Theory progress in the calculation of hadronic contributions to the $(g - 2)_\mu$ 

Gilberto Colangelo

New physics at the low-energy precision frontier
LPT Orsay, 18.9.2019
Outline

Introduction: $(g - 2)_\mu$ and hadronic contributions

Hadronic Vacuum Polarization contribution to $(g - 2)_\mu$

Master Formula

A dispersion relation for HLbL

Conclusions
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Introduction: \((g - 2)_\mu\) and hadronic contributions

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Conclusions
Status of \((g - 2)_\mu\), experiment vs SM

Davier, Hoecker, Malaescu, Zhang 2019

- J 2018: \(-31.5 \pm 4.4\) (4.1σ)
- KNT 2018: \(-27.0 \pm 3.6\) (3.7σ)
- DHMZ 2019: \(-26.2 \pm 4.8\) (3.3σ)
- BNL-E821: \(0 \pm 6.3\)

Systematic uncertainty
SM predictions
Experiment

\(a_\mu - a_\mu^{\text{exp}} [\times 10^{-10}]\)
Status of $(g - 2)_\mu$, experiment vs SM

Keshavarzi, Nomura, Teubner, 2018 (KNT18)

Fermilab experiment’s goal: error $\times 1/4$, should be matched by theory:

$\Rightarrow$ Muon “$(g - 2)$ Theory Initiative” lead by A. El-Khadra and C. Lehner
### Status of $(g - 2)_\mu$, experiment vs SM

<table>
<thead>
<tr>
<th></th>
<th>$a_\mu \times 10^{-11}$</th>
<th>$\Delta a_\mu \times 10^{-11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>experiment</strong></td>
<td>116 592 089</td>
<td>63.</td>
</tr>
<tr>
<td>QED $\mathcal{O}(\alpha)$</td>
<td>116 140 973.21</td>
<td>0.03</td>
</tr>
<tr>
<td>QED $\mathcal{O}(\alpha^2)$</td>
<td>413 217.63</td>
<td>0.01</td>
</tr>
<tr>
<td>QED $\mathcal{O}(\alpha^3)$</td>
<td>30 141.90</td>
<td>0.00</td>
</tr>
<tr>
<td>QED $\mathcal{O}(\alpha^4)$</td>
<td>381.01</td>
<td>0.02</td>
</tr>
<tr>
<td>QED $\mathcal{O}(\alpha^5)$</td>
<td>5.09</td>
<td>0.01</td>
</tr>
<tr>
<td>QED total</td>
<td>116 584 718.97</td>
<td>0.07</td>
</tr>
<tr>
<td><strong>electroweak, total</strong></td>
<td>153.6</td>
<td>1.0</td>
</tr>
<tr>
<td>HVP (LO) [KNT 18]</td>
<td>6 932.7</td>
<td>24.6</td>
</tr>
<tr>
<td>HVP (NLO) [KNT 18]</td>
<td>-98.2</td>
<td>0.4</td>
</tr>
<tr>
<td>HLbL [update of Glasgow consensus–KNT 18]</td>
<td>98.0</td>
<td>26.0</td>
</tr>
<tr>
<td>HVP (NNLO) [Kurz, Liu, Marquard, Steinhauser 14]</td>
<td>12.4</td>
<td>0.1</td>
</tr>
<tr>
<td>HLbL (NLO) [GC, Hoferichter, Nyffeler, Passera, Stoffer 14]</td>
<td>3.0</td>
<td>2.0</td>
</tr>
<tr>
<td><strong>theory</strong></td>
<td>116 591 820.5</td>
<td>35.6</td>
</tr>
</tbody>
</table>
Status of $(g - 2)_\mu$, experiment vs SM

$$a^\text{exp}_\mu - a^\text{SM}_\mu = 268.5 \pm 72.4 \quad [3.7\sigma]$$

Keshavarzi, Nomura, Teubner, 2018
Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved
Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved

basic principles: unitarity and analyticity

direct relation to experiment: \( \sigma_{\text{tot}}(e^+ e^- \rightarrow \gamma^* \rightarrow \text{hadrons}) \)

alternative approach: lattice (ETMC, Mainz, HPQCD, BMW, RBC/UKQCD)
Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved
- Hadronic light-by-light (HLbL) is more problematic:
  - 4-point function of EM currents in QCD
  - “it cannot be expressed in terms of measurable quantities”
  - until recently, only model calculations
  - lattice QCD is making fast progress
Muon $g - 2$ Theory Initiative

Steering Committee:
GC
Michel Davier
Simon Eidelman
Aida El-Khadra (co-chair)
Christoph Lehner (co-chair)
Tsutomu Mibe (J-PARC E34 experiment)
Andreas Nyffeler
Lee Roberts (Fermilab E989 experiment)
Thomas Teubner

Workshops:

- First plenary meeting, Q-Center (Fermilab), 3-6 June 2017
- HVP WG workshop, KEK (Japan), 12-14 February 2018
- HLbL WG workshop, U. of Connecticut, 12-14 March 2018
- Second plenary meeting, Mainz, 18-22 June 2018
- Third plenary meeting, Seattle, 9-13 September 2019
Take-home message of this talk

1. There is significant progress in the theoretical evaluation of hadronic contribution to \((g - 2)_\mu\)

2. Dispersion relations (DR) play an important role in these improvements
   - for HVP DR have been applied since many years but it can be pushed even further
   - for HLbL this is a recent development and is leading to a significant reduction of the uncertainty

3. Lattice—though not yet competitive—is catching up
Outline

Introduction: \((g - 2)_\mu\) and hadronic contributions

Hadronic Vacuum Polarization contribution to \((g - 2)_\mu\)

Master Formula
A dispersion relation for HLbL

Conclusions
Calculating the HVP contribution

- HVP can be calculated with a data-driven approach
- basic principles: unitarity and analyticity
- direct relation to experiment: $\sigma_{\text{tot}}(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})$
- dedicated $e^+e^-$ program: BaBar, Belle, BESIII, CMD3, KLOE2, SND
HVP, gauge invariance and analyticity

\[ \Pi_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle 0| T j_\mu(x) j_\nu(0) |0 \rangle = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2) \]

where \( j^\mu(x) = \sum_i Q_i \bar{q}_i(x) \gamma^\mu q_i(x) \), \( i = u, d, s \) is the em current

- Lorentz invariance: 2 structures
- gauge invariance: reduction to 1 structure
- Lorentz-tensor defined in such a way that the function \( \Pi(q^2) \) does not have kinematic singularities or zeros
- \( \hat{\Pi}(q^2) := \Pi(q^2) - \Pi(0) \) satisfies

\[ \hat{\Pi}(q^2) = \frac{q^2}{\pi} \int_{4M^2_{\pi}}^{\infty} dt \frac{\text{Im} \hat{\Pi}(t)}{t(t - q^2)} \]
Unitarity relation for HVP

For HVP the unitarity relation is simple and looks the same for all possible intermediate states

\[ \text{Im} \Pi(q^2) \propto \sigma(e^+ e^- \to \text{hadrons}) = \sigma(e^+ e^- \to \mu^+ \mu^-) R(q^2) \]
Unitarity relation for HVP

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\[ a^\text{hvp}_\mu = \frac{\alpha^2}{3\pi^2} \int_{s_{\text{th}}}^{\infty} \frac{ds}{s} K(s) R(s) \]

\( K(s) \) known, depends on \( m_\mu \) and \( K(s) \sim \frac{1}{s} \) for large \( s \)
<table>
<thead>
<tr>
<th>Channel</th>
<th>This work (KNT18)</th>
<th>DHMZ17 [77]</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Channel</td>
<td>Data based channels ((\sqrt{s} \leq 1.8 \text{ GeV}))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\pi^0\gamma) (data + ChPT)</td>
<td>4.58 ± 0.10</td>
<td>4.29 ± 0.10</td>
<td>0.29 ± 0.14</td>
</tr>
<tr>
<td>(\pi^+\pi^-) (data + ChPT)</td>
<td>503.74 ± 1.96</td>
<td>507.14 ± 2.58</td>
<td>−3.40 ± 3.24</td>
</tr>
<tr>
<td>(\pi^+\pi^-\pi^0) (data + ChPT)</td>
<td>47.70 ± 0.89</td>
<td>46.20 ± 1.45</td>
<td>1.50 ± 1.70</td>
</tr>
<tr>
<td>(\pi^+\pi^-\pi^0\pi^0)</td>
<td>13.99 ± 0.19</td>
<td>13.68 ± 0.31</td>
<td>0.31 ± 0.36</td>
</tr>
<tr>
<td>((2\pi^+2\pi^-\pi^0)_{\text{no } \eta})</td>
<td>18.15 ± 0.74</td>
<td>18.03 ± 0.54</td>
<td>0.12 ± 0.92</td>
</tr>
<tr>
<td>(3\pi^+3\pi^-)</td>
<td>0.10 ± 0.01</td>
<td>0.11 ± 0.01</td>
<td>−0.01 ± 0.01</td>
</tr>
<tr>
<td>((2\pi^+2\pi^-2\pi^0)_{\text{no } \eta\omega})</td>
<td>0.77 ± 0.11</td>
<td>0.72 ± 0.17</td>
<td>0.05 ± 0.20</td>
</tr>
<tr>
<td>(K^+K^-)</td>
<td>23.00 ± 0.22</td>
<td>22.81 ± 0.41</td>
<td>0.19 ± 0.47</td>
</tr>
<tr>
<td>(K^0\bar{K}^0)</td>
<td>13.04 ± 0.19</td>
<td>12.82 ± 0.24</td>
<td>0.22 ± 0.31</td>
</tr>
<tr>
<td>(K^0\bar{K}^0)</td>
<td>2.44 ± 0.11</td>
<td>2.45 ± 0.15</td>
<td>−0.01 ± 0.19</td>
</tr>
<tr>
<td>(K^0\bar{K}^0)</td>
<td>0.86 ± 0.05</td>
<td>0.85 ± 0.05</td>
<td>0.01 ± 0.07</td>
</tr>
<tr>
<td>(\eta\gamma) (data + ChPT)</td>
<td>0.70 ± 0.02</td>
<td>0.65 ± 0.02</td>
<td>0.05 ± 0.03</td>
</tr>
<tr>
<td>(\eta\pi^+\pi^-)</td>
<td>1.18 ± 0.05</td>
<td>1.18 ± 0.07</td>
<td>0.00 ± 0.09</td>
</tr>
<tr>
<td>((\eta\pi^+\pi^-\pi^0)_{\text{no } \omega})</td>
<td>0.48 ± 0.12</td>
<td>0.39 ± 0.12</td>
<td>0.09 ± 0.17</td>
</tr>
<tr>
<td>(\eta\pi^+\pi^-\pi^0\pi^0)</td>
<td>0.03 ± 0.01</td>
<td>0.03 ± 0.01</td>
<td>0.00 ± 0.01</td>
</tr>
<tr>
<td>(\eta\omega)</td>
<td>0.29 ± 0.02</td>
<td>0.32 ± 0.03</td>
<td>−0.03 ± 0.04</td>
</tr>
<tr>
<td>(\omega\rightarrow\pi^0\gamma\pi^0)</td>
<td>0.87 ± 0.02</td>
<td>0.94 ± 0.03</td>
<td>−0.07 ± 0.04</td>
</tr>
<tr>
<td>((\eta\phi)_{\text{no } \phi\rightarrow K^0\bar{K}^0})</td>
<td>0.22 ± 0.02</td>
<td>0.36 ± 0.03</td>
<td>−0.14 ± 0.04*</td>
</tr>
<tr>
<td>(\phi\rightarrow\text{unaccounted})</td>
<td>0.04 ± 0.04</td>
<td>0.05 ± 0.00</td>
<td>−0.01 ± 0.04</td>
</tr>
<tr>
<td>(\eta\omega\pi^0)</td>
<td>0.10 ± 0.05</td>
<td>0.06 ± 0.04</td>
<td>0.04 ± 0.06</td>
</tr>
<tr>
<td>(K^0\bar{K}^0\eta)</td>
<td>0.12 ± 0.02</td>
<td>0.01 ± 0.01</td>
<td>0.11 ± 0.02*</td>
</tr>
</tbody>
</table>

Estimated contributions \((\sqrt{s} \leq 1.8 \text{ GeV})\):

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</thead>
<tbody>
<tr>
<td>((\pi^+\pi^-\pi^0)_{\text{no } \eta})</td>
<td>0.40 ± 0.04</td>
<td>0.35 ± 0.04</td>
<td>0.05 ± 0.06</td>
</tr>
<tr>
<td>((\pi^+\pi^-4\pi^0)_{\text{no } \eta})</td>
<td>0.12 ± 0.12</td>
<td>0.11 ± 0.11</td>
<td>0.01 ± 0.16</td>
</tr>
<tr>
<td>(K\bar{K}3\pi)</td>
<td>−0.02 ± 0.01</td>
<td>−0.03 ± 0.02</td>
<td>0.01 ± 0.02</td>
</tr>
<tr>
<td>(\omega\rightarrow\text{npp}2\pi)</td>
<td>0.08 ± 0.01</td>
<td>0.08 ± 0.01</td>
<td>0.00 ± 0.01</td>
</tr>
<tr>
<td>(\omega\rightarrow\text{npp}3\pi)</td>
<td>0.10 ± 0.02</td>
<td>0.36 ± 0.01</td>
<td>−0.26 ± 0.02</td>
</tr>
<tr>
<td>(\omega\rightarrow\text{npp}K\bar{K})</td>
<td>0.00 ± 0.00</td>
<td>0.01 ± 0.00</td>
<td>−0.01 ± 0.00</td>
</tr>
<tr>
<td>(\eta\pi^+\pi^-2\pi^0)</td>
<td>0.03 ± 0.01</td>
<td>0.03 ± 0.01</td>
<td>0.00 ± 0.01</td>
</tr>
</tbody>
</table>

Other contributions:

<table>
<thead>
<tr>
<th>Channel</th>
<th>This work (KNT18)</th>
<th>DHMZ17 [77]</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(J/\psi)</td>
<td>6.26 ± 0.19</td>
<td>6.28 ± 0.07</td>
<td>−0.02 ± 0.20</td>
</tr>
<tr>
<td>(\psi')</td>
<td>1.58 ± 0.04</td>
<td>1.57 ± 0.03</td>
<td>0.01 ± 0.05</td>
</tr>
<tr>
<td>(\Upsilon(1S - 4S))</td>
<td>0.09 ± 0.00</td>
<td>-</td>
<td>0.09 ± 0.00**</td>
</tr>
</tbody>
</table>

Contributions by energy region:

<table>
<thead>
<tr>
<th>Energy Region</th>
<th>This work (KNT18)</th>
<th>DHMZ17 [77]</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.8 \leq \sqrt{s} \leq 3.7 \text{ GeV})</td>
<td>34.54 ± 0.56 (data)</td>
<td>33.45 ± 0.65 (pQCD)**</td>
<td>1.09 ± 0.86</td>
</tr>
<tr>
<td>(3.7 \leq \sqrt{s} \leq 5.0 \text{ GeV})</td>
<td>7.33 ± 0.11 (data)</td>
<td>7.29 ± 0.03 (data)</td>
<td>0.04 ± 0.11</td>
</tr>
<tr>
<td>(5.0 \leq \sqrt{s} \leq 9.3 \text{ GeV})</td>
<td>6.62 ± 0.10 (data)</td>
<td>6.86 ± 0.04 (pQCD)</td>
<td>−0.24 ± 0.11</td>
</tr>
<tr>
<td>(9.3 \leq \sqrt{s} \leq 12.0 \text{ GeV})</td>
<td>1.12 ± 0.01 (data+pQCD)</td>
<td>1.21 ± 0.01 (pQCD)</td>
<td>−0.09 ± 0.02**</td>
</tr>
<tr>
<td>(12.0 \leq \sqrt{s} \leq 40.0 \text{ GeV})</td>
<td>1.64 ± 0.00 (pQCD)</td>
<td>1.64 ± 0.00 (pQCD)</td>
<td>0.00 ± 0.00</td>
</tr>
<tr>
<td>(&gt; 40.0 \text{ GeV})</td>
<td>0.16 ± 0.00 (pQCD)</td>
<td>0.16 ± 0.00 (pQCD)</td>
<td>0.00 ± 0.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>693.3 ± 2.5</td>
<td>693.1 ± 3.4</td>
<td>0.2 ± 4.2</td>
</tr>
</tbody>
</table>
Keshavarzi-Nomura-Teubner (KNT18): some remarks

- Completely new analysis: all data reanalyzed anew
- Thorough analysis of KLOE data in collaboration with the KLOE people to calculate and release a new correlation matrix for all KLOE data
- Integration by trapezoidal rule (as always)

KLOE+KNT 17
KNT18: $2\pi$ channel

![Graph showing the fit of $\sigma^0(e^+e^- \rightarrow \pi^+\pi^-)$ data for various experiments.](graph)

- BaBar (09)
- CMD-2 (03)
- SND (04)
- CMD-2 (06)
- KLOE combination
- BESIII (15)

The graph displays the fit of all $\pi^+\pi^-$ data across different experiments, indicating the measured cross-sections as a function of $\sqrt{s}$.
KNT18: $2\pi$ channel

Fit of all $\pi^+\pi^-$ data: $369.41 \pm 1.32$

Direct scan only: $370.77 \pm 2.61$

KLOE combination: $366.88 \pm 2.15$

BaBar (09): $376.71 \pm 2.72$

BESIII (15): $368.15 \pm 4.22$
KNT18: $2\pi$ channel

$$a_{\mu} \pi^+\pi^- (0.6 \leq \sqrt{s} \leq 0.9 \text{ GeV}) = (369.41 \pm 1.32) \times 10^{-10}$$

$$\chi^2_{\text{min}}/\text{d.o.f.} = 1.30$$
KNT18: final results

\[ a_\mu^{\text{had, LO VP} \times 10^{10}} \]

- DEHZ03: 696.3 ± 7.2
- HMNT03: 692.4 ± 6.4
- DEHZ06: 690.9 ± 4.4
- HMNT06: 689.4 ± 4.6
- FJ06: 692.1 ± 5.6
- DHMZ10: 692.3 ± 4.2
- JS11: 690.8 ± 4.7
- HLMNT11: 694.9 ± 4.3
- FJ17: 688.1 ± 4.1
- DHMZ17: 693.1 ± 3.4
- KNT18: 693.3 ± 2.5
KNT18: final results
Some remarks:

- improved extrapolation in the low-energy region (analyticity)
- conservative approach in combining KLOE and BABAR
- update to the newest data available (in different channels)
Another illustration of the problem in the $\pi\pi$ channel

$e^+e^- \rightarrow \pi^+\pi^-$

Relative difference to Fit (all data)

Combined Fit (all data)  Fit (without KLOE)  Fit (without BABAR)
Davier-Hoecker-Malaescu-Zhang (DHMZ19)

Another illustration of the problem in the $\pi\pi$ channel

$\alpha_{\mu} (\pi^+\pi^-, 0.6 - 0.9 \text{ GeV}) \times 10^{-10}$

- CLEO: $376.9 \pm 6.3$
- SND: $371.7 \pm 5.0$
- BESIII: $368.2 \pm 4.2$
- CMD-2: $372.4 \pm 3.0$
- BABAR: $376.7 \pm 2.7$
- KLOE: $366.9 \pm 2.1$
- CLEO: $6.3 \pm 376.9$
Davier-Hoecker-Malaescu-Zhang (DHMZ19)

Final result:

- J 2018: $-31.5 \pm 4.4 \ (4.1\sigma)$
- KNT 2018: $-27.0 \pm 3.6 \ (3.7\sigma)$
- DHMZ 2019: $-26.2 \pm 4.8 \ (3.3\sigma)$

$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM predictions}} \ [ \times 10^{-10} ]$

Systematic uncertainty

BNL-E821: $0 \pm 6.3$
Some remarks on KNT18 vs DHMZ19

- the two latest analyses of HVP based on the dispersive approach substantially agree on the final number
- they differ in the evaluation of the $2\pi$ contribution ($\Delta_{\pi\pi} = 4.06 \pm 3.8$)
- for the $2\pi$ channel there is a long-standing discrepancy between KLOE and BABAR, the most precise datasets
Some remarks on KNT18 vs DHMZ19

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- they differ in the evaluation of the $2\pi$ contribution ($\Delta_{\pi\pi} = 4.06 \pm 3.8$)
- for the $2\pi$ channel there is a long-standing discrepancy between KLOE and BABAR, the most precise datasets
- analyze the $2\pi$ contribution with more theory input
The $2\pi$ contribution

For HVP the unitarity relation is simple and looks the same for all possible intermediate states

$$\text{Im} \Pi(q^2) \propto \sigma(e^+e^- \rightarrow \text{hadrons})$$

which implies

$$\bar{\Pi}(q^2) = \sum_i \frac{q^2}{\pi} \int_{4M^2_\pi}^{\infty} dt \frac{\sigma(e^+e^- \rightarrow H_i(t))}{4\pi\alpha(t - q^2)}$$
The $2\pi$ contribution

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which implies

$$\bar{\Pi}_{2\pi}(q^2) = \frac{q^2}{\pi} \int_{4M^2_{\pi}}^{\infty} dt \frac{\sigma(e^+e^- \rightarrow 2\pi)}{4\pi\alpha(t-q^2)}$$

$$(H_i(t) = 2\pi)$$

de Trocóniz, Ynduráin (01,04), Leutwyler, GC (02,03), Anthanarayan et al. (13,16)
The $2\pi$ contribution

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\[ (H_i(t) = 2\pi) \]

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$$(H_i(t) = 2\pi)$$

de Trocóniz, Ynduráin (01,04), Leutwyler, GC (02,03), Anthanarayan et al. (13,16)

The pion vector form factor $F_\pi^V(t)$ also satisfies a dispersion relation which relates it to the P wave $\pi\pi$ phase shift

Most phenomenological analyses do not use this property
Results for \((g - 2)_\mu\)

Based on: dispersive representation of \(F^\pi_V(t)\) which depends on a small number of parameters which are fit to data

\[a^\mu_{HVP,\pi\pi} \leq 0.63\text{GeV} = 132.8(0.4)(1.0) \cdot 10^{-10}\]

[in agreement with 132.9(8) \(\text{Ananthanarayan et al. (16)}\)]

\(\text{Full range:}\)

\[a^\mu_{HVP,\pi\pi} \leq 1\text{GeV} = 495.0(1.5)(2.1) \cdot 10^{-10}\]
Results for $a_{\mu}^{\pi\pi}|_{\leq1\text{ GeV}}$ from the VFF fits to single experiments and combinations

- SND
- CMD-2
- BaBar
- KLOE
- energy scan
- all $e^+e^-$
- all $e^+e^-$, NA7

$10^{10} \times a_{\mu}^{\pi\pi}|_{\leq1\text{ GeV}}$
Results for \((g - 2)_{\mu}\)

Uncertainties on \(a_{\mu}^{\pi\pi}\left|_{\leq 1\text{ GeV}}\) in combined fit to all experiments
### Results for \((g - 2)_\mu\)

#### Comparison to other recent analyses

<table>
<thead>
<tr>
<th>Energy range</th>
<th>our work</th>
<th>KNT18</th>
<th>DHMZ17</th>
<th>ACD18</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\leq 0.6) GeV</td>
<td>110.1(9)</td>
<td>108.7(9)</td>
<td>110.2(1.0)</td>
<td></td>
</tr>
<tr>
<td>(\leq 0.7) GeV</td>
<td>214.8(1.7)</td>
<td>213.0(1.2)</td>
<td>214.7(1.3)</td>
<td></td>
</tr>
<tr>
<td>(\leq 0.8) GeV</td>
<td>413.2(2.3)</td>
<td>411.6(1.7)</td>
<td>414.0(1.9)</td>
<td></td>
</tr>
<tr>
<td>(\leq 0.9) GeV</td>
<td>479.8(2.6)</td>
<td>478.1(1.9)</td>
<td>481.2(2.2)</td>
<td></td>
</tr>
<tr>
<td>(\leq 1.0) GeV</td>
<td>495.0(2.6)</td>
<td>493.4(1.9)</td>
<td>496.7(2.2)</td>
<td></td>
</tr>
<tr>
<td>([0.6, 0.7]) GeV</td>
<td>104.7(7)</td>
<td>104.3(5)</td>
<td>104.5(5)</td>
<td></td>
</tr>
<tr>
<td>([0.7, 0.8]) GeV</td>
<td>198.3(9)</td>
<td>198.6(8)</td>
<td>199.3(9)</td>
<td></td>
</tr>
<tr>
<td>([0.8, 0.9]) GeV</td>
<td>66.6(4)</td>
<td>66.5(3)</td>
<td>67.2(4)</td>
<td></td>
</tr>
<tr>
<td>([0.9, 1.0]) GeV</td>
<td>15.3(1)</td>
<td>15.3(1)</td>
<td>15.5(1)</td>
<td></td>
</tr>
<tr>
<td>([\sqrt{0.1}, \sqrt{0.95}]) GeV</td>
<td>490.7(2.6)</td>
<td>489.0(1.9)</td>
<td>492.2(2.2)</td>
<td></td>
</tr>
</tbody>
</table>

KNT18 = Keshavarzi et al. (18), DHMZ17 = Davier et al. (17), ACD18 = Ananthanarayan et al. (18)
Latest results of Lattice HVP calculations (2019)

- **Fermilab/HPQCD/MILC 2019**
- **ETMC, 1808.00887**
- **RBC/UKQCD 1801.07224**
- **BMW, 1711.04980**
- **Mainz/CLS, 1705.01775**
- **HPQCD/RV 1601.03071**

- **Keshavarzi et al., 1802.02995**
- **Davier et al., 1706.09436**
- **Jegerlehner, 1705.00263**
- **Benayoun et al., 1507.02943**

*no new physics*

$L_{\beta\beta}$ to $(g - 2)_{\mu}$
Latest results of Lattice HVP calculations (2019)

- Mainz/CLS 17 ($N_f = 2$)
- BMW 17
- RBC/UKQCD 18
- ETMC 19
- PACS 19
- FNAL-HPQCD-MILC 19
- This work

Red band=KNT18

Gérardin et al. (Mainz) (19)
Conclusions: HVP

- precision in the dispersive evaluation of the HVP contribution has reached a stunning 0.4%
- the dominant $\pi\pi$ contribution could become even more precise were it not for an experimental discrepancy between KLOE and BABAR
- ⇒ use more theory: analyticity and unitarity relate the $\pi\pi$ contribution and the well known $\pi\pi$ P-wave phase shift:
  \[
a^\text{HVP,}\pi\pi_{|_{\leq 1\text{GeV}}} = 495.0(2.6) \cdot 10^{-10}
\]
  [KNT18: 493.4(1.9) and DHMZ17: 496.7(2.2)]
- At the Seattle Workshop a consensus has been achieved on how to merge all recent analyses into a single number
Outline

Introduction: $(g - 2)_\mu$ and hadronic contributions

Hadronic Vacuum Polarization contribution to $(g - 2)_\mu$

Master Formula

A dispersion relation for HLbL

Conclusions
Calculating the HLbL contribution is complicated

- 4-point function of em currents in QCD

- until recently, only model calculations

- a data-driven approach like for HVP seemed hopeless
  “it cannot be expressed in terms of measurable quantities”

- lattice QCD is an alternative and is making fast progress
Different model-based evaluations of HLbL

<table>
<thead>
<tr>
<th>Contribution</th>
<th>BPaP(96)</th>
<th>HKS(96)</th>
<th>KnN(02)</th>
<th>MV(04)</th>
<th>BP(07)</th>
<th>PdRV(09)</th>
<th>N/JN(09)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0$, $\eta$, $\eta'$</td>
<td>$85\pm13$</td>
<td>$82.7\pm6.4$</td>
<td>$83\pm12$</td>
<td>$114\pm10$</td>
<td>$-$</td>
<td>$-$</td>
<td>$114\pm13$</td>
</tr>
<tr>
<td>$\pi$, $K$ loops</td>
<td>$-19\pm13$</td>
<td>$-4.5\pm8.1$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-19\pm19$</td>
<td>$-19\pm13$</td>
</tr>
<tr>
<td>&quot; + subl. in $N_C$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$0\pm10$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>axial vectors</td>
<td>$2.5\pm1.0$</td>
<td>$1.7\pm1.7$</td>
<td>$-$</td>
<td>$22\pm5$</td>
<td>$-$</td>
<td>$15\pm10$</td>
<td>$22\pm5$</td>
</tr>
<tr>
<td>scalars</td>
<td>$-6.8\pm2.0$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-7\pm7$</td>
</tr>
<tr>
<td>quark loops</td>
<td>$21\pm3$</td>
<td>$9.7\pm11.1$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$2.3$</td>
</tr>
<tr>
<td>total</td>
<td>$83\pm32$</td>
<td>$89.6\pm15.4$</td>
<td>$80\pm40$</td>
<td>$136\pm25$</td>
<td>$110\pm40$</td>
<td>$105\pm26$</td>
<td>$116\pm39$</td>
</tr>
</tbody>
</table>

Legenda: B=Bijnens, Pa=Pallante, P=Prades, H=Hayakawa, K=Kinoshita, S=Sanda, Kn=Knecht, N=Nyffeler, M=Melnikhov, V=Vainshtein, dR=de Rafael, J=Jegerlehner

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, i.e. two-pion cuts ($K$s are subdominant)
- heavier single-particle poles decreasingly important
## Different model-based evaluations of HLbL

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<td>$22 \pm 5$</td>
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<tr>
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<td>$-\quad$</td>
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<td>$-\quad$</td>
<td>$-\quad$</td>
<td>$-7 \pm 7$</td>
<td>$-7 \pm 2$</td>
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<tr>
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<td>$-\quad$</td>
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- pseudoscalar pole contributions most important
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- heavier single-particle poles decreasingly important
Advantages of the dispersive approach

- model independent
- **unambiguous definition** of the various contributions
- makes a data-driven evaluation possible (in principle)

First attempts:
- GC, Hoferichter, Procura, Stoffer (14)
- Pauk, Vanderhaeghen (14)

- similar philosophy, with a different implementation: Schwinger sum rule

- why hasn’t this been adopted before?
The HLbL tensor

HLbL tensor:

\[ \Pi_{\mu \nu \lambda \sigma} = i^3 \int dx \int dy \int dz \, e^{-i(x \cdot q_1 + y \cdot q_2 + z \cdot q_3)} \langle 0 | T \{ j^{\mu}(x) j^{\nu}(y) j^{\lambda}(z) j^{\sigma}(0) \} | 0 \rangle \]

\[ q_4 = k = q_1 + q_2 + q_3 \quad k^2 = 0 \]

General Lorentz-invariant decomposition:

\[ \Pi_{\mu \nu \lambda \sigma} = g^{\mu \nu} g^{\lambda \sigma} \Pi^1 + g^{\mu \lambda} g^{\nu \sigma} \Pi^2 + g^{\mu \sigma} g^{\nu \lambda} \Pi^3 + \sum_{i,j,k,l} q_i^{\mu} q_j^{\nu} q_k^{\lambda} q_l^{\sigma} \Pi_{ijkl}^4 + \ldots \]

consists of 138 scalar functions \{\Pi^1, \Pi^2, \ldots\}, but in \( d = 4 \) only

136 are linearly independent

Eichmann et al. (14)

Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

⇒ Apply the Bardeen-Tung (68) method + Tarrach (75) addition
Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

- 43 basis tensors (BT)
- 11 additional ones (T)
- of these 54 only 7 are distinct structures
- all remaining 47 can be obtained by crossing transformations of these 7: manifest crossing symmetry
- the dynamical calculation needed to fully determine the LbL tensor concerns these 7 scalar amplitudes

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_{i}^{\mu\nu\lambda\sigma} \Pi_{i}$$

GC, Hoferichter, Procura, Stoffer (2015)

in $d = 4$: 41= no. of helicity amplitudes to guarantee basis completeness everywhere
Master Formula

\[ a_{\mu}^{\text{HLbL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p)\hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_\mu^2][(p - q_2)^2 - m_\mu^2]} \]

- \( \hat{T}_i \): known kernel functions
- \( \hat{\Pi}_i \): linear combinations of the \( \Pi_i \)
- the \( \Pi_i \) are amenable to a dispersive treatment: their imaginary parts are related to measurable subprocesses
- 5 integrals can be performed with Gegenbauer polynomial techniques

GC, Hoferichter, Procura, Stoffer (2015)
Master Formula

After performing the 5 integrations:

$$a_{\mu}^{\text{HLbL}} = \frac{2\alpha^3}{48\pi^2} \int_0^\infty dQ_1^4 \int_0^\infty dQ_2^4 \int_{-1}^1 d\tau \sqrt{1 - \tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

where $Q_i^\mu$ are the Wick-rotated four-momenta and $\tau$ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.
Setting up the dispersive calculation

We split the HLbL tensor as follows:

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\pi\text{-box}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots \]

Pion pole: imaginary parts = \( \delta \)-functions
Projection on the BTT basis: easy ✓
Our master formula = explicit expressions in the literature ✓
Input: pion transition form factor
First results of direct lattice calculations

Hoferichter et al. (18)
Gerardin, Meyer, Nyffeler (16,19)
Setting up the dispersive calculation

We split the HLbL tensor as follows:

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0 \text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\pi \text{-box}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots \]

\[ \equiv F_V^{\pi}(q_1^2)F_V^{\pi}(q_2^2)F_V^{\pi}(q_3^2) \times \left[ \begin{array}{c} + \\ + \\ + \end{array} \right] \]
Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\pi\text{-box}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

The "rest" with $2\pi$ intermediate states has cuts only in one channel and will be calculated dispersively after partial-wave expansion.
Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Cuts with states $\neq \pi$ or $2\pi$ are neglected

$\eta, \eta'$ or the $K$-loop contribution can be calculated analogously
Pion-pole contribution

Latest complete analyses:

- **Dispersive calculation of the pion TFF**
  \[ a_{\mu}^{\pi^0} = 63.0^{+2.7}_{-2.1} \times 10^{-11} \]

- **Padé-Canterbury approximants**
  \[ a_{\mu}^{\pi^0} = 63.6(2.7) \times 10^{-11} \]

- **Lattice**
  \[ a_{\mu}^{\pi^0} = 62.3(2.3) \times 10^{-11} \]
Pion-box contribution

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\mu-\text{pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots \]
Pion-box contribution

The only ingredient needed for the pion-box contribution is the vector form factor

$$\hat{\Pi}_{\pi}^{\text{box}} = F^V_\pi(q_1^2) F^V_\pi(q_2^2) F^V_\pi(q_3^2) \frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \, I_i(x, y),$$

where

$$I_1(x, y) = \frac{8xy(1 - 2x)(1 - 2y)}{\Delta_{123}\Delta_{23}},$$

and analogous expressions for $I_{4,7,17,39,54}$ and

$$\Delta_{123} = M_{\pi}^2 - xyq_1^2 - x(1 - x - y)q_2^2 - y(1 - x - y)q_3^2,$$

$$\Delta_{23} = M_{\pi}^2 - x(1 - x)q_2^2 - y(1 - y)q_3^2.$$
Pion-box contribution

Uncertainties are negligibly small:

\[ a_{\mu}^{\text{FsQED}} = -15.9(2) \times 10^{-11} \]
First evaluation of $S$-wave $2\pi$-rescattering

Omnès solution for $\gamma^*\gamma^* \rightarrow \pi\pi$ provides the following:

\[
\begin{align*}
&\quad \quad = \quad + \\
\quad \text{recursive PWE, no LHC}
\end{align*}
\]

$S$-wave contribution below $\sim 1$ GeV:

\[
a_{\mu, J=0}^{\pi\pi, \pi\text{-pole LHC}} = -8(1) \times 10^{-11}
\]
Two-pion contribution to $(g - 2)_\mu$ from HLbL

Two-pion contributions to HLbL:

\[
\begin{align*}
\text{pion box} & \quad + \quad \text{rescattering contribution} \\
\end{align*}
\]

\[
a_\mu^{\pi-\text{box}} + a_\mu^{\pi\pi,\pi-\text{pole LHC}} = -24(1) \cdot 10^{-11}
\]

Contributions from above 1 GeV and higher partial waves are work in progress
Contributions to $10^{11} \cdot a^{\text{HLbL}}_{\mu}$

- Pseudoscalar poles
  
- Pion box ($\text{kaon box} \sim -0.5$)

- S-wave $\pi\pi$ rescattering

Partial total: $69.9 \pm 4.1$

- Scalars and tensors with $M_R > 1$ GeV
  
- Axial vectors

- Short-distance contribution

Central value: $86 \pm XX$

Uncertainties added in quadrature: $XX = 14$

Uncertainties added linearly: $XX = 25$
## Improvements obtained with the dispersive approach

<table>
<thead>
<tr>
<th>Contribution</th>
<th>PdRV(09)</th>
<th>N/JN(09)</th>
<th>J(17)</th>
<th>White Paper</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0, \eta, \eta'$-poles</td>
<td>$114 \pm 13$</td>
<td>$99 \pm 16$</td>
<td>$95.45 \pm 12.40$</td>
<td>$93.8 \pm 4.0$</td>
</tr>
<tr>
<td>$\pi, K$-loop/box</td>
<td>$-19 \pm 19$</td>
<td>$-19 \pm 13$</td>
<td>$-20 \pm 5$</td>
<td>$-16.4 \pm 0.2$</td>
</tr>
<tr>
<td>S-wave $\pi\pi$</td>
<td>$-7 \pm 7$</td>
<td>$-7 \pm 2$</td>
<td>$-5.98 \pm 1.20$</td>
<td>$-8 \pm 1$</td>
</tr>
<tr>
<td>scalars</td>
<td>$15 \pm 10$</td>
<td>$22 \pm 5$</td>
<td>$7.55 \pm 2.71$</td>
<td>$8 \pm 8$</td>
</tr>
<tr>
<td>tensors</td>
<td>$2.3$</td>
<td>$21 \pm 3$</td>
<td>$22.3 \pm 5.0$</td>
<td>$10 \pm 10$</td>
</tr>
<tr>
<td>total</td>
<td>$105 \pm 26$</td>
<td>$116 \pm 39$</td>
<td>$100.4 \pm 28.2$</td>
<td>$86 \pm XX$</td>
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</table>

HLbL in units of $10^{-11}$.

PdRV = Prades, de Rafael, Vainshtein (“Glasgow consensus”);
N = Nyffeler; J = Jegerlehner
Outline

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Hadronic Vacuum Polarization contribution to \((g - 2)_\mu\)

Hadronic light-by-light contribution to \((g - 2)_\mu\)

Master Formula

A dispersion relation for HLbL

Conclusions
Conclusions: HLbL

- The HLbL contribution to \((g - 2)_\mu\) can be expressed in terms of measurable quantities in a dispersive approach.

- **master formula**: HLbL contribution to \(a_\mu\) as triple-integral over scalar functions which satisfy dispersion relations.

- The relevant measurable quantity entering the dispersion relation depends on the intermediate state:
  - **single-pion contribution**: pion transition form factor.
  - **pion-box contribution**: pion vector form factor.
  - **2-pion rescattering**: \(\gamma^*(\gamma^(*) \rightarrow \pi\pi\) helicity amplitudes.

- These three contributions (S-wave for the latter) have been calculated with remarkably small uncertainties.

- Work on calculating other contributions and estimating missing pieces is in progress.
Final conclusions

- The $\sim 3.5 \sigma$ discrepancy between SM and measurement for $(g - 2)_\mu$ continues to stay.

- The Fermilab Muon $g - 2$ experiment aims to reduce the experimental uncertainty by a factor four, potentially leading to a $7\sigma$ discrepancy.

- I have reviewed current theoretical efforts to improve the calculations of the hadronic contributions in a model-independent way based on dispersive approaches.

- The goal of matching the experimental reduction of the uncertainty with a similar reduction on the theory side is being achieved.