Multiplet bases – why and why not?

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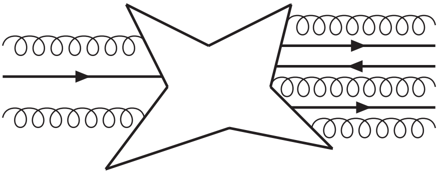
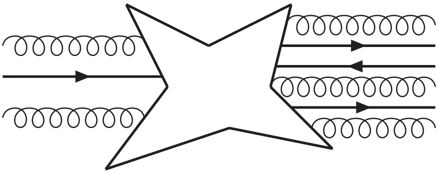
You know nothing
about parton showers and resummation,
Jon Snow.


I drink and I know things
...about multiplet bases.

- QCD process

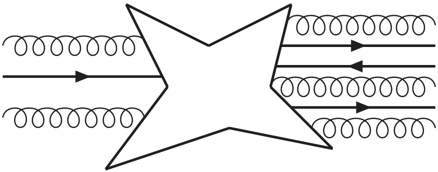


- QCD process



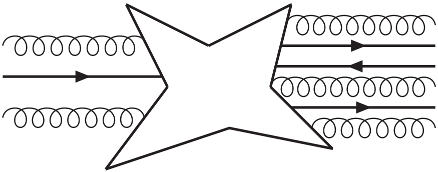
- colour structure: invariant tensor

- QCD process



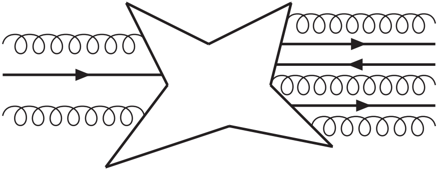
- colour structure: invariant tensor
- total amplitude as linear combination of colour structures requires basis of invariant tensors

- QCD process

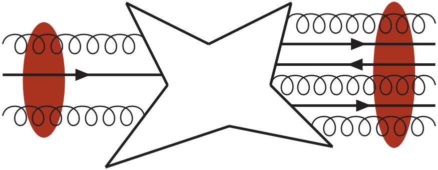


- colour structure: invariant tensor
- total amplitude as linear combination of colour structures requires basis of invariant tensors
- construct a multiplet basis

Constructing multiplet bases

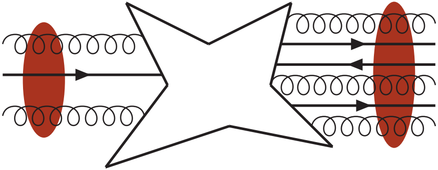


Constructing multiplet bases



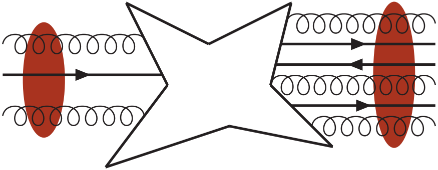
- project to irreps on both sides

Constructing multiplet bases



- project to irreps on both sides
- Schur's Lemma: colour structure maps only irreps to equivalent irreps
- construct transition operators

Constructing multiplet bases



- project to irreps on both sides
- Schur's Lemma: colour structure maps only irreps to equivalent irreps
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Most work goes into first step: constructing projectors.

Multiplet bases are. . .

- orthogonal & minimal.

Multiplet bases are...

- orthogonal & minimal.

...as opposed to trace or colour flow bases which are

- non-orthogonal & overcomplete.

Illustration: soft anomalous dimension matrix for $gg \to gg$[1]

$$\frac{\alpha_s C_A}{\pi}\begin{pmatrix} T & 0 & 0 & 0 & 0 & 0 & -\frac{U}{N_c} & \frac{T-U}{N_c} & 0 \\ 0 & U & 0 & 0 & 0 & 0 & 0 & \frac{U-T}{N_c} & -\frac{T}{N_c} \\ 0 & 0 & T & 0 & 0 & 0 & -\frac{U}{N_c} & \frac{T-U}{N_c} & 0 \\ 0 & 0 & 0 & (T+U) & 0 & 0 & \frac{U}{N_c} & 0 & \frac{T}{N_c} \\ 0 & 0 & 0 & 0 & U & 0 & 0 & \frac{U-T}{N_c} & -\frac{T}{N_c} \\ 0 & 0 & 0 & 0 & 0 & (T+U) & \frac{U}{N_c} & 0 & \frac{T}{N_c} \\ \frac{T-U}{N_c} & 0 & \frac{T-U}{N_c} & \frac{T}{N_c} & 0 & \frac{T}{N_c} & 2T & 0 & 0 \\ -\frac{U}{N_c} & -\frac{T}{N_c} & -\frac{U}{N_c} & 0 & -\frac{T}{N_c} & 0 & 0 & 0 & 0 \\ 0 & \frac{U-T}{N_c} & 0 & \frac{U}{N_c} & \frac{U-T}{N_c} & \frac{U}{N_c} & 0 & 0 & 2U \end{pmatrix}$$

9×9

trace basis

$$\begin{pmatrix} \frac{\alpha_s}{\pi}\begin{pmatrix} N_c T & 0 & 0 \\ 0 & N_c U & 0 \\ 0 & 0 & N_c\,(T+U) \end{pmatrix} & \\ & \frac{\alpha_s}{\pi}\begin{pmatrix} 6T & 0 & -6U & 0 & 0 \\ 0 & 3T+\frac{3U}{2} & -\frac{3U}{2} & -3U & 0 \\ -\frac{3U}{4} & -\frac{3U}{2} & 3T+\frac{3U}{2} & 0 & -\frac{9U}{4} \\ 0 & -\frac{6U}{5} & 0 & 3U & -\frac{9U}{5} \\ 0 & 0 & -\frac{2U}{3} & -\frac{4U}{3} & -2T+4U \end{pmatrix} \end{pmatrix}$$

8×8

multiplet basis

[1] N. Kidonakis, G. Oderda & G. Sterman, Nucl. Phys. **B531** (1998) 365–402, hep-ph/9803241.

Illustration: soft anomalous dimension matrix for $gg \to gg$[1]

$$
\frac{\alpha_s C_A}{\pi}
\begin{pmatrix}
T & 0 & 0 & 0 & 0 & 0 & -\frac{U}{N_c} & \frac{T-U}{N_c} & 0 \\
0 & U & 0 & 0 & 0 & 0 & 0 & \frac{U-T}{N_c} & -\frac{T}{N_c} \\
0 & 0 & T & 0 & 0 & 0 & -\frac{U}{N_c} & \frac{T-U}{N_c} & 0 \\
0 & 0 & 0 & (T+U) & 0 & 0 & \frac{U}{N_c} & 0 & \frac{T}{N_c} \\
0 & 0 & 0 & 0 & U & 0 & 0 & \frac{U-T}{N_c} & -\frac{T}{N_c} \\
0 & 0 & 0 & 0 & 0 & (T+U) & \frac{U}{N_c} & 0 & \frac{T}{N_c} \\
\frac{T-U}{N_c} & 0 & \frac{T-U}{N_c} & \frac{T}{N_c} & 0 & \frac{T}{N_c} & 2T & 0 & 0 \\
-\frac{U}{N_c} & -\frac{T}{N_c} & -\frac{U}{N_c} & 0 & -\frac{T}{N_c} & 0 & 0 & 0 & 0 \\
0 & \frac{U-T}{N_c} & 0 & \frac{U}{N_c} & \frac{U-T}{N_c} & \frac{U}{N_c} & 0 & 0 & 2U
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{\alpha_s}{\pi}\begin{pmatrix} N_c T & 0 & 0 \\ 0 & N_c U & 0 \\ 0 & 0 & N_c(T+U) \end{pmatrix} & \\
 & \frac{\alpha_s}{\pi}\begin{pmatrix}
6T & 0 & -6U & 0 & 0 \\
0 & 3T+\frac{3U}{2} & -\frac{3U}{2} & -3U & 0 \\
-\frac{3U}{4} & -\frac{3U}{2} & 3T+\frac{3U}{2} & 0 & -\frac{9U}{4} \\
0 & -\frac{6U}{5} & 0 & 3U & -\frac{9U}{5} \\
0 & 0 & -\frac{2U}{3} & -\frac{4U}{3} & -2T+4U
\end{pmatrix}
\end{pmatrix}
$$

9×9

trace basis

8×8

multiplet basis

...for $ggg \to ggg$ (or $gggg \to gggg$) we'd have

265×265 $\qquad$ 145×145

$(14833 \times 14833$ $\qquad$ $3598 \times 3598)$

[1] N. Kidonakis, G. Oderda & G. Sterman, Nucl. Phys. **B531** (1998) 365–402, hep-ph/9803241.

constructing multiplet bases

- gluons only
  SK & M. Sjödahl, JHEP **09** (2012) 124, arXiv:1207.0609
- quarks, anti-quarks & gluons
  SK & M. Sjödahl, JHEP **09** (2012) 124, arXiv:1207.0609
  M. Sjödahl & J. Thorén, JHEP **11** (2018) 198, arXiv:1809.05002
- quarks (or anti-quarks) only $\rightsquigarrow$ Hermitian Young operators
  SK & M. Sjödahl, J. Math. Phys. **55** (2014) 021702, arXiv:1307.6147
  J. Alcock-Zeilinger & H. Weigert, J. Math. Phys. **58** (2017) 051702, arXiv:1610.10088
  J. Alcock-Zeilinger & H. Weigert, J. Math. Phys. **58** (2017) 051703, arXiv:1610.08802
- quarks and anti-quarks, singlets only
  J. Alcock-Zeilinger & H. Weigert, arXiv:1812.11223

working with multiplet bases

- expanding into multiplet bases $\rightsquigarrow$ Wigner 3j & 6j coefficients
  M. Sjödahl & J. Thorén, JHEP **09** (2015) 055, arXiv:1507.03814
- download bases for up to 7 external partons
  SK & M. Sjödahl, JHEP **09** (2012) 124, arXiv:1207.0609
  M. Sjödahl & J. Thorén, JHEP **09** (2015) 055, arXiv:1507.03814
- download 6j coefficients
  [M. Sjödahl & J. Thorén, JHEP **09** (2015) 055, arXiv:1507.03814]
  M. Sjödahl & J. Thorén, JHEP **11** (2018) 198, arXiv:1809.05002
- software
  M. Sjödahl: ColorMath/ColorFull (Mathematica/C++ packages)

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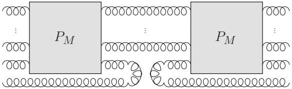
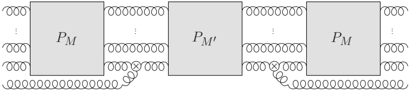
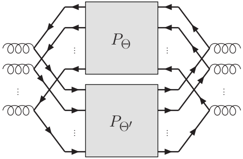
Resources

gluons only

- first occurrence n_f
  $$M' \subset M \otimes A \qquad \Rightarrow \qquad |n_f(M) - n_f(M')| \leq 1$$
- rules for different cases
  - $P_{M'} = \frac{\dim M'}{\dim M}$ 

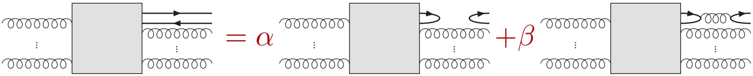
  - $P_{M'} = \gamma$ 

  - $T =$ 

SK & M. Sjödahl, JHEP **09** (2012) 124, `arXiv:1207.0609`

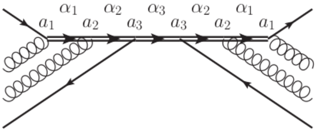
quarks, anti-quarks & gluons – two strategies

- based on gluon projectors



SK & M. Sjödahl, JHEP **09** (2012) 124, `arXiv:1207.0609`

- recursive construction, arbitrary parton order



M. Sjödahl & J. Thorén, JHEP **11** (2018) 198, `arXiv:1809.05002`

quarks only (or anti-quarks only)

- (conventional) Young operators are not Hermitian
  $\rightsquigarrow$ do not yield orthogonal bases

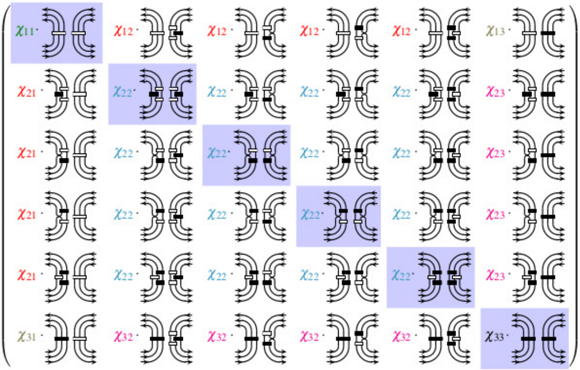
$Y_{\substack{1\,2\\3}} = \frac{4}{3}$  , $Y_{\substack{1\,3\\2}} = \frac{4}{3}$ 

- Hermitian Young operators

$P_{\substack{1\,2\\3}} = \frac{4}{3}$  , $P_{\substack{1\,3\\2}} = \frac{4}{3}$ 

SK & M. Sjödahl, J. Math. Phys. **55** (2014) 021702, arXiv:1307.6147
J. Alcock-Zeilinger & H. Weigert, J. Math. Phys. **58** (2017) 051702, arXiv:1610.10088
J. Alcock-Zeilinger & H. Weigert, J. Math. Phys. **58** (2017) 051703, arXiv:1610.08802

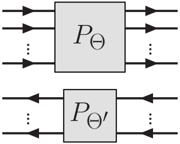
quarks & anti-quarks, singlets only – bending lines



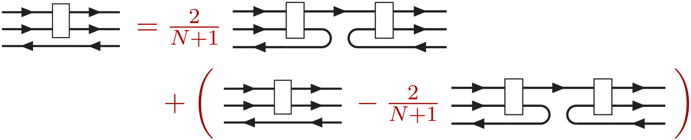
J. Alcock-Zeilinger & H. Weigert, arXiv:1812.11223

quarks & anti-quarks – non-singlets

- apply two Hermitian Young operators

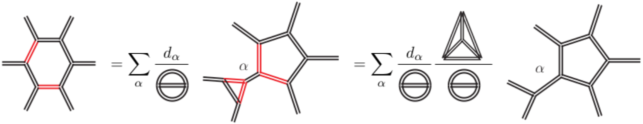


- further decompose by subtracting contractions, e.g.

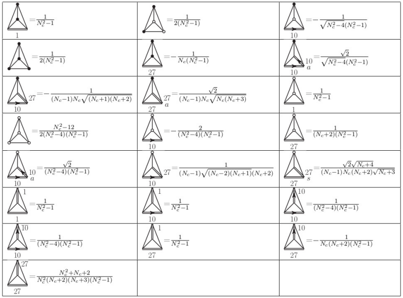


work in progress (with J. Alcock-Zeilinger)

expanding into multiplet bases – 3j & 6j coefficients



calculate 6j coefficients using multiplet bases



1: $=\frac{1}{N_c^2-1}$	$=\frac{1}{2(N_c^2-1)}$	10: $=-\frac{1}{\sqrt{N_c^2-4}(N_c^2-1)}$
$=\frac{1}{2(N_c^2-1)}$	27: $=-\frac{1}{N_c(N_c^2-1)}$	10, 10 a: $=\frac{\sqrt{2}}{\sqrt{N_c^2-4}(N_c^2-1)}$
10, 27: $=-\frac{1}{(N_c-1)N_c\sqrt{(N_c+1)(N_c+2)}}$	27, 27 a: $=\frac{\sqrt{2}}{(N_c-1)N_c\sqrt{N_c(N_c+3)}}$	1: $=\frac{1}{N_c^2-1}$
$=\frac{N_c^2-12}{2(N_c^2-4)(N_c^2-1)}$	10: $=-\frac{2}{(N_c^2-4)(N_c^2-1)}$	27: $=\frac{1}{(N_c+2)(N_c^2-1)}$
10, 10 a: $=\frac{\sqrt{2}}{(N_c^2-4)(N_c^2-1)}$	10, 27: $=\frac{1}{(N_c-1)\sqrt{(N_c-2)(N_c+1)}(N_c+2)}$	27, 27 s: $=\frac{\sqrt{2}\sqrt{N_c+4}}{(N_c-1)N_c(N_c+2)\sqrt{N_c+3}}$
1, 1: $=\frac{1}{N_c^2-1}$	10, 1: $=\frac{1}{N_c^2-1}$	10, 10: $=\frac{1}{(N_c^2-4)(N_c^2-1)}$
10, 10: $=\frac{1}{(N_c^2-4)(N_c^2-1)}$	27, 1: $=\frac{1}{N_c^2-1}$	27, 10: $=-\frac{1}{N_c(N_c+2)(N_c^2-1)}$
27, 27: $=\frac{N_c^2+N_c+2}{N_c^2(N_c+2)(N_c+3)(N_c^2-1)}$		

M. Sjödahl & J. Thorén, JHEP **09** (2015) 055, `arXiv:1507.03814`

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