Constructing Hermitian Young Operators

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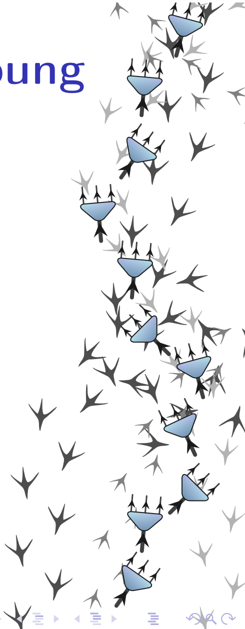


Outline

Irreducible representations of SU(N) on $V^{\otimes n}$, V a vector space.

1. Background
   - Focus: Invariant theory, Young projection operators
   - Birdtracks: graphical representation of invariants
2. Modern results
   - Hermitian projection operators of SU(N): KS & MOLD
   - Unitary transition operators between equivalent representations

Representations of S_n and SU(N) on $V^{\otimes n}$

- For any $\boldsymbol{v} \in V^{\otimes n}$, the action of $\rho \in S_n$ on the components $v^{i_1 i_2 \dots i_n}$ of $\boldsymbol{v}$ is defined as

$$\rho(v^{i_1 i_2 \dots i_n}) := v^{i_{\rho^{-1}(1)} i_{\rho^{-1}(2)} \cdots i_{\rho^{-1}(n)}} \qquad (1)$$

- For any $U \in \mathrm{SU}(N)$ with defining representation $\gamma(U)$ on V, the induced product representation on $V^{\otimes n}$ (also denoted by U) is

$$(U\boldsymbol{v})^{i_1 \dots i_k} := \gamma(U)^{i_1}{}_{j_1} \cdots \gamma(U)^{i_k}{}_{j_k} v^{j_1 \dots j_k} \ . \qquad (2)$$

These actions commute on $V^{\otimes n}$, so every $\rho \in S_n$ is an *invariant* of SU(N),

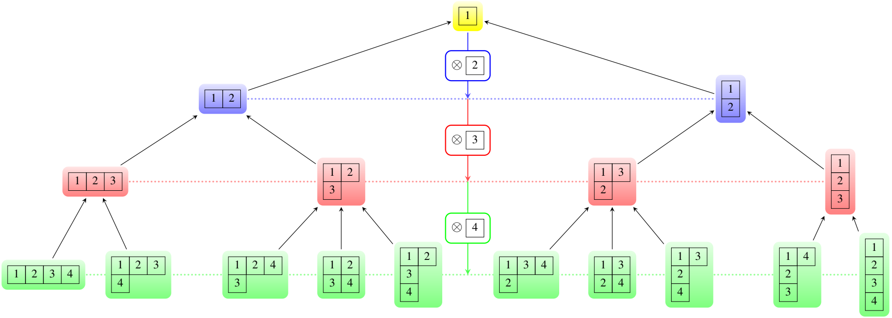
$$U \rho U^{\dagger} = \rho \qquad (3)$$

Elements of S_n are *all* the invariants of SU(N) on $V^{\otimes n}$

Ancestry of Young tableaux over $V^{\otimes 4}$

add box i to a tableau in $\mathcal{Y}_{i-1}$

- tableau is left-aligned and top-aligned
- numbers increase in each row
- numbers increae in each column



Young projection operators

Young projector e_Θ is quasi-idempotent

- Antisymmetrize over numbers in the same row
- Symmetrize over numbers in the same column

$$\Theta = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$$

$$\begin{aligned} e_\Theta &= \mathbf{S}_\Theta \mathbf{A}_\Theta \\ &= \underbrace{\frac{1}{2}\left(\mathrm{id}_5 + (12)\right) \cdot \frac{1}{2}\left(\mathrm{id}_5 + (34)\right)}_{\mathbf{S}_\Theta} \\ &\quad \times \underbrace{\frac{1}{6}\left(\mathrm{id}_5 - (13) - (15) - (35) + (135) + (153)\right) \cdot \frac{1}{2}\left(\mathrm{id}_5 - (24)\right)}_{\mathbf{A}_\Theta} \end{aligned}$$

Young projection operators

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- Antisymmetrize over numbers in the same row
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$$\Theta = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \longrightarrow \quad Y_\Theta = \alpha_\Theta e_\theta \text{ is idempotent}$$

$$\begin{aligned} e_\Theta &= \mathbf{S}_\Theta \mathbf{A}_\Theta \\ &= \underbrace{\frac{1}{2}(\mathrm{id}_5 + (12)) \cdot \frac{1}{2}(\mathrm{id}_5 + (34))}_{\mathbf{S}_\Theta} \\ &\quad \times \underbrace{\frac{1}{6}(\mathrm{id}_5 - (13) - (15) - (35) + (135) + (153)) \cdot \frac{1}{2}(\mathrm{id}_5 - (24))}_{\mathbf{A}_\Theta} \end{aligned}$$

- Acting permutation on tensor components amounts to acting a product of Kronecker deltas:

$$(123)v^{i_1 i_2 i_3} = v^{i_3 i_1 i_2} = \delta^{i_2}_{\ j_1}\,\delta^{i_3}_{\ j_2}\,\delta^{i_1}_{\ j_3}\,v^{j_1 j_2 j_3}$$

Representations of SU(N) and Young projection operators

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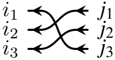
$$\delta^{i_2}_{j_1}\,\delta^{i_3}_{j_2}\,\delta^{i_1}_{j_3} = \quad$$ 

$$= (123)\ .$$

P. Cvitanović: "Group Theory - Birdtracks, Lie's and Exceptional Groups"

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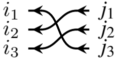

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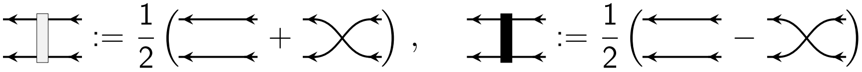
P. Cvitanović: "Group Theory - Birdtracks, Lie's and Exceptional Groups"

- Symmetrizers and antisymmetrizers



Representations of SU(N) and Young projection operators

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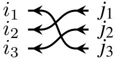
$$\delta^{i_2}{}_{j_1}\,\delta^{i_3}{}_{j_2}\,\delta^{i_1}{}_{j_3} =$$



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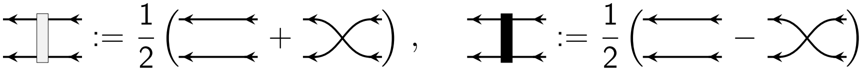
P. Cvitanović: "Group Theory - Birdtracks, Lie's and Exceptional Groups"

- Symmetrizers and antisymmetrizers

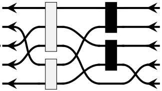


- Young projection operators from tableaux

$\Theta =$

1	3	4
2	5	

$\longrightarrow$ $Y_\Theta := \underbrace{2}_{\alpha_\Theta}$



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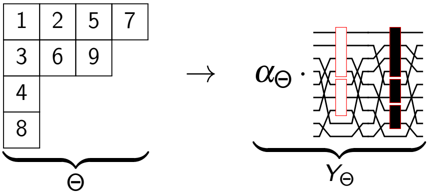
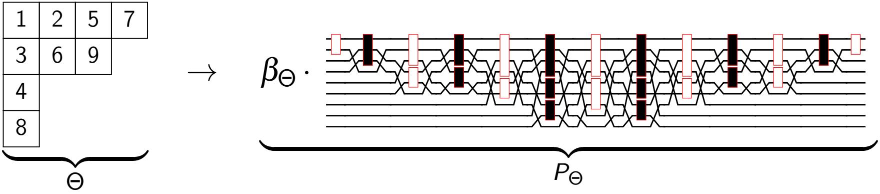
A bigger example:



Wishlist of Properties:	Young Y_Θ	
Idempotency: $Y_\Theta \cdot Y_\Theta = Y_\Theta$	✓	
Transversality: $Y_\Theta \cdot Y_\Phi = \delta_{\Theta\Phi} Y_\Theta$	✗	
Completeness: $\sum_{\Theta \in \mathcal{Y}_n} Y_\Theta = \mathrm{id}_n$	✓	

Multiplets from Tableaux

A bigger example:



J. A-Z and H. Weigert, arXiv:1610.10088 (based on Keppeler & Sjödahl, arXiv:1307.6147)

Wishlist of Properties:	Young Y_Θ	Herm. Young P_Θ
Idempotency: $Y_\Theta \cdot Y_\Theta = Y_\Theta$	✓	✓
Transversality: $Y_\Theta \cdot Y_\Phi = \delta_{\Theta\Phi} Y_\Theta$	✘	✓
Completeness: $\sum_{\Theta \in \mathcal{Y}_n} Y_\Theta = \mathrm{id}_n$	✓	✓

$\Longrightarrow$ Replace Y_Θ by P_Θ

Parent Map & Ancestor tableaux

For any Young tableau Θ consisting of n boxes, $\Theta \in \mathcal{Y}_n$, parent tableau $\Theta_{(1)}$ is obtained by removing $\boxed{n}$

$$\Theta = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \longrightarrow \Theta_{(1)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

Parent map: $\pi : \mathcal{Y}_n \to \mathcal{Y}_{n-1}$, $\pi(\Theta) = \Theta_{(1)}$

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Parent map: $\pi : \mathcal{Y}_n \to \mathcal{Y}_{n-1}$, $\pi(\Theta) = \Theta_{(1)}$

Repeated application:

$$\underbrace{\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}}_{\Theta} \xrightarrow{\pi} \underbrace{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}}_{\Theta_{(1)}} \xrightarrow{\pi} \underbrace{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}}_{\Theta_{(2)}} \xrightarrow{\pi} \underbrace{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}}_{\Theta_{(3)}}$$

$\Theta_{(m)}$ is the ancestor of Θ m generations back.

Iterative construction using tableau ancestry:

1. If $\Theta \in \mathcal{Y}_2$ (2 boxes),
$$P_\Theta = Y_\Theta \ .$$
2. If $\Theta \in \mathcal{Y}_n$ with $n > 2$ (> 2 boxes),
$$P_\Theta = \underline{P_{\Theta_{(1)}}}\, Y_\Theta\, \underline{P_{\Theta_{(1)}}}$$

($P_{\Theta_{(1)}}$ embedded in $V^{\otimes n}$).

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($\underline{P_{\Theta_{(1)}}}$ embedded in $V^{\otimes n}$).

Properties of KS operators:

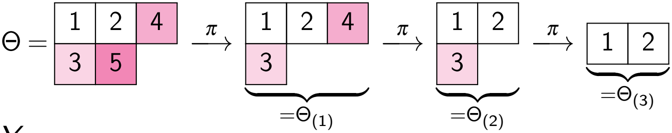
- idempotent
- complete
- transversal

KS Projector Example

$$\Theta = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$$

KS Projector Example

S. Keppeler and M. Sjödahl, arXiv:1307.6147



$$P_{\Theta_{(3)}} = Y_{\Theta_{(3)}}$$

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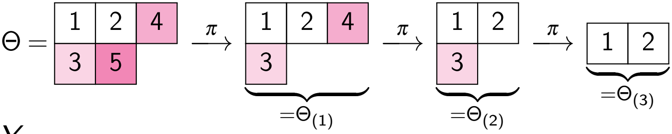
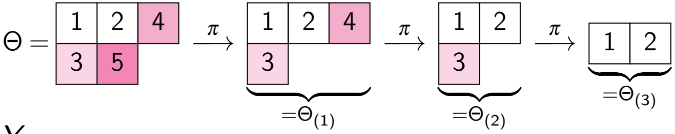


$$P_{\Theta_{(3)}} = Y_{\Theta_{(3)}}$$

$$P_{\Theta_{(2)}} = \underbrace{Y_{\Theta_{(3)}}}_{P_{\Theta_{(3)}}} Y_{\Theta_{(2)}} \underbrace{Y_{\Theta_{(3)}}}_{P_{\Theta_{(3)}}}$$

KS Projector Example



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$$P_{\Theta_{(1)}} = \underbrace{Y_{\Theta_{(3)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}}}_{=P_{\Theta_{(2)}}} Y_{\Theta_{(1)}} \underbrace{Y_{\Theta_{(3)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}}}_{=P_{\Theta_{(2)}}}$$

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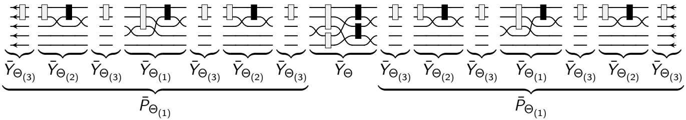
$$P_{\Theta} = \underbrace{Y_{\Theta_{(3)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}} Y_{\Theta_{(1)}} Y_{\Theta_{(3)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}}}_{=P_{\Theta_{(1)}}} Y_{\Theta} \underbrace{Y_{\Theta_{(3)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}} Y_{\Theta_{(1)}} Y_{\Theta_{(3)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}}}_{=P_{\Theta_{(1)}}}$$

$$= \frac{128}{9} \cdot$$

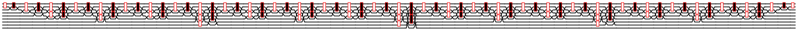


MOLD advantage

Young tableau

$$\Theta = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 6 \\ \cline{1-2} 5 & 8 \\ \cline{1-2} 9 \\ \cline{1-1}\end{array}$$

- Keppeler-Sjödahl operator: (arXiv:1307.6147)



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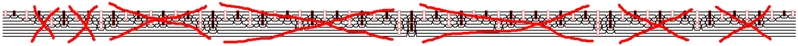

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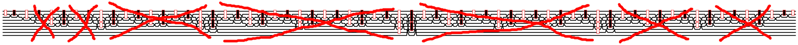

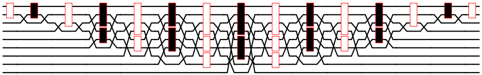

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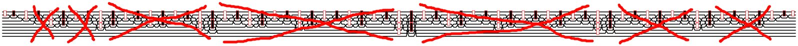
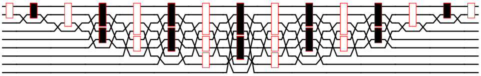


- MOLD operator: (arXiv:1610.10088)



Lexically ordered tableau: We say that a tableau is lexically ordered if wither the row-word (read row-wise from top to bottom) or the column-word (read column-wise from left to right) is in lexical order

1	5	7	8
2	6		
3			
4			

ordered

1	2	4	6
3	7		
5			
8			

not ordered

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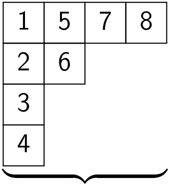
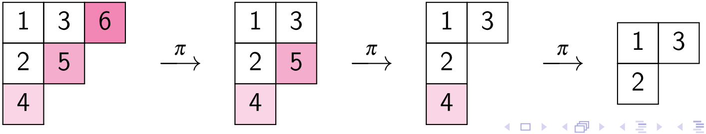


Measure of Lexical disorder (MOLD) is the minimum number of applications of π needed to arrive at a lexically ordered Young tableau.



Utilize the partial order of a tableau to obtain Hermitian projector:

For a tableau $\Theta \in \mathcal{Y}_n$ with MOLD m

1. Find the ancestor $\Theta_{(m)}$
2. If $\Theta_{(m)}$ is
   - row-ordered, start with the symmetrizer $\mathbf{S}_{\Theta_{(m)}}$
   - column-ordered, start with the antisymmetrizer $\mathbf{S}_{\Theta_{(m)}}$

   on the outside of the operator
3. Moving inwards, alternate symmetrizers and antisymmetrizers, always going up one generation until $\Theta_{(1)}$
4. The central part of the operator is either

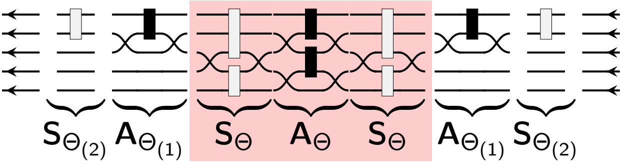
$$\mathbf{S}_\Theta \mathbf{A}_\Theta \mathbf{S}_\Theta \quad \text{or} \quad \mathbf{A}_\Theta \mathbf{S}_\Theta \mathbf{A}_\Theta$$

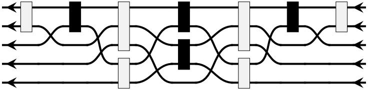
   keeping alternation between symmetrizers and antisymmetrizer

Utilize the partial order of a tableau:

$$\Theta := \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array} \xrightarrow{\pi} \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & & \\ \cline{1-1}\end{array} \xrightarrow{\pi} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

$$P_\Theta = \beta_\Theta \cdot \mathbf{S}_{\Theta_{(2)}} \mathbf{A}_{\Theta_{(1)}} \mathbf{S}_\Theta \mathbf{A}_\Theta \mathbf{S}_\Theta \mathbf{A}_{\Theta_{(1)}} \mathbf{S}_{\Theta_{(2)}}$$

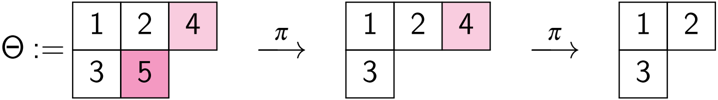
$= \beta_\Theta \cdot$ 

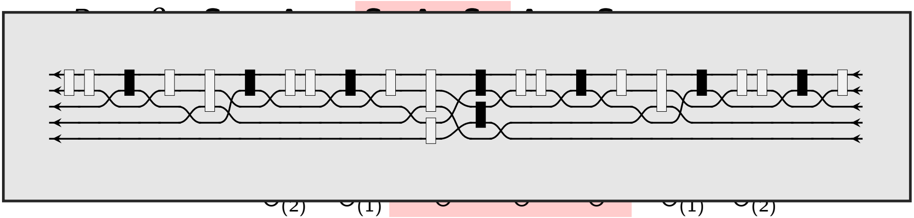
$= \beta_\Theta \cdot$  .

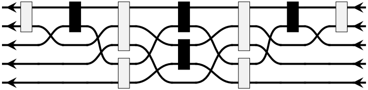
MOLD construction

J. A-Z and H. Weigert, arXiv:1610.10088

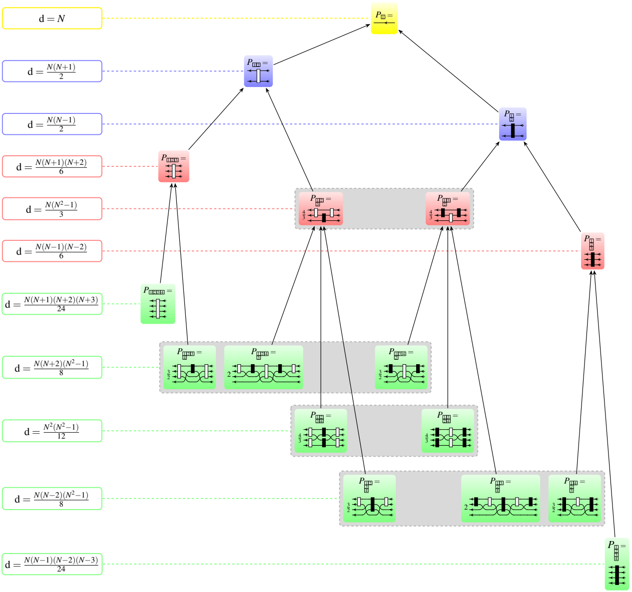
Utilize the partial order of a tableau:





$= \beta_\Theta \cdot$  .

Ancestry of Hermitian Young projection operators over $V^{\otimes 4}$



Schur's Lemma:

Let V_i and V_j be two irreducible G-modules of a group G. Let $T_{ij} : V_j \to V_i$ be a G-homomorphism. Then

1. *T_{ij} is a G-isomorphism if and only if V_i and V_j carry equivalent representations of G, or*
2. *T_{ij} is the zero map.*

T_{ij} is called the ***transition operator****.*

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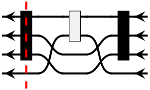
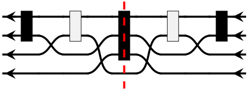
*T_{ij} is called the **transition operator**.*

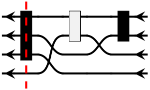
- Each projection operator $P_i : V_i \to V_i$ defines an irreducible SU(N)-module V_i on $V^{\otimes k}$
- Projection operators corresponding to Young tableaux of the same shape correspond to equivalent irreducible representations of SU(N)

Transition operators $T_{\Theta\Phi}$

J. A-Z and H. Weigert, arXiv:1610.08802

$$\Theta = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad \Phi = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

$P_\Theta = \frac{3}{2} \cdot$  and $P_\Phi = 2 \cdot$ 

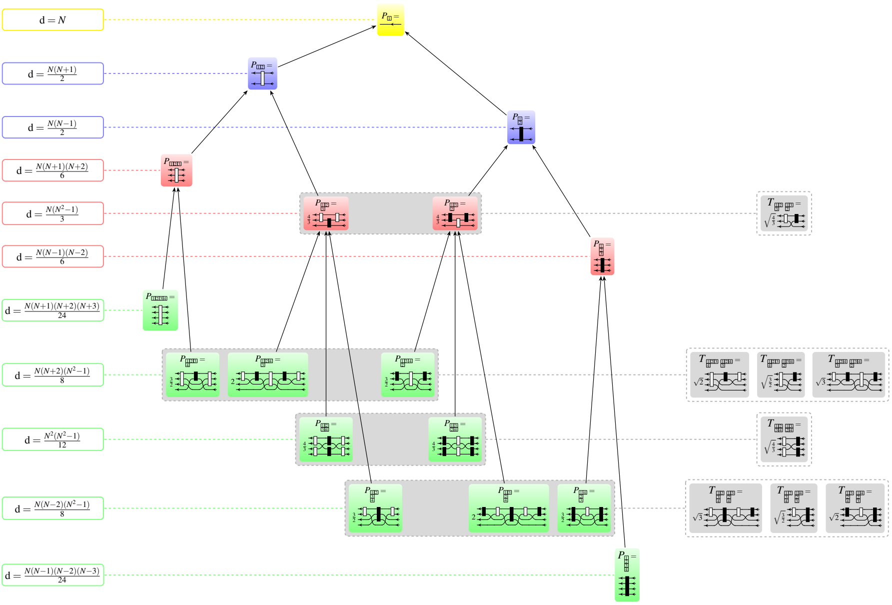
$T_{\Theta\Phi} = \sqrt{2} \cdot$  .

Properties:

- $T_{\Theta\Phi} P_\Phi = T_{\Theta\Phi} = P_\Theta T_{\Theta\Phi}$
- $T^\dagger_{\Theta\Phi} T_{\Theta\Phi} = P_\Phi$
- $T_{\Theta\Phi} T^\dagger_{\Theta\Phi} = P_\Theta$

where $T^\dagger_{\Theta\Phi} = T_{\Phi\Theta}$

Ancestry of Hermitian Young projection operators over $V^{\otimes 4}$



Summary:

- Representation theory of SU(N) on $V^{\otimes n}$ is well-developed, but still open questions
- MOLD: efficient construction algorithm for Hermitian Young projection operators and unitary transition operators (in the birdtrack formalism)

Outlook:

- Projection and transition operators of SU(N) on mixed product spaces (e.g. $V^{\otimes m} \otimes (V^*)^{\otimes n}$)?
- Representations of SO(N) and other Lie groups?

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