Theory progress in the calculation of hadronic contributions to the $(g-2)_\mu$

Gilberto Colangelo

UNIVERSITÄT BERN

AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

New physics at the low-energy precision frontier
LPT Orsay, 18.9.2019

Outline

Introduction: $(g-2)_\mu$ and hadronic contributions

Hadronic Vacuum Polarization contribution to $(g-2)_\mu$

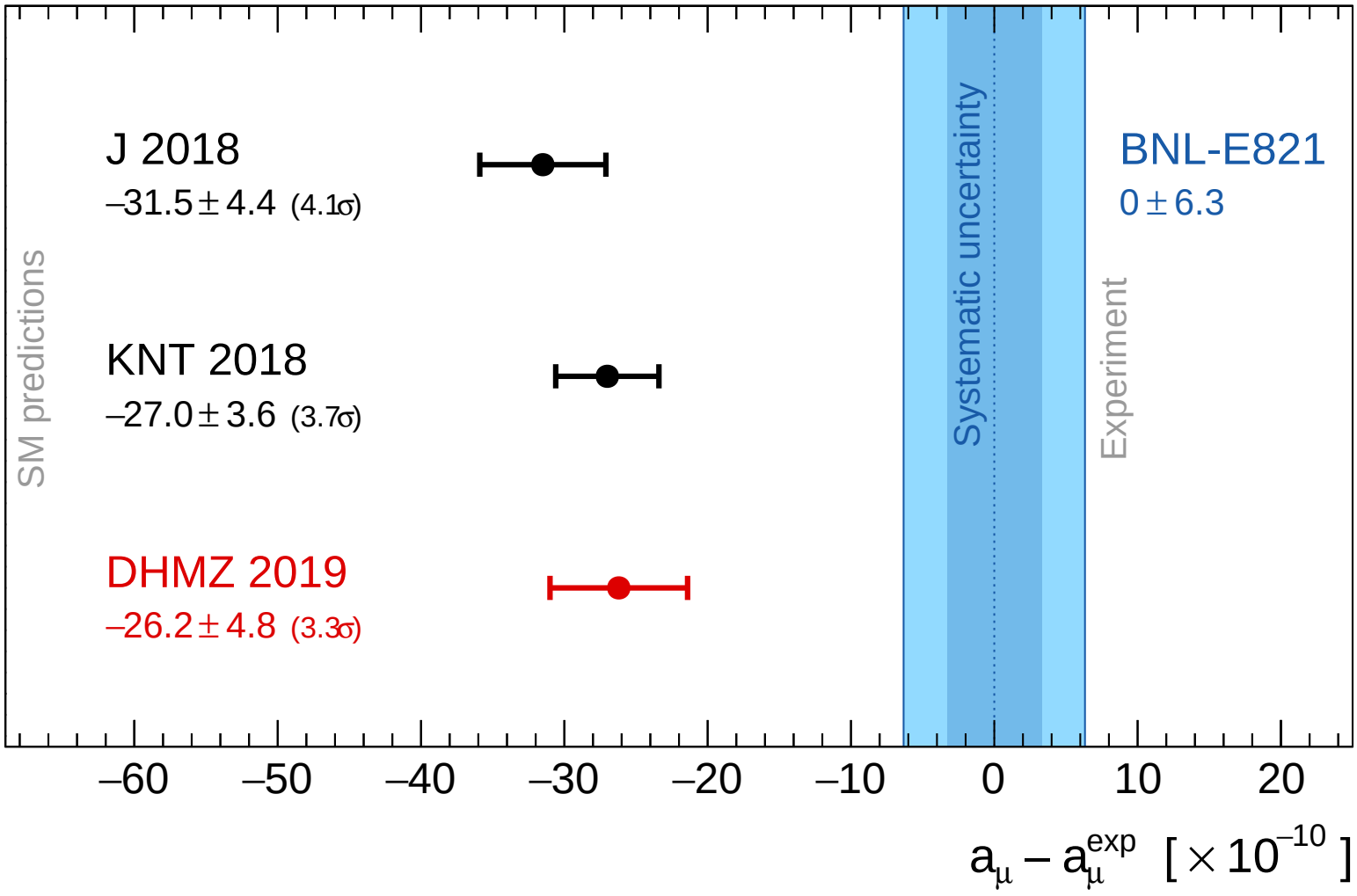
Hadronic light-by-light contribution to $(g-2)_\mu$
- Master Formula
- A dispersion relation for HLbL

Conclusions

Outline

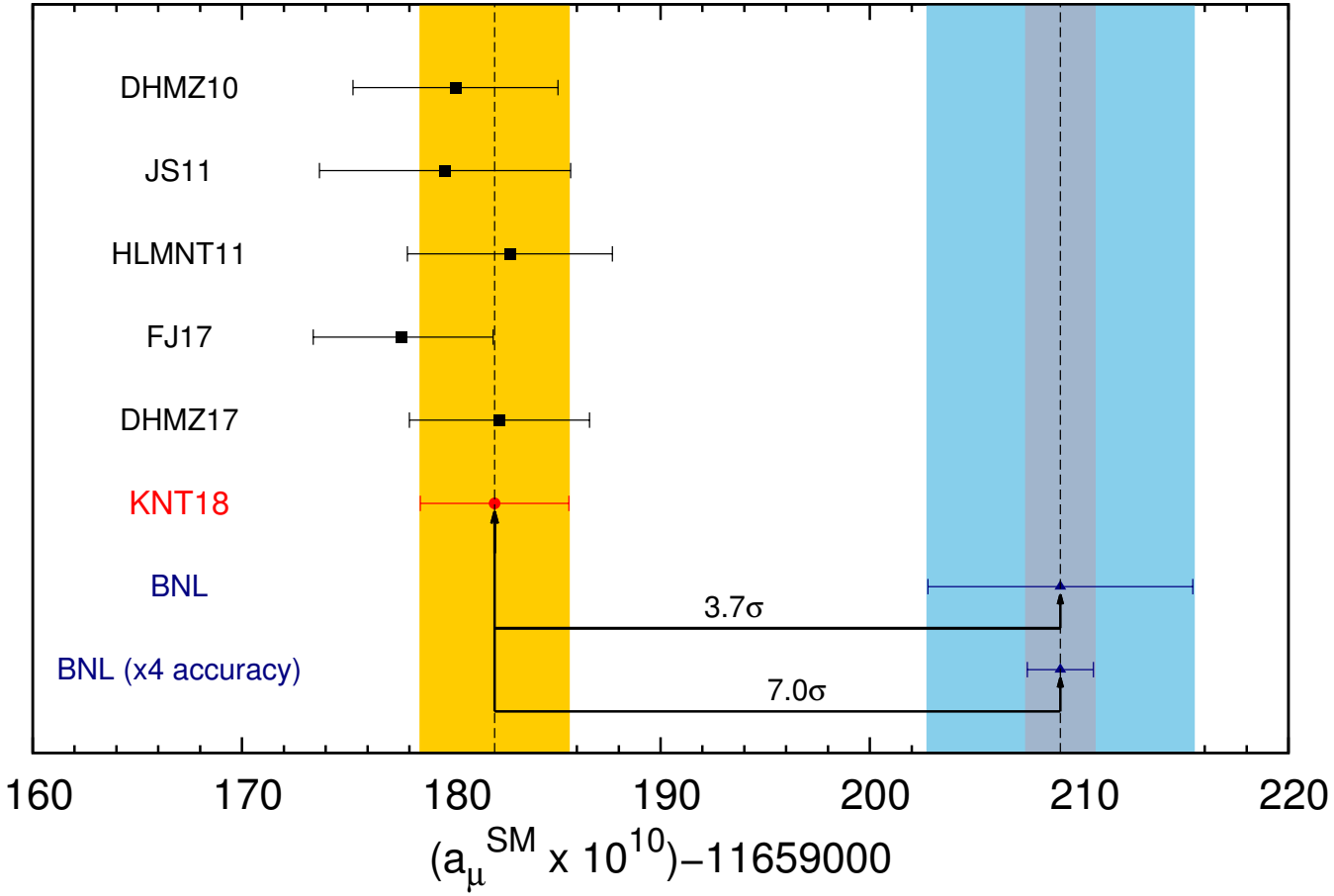
Status of $(g-2)_\mu$, experiment vs SM

Davier, Hoecker, Malaescu, Zhang 2019



Status of $(g-2)_\mu$, experiment vs SM

Keshavarzi, Nomura, Teubner, 2018 (KNT18)



Fermilab experiment's goal: error $\times 1/4$, should be matched by theory:

$\Rightarrow$ Muon "$(g-2)$ Theory Initiative" lead by A. El-Khadra and C. Lehner

Status of $(g-2)_\mu$, experiment vs SM

KNT 18

	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.97	0.07
electroweak, total	153.6	1.0
HVP (LO) [KNT 18]	6 932.7	24.6
HVP (NLO) [KNT 18]	−98.2	0.4
HLbL [update of Glasgow consensus–KNT 18]	98.0	26.0
HVP (NNLO) [Kurz, Liu, Marquard, Steinhauser 14]	12.4	0.1
HLbL (NLO) [GC, Hoferichter, Nyffeler, Passera, Stoffer 14]	3.0	2.0
theory	116 591 820.5	35.6

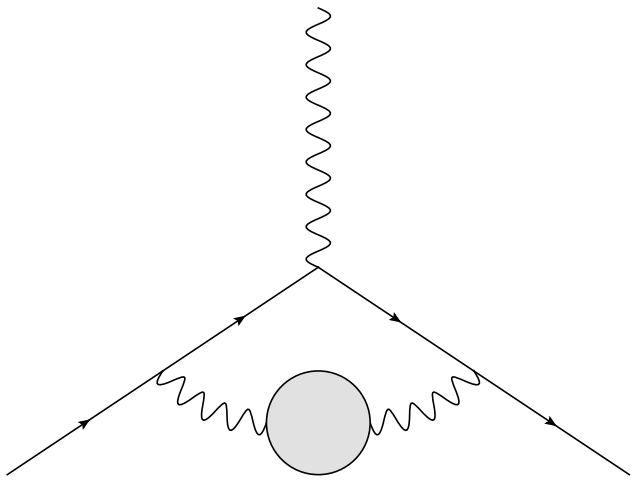
Status of $(g-2)_\mu$, experiment vs SM

$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = 268.5 \pm 72.4 \qquad [3.7\sigma]$$

Keshavarzi, Nomura, Teubner, 2018

Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved



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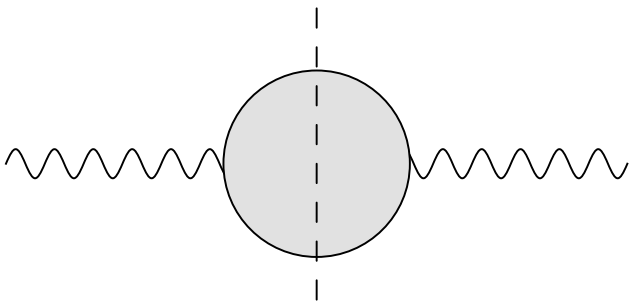
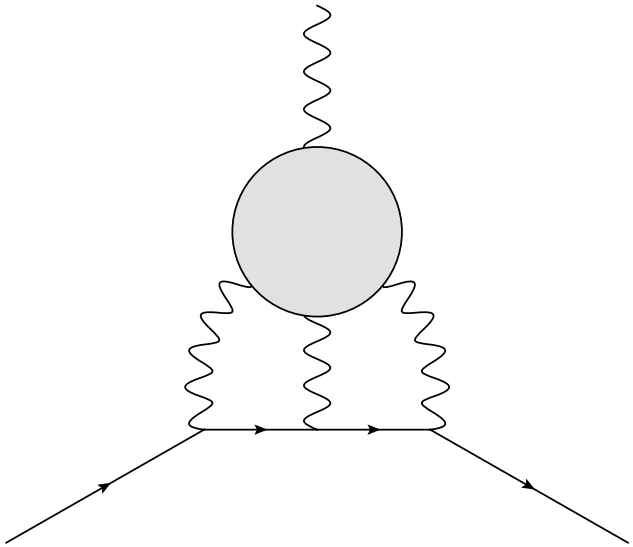


  - basic principles: unitarity and analyticity
  - direct relation to experiment: $\sigma_{\rm tot}(e^+e^- \to \gamma^* \to \text{hadrons})$
  - dedicated e^+e^- program: BaBar, Belle, BESIII, CMD3, KLOE2, SND
  - alternative approach: lattice (ETMC, Mainz, HPQCD, BMW, RBC/UKQCD)

Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved
- Hadronic light-by-light (HLbL) is more problematic:



  - 4-point fct. of em currents in QCD
  - "*it cannot be expressed in terms of measurable quantities*"
  - until recently, only model calculations
  - lattice QCD is making fast progress

Muon $g-2$ Theory Initiative

Steering Committee:
GC
Michel Davier
Simon Eidelman
Aida El-Khadra (co-chair)
Christoph Lehner (co-chair)
Tsutomu Mibe (J-PARC E34 experiment)
Andreas Nyffeler
Lee Roberts (Fermilab E989 experiment)
Thomas Teubner

Workshops:

- First plenary meeting, Q-Center (Fermilab), 3-6 June 2017
- HVP WG workshop, KEK (Japan), 12-14 February 2018
- HLbL WG workshop, U. of Connecticut, 12-14 March 2018
- Second plenary meeting, Mainz, 18-22 June 2018
- Third plenary meeting, Seattle, 9-13 September 2019

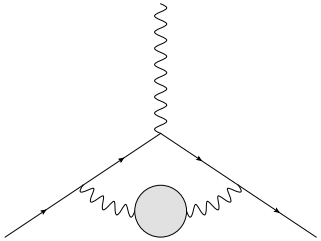
Take-home message of this talk

1. There is significant progress in the theoretical evaluation of hadronic contribution to $(g-2)_\mu$
2. Dispersion relations (DR) play an important role in these improvements
   - for HVP DR have been applied since many years
     but it can be pushed even further
   - for HLbL this is a recent development and is leading to
     a significant reduction of the uncertanty
3. Lattice—though not yet competitive—is catching up

Outline

Calculating the HVP contribution

- HVP can be calculated with a data-driven approach



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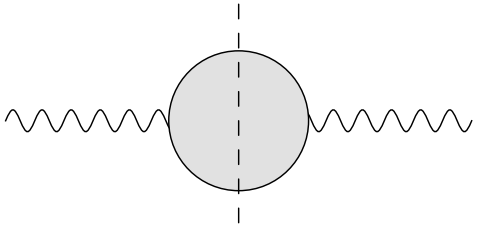


- direct relation to experiment: $\sigma_{\rm tot}(e^+e^- \to \gamma^* \to \text{hadrons})$
- dedicated e^+e^- program: BaBar, Belle, BESIII, CMD3, KLOE2, SND

HVP, gauge invariance and analyticity

$$\Pi_{\mu\nu}(q) = i\int d^4x e^{iqx}\langle 0|Tj_\mu(x)j_\nu(0)|0\rangle = \left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi(q^2)$$

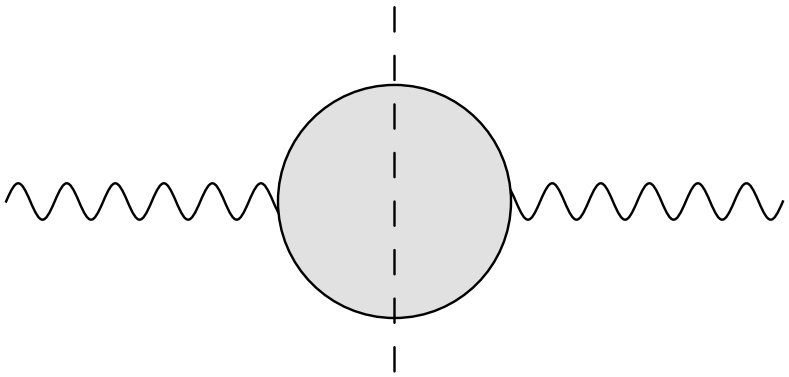
where $j^\mu(x) = \sum_i Q_i \bar{q}_i(x)\gamma^\mu q_i(x)$, $i = u, d, s$ is the em current

- Lorentz invariance: 2 structures
- gauge invariance: reduction to 1 structure
- Lorentz-tensor defined in such a way that the function $\Pi(q^2)$ does not have kinematic singularities or zeros
- $\hat{\Pi}(q^2) := \Pi(q^2) - \Pi(0)$ satisfies

$$\hat{\Pi}(q^2) = \frac{q^2}{\pi}\int_{4M_\pi^2}^{\infty} dt \frac{\mathrm{Im}\hat{\Pi}(t)}{t(t-q^2)}$$

Unitarity relation for HVP

For HVP the unitarity relation is simple and looks the same for all possible intermediate states



$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons}) = \sigma(e^+e^- \to \mu^+\mu^-)R(q^2)$$

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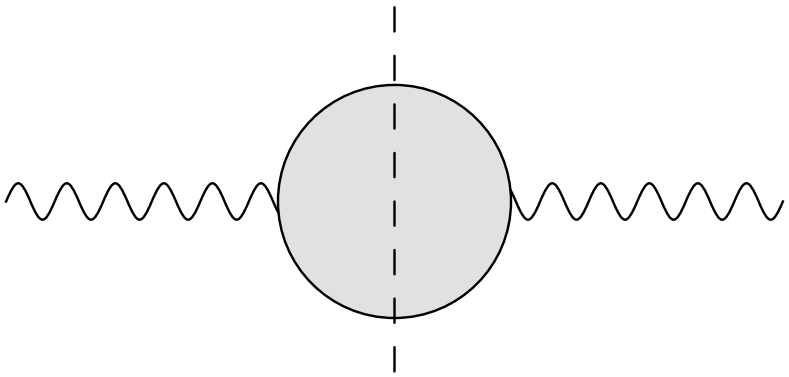


$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons}) = \sigma(e^+e^- \to \mu^+\mu^-)R(q^2)$$

$$a_\mu^{\mathrm{hvp}} = \frac{\alpha^2}{3\pi^2}\int_{s_{th}}^{\infty}\frac{ds}{s}K(s)R(s)$$

$K(s)$ known, depends on m_μ and $K(s) \sim \frac{1}{s}$ for large s

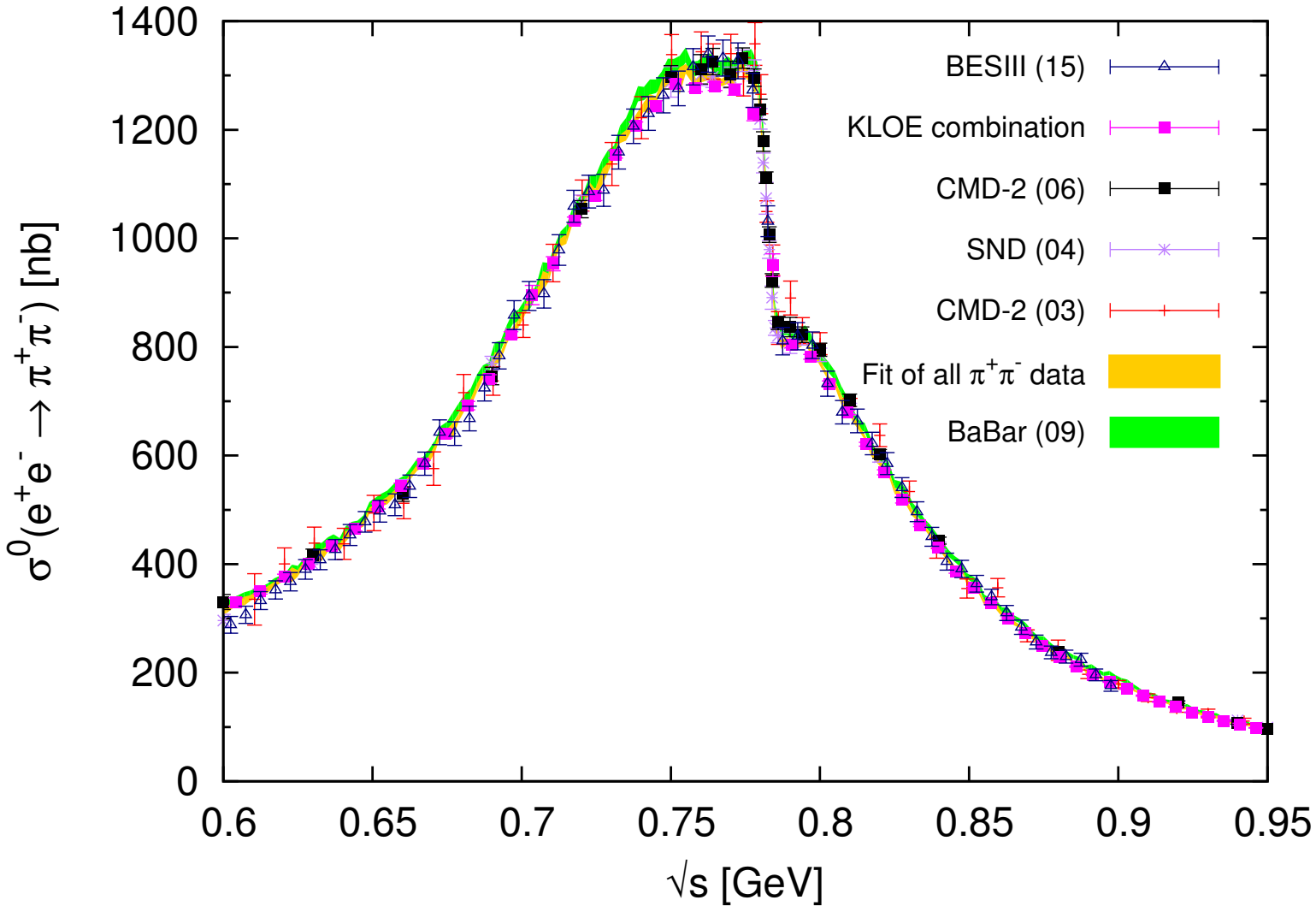
Uni

Channel	This work (KNT18)	DHMZ17 [77]	Difference
Data based channels ($\sqrt{s} \leq 1.8$ GeV)			
$\pi^0\gamma$ (data + ChPT)	4.58 ± 0.10	4.29 ± 0.10	0.29 ± 0.14
$\pi^+\pi^-$ (data + ChPT)	503.74 ± 1.96	507.14 ± 2.58	-3.40 ± 3.24
$\pi^+\pi^-\pi^0$ (data + ChPT)	47.70 ± 0.89	46.20 ± 1.45	1.50 ± 1.70
$\pi^+\pi^-\pi^+\pi^-$	13.99 ± 0.19	13.68 ± 0.31	0.31 ± 0.36
$\pi^+\pi^-\pi^0\pi^0$	18.15 ± 0.74	18.03 ± 0.54	0.12 ± 0.92
$(2\pi^+2\pi^-\pi^0)_{\text{no }\eta}$	0.79 ± 0.08	0.69 ± 0.08	0.10 ± 0.11
$3\pi^+3\pi^-$	0.10 ± 0.01	0.11 ± 0.01	-0.01 ± 0.01
$(2\pi^+2\pi^-2\pi^0)_{\text{no }\eta\omega}$	0.77 ± 0.11	0.72 ± 0.17	0.05 ± 0.20
K^+K^-	23.00 ± 0.22	22.81 ± 0.41	0.19 ± 0.47
$K^0_S K^0_L$	13.04 ± 0.19	12.82 ± 0.24	0.22 ± 0.31
$KK\pi$	2.44 ± 0.11	2.45 ± 0.15	-0.01 ± 0.19
$KK2\pi$	0.86 ± 0.05	0.85 ± 0.05	0.01 ± 0.07
$\eta\gamma$ (data + ChPT)	0.70 ± 0.02	0.65 ± 0.02	0.05 ± 0.03
$\eta\pi^+\pi^-$	1.18 ± 0.05	1.18 ± 0.07	0.00 ± 0.09
$(\eta\pi^+\pi^-\pi^0)_{\text{no }\omega}$	0.48 ± 0.12	0.39 ± 0.12	0.09 ± 0.17
$\eta 2\pi^+2\pi^-$	0.03 ± 0.01	0.03 ± 0.01	0.00 ± 0.01
$\eta\omega$	0.29 ± 0.02	0.32 ± 0.03	-0.03 ± 0.04
$\omega(\to \pi^0\gamma)\pi^0$	0.87 ± 0.02	0.94 ± 0.03	-0.07 ± 0.04
$(\eta\phi)_{\text{no }\phi\to K^0\bar{K}^0}$	0.22 ± 0.02	0.36 ± 0.03	-0.14 ± 0.04*
$\phi \to$ unaccounted	0.04 ± 0.04	0.05 ± 0.00	-0.01 ± 0.04
$\eta\omega\pi^0$	0.10 ± 0.05	0.06 ± 0.04	0.04 ± 0.06
$K^0_S K^0_L \eta$	0.12 ± 0.02	0.01 ± 0.01	0.11 ± 0.02*
Estimated contributions ($\sqrt{s} \leq 1.8$ GeV)			
$(\pi^+\pi^-3\pi^0)_{\text{no }\eta}$	0.40 ± 0.04	0.35 ± 0.04	0.05 ± 0.06
$(\pi^+\pi^-4\pi^0)_{\text{no }\eta}$	0.12 ± 0.12	0.11 ± 0.11	0.01 ± 0.16
$KK3\pi$	-0.02 ± 0.01	-0.03 ± 0.02	0.01 ± 0.02
$\omega(\to \text{npp})2\pi$	0.08 ± 0.01	0.08 ± 0.01	0.00 ± 0.01
$\omega(\to \text{npp})3\pi$	0.10 ± 0.02	0.36 ± 0.01	-0.26 ± 0.02
$\omega(\to \text{npp})KK$	0.00 ± 0.00	0.01 ± 0.00	-0.01 ± 0.00
$\eta\pi^+\pi^-2\pi^0$	0.03 ± 0.01	0.03 ± 0.01	0.00 ± 0.01
Other contributions			
J/ψ	6.26 ± 0.19	6.28 ± 0.07	-0.02 ± 0.20
ψ'	1.58 ± 0.04	1.57 ± 0.03	0.01 ± 0.05
$\Upsilon(1S-4S)$	0.09 ± 0.00	-	0.09 ± 0.00**
Contributions by energy region			
$1.8 \leq \sqrt{s} \leq 3.7$ GeV	34.54 ± 0.56 (data)	33.45 ± 0.65 (pQCD)***	1.09 ± 0.86
$3.7 \leq \sqrt{s} \leq 5.0$ GeV	7.33 ± 0.11 (data)	7.29 ± 0.03 (data)	0.04 ± 0.11
$5.0 \leq \sqrt{s} \leq 9.3$ GeV	6.62 ± 0.10 (data)	6.86 ± 0.04 (pQCD)	-0.24 ± 0.11
$9.3 \leq \sqrt{s} \leq 12.0$ GeV	1.12 ± 0.01 (data+pQCD)	1.21 ± 0.01 (pQCD)	-0.09 ± 0.02**
$12.0 \leq \sqrt{s} \leq 40.0$ GeV	1.64 ± 0.00 (pQCD)	1.64 ± 0.00 (pQCD)	0.00 ± 0.00
> 40.0 GeV	0.16 ± 0.00 (pQCD)	0.16 ± 0.00 (pQCD)	0.00 ± 0.00
Total	693.3 ± 2.5	693.1 ± 3.4	0.2 ± 4.2

Keshavarzi-Nomura-Teubner (KNT18): some remarks

- Completely new analysis: all data reanalyzed anew
- Thorough analysis of KLOE data in collaboration with the KLOE people to calculate and release a new correlation matrix for all KLOE data KLOE+KNT 17
- Integration by trapezoidal rule (as always)

KNT18: 2π channel



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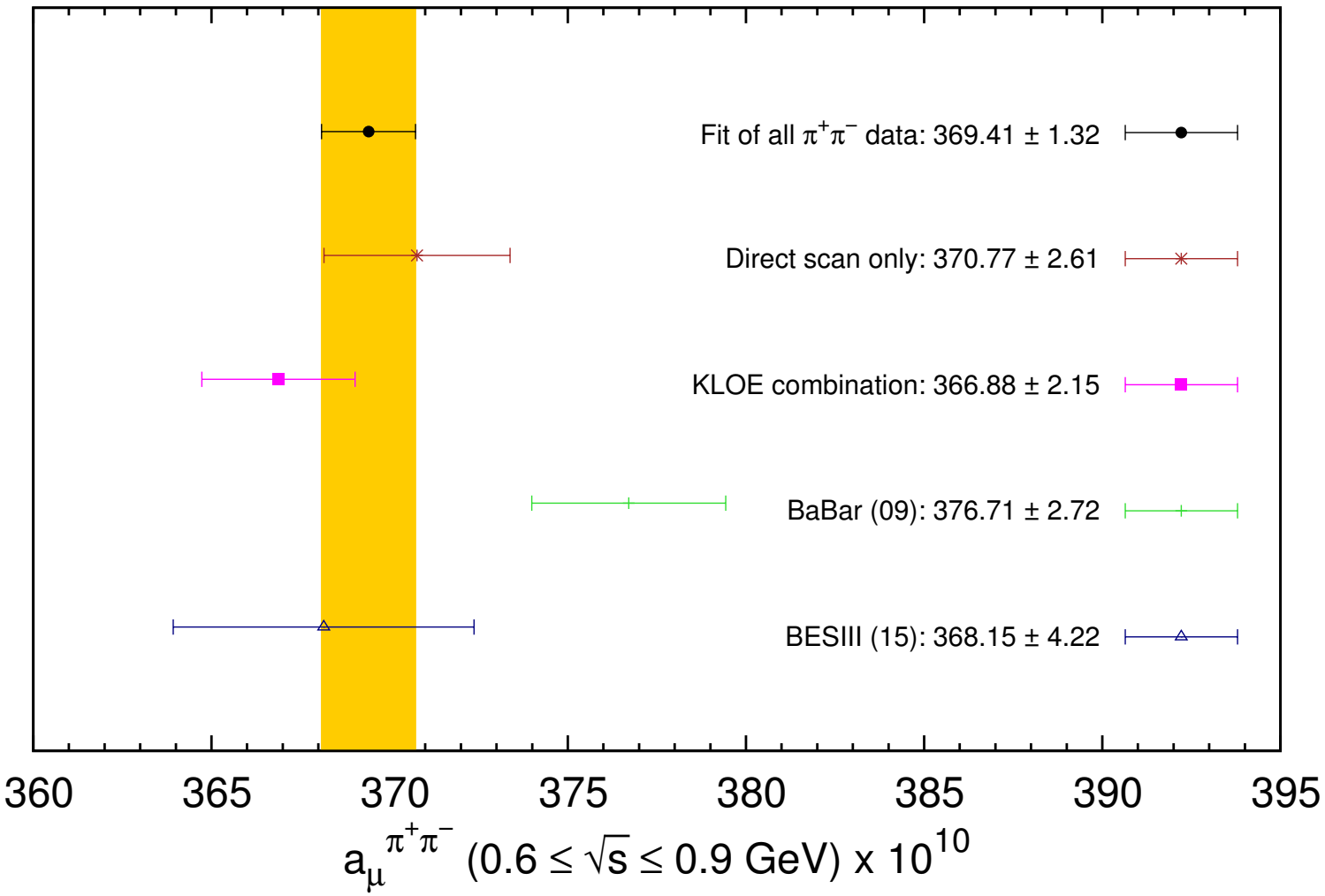

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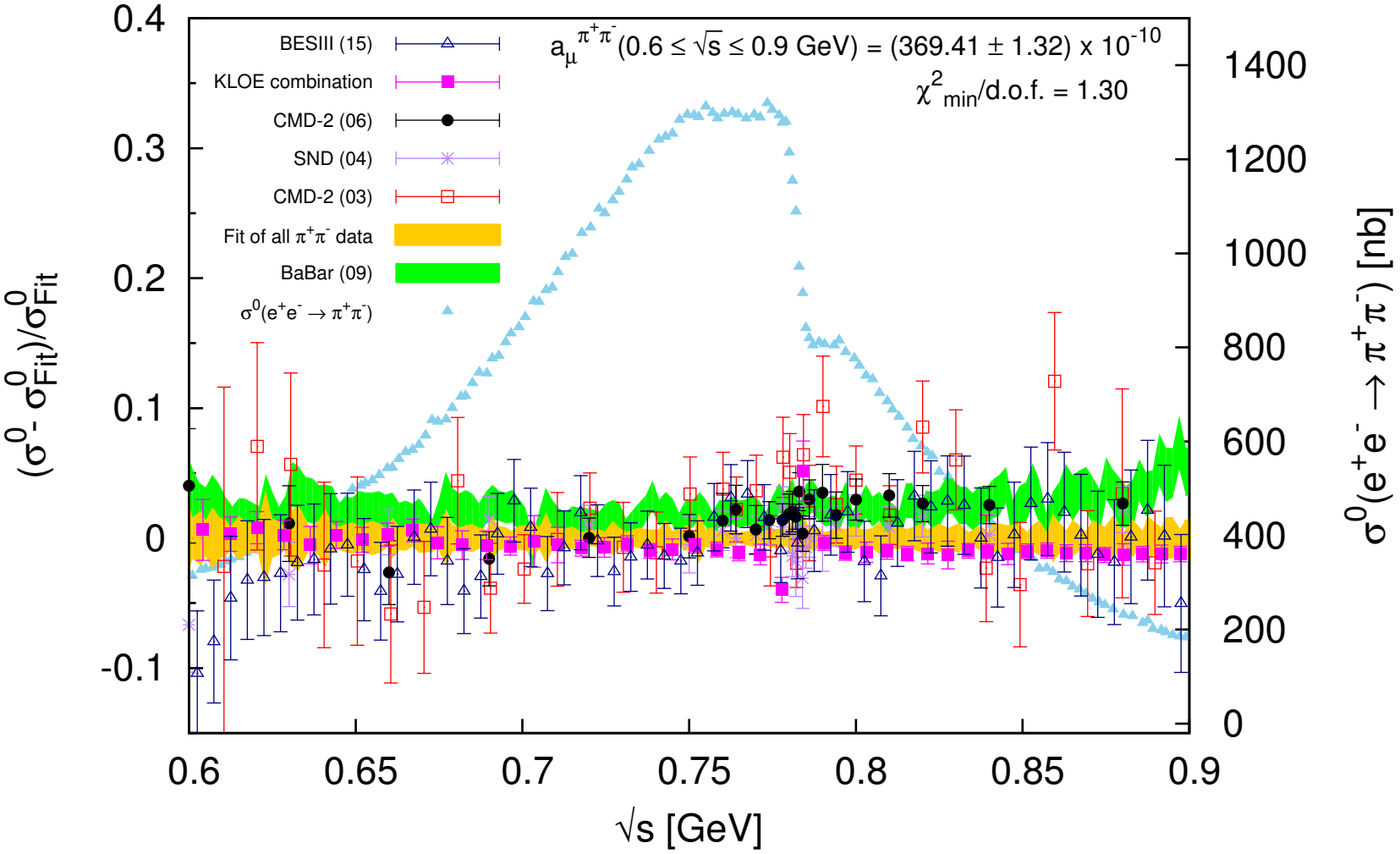
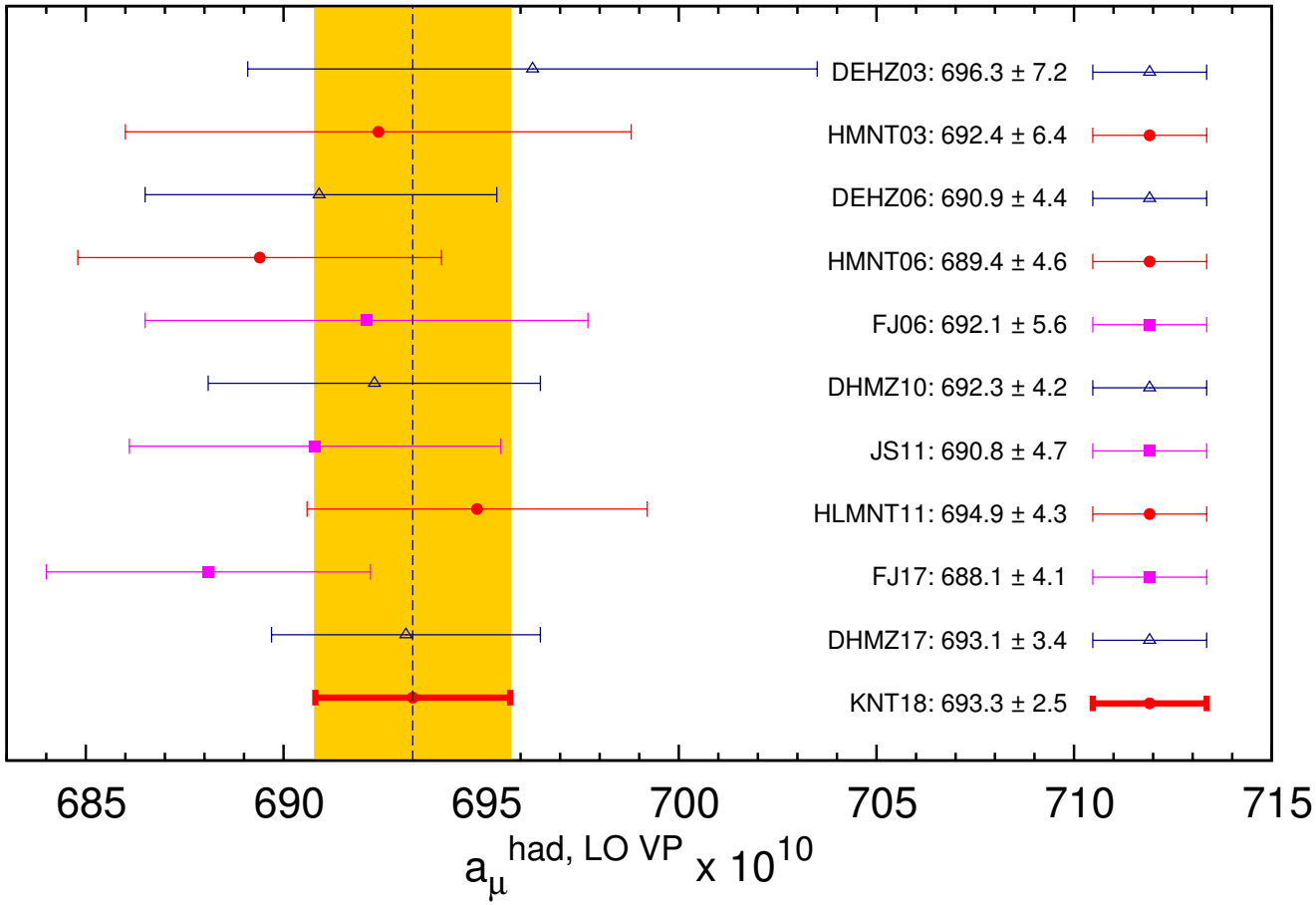


KNT18: final results



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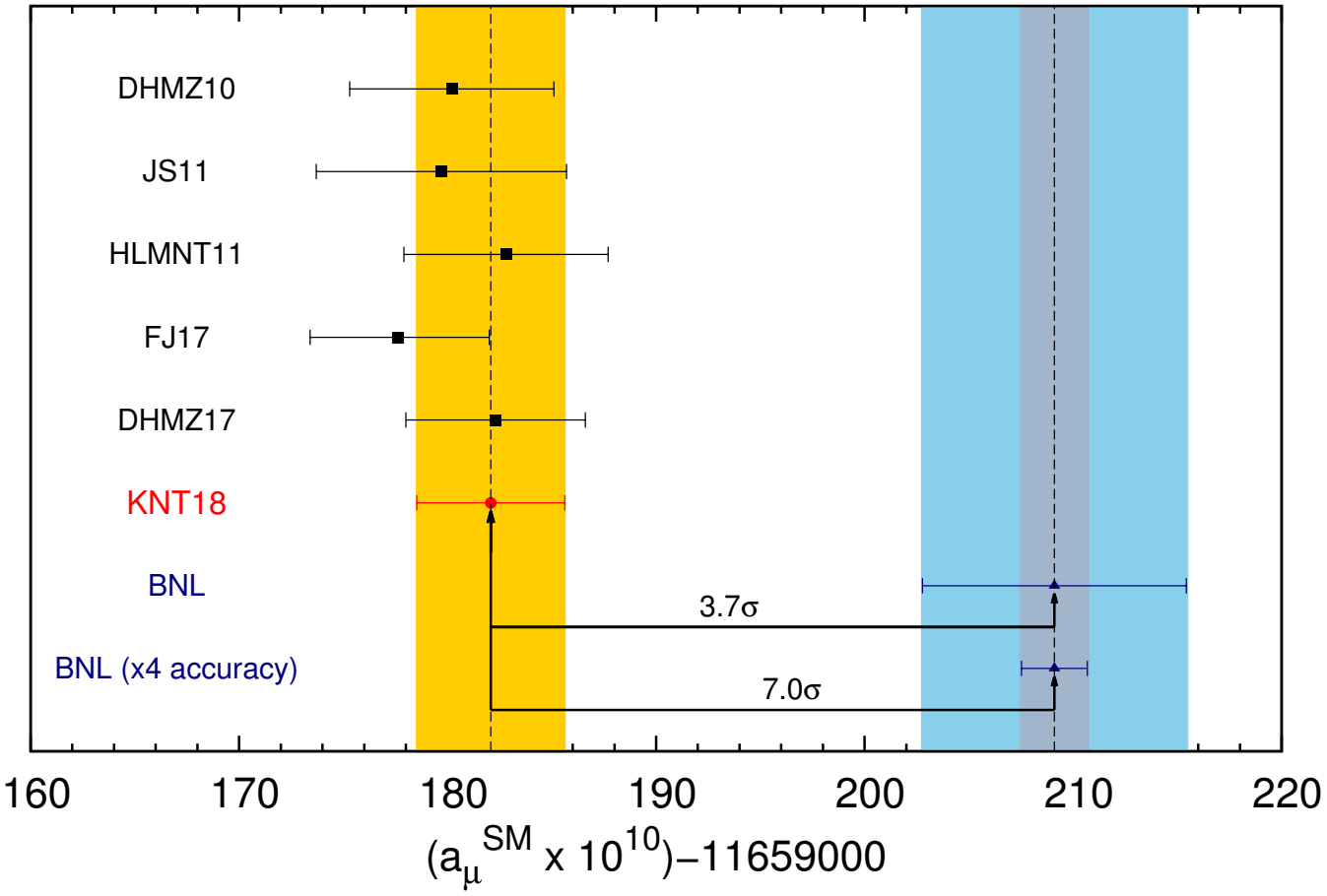


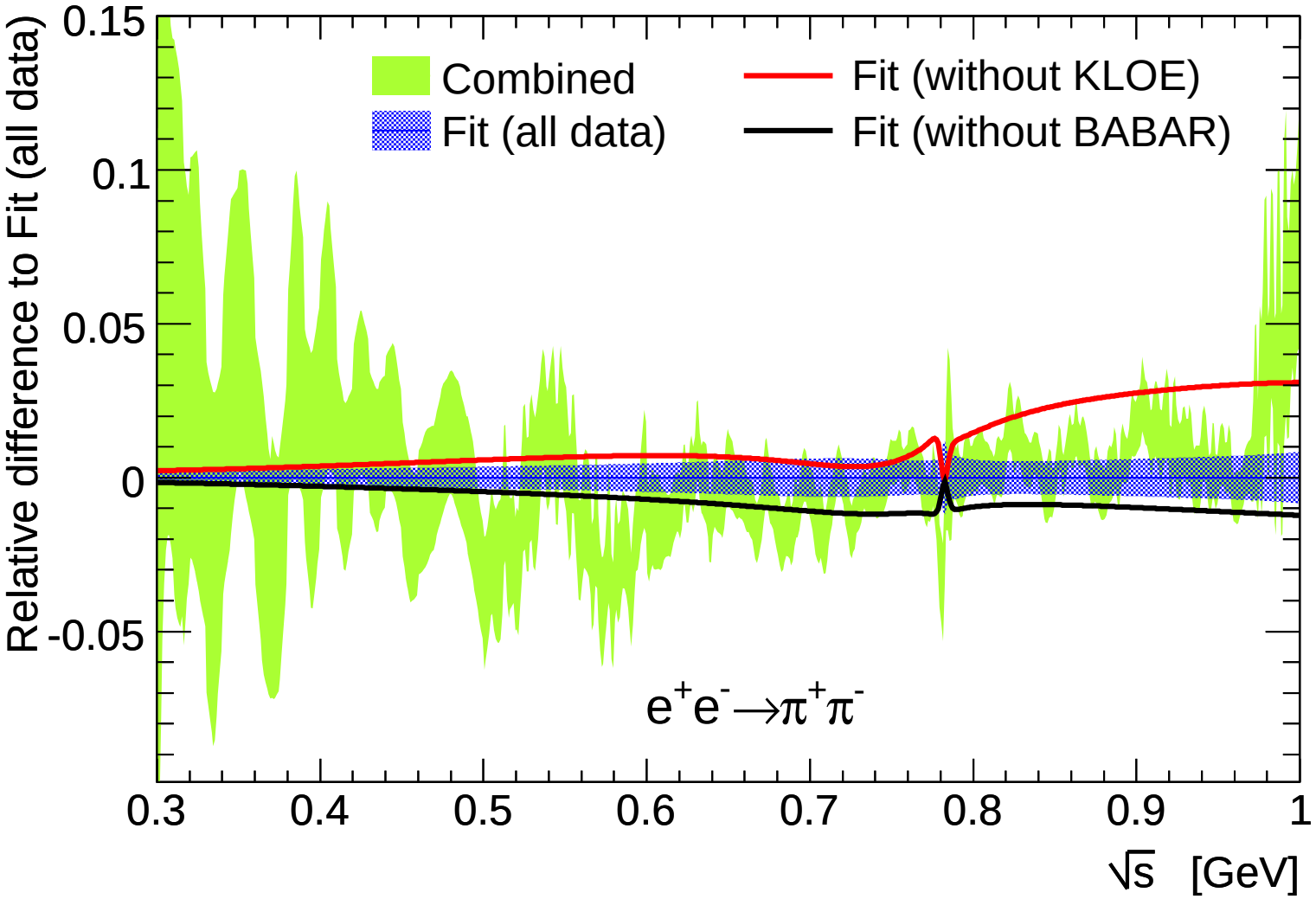
Davier-Hoecker-Malaescu-Zhang (DHMZ19)

Some remarks:

- improved extrapolation in the low-energy region (analyticity)
- conservative approach in combining KLOE and BABAR
- update to the newest data available (in different channels)

Davier-Hoecker-Malaescu-Zhang (DHMZ19)

Another illustration of the problem in the $\pi\pi$ channel



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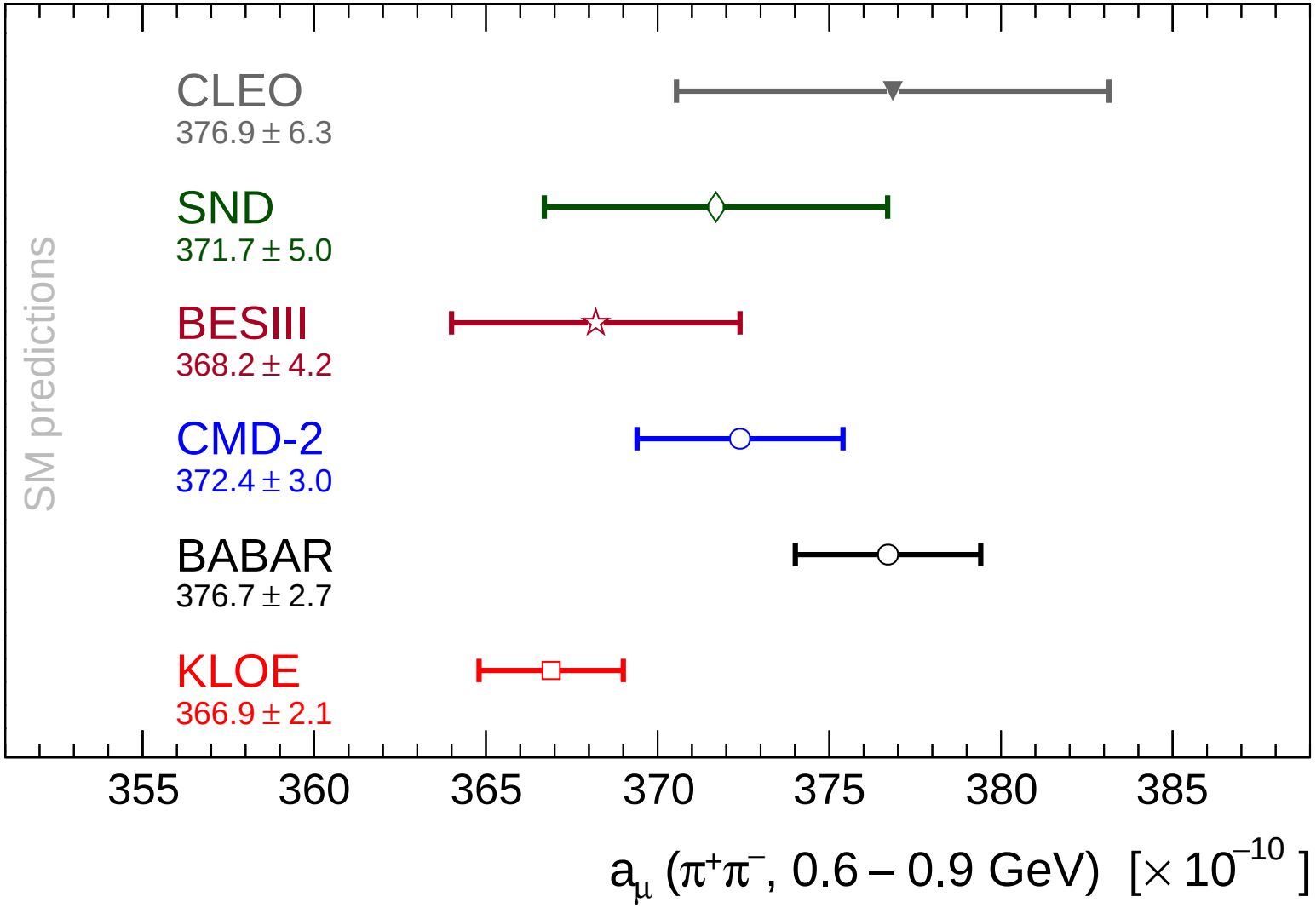
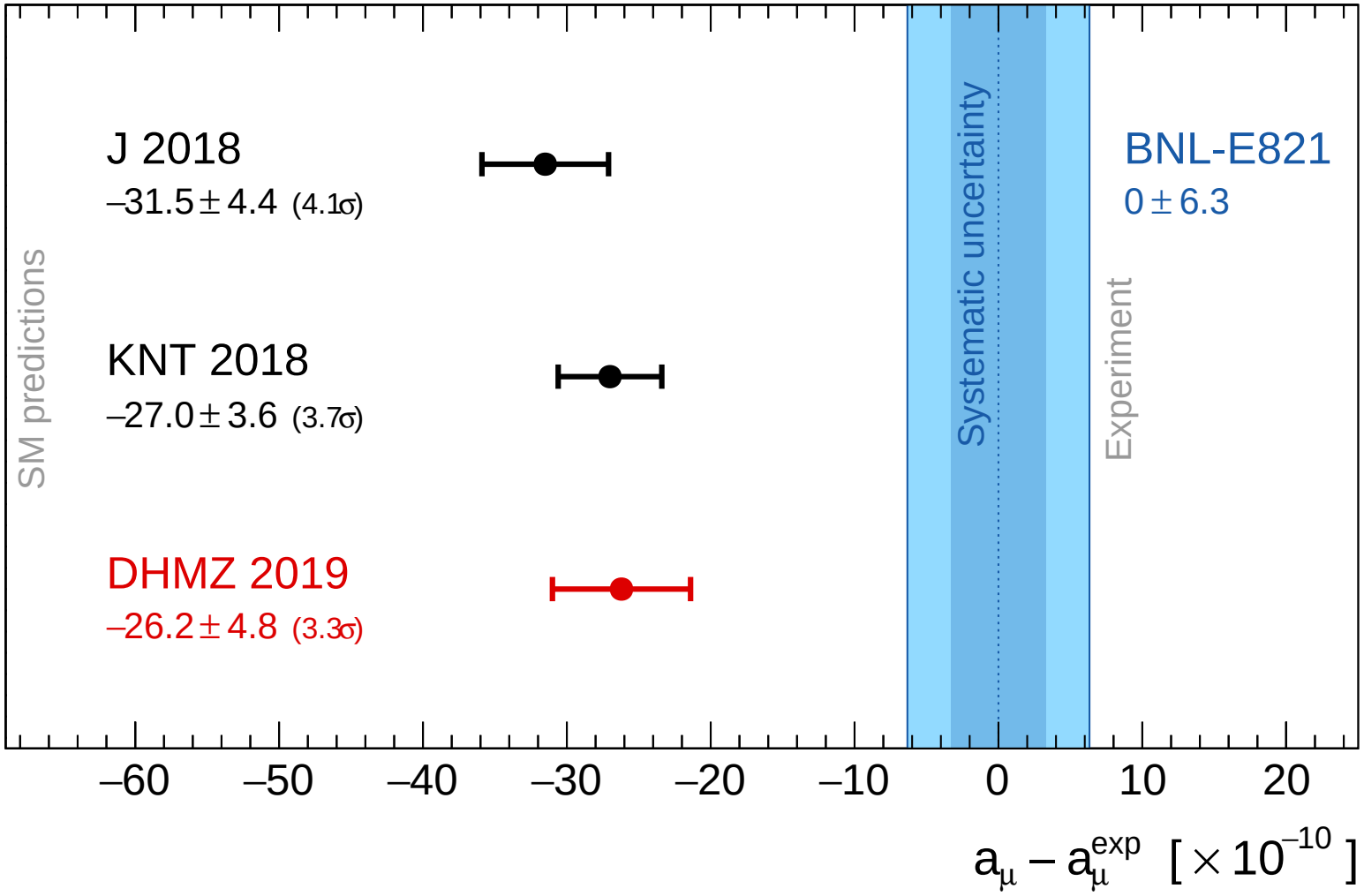


Davier-Hoecker-Malaescu-Zhang (DHMZ19)

Final result:



Some remarks on KNT18 vs DHMZ19

- the two latest analyses of HVP based on the dispersive approach substantially agree on the final number
- they differ in the evaluation of the 2π contribution ($\Delta_{\pi\pi} = 4.06 \pm 3.8$)
- for the 2π channel there is a long-standing discrepancy between KLOE and BABAR, the most precise datasets

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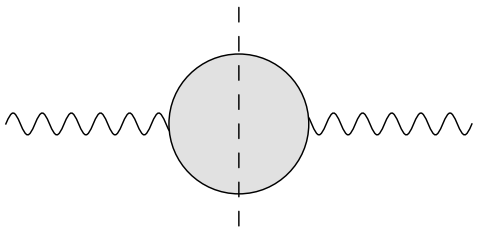
- the two latest analyses of HVP based on the dispersive approach substantially agree on the final number
- they differ in the evaluation of the 2π contribution ($\Delta_{\pi\pi} = 4.06 \pm 3.8$)
- for the 2π channel there is a long-standing discrepancy between KLOE and BABAR, the most precise datasets
- $\Rightarrow$ analyze the 2π contribution with more theory input

The 2π contribution

For HVP the unitarity relation is simple and looks the same for all possible intermediate states



$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons})$$

which implies ($H_i(t) = i$-th hadronic state)

$$\bar{\Pi}(q^2) = \sum_i \frac{q^2}{\pi} \int_{4M_\pi^2}^{\infty} dt \frac{\sigma(e^+e^- \to H_i(t))}{4\pi\alpha(t-q^2)}$$

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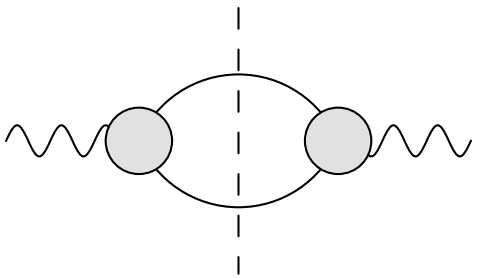

$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons})$$

which implies $(H_i(t) = 2\pi)$

$$\bar{\Pi}_{2\pi}(q^2) = \frac{q^2}{\pi}\int_{4M_\pi^2}^{\infty} dt \frac{\sigma(e^+e^- \to 2\pi)}{4\pi\alpha(t-q^2)}$$

de Trocóniz, Ynduráin (01,04), Leutwyler, GC (02,03), Anthanarayan et al. (13,16)

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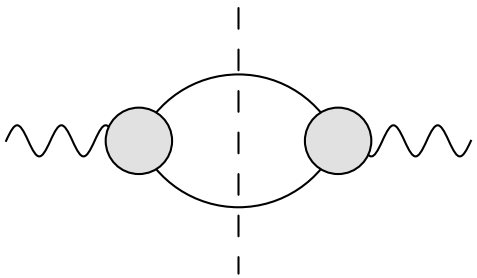

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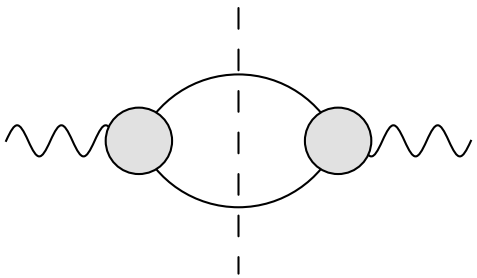

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de Trocóniz, Ynduráin (01,04), Leutwyler, GC (02,03), Anthanarayan et al. (13,16)

The pion vector form factor $F_V^\pi(t)$ also satisfies a dispersion relation which relates it to the P wave $\pi\pi$ phase shift

Most phenomenological analyses do not use this property

Results for $(g-2)_\mu$

Based on: dispersive representation of $F_V^\pi(t)$ which depends on a small number of parameters which are fit to data

GC, Hoferichter, Stoffer (18)

- Low energy:

$$a_\mu^{\mathrm{HVP},\pi\pi}{}_{|\leq 0.63\mathrm{GeV}} = 132.8(0.4)(1.0)\cdot 10^{-10}$$

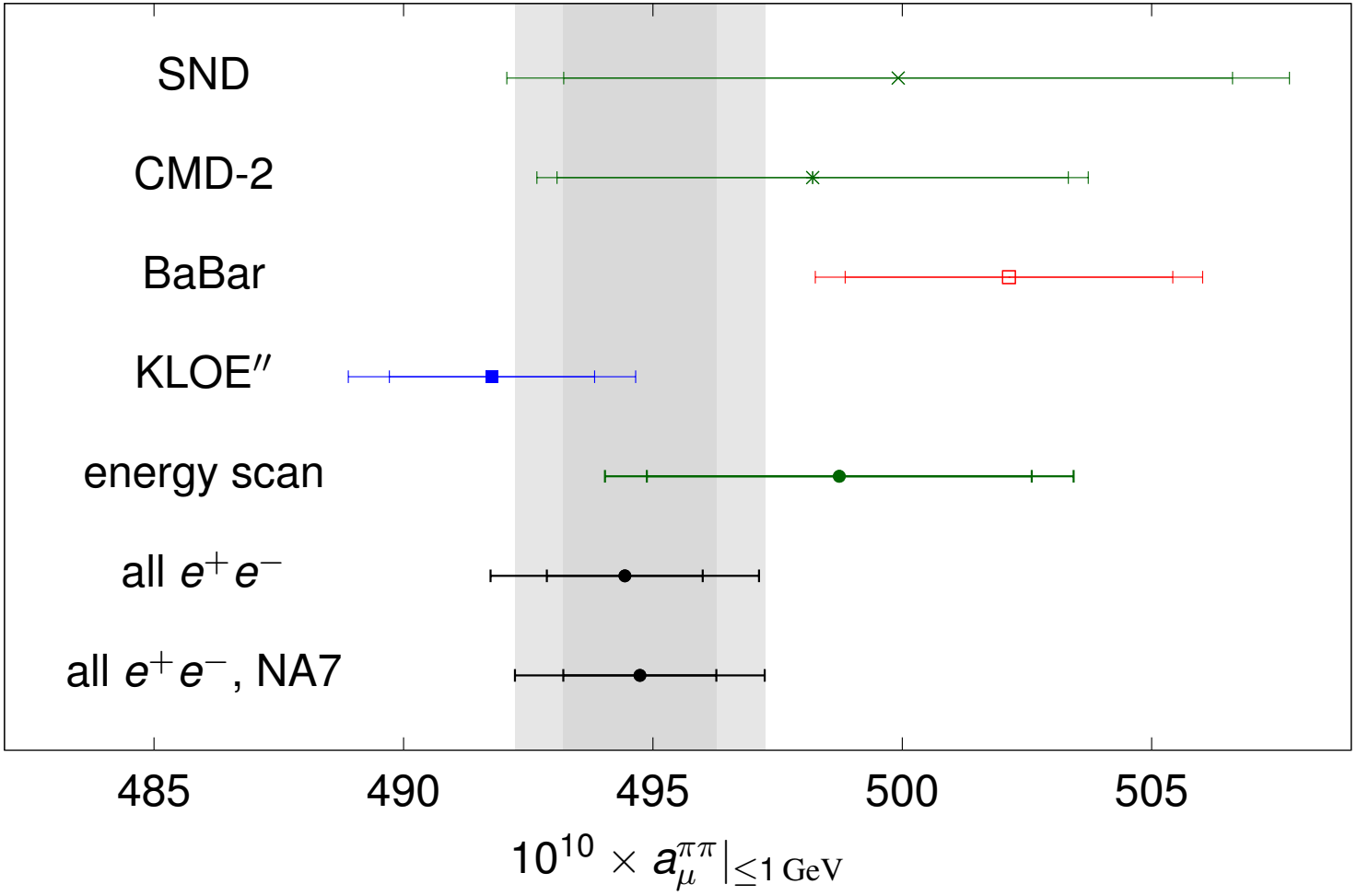
[in agreement with 132.9(8) Ananthanarayan et al. (16)]

- Full range:

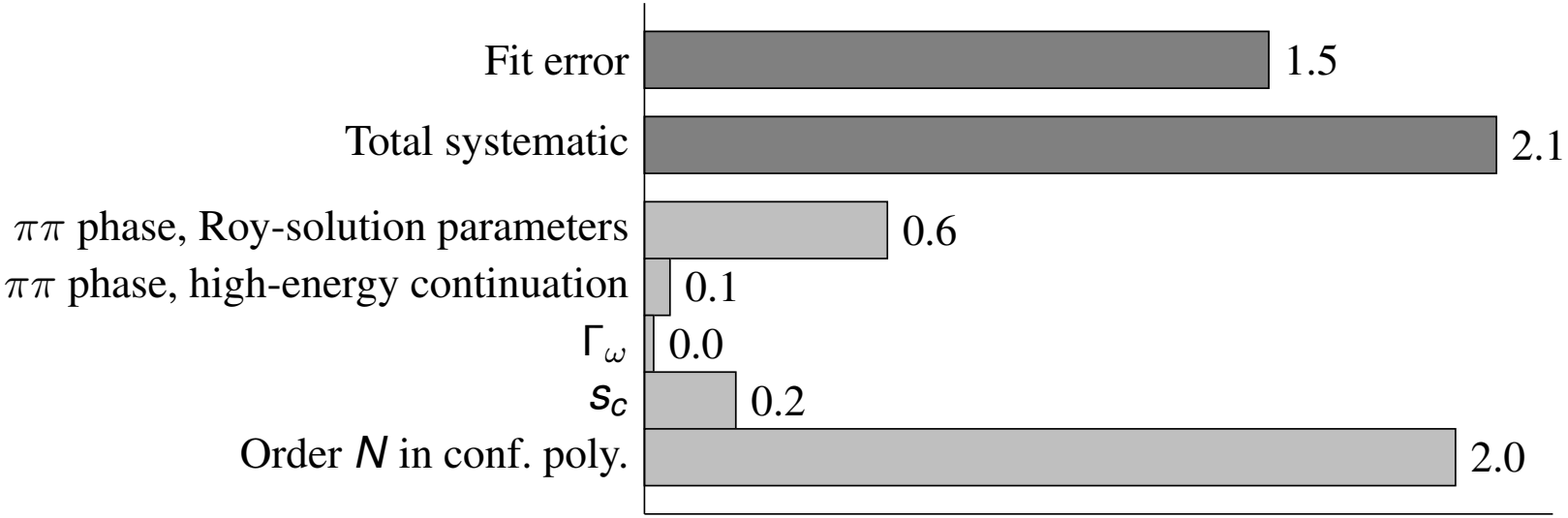
$$a_\mu^{\mathrm{HVP},\pi\pi}{}_{|\leq 1\mathrm{GeV}} = 495.0(1.5)(2.1)\cdot 10^{-10}$$

Results for $(g-2)_\mu$

Result for $a_\mu^{\pi\pi}|_{\leq 1\,\mathrm{GeV}}$ from the VFF fits to single experiments and combinations



Results for $(g-2)_\mu$



Uncertainties on $a_\mu^{\pi\pi}|_{\leq 1\,\mathrm{GeV}}$ in combined fit to all experiments

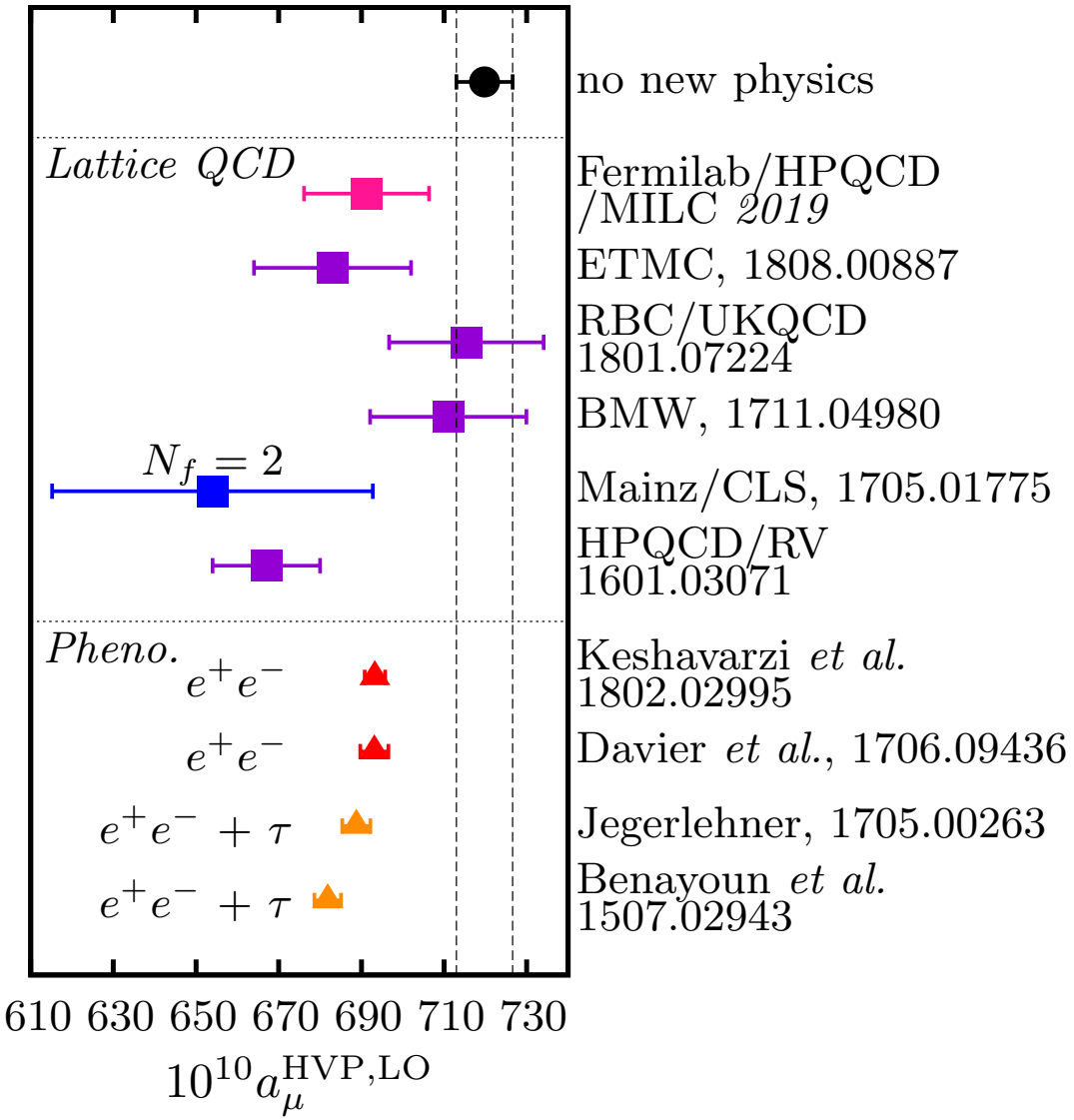
Results for $(g-2)_\mu$

Comparison to other recent analyses

Energy range	our work	KNT18	DHMZ17	ACD18
$\leq 0.6\,\mathrm{GeV}$	110.1(9)	108.7(9)	110.2(1.0)	
$\leq 0.7\,\mathrm{GeV}$	214.8(1.7)	213.0(1.2)	214.7(1.3)	
$\leq 0.8\,\mathrm{GeV}$	413.2(2.3)	411.6(1.7)	414.0(1.9)	
$\leq 0.9\,\mathrm{GeV}$	479.8(2.6)	478.1(1.9)	481.2(2.2)	
$\leq 1.0\,\mathrm{GeV}$	495.0(2.6)	493.4(1.9)	496.7(2.2)	
$[0.6, 0.7]\,\mathrm{GeV}$	104.7(7)	104.3(5)	104.5(5)	
$[0.7, 0.8]\,\mathrm{GeV}$	198.3(9)	198.6(8)	199.3(9)	
$[0.8, 0.9]\,\mathrm{GeV}$	66.6(4)	66.5(3)	67.2(4)	
$[0.9, 1.0]\,\mathrm{GeV}$	15.3(1)	15.3(1)	15.5(1)	
$\leq 0.63\,\mathrm{GeV}$	132.8(1.1)	131.2(1.1)	132.8(1.1)	132.9(8)
$[0.6, 0.9]\,\mathrm{GeV}$	369.6(1.7)	369.4(1.3)	371.0(1.5)	
$\left[\sqrt{0.1}, \sqrt{0.95}\right]\,\mathrm{GeV}$	490.7(2.6)	489.0(1.9)	492.2(2.2)	

KNT18 = Keshavarzi et al. (18), DHMZ17 = Davier et al. (17), ACD18 = Ananthanarayan et al. (18)

Latest results of Lattice HVP calculations (2019)



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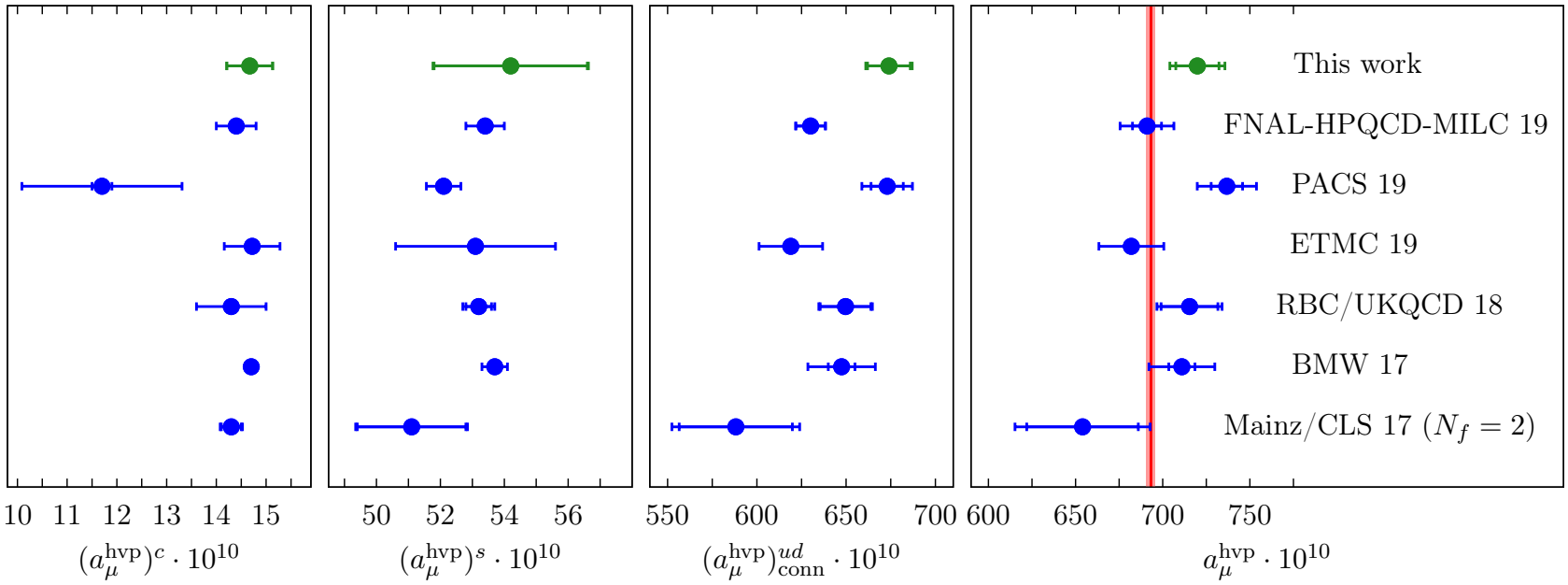


Red band=KNT18

Gérardin et al. (Mainz) (19)

Conclusions: HVP

- precision in the dispersive evaluation of the HVP contribution has reached a stunning 0.4%
- the dominant $\pi\pi$ contribution could become even more precise were it not for an experimental discrepancy between KLOE and BABAR
- $\Rightarrow$ use more theory: analyticity and unitarity relate the $\pi\pi$ contribution and the well known $\pi\pi$ P-wave phase shift:

$$a_\mu^{\mathrm{HVP},\pi\pi}\Big|_{\leq 1\mathrm{GeV}} = 495.0(2.6)\cdot 10^{-10}$$

[KNT18: 493.4(1.9) and DHMZ17: 496.7(2.2)]

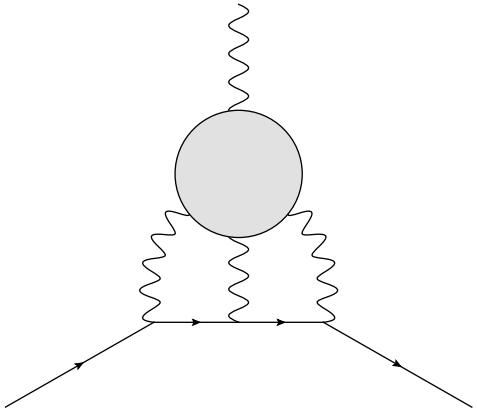
- At the Seattle Workshop a consensus has been achieved on how to merge all recent analyses into a single number

Outline

Calculating the HLbL contribution

Calculating the HLbL contribution is complicated

- 4-point function of em currents in QCD



- until recently, only model calculations
- a data-driven approach like for HVP seemed hopeless
  "*it cannot be expressed in terms of measurable quantities*"
- lattice QCD is an alternative and is making fast progress

Different model-based evaluations of HLbL

Jegerlehner-Nyffeler 2009

Contribution	BPaP(96)	HKS(96)	KnN(02)	MV(04)	BP(07)	PdRV(09)	N/JN(09)
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
" " + subl. in N_c	–	–	–	0±10	–	–	–
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	–	15±10	22± 5
scalars	−6.8±2.0	–	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	110±40	105±26	116±39

Legenda: B=Bijnens Pa=Pallante P=Prades H=Hayakawa K=Kinoshita S=Sanda Kn=Knecht N=Nyffeler M=Melnikhov V=Vainshtein dR=de Rafael J=Jegerlehner

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, *i.e.* two-pion cuts (*Ks are subdominant*)
- heavier single-particle poles decreasingly important

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Advantages of the dispersive approach

- model independent
- unambiguous definition of the various contributions
- makes a data-driven evaluation possible (in principle)
- First attempts: GC, Hoferichter, Procura, Stoffer (14)
  Pauk, Vanderhaeghen (14)
- similar philosophy, with a different implementation: Schwinger sum rule Hagelstein, Pascalutsa (17)
- why hasn't this been adopted before?

The HLbL tensor

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz\, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0| T\{j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0)\}|0\rangle$$

$$q_4 = k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

General Lorentz-invariant decomposition:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 + \sum_{i,j,k,l} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} + \ldots$$

consists of 138 scalar functions $\{\Pi^1, \Pi^2, \ldots\}$, but in $d = 4$ only 136 are linearly independent Eichmann *et al.* (14)

Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

$\Rightarrow$ Apply the Bardeen-Tung (68) method+Tarrach (75) addition

Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

GC, Hoferichter, Procura, Stoffer (2015)

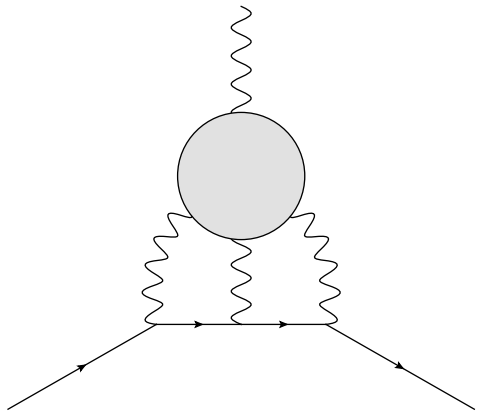
- 43 basis tensors (BT) — in $d = 4$: 41=no. of helicity amplitudes
- 11 additional ones (T) — to guarantee basis completeness everywhere
- of these 54 only 7 are distinct structures
- all remaining 47 can be obtained by crossing transformations of these 7: manifest crossing symmetry
- the dynamical calculation needed to fully determine the LbL tensor concerns these 7 scalar amplitudes

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

Master Formula

$$a_\mu^{\mathrm{HLbL}} = -e^6 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p)\hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known kernel functions
- $\hat{\Pi}_i$: linear combinations of the Π_i
- the Π_i are amenable to a dispersive treatment: their imaginary parts are related to measurable subprocesses
- 5 integrals can be performed with Gegenbauer polynomial techniques



GC, Hoferichter, Procura, Stoffer (2015)

Master Formula

After performing the 5 integrations:

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{48\pi^2} \int_0^\infty dQ_1^4 \int_0^\infty dQ_2^4 \int_{-1}^{1} d\tau \sqrt{1-\tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

where Q_i^μ are the Wick-rotated four-momenta and τ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

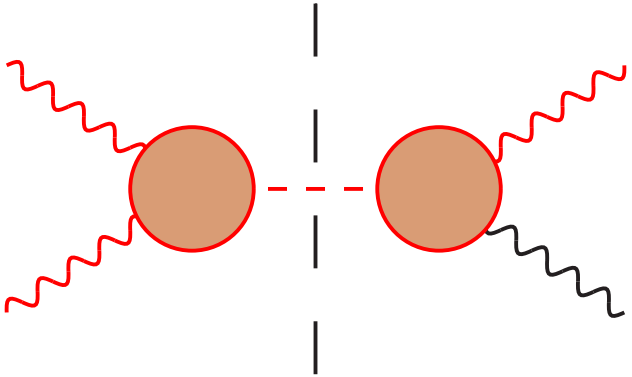
The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

GC, Hoferichter, Procura, Stoffer (2015)

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion pole: imaginary parts = δ-functions
Projection on the BTT basis: easy ✓
Our master formula=explicit expressions in the literature ✓
Input: pion transition form factor Hoferichter et al. (18)
First results of direct lattice calculations Gerardin, Meyer, Nyffeler (16,19)

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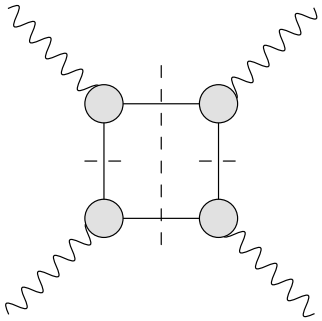
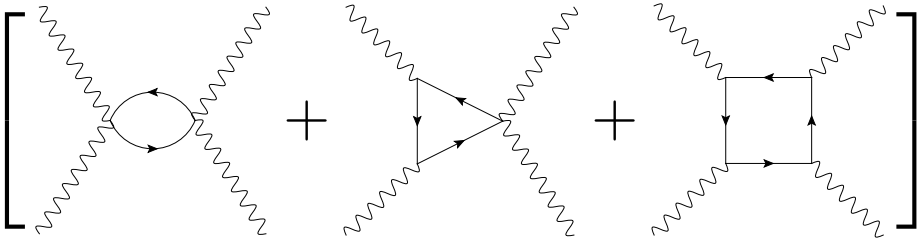


$$\equiv F^V_\pi(q_1^2)F^V_\pi(q_2^2)F^V_\pi(q_3^2)$$

$\times$



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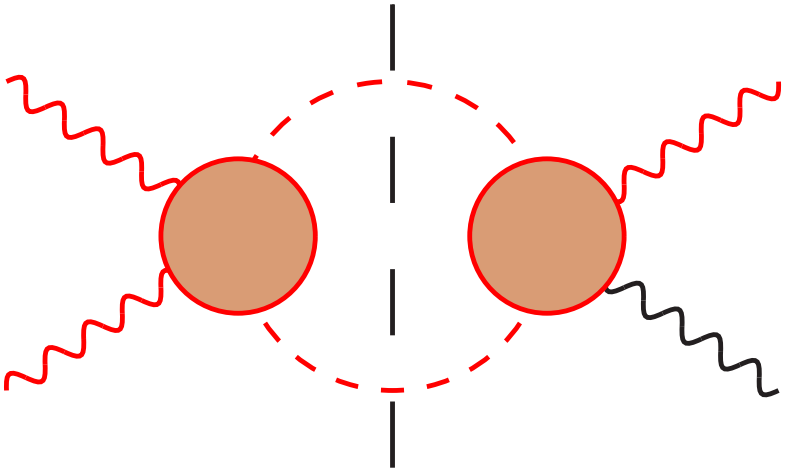


The "rest" with 2π intermediate states has cuts only in one channel and will be
calculated dispersively after partial-wave expansion

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Cuts with states $\neq \pi$ or 2π are neglected

η, η' or the K-loop contribution can be calculated analogously

Pion-pole contribution

Latest complete analyses:

- Dispersive calculation of the pion TFF Hoferichter et al. (18)

$$a_\mu^{\pi^0} = 63.0^{+2.7}_{-2.1} \times 10^{-11}$$

- Padé-Canterbury approximants Masjuan & Sanchez-Puertas (17)

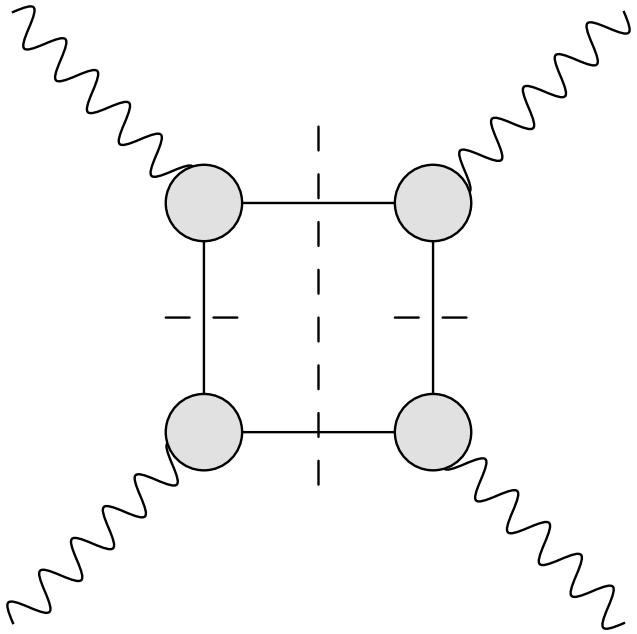
$$a_\mu^{\pi^0} = 63.6(2.7) \times 10^{-11}$$

- Lattice Gérardin, Meyer, Nyffeler (19)

$$a_\mu^{\pi^0} = 62.3(2.3) \times 10^{-11}$$

Pion-box contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion-box contribution

The only ingredient needed for the pion-box contribution is the vector form factor

$$\hat{\Pi}_i^{\pi\text{-box}} = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\frac{1}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy\, I_i(x,y),$$

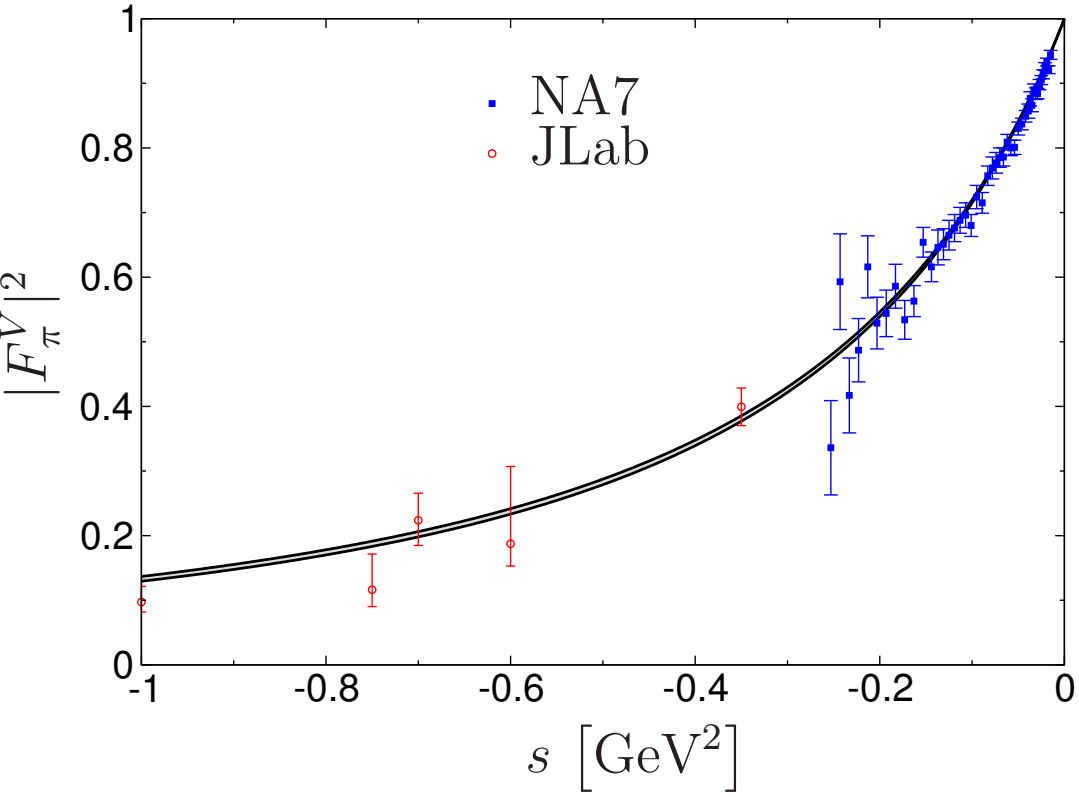
where

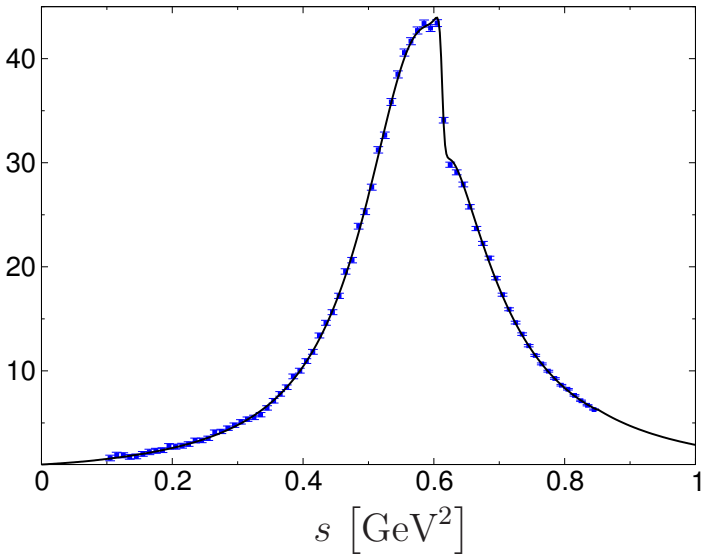
$$I_1(x,y) = \frac{8xy(1-2x)(1-2y)}{\Delta_{123}\Delta_{23}},$$

and analogous expressions for $I_{4,7,17,39,54}$ and

$$\Delta_{123} = M_\pi^2 - xyq_1^2 - x(1-x-y)q_2^2 - y(1-x-y)q_3^2,$$
$$\Delta_{23} = M_\pi^2 - x(1-x)q_2^2 - y(1-y)q_3^2$$

Pion-box contribution



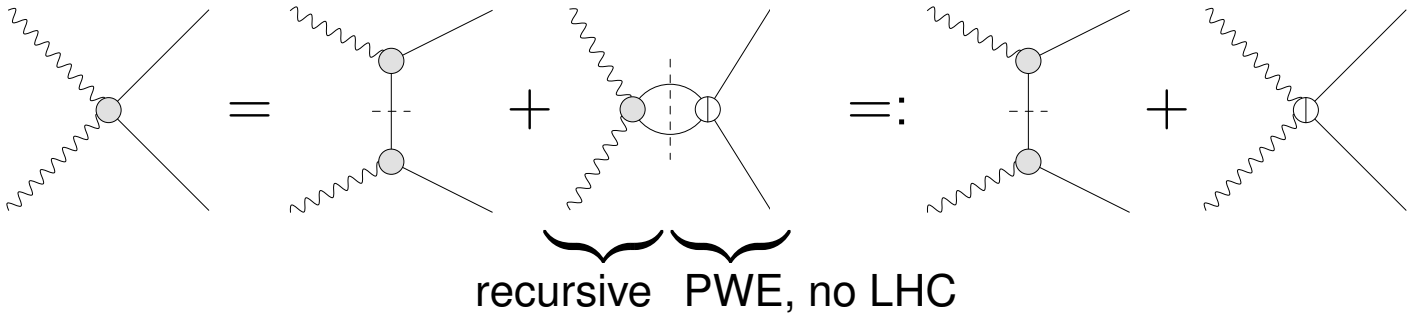


Uncertainties are negligibly small:

$$a_\mu^{\mathrm{FsQED}} = -15.9(2) \cdot 10^{-11}$$

First evaluation of S- wave 2π-rescattering

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ provides the following:

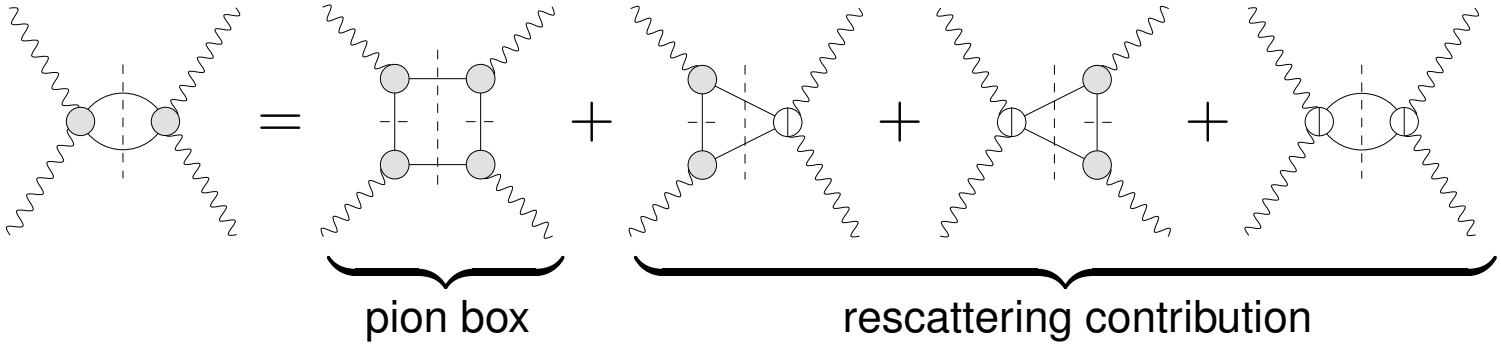


S-wave contribution below $\sim$ 1 GeV:

$$a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$$

Two-pion contribution to $(g-2)_\mu$ from HLbL

Two-pion contributions to HLbL:



$$a_\mu^{\pi-\mathrm{box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -24(1)\cdot 10^{-11}$$

Contributions from above 1 GeV and higher partial waves are work in progress

White Paper Summary of HLbL (very preliminary!)

Contributions to $10^{11} \cdot a_\mu^{\mathrm{HLbL}}$

▶ Pseudoscalar poles	$= 93.8^{+4.0}_{-3.6}$
▶ pion box (kaon box ~ -0.5)	$= -15.9(2)$
▶ S-wave $\pi\pi$ rescattering	$= -8(1)$
Partial total:	69.9 ± 4.1
▶ scalars and tensors with $M_R > 1$ GeV	$\sim -2(3)$
▶ axial vectors	$\sim 8(8)$
▶ short-distance contribution	$\sim 10(10)$

Central value:	$86 \pm XX$
Uncertainties added in quadrature:	$XX = 14$
Uncertainties added linearly:	$XX = 25$

Improvements obtained with the dispersive approach

Contribution	PdRV(09)	N/JN(09)	J(17)	White Paper
π^0, η, η'-poles	114 ± 13	99 ± 16	95.45 ± 12.40	93.8 ± 4.0
π, K-loop/box	-19 ± 19	-19 ± 13	-20 ± 5	-16.4 ± 0.2
S-wave $\pi\pi$	–	–	–	-8 ± 1
scalars	-7 ± 7	-7 ± 2	-5.98 ± 1.20	$\}\ -2 \pm 3$
tensors	–	–	1.1 ± 0.1	
axials	15 ± 10	22 ± 5	7.55 ± 2.71	8 ± 8
q-loops / SD	2.3	21 ± 3	22.3 ± 5.0	10 ± 10
total	105 ± 26	116 ± 39	100.4 ± 28.2	$86 \pm XX$

HLbL in units of 10^{-11}.

PdRV = Prades, de Rafael, Vainshtein ("Glasgow consensus");

N = Nyffeler; J = Jegerlehner

Outline

Conclusions: HLbL

- The HLbL contribution to $(g-2)_\mu$ can be expressed in terms of measurable quantities in a dispersive approach
- master formula: HLbL contribution to a_μ as triple-integral over scalar functions which satisfy dispersion relations
- the relevant measurable quantity entering the dispersion relation depends on the intermediate state:
  - single-pion contribution: pion transition form factor
  - pion-box contribution: pion vector form factor
  - 2-pion rescattering: $\gamma^*\gamma^{(*)} \to \pi\pi$helicity amplitudes

  these three contributions (S-wave for the latter) have been calculated with remarkably small uncertainties
- work on calculating other contributions and estimating missing pieces is in progress

Final conclusions

- The $\sim$3.5 σ discrepancy between SM and measurement for $(g-2)_\mu$ continues to stay
- The Fermilab Muon $g-2$ experiment aims to reduce the experimental uncertainty by a factor four, potentially leading to a 7σ discrepancy
- I have reviewed current theoretical efforts to improve the calculations of the hadronic contributions in a model-independent way based on dispersive approaches
- The goal of matching the experimental reduction of the uncertainty with a similar reduction on the theory side is being achieved