Color Evolution in Parton Showers

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Perturbative Cross Section

The main focus of this workshop is to calculate the pQCD cross sections as precise as possible, thus we have a pretty integral

$$
\begin{aligned}
\sigma[O_J] = \sum_m \frac{1}{m!} \sum_{\{a,b,f_1,\dots,f_m\}} \int_0^1 d\eta_\mathrm{a} \overbrace{\int_{\eta_\mathrm{a}}^1 \frac{dz}{z}\, \mathit{\Gamma}^{-1}_{aa'}(z,\mu^2)\, f_{a'/A}(\eta_\mathrm{a}/z,\mu^2)}^{\text{Bare PDF}} \\
\times \int_0^1 d\eta_\mathrm{b} \int_{\eta_\mathrm{b}}^1 \frac{d\bar{z}}{\bar{z}}\, \mathit{\Gamma}^{-1}_{bb'}(\bar{z},\mu^2)\, f_{b'/A}(\eta_\mathrm{b}/\bar{z},\mu^2) \\
\times \int d\phi(\eta_\mathrm{a}\eta_\mathrm{b}s,\{p,f\}_m) \big\langle M(\{p,f\}_m)\big| \underbrace{O_J(\{p,f\}_m)}_{\text{IR safe measurement operator}} \overbrace{\big|M(\{p,f\}_m)\big\rangle}^{\text{Partonic matrix element}} \\
+ \mathcal{O}\left(\frac{\Lambda^2_{QCD}}{\mu_J^2}\right)
\end{aligned}
$$

Error of the factorization
(Cannot be beaten by calculating higher and higher order.)

and here the MSbar parton in parton renormalised PDF is

$$
\mathit{\Gamma}_{aa'}(z,\mu^2) = \delta(1-z)\delta_{aa'} - \frac{\alpha_\mathrm{s}(\mu^2)}{2\pi}\frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} P_{aa'}(z) + \cdots \quad .
$$

Statistical Space

Introducing the statistical space we can represent the QCD density operator as a vector

$$\sigma[O_J] = \underbrace{(1|}_{\text{All the initial and final state sums and integrals}} \mathcal{O}_J \overbrace{\left[\mathcal{F}(\mu^2)\circ \mathcal{Z}_F(\mu^2)\right]}^{\text{Bare PDFs for both incoming hadrons}} \underbrace{|\rho(\mu^2))}_{|M\rangle\langle M|}$$

Bare PDFs for both incoming hadrons

All the initial and final state sums and integrals

QCD density operator
Describes the fully exclusive partonic final states.

The physical cross section is RG invariant as well as the QCD density operator and the bare PDF.

$$\mu^2 \frac{d}{d\mu^2}|\rho(\mu^2)) = \mu^2 \frac{d}{d\mu^2}\left[\mathcal{F}(\mu^2)\circ \mathcal{Z}_F(\mu^2)\right] = 0 + \mathcal{O}(\alpha_s^{k+1})$$

Perturbative expansion of the density operator

$$|\rho(\mu^2)) = \sum_{n=0}^{k}\left[\frac{\alpha_s(\mu^2)}{2\pi}\right]^n \sum_{\substack{n_R=0 \\ n_R+n_V=n}}^{n}\sum_{n_V=0}^{n} |\rho^{(n_R,n_V)}(\mu^2))$$

Number of real radiations (n_R)

Number of loops (n_V)

pQCD Cross Sections

ZN, D. Soper, **Phys.Rev. D98** (2018) no.1, 014034

Singularities cancel each other here

Subtractions

$$\sigma[O_J] = \overbrace{\big(1\big|\,\big[\mathcal{F}(\mu^2)\circ\mathcal{Z}_F(\mu^2)\big]\,\mathcal{D}(\mu^2)}\;\underbrace{\mathcal{D}^{-1}(\mu^2)\,\mathcal{O}_J\,\big|\rho(\mu^2)\big)}_{=|\rho_{\rm H}(\mu^2))}$$

Hard part, finite in d=4 dimension

$$+\,\mathcal{O}(\alpha_{\rm s}^{k+1}L^{2k+2})$$
$$+\,\mathcal{O}(\Lambda_{QCD}^2/\mu_J^2)$$

By solving the corresponding renormalization group equation

Unitary shower

Hopefully n << 2k+1

$$\sigma[O_J] = \big(1\big|\mathcal{O}_J\,\overbrace{\mathcal{U}(\mu_{\rm f}^2,\mu^2)}\,\underbrace{\mathcal{U}_{\mathcal{V}}(\mu_{\rm f}^2,\mu^2)}\,\mathcal{F}(\mu^2)\,\big|\rho_{\rm H}(\mu^2)\big) + \mathcal{O}(\overbrace{\alpha_{\rm s}^{k+1}L^n}) + \mathcal{O}(\mu_{\rm f}^2/\mu_J^2)$$

Resummation of threshold effects

Unitary Shower Operator

Here we focus on the the unitary shower

$$\mathcal{U}(\mu_{\mathrm{f}}^2, \mu_{\mathrm{H}}^2) = \mathbb{T}\exp\left(\int_{\mu_{\mathrm{f}}^2}^{\mu_{\mathrm{H}}^2} \frac{d\mu^2}{\mu^2}\,\frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\left[\mathcal{S}^{(1)}(\mu^2) + \frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\mathcal{S}^{(2)}(\mu^2) + \cdots\right]\right)$$

Here we are interested only at first order level:

$$\frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\mathcal{S}^{(1)}(\mu^2) = \overbrace{\mathcal{H}(\mu^2)}^{\text{Real emissions}} - \underbrace{\mathcal{V}_{\mathrm{Re}}(\mu^2)}_{\text{Inclusive splitting operator}} - \overbrace{\mathrm{i}\pi\mathcal{V}_{\mathrm{i}\pi}(\mu^2)}^{\text{Imaginary part of the 1-loop contributions}}$$

Unitary condition tells us:

$$\big(1\big|\mathcal{S}^{(1)}(\mu^2) = \big(1\big|\big[\mathcal{H}(\mu^2) - \mathcal{V}_{\mathrm{Re}}(\mu^2)\big] = \big(1\big|\mathcal{V}_{\mathrm{i}\pi}(\mu^2) = 0$$

Evolution Equation

The shower operator obeys the following integral equation:

$$\mathcal{U}(\mu_2^2, \mu_1^2) = \mathcal{N}(\mu_2^2, \mu_1^2) + \int_{\mu_2^2}^{\mu_1^2} \frac{d\mu^2}{\mu^2} \mathcal{U}(\mu_2^2, \mu^2) \mathcal{H}(\mu^2) \mathcal{N}(\mu^2, \mu_1^2)$$

where the no-splitting (**Sudakov**) operator is

$$\mathcal{N}(\mu_2^2, \mu_1^2) = \mathbb{T} \exp\left\{ - \int_{\mu_2^2}^{\mu_1^2} \frac{d\mu^2}{\mu^2} \underbrace{\left[\mathcal{V}_{\mathrm{Re}}(\mu^2) + \mathrm{i}\pi \mathcal{V}_{\mathrm{i}\pi}(\mu^2)\right]} \right\}$$

This is not a diagonal operator and it is impossible to diagonalise when we have ~*O(10)* partons.

Let's define a systematic approximation!

LC+ Approximation

ZN, D. Soper, **JHEP06** (2012) 044

The **real splittings** are described by

$$\mathcal{H}(\mu^2)\big|\{p,f,c',c\}_m\big)$$
$$\propto \sum_{l,k} H_{lk}(\mu^2)\big|\{p,f\}_m\big)\left\{t_l^\dagger\big|\{c\}_m\big\rangle\big\langle\{c'\}_m\big|t_k + t_k^\dagger\big|\{c\}_m\big\rangle\big\langle\{c'\}_m\big|t_l\right\}$$

The index l always represents the emitter parton and the emitted parton can be collinear only with l.

The **inclusive splitting operator** is

$$\mathcal{V}_{\rm Re}(\mu^2)\big|\{p,f,c',c\}_m\big)$$
$$\propto \sum_{l,k} V_{lk}(\mu^2)\big|\{p,f\}_m\big)\left\{\big|\{c\}_m\big\rangle\big\langle\{c'\}_m\big|[t_k\cdot t_l^\dagger] + [t_l\cdot t_k^\dagger]\big|\{c\}_m\big\rangle\big\langle\{c'\}_m\big|\right\}$$

We need an approximation (only in the color space) that

- **can handle color interference** contributions
- is as minimal approximation as possible
- is **exact in the collinear and soft-collinear** regions
- makes some **harm only in the wide angle soft** region
- preserves unitary

LC+ Approximation

We insert a projection **only on the spectator side**

$$t_k^\dagger \big|\{c\}_m\big\rangle \longrightarrow C(l, m+1) t_k^\dagger \big|\{c\}_m\big\rangle$$
$$\big\langle\{c'\}_m\big| t_k \longrightarrow \big\langle\{c'\}_m\big| t_k C(l, m+1)^\dagger$$

The **operator** ***C(l, m+1)*** is defined by it action on the basis states:

$$C(l, m+1)\big|\{\hat{c}\}_{m+1}\big\rangle = \begin{cases} \big|\{\hat{c}\}_{m+1}\big\rangle & \text{if } l \text{ and } m+1 \text{ are color connected in } \{\hat{c}\}_{m+1} \\ 0 & \text{otherwise} \end{cases}$$

(In string basis l and m+1 are color connected when they are next to each other along the fermion line.)

In the inclusive splitting operator, the color simplifies a lot:

$$[t_l \cdot t_k^\dagger]\big|\{c\}_m\big\rangle \longrightarrow [t_l \cdot C(l, m+1) t_k^\dagger]\big|\{c\}_m\big\rangle = \big|\{c\}_m\big\rangle \frac{t_l^2}{1 + \delta_{\mathrm{g} f_l}}$$

$$\big\langle\{c'\}_m\big|[t_k \cdot t_l^\dagger] \longrightarrow \big\langle\{c'\}_m\big|[t_k C(l, m+1)^\dagger \cdot t_l^\dagger] = \frac{t_l^2}{1 + \delta_{\mathrm{g} f_l}} \big\langle\{c'\}_m\big|$$

LC+ Approximation

In LC+ approximation every basis state is eigenstate of the inclusive splitting operator

$$\mathcal{V}^{\mathrm{LC+}}(\mu^2)\big|\{p,f,c',c\}_m\big) = \lambda(\{p,f,c',c\}_m,\mu^2)\big|\{p,f,c',c\}_m\big)$$

and we have **Sudakov factor** instead of Sudakov operator

$$\mathcal{N}^{\mathrm{LC+}}(\mu_2^2,\mu_1^2)\big|\{p,f,c',c\}_m\big) = \exp\left\{-\int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\lambda(\{p,f,c',c\}_m,\mu^2)\right\}\big|\{p,f,c',c\}_m\big)$$

Based on this we can define the **LC+ parton shower** and its evolution equation is

$$\mathcal{U}^{\mathrm{LC+}}(\mu_2^2,\mu_1^2) = \mathcal{N}^{\mathrm{LC+}}(\mu_2^2,\mu_1^2) + \int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\,\mathcal{U}^{\mathrm{LC+}}(\mu_2^2,\mu^2)\mathcal{H}^{\mathrm{LC+}}(\mu^2)\mathcal{N}^{\mathrm{LC+}}(\mu^2,\mu_1^2)$$

Beyond LC+

ZN, D. Soper, **Phys.Rev. D99** (2019) no.5, 054009

Now we can define the operators of the soft wide angle emissions

$$\mathcal{H}(\mu^2) = \mathcal{H}^{\mathrm{LC+}}(\mu^2) + \Delta\mathcal{H}(\mu^2)$$
$$\mathcal{V}(\mu^2) = \mathcal{V}_{\mathrm{Re}}(\mu^2) + \mathrm{i}\pi\mathcal{V}_{\mathrm{i}\pi}(\mu^2) = \mathcal{V}^{\mathrm{LC+}}(\mu^2) + \Delta\mathcal{V}(\mu^2)$$

With these the full shower is

$$\mathcal{U}(\mu_2^2, \mu_1^2) = \mathcal{U}^{\mathrm{LC+}}(\mu_2^2, \mu_1^2) + \int_{\mu_2^2}^{\mu_1^2} \frac{d\mu^2}{\mu^2} \mathcal{U}(\mu_2^2, \mu^2) \left[\Delta\mathcal{H}(\mu^2) - \Delta\mathcal{V}(\mu^2)\right] \mathcal{U}^{\mathrm{LC+}}(\mu^2, \mu_1^2)$$

One can expand this in terms of the soft wide angle operators at a given order. In principle this is what we want, but this form is not efficient for implementation.
Let's **try something else**:

$$\mathcal{N}(\mu_2^2, \mu_1^2) = \underbrace{\mathcal{X}(\mu_2^2, \mu_1^2)}\, \mathcal{N}^{\mathrm{LC+}}(\mu_2^2, \mu_1^2)$$

Hopefully it is **simple enough** to deal with it perturbatively.

Beyond LC+

The evolution equation is

$$\mathcal{U}(\mu_2^2,\mu_1^2) = \mathcal{X}(\mu_2^2,\mu_1^2)\mathcal{N}^{\text{LC+}}(\mu_2^2,\mu_1^2) + \int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\,\mathcal{U}(\mu_2^2,\mu^2)\left[\mathcal{H}^{\text{LC+}}(\mu^2) + \Delta\mathcal{H}(\mu^2)\right]\mathcal{X}(\mu^2,\mu_1^2)\mathcal{N}^{\text{LC+}}(\mu^2,\mu_1^2)$$

When we iterate this equation we can **control the number of ΔH** operator **insertions**.

The X operator obeys its evolution equation:

$$\mathcal{X}(\mu_2^2,\mu_1^2) = 1 - \int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\,\mathcal{X}(\mu_2^2,\mu^2)\,\mathcal{N}^{\text{LC+}}(\mu_2^2,\mu^2)\Delta\mathcal{V}(\mu^2)\,\mathcal{N}^{\text{LC+}}(\mu_2^2,\mu^2)^{-1}$$

It is not immediately obvious but this operator depends **only pure soft** contributions

$$\mathcal{X}(\mu_2^2,\mu_1^2) = 1 - \int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\,\mathcal{X}(\mu_2^2,\mu^2)\,\mathcal{N}^{\text{LC+}}_{\text{soft}}(\mu_2^2,\mu^2)\Delta\mathcal{V}(\mu^2)\,\mathcal{N}^{\text{LC+}}_{\text{soft}}(\mu_2^2,\mu^2)^{-1}$$

and the **Sudakov factor** is

$$\mathcal{N}^{\text{LC+}}_{\text{soft}}(\mu_2^2,\mu_1^2)\big|\{p,f,c',c\}_m\big) = \exp\left\{-\int_{\mu_2^2}^{\mu_1^2}\frac{d\mu^2}{\mu^2}\,\underbrace{\lambda_{\text{soft}}(\{p,f,c',c\}_m,\mu^2)}\right\}\big|\{p,f,c',c\}_m\big)$$

It is rather **simple** and can be computed "***quasi-analytically***".

Beyond LC+

Expanding the shower operator in terms of ΔH and ΔV operators, we can write

$$\mathcal{U}(\mu_2^2,\mu_1^2) = \mathcal{U}^{\rm LC+}(\mu_2^2,\mu_1^2) + \underbrace{\mathcal{U}^{(1)}(\mu_2^2,\mu_1^2)}_{\sim\mathcal{O}([\Delta\mathcal{H}(\mu^2)-\Delta\mathcal{V}(\mu^2)])} + \overbrace{\mathcal{U}^{(2)}(\mu_2^2,\mu_1^2)}^{\sim\mathcal{O}([\Delta\mathcal{H}(\mu^2)-\Delta\mathcal{V}(\mu^2)]^2)} + \dots$$

This expansion is systematic and the **unitary condition** is satisfied term by term,

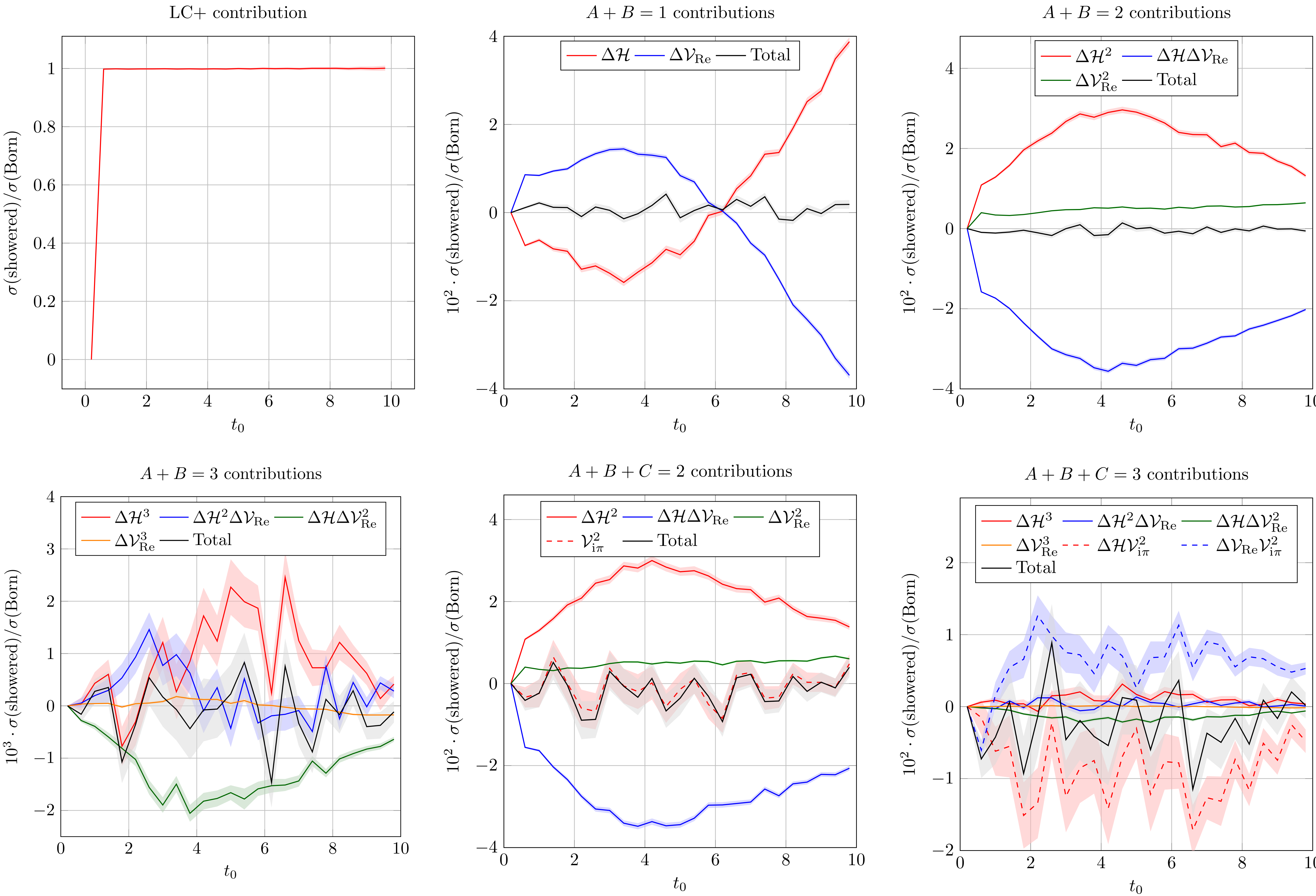
$$\big(1\big|\mathcal{U}^{\rm LC+}(\mu_2^2,\mu_1^2) = \big(1\big|$$

and for the corrections

$$\big(1\big|\mathcal{U}^{(k)}(\mu_2^2,\mu_1^2) = 0 \qquad \text{for } k = 1,2,3,\dots$$

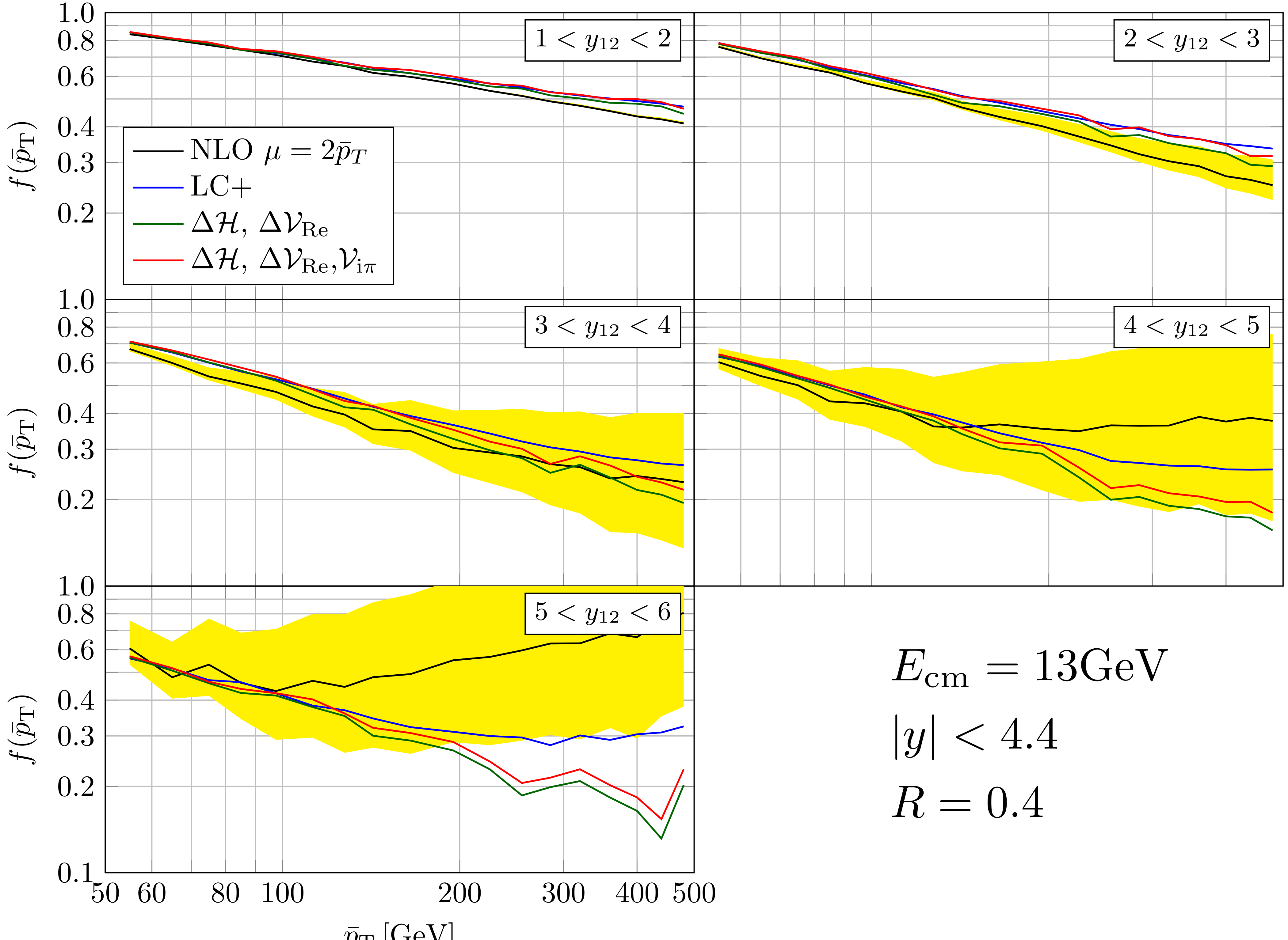
We can test this numerically!

Unitary Test



Rapidity Gap Survival

ZN, D. Soper, **arXiv:1905.07176**

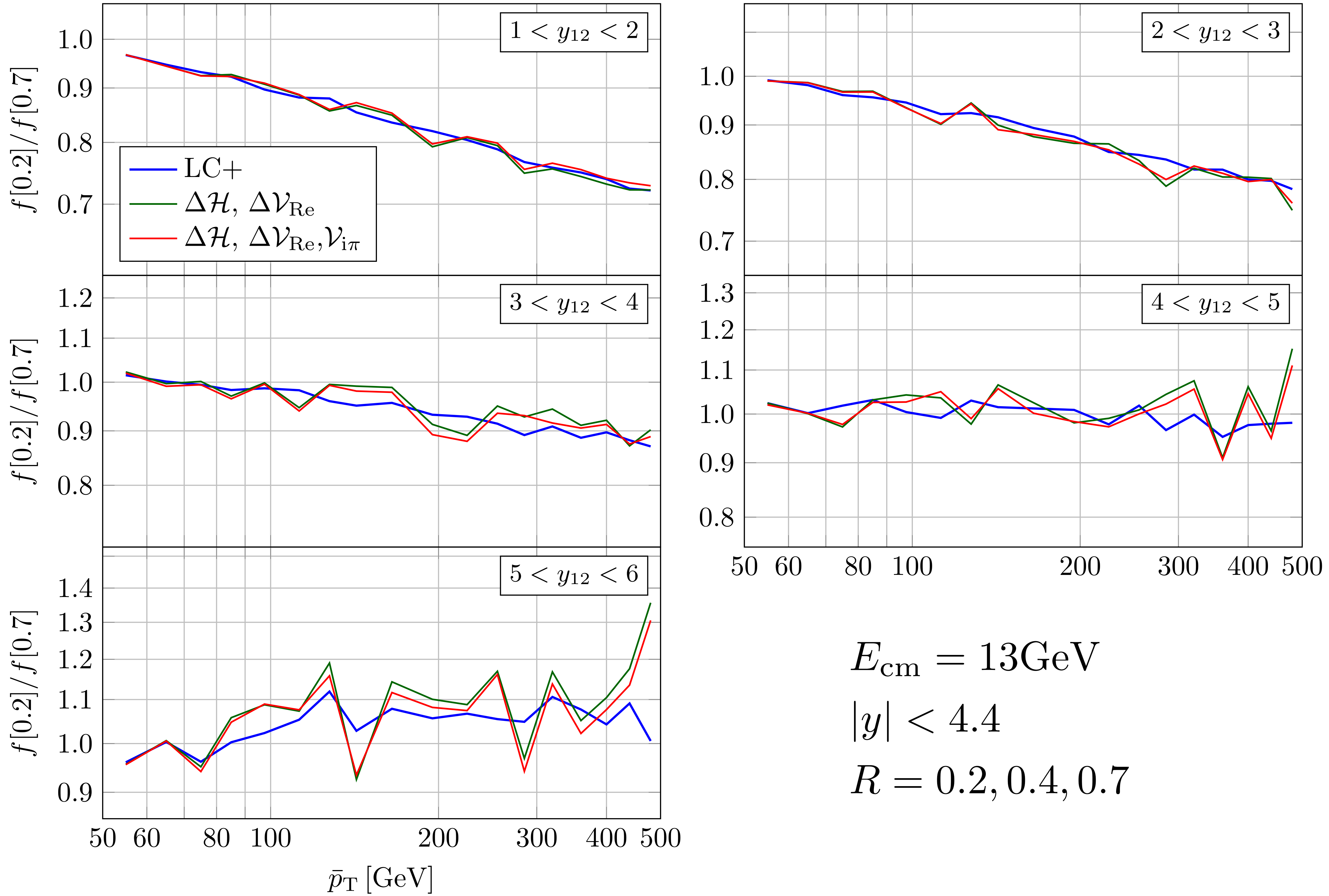


$E_{\mathrm{cm}} = 13\mathrm{GeV}$

$|y| < 4.4$

$R = 0.4$

Rapidity Gap Survival



$E_{\text{cm}} = 13\text{GeV}$

$|y| < 4.4$

$R = 0.2, 0.4, 0.7$

Summary

- **Deductor** is designed to do a better job with color, spin and summation of large logarithms compared.
  - Lambda, kT and angular ordering
  - LC+ color treatment. It allows us to do color evolution at amplitude level.
  - **Wide angle soft gluon effects perturbatively.**
  - Threshold log summation
  - Spin correlations are not yet computed
- Next version is available soon
  - ***Fully exponentiated Glauber (Coulomb) gluon effects***
- It is available from

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http://www.desy.de/~znagy/deductor
http://pages.uoregon.edu/soper/deductor
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