, simplify traces of γ^μ
- Give each external particle an explicit helicity
  - Now called a Spinor-helicity diagram
  - Now diagram is a complex number - easy to square
- Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
- Dirac spinors reducible into two irreps of diff chirality
  - Weyl rep of Dirac algebra naturally separates the two irreps
  - Massless particles chirality $\sim$ helicity

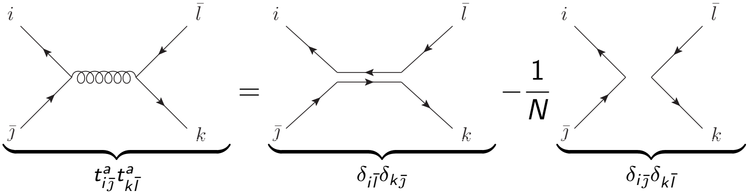
Calculating an Amplitude: the Spinor-Helicity Pieces

- Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
  - $\gamma^{\mu} = \begin{pmatrix} 0 & \sqrt{2}\tau^{\mu,\dot{\alpha}\beta} \\ \sqrt{2}\bar{\tau}^{\mu}_{\alpha\dot{\beta}} & 0 \end{pmatrix}$, $\sqrt{2}\tau^{\mu,\dot{\alpha}\beta} = \sigma^{\mu,\dot{\alpha}\beta}$
  - $v(p) = \begin{pmatrix} \tilde{\lambda}_p^{\dot{\alpha}} \\ \lambda_{p,\alpha} \end{pmatrix}$, $\bar{u}(p) = \begin{pmatrix} \tilde{\lambda}_{p,\dot{\alpha}} & \lambda_p^{\alpha} \end{pmatrix}$
  - $\varepsilon_+^{\mu}(p,r) = \frac{\tilde{\lambda}_{p,\dot{\alpha}}\tau^{\mu,\dot{\alpha}\beta}\lambda_{r,\beta}}{\langle rp \rangle}$, $\varepsilon_-^{\mu}(p,r) = \frac{\lambda_p^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}_r^{\dot{\beta}}}{[pr]}$
- Final result in terms of inner products:
  - $\lambda_i^{\alpha}\lambda_{j\alpha} \equiv \langle ij \rangle$, $\tilde{\lambda}_{i,\dot{\beta}}\tilde{\lambda}_j^{\dot{\beta}} \equiv [ij]$, $\langle ij \rangle, [ij] \sim \sqrt{s_{ij}}$
  - e.g. 

$= \frac{2ie^2}{s_{e^+e^-}}[e^-\mu^+]\langle\mu^- e^+\rangle$

Define Problem

- Can we still improve on this?
    - Deriving spinor inner products $\langle ij \rangle$, $[kl]$ requires at least 2 steps
        - Re-write every object as spinors
        - Use Fierz identity $\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tau_{\mu}^{\dot{\alpha}\beta} = \delta_{\alpha}^{\ \beta}\delta^{\dot{\alpha}}_{\ \dot{\beta}}$
    - Not intuitive which inner products we obtain
- In SU(N) use graphical reps for calculations, e.g. Fierz id.



- Spinor-helicity $\equiv su(2) \oplus su(2)$
    - Can we do the same?

Creating a Helicity Flow for QED: Part 1

- Key difference:
  - Colour $\equiv$ single $SU(N)$: generators $t^a \to \delta$'s
  - Spinor-hel $\equiv su(2) \oplus su(2)$: $\tau^\mu, \bar{\tau}^\mu, \lambda, \tilde{\lambda}, \varepsilon^\mu_\pm, \to \langle ij\rangle, [kl]$

- First step: Spinors and their inner products
  - $\lambda_i^\alpha \lambda_{j,\alpha} = \langle ij\rangle = -\langle ji\rangle =$ 

  - $\tilde{\lambda}_{i,\dot{\beta}} \tilde{\lambda}_j^{\dot{\beta}} = [ij] = -[ji] =$ 

  - $\lambda_{j,\alpha} =$ 

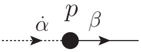
 , $\lambda_i^\alpha =$ 

 , $\delta_\alpha^{\ \beta} =$ 

  - $\tilde{\lambda}_{i,\dot{\alpha}} =$ 

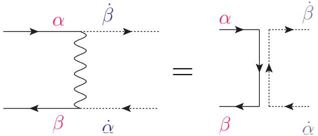
 , $\tilde{\lambda}_j^{\dot{\alpha}} =$ 

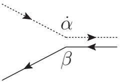
 , $\delta^{\dot{\beta}}_{\ \dot{\alpha}} =$ 

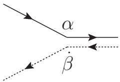
- Second step: Fermion propagators
  - $\not{p} = \sqrt{2} p^\mu \tau_\mu^{\dot{\alpha}\beta} \overset{p^2=0}{=} \tilde{\lambda}_p^{\dot{\alpha}} \lambda_p^\beta =$ 

Creating a Helicity Flow for QED: Part 2

- Third step: Vertices and vector propagators
  - vertices $\frac{\gamma^\mu}{\sqrt{2}} \to \tau^\mu, \bar{\tau}^\mu$ contracted with vector propagator $g_{\mu\nu}$
  - Fierz identity with indices: $\bar{\tau}^\mu_{\alpha\dot{\beta}} \tau_\mu^{\dot{\alpha}\beta} = \delta_\alpha{}^\beta \delta^{\dot{\alpha}}{}_{\dot{\beta}}$
  - Fierz identity with flow:

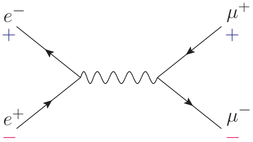


  - $\Rightarrow \tau^{\mu,\dot{\alpha}\beta} =$ 

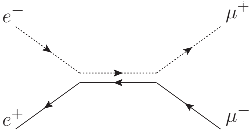
 , $\bar{\tau}^\mu_{\alpha\dot{\beta}} =$ 

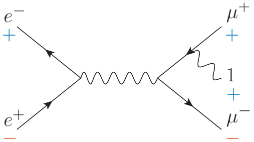
  - $\Rightarrow g_{\mu\nu} =$  , or 
  - Fierz identity already utilised in flow rule

Simple QED Examples

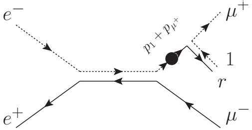


$$= \frac{2ie^2}{s_{e^+e^-}}$$



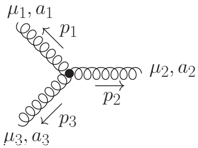
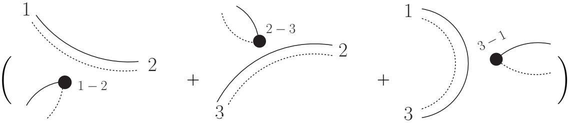


$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}$$



- Immediately read off inner products
- Regular spinor-hel requires a few steps

The Non-abelian Massless QCD Flow Vertices

<table>
<tr><th>Feynman</th><th>Flow</th></tr>
<tr><td>
</td><td>$-\frac{g_s f^{abc}}{2}$ 
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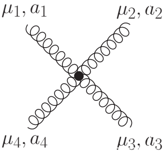
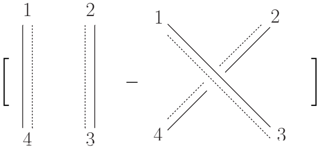
<tr><td>
</td><td>$ig_s^2 \sum_{Z(2,3,4)} f^{a_1 a_2 b} f^{b a_4 a_3}$ 
</td></tr>
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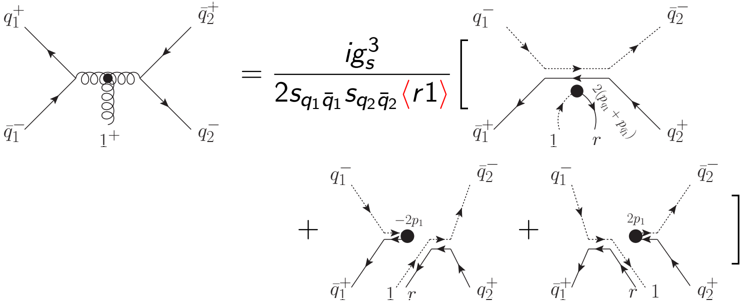
QCD Example: $q_1\bar{q}_1 \to q_2\bar{q}_2 g$

- Triple-gluon vertex provides new structures



$$= \frac{ig_s^3}{2 s_{q_1\bar{q}_1} s_{q_2\bar{q}_2} \langle r1 \rangle} \Big[\ldots + \ldots + \ldots \Big]$$

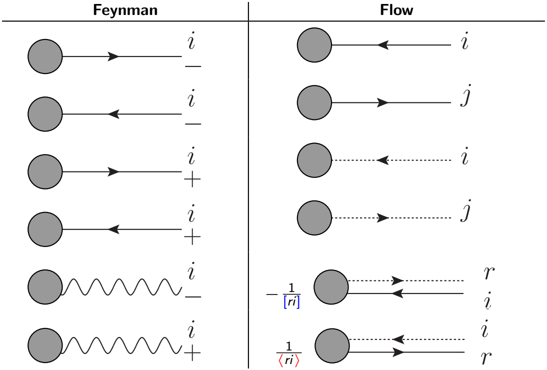
Summary

- Helicity flow allows for single-line calculation of Feynman diagram
- Also gives transparent/intuitive picture of inner products
- In contrast, spinor-hel method:
  - Requires multiple steps
  - Final result intransparent/unintuitive
- Massless QED and QCD tree-level done
- Useful for any generator using diagrams to avoid dealing with Lorentz algebra

Outlook

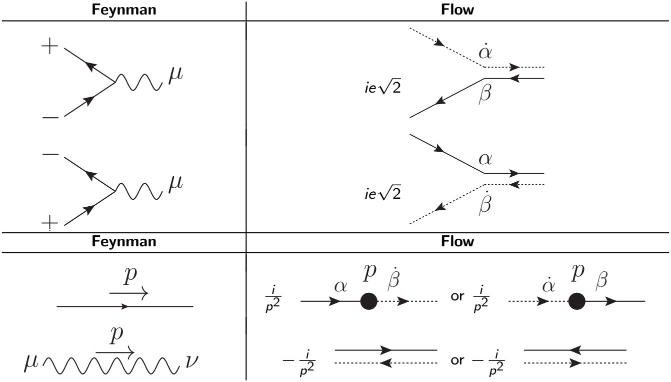
- Initial paper coming soon
- Complete the SM at tree level
- Loop calculations
- Applications within generator(s)
- Amplitude-level calculations

The QED Flow Rules: External Particles



Everything already Fierzed, in terms of spinors

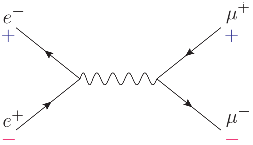
The QED Flow Rules: Vertices and Propagators



Everything already Fierzed, in terms of spinors

Simplest QED Example

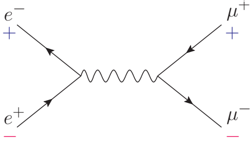
- Regular spinor-helicity $\equiv$ easy



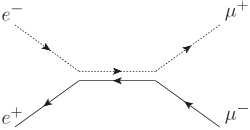
$$= \frac{2ie^2}{s_{e^+e^-}} (\tilde{\lambda}_{e^-,\dot{\alpha}} \tau_{\mu}^{\dot{\alpha}\beta} \lambda_{e^+,\beta})(\lambda_{\mu^-}^{\alpha} \bar{\tau}^{\mu}_{\alpha\dot{\beta}} \tilde{\lambda}_{\mu^+}^{\dot{\beta}})$$

$$= \frac{2ie^2}{s_{e^+e^-}} \tilde{\lambda}_{e^-,\dot{\alpha}} \tilde{\lambda}_{\mu^+}^{\dot{\alpha}} \lambda_{\mu^-}^{\beta} \lambda_{e^+,\beta} \equiv \frac{2ie^2}{s_{e^+e^-}} [e^- \mu^+] \langle \mu^- e^+ \rangle$$

- Helicity flow $\equiv$ super easy and intuitive

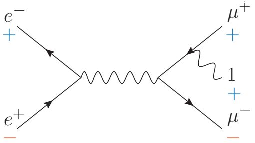


$$= \frac{2ie^2}{s_{e^+e^-}}$$



Next Simplest QED Example

- Regular spinor-helicity $\equiv$ easy



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_\mu^{\dot{\alpha}\beta}\lambda_{e^+,\beta}\right)\left(\lambda_{\mu^-}^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}(\not{p}_1+\not{p}_{\mu^+})^{\dot{\beta}\delta}\not{\epsilon}^{+}_{\delta\dot{\gamma}}(1,r)\tilde{\lambda}^{\dot{\gamma}}_{\mu^+}\right)$$

$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_\mu^{\dot{\alpha}\beta}\lambda_{e^+,\beta}\right)\tilde{\lambda}_{1,\dot{\delta}}\tilde{\lambda}^{\dot{\delta}}_{\mu^+}$$

$$\times\left(\lambda_{\mu^-}^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}_1^{\dot{\beta}}\lambda_1^{\delta}\lambda_{r,\delta}+\lambda_{\mu^-}^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}_{\mu^+}^{\dot{\beta}}\lambda_{\mu^+}^{\delta}\lambda_{r,\delta}\right)$$

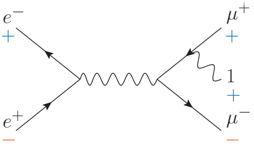
$$\sim \lambda_{\mu^-}^{\beta}\lambda_{e^+,\beta}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}_1^{\dot{\alpha}}\lambda_1^{\delta}\lambda_{r,\delta}+\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}_{\mu^+}^{\dot{\alpha}}\lambda_{\mu^+}^{\delta}\lambda_{r,\delta}\right)\tilde{\lambda}_{1,\dot{\delta}}\tilde{\lambda}^{\dot{\delta}}_{\mu^+}$$

Correct Answer

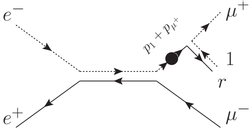
$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\left([e^-1]\langle 1r\rangle+[e^-\mu^+]\langle\mu^+r\rangle\right)[1\mu^+]\langle\mu^-e^+\rangle$$

Next Simplest QED Example

- Helicity flow $\equiv$ super easy and intuitive



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}\, s_{\mu^+1} \langle r1 \rangle}$$



- Immediately read off inner products

Correct Answer

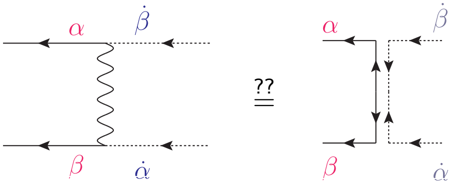
$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}\, s_{\mu^+1} \langle r1 \rangle} \Big([e^- 1] \langle 1 r \rangle + [e^- \mu^+] \langle \mu^+ r \rangle \Big) [1 \mu^+] \langle \mu^- e^+ \rangle$$

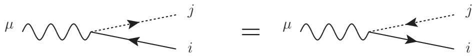
Calculating A_i: the Spinor-Helicity Method

- Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
- Weyl representation of Dirac algebra naturally separates the two reps
  - $\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^{\mu,\dot{\alpha}\beta} \\ \sqrt{2}\bar{\tau}^\mu_{\alpha\dot{\beta}} & 0 \end{pmatrix}$, $\sqrt{2}\tau^\mu = (1, \vec{\sigma})$, $\sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma})$
  - $\mathrm{Tr}(\tau^\mu \bar{\tau}^\mu) = g^{\mu\nu}$, $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $P_\pm = \frac{1}{2}(1 \pm \gamma^5)$
  - $u(p) = \begin{pmatrix} u_-(p) \\ u_+(p) \end{pmatrix} = \begin{pmatrix} v_+(p) \\ v_-(p) \end{pmatrix} = \begin{pmatrix} \tilde{\lambda}^{\dot{\alpha}}_p \\ \lambda_{p,\alpha} \end{pmatrix}$, $\bar{u}(p) = \begin{pmatrix} \tilde{\lambda}_{p,\dot{\alpha}} & \lambda^\alpha_p \end{pmatrix}$
- Final result in terms of inner products:
  - $\lambda^\alpha_i \lambda_{j\alpha} \equiv \langle ij \rangle$, $\tilde{\lambda}_{i,\dot{\beta}} \tilde{\lambda}^{\dot{\beta}}_j \equiv [ij]$, $\langle ij \rangle, [ij] \sim \sqrt{s_{ij}}$
- $\varepsilon^\mu_+(p,r) = \dfrac{\tilde{\lambda}_{p,\dot{\alpha}} \tau^{\mu,\dot{\alpha}\beta} \lambda_{r,\beta}}{\langle rp \rangle}$, $\varepsilon^\mu_-(p,r) = \dfrac{\lambda^\alpha_p \bar{\tau}^\mu_{\alpha\dot{\beta}} \tilde{\lambda}^{\dot{\beta}}_r}{[pr]}$
- No complicated traces of γ matrices, rather simple identities like:
  - $(\tilde{\lambda}_{i,\dot{\alpha}} \tau^{\dot{\alpha}\beta}_\mu \lambda_{j,\beta})(\lambda^\gamma_k \bar{\tau}^\mu_{\gamma\dot{\delta}} \tilde{\lambda}^{\dot{\delta}}_l) = \lambda^\beta_i \lambda_{k\beta} \tilde{\lambda}_{l,\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}}_j = \langle ik \rangle [lj]$

Fun with Arrows and the Fierz Identity

- Sometimes have to contract $\tau^{\mu}\tau_{\mu}$ or $\bar{\tau}^{\mu}\bar{\tau}_{\mu}$
- This would lead to arrows pointing towards each other, e.g.



- To fix, use charge conservation of a current:
  - $\lambda_i^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}_j^{\dot{\beta}} = \tilde{\lambda}_{j,\dot{\alpha}}\tau^{\mu,\dot{\alpha}\beta}\lambda_{i,\beta}$
- Or in pictures:
  - 

How to Calculate a (Massless) Scattering Amplitude

- QCD often factorise colour, use helicity basis for kinematics

$$\mathcal{M}_h\left(1^{h_1},\ldots,n^{h_n}\right)=\sum_i C_i A_i\left(p_1^{h_1},\ldots,p_n^{h_n}\right)$$

- $C_i \equiv$ colour factor
  - QED: $C_i = 1$
- $A_i \equiv$ kinematic amplitude
  - Cross incoming particles to outgoing
  - Each particle j is given a specific helicity h_j
  - Since massless, helicity $\sim$ chirality