LONGITUDINAL DYNAMICS

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CERN, BE-OP

Basics of Accelerator Physics and Technology
ESI, Archamps, 7-11 October 2019

Scope and Summary of the 2 lectures:

The goal of an accelerator is to provide a stable particle beam.

The particles nevertheless perform transverse betatron oscillations.
We will see that they also perform **oscillations** in the longitudinal plane and in energy.

We will look at the stability of these oscillations, and their dynamics.

- Acceleration methods
- Accelerating structures
- Linac: Phase Stability +
          Energy-Phase oscillations
- Circular accelerators:
          Cyclotron / Synchrotron
- Stability in a Synchrotron
- Longitudinal Phase Space Motion
- Bunch and Bucket
- Injection Matching + Filamentation
- RF manipulations in the PS

More related lectures:

- Linacs   – Alessandra Lombardi
- RF Systems   - myself

The CERN Accelerator Complex
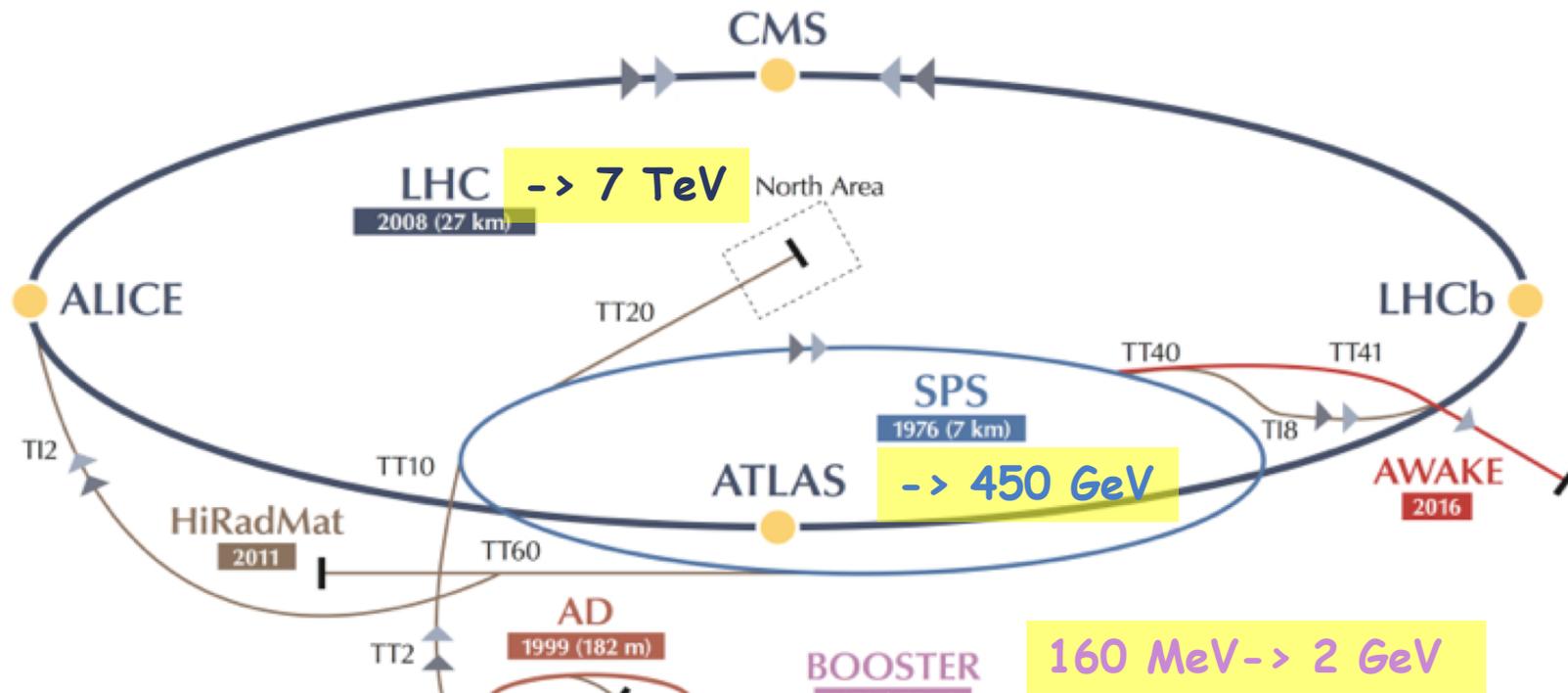
- Linear accelerators scale in size and cost(!) ~linearly with the energy.
- Circular accelerators can each turn reuse
  - the **accelerating system**
  - the vacuum chamber   4
  - the bending/focusing magnets
  - beam instrumentation, …
-> economic solution to reach higher particle energies
But **each accelerator** has a **limited energy range.**

Particle types and acceleration

The accelerating system will depend upon the evolution of the particle velocity:
• electrons reach a constant velocity (~speed of light) at relatively low energy
• heavy particles reach a constant velocity only at very high energy
    -> we need different types of resonators, optimized for different velocities
    -> the **revolution frequency will vary**, so the RF frequency will be changing
    -> magnetic field needs to follow the momentum increase

Particle rest mass m_0:
electron   0.511 MeV
proton   938 MeV
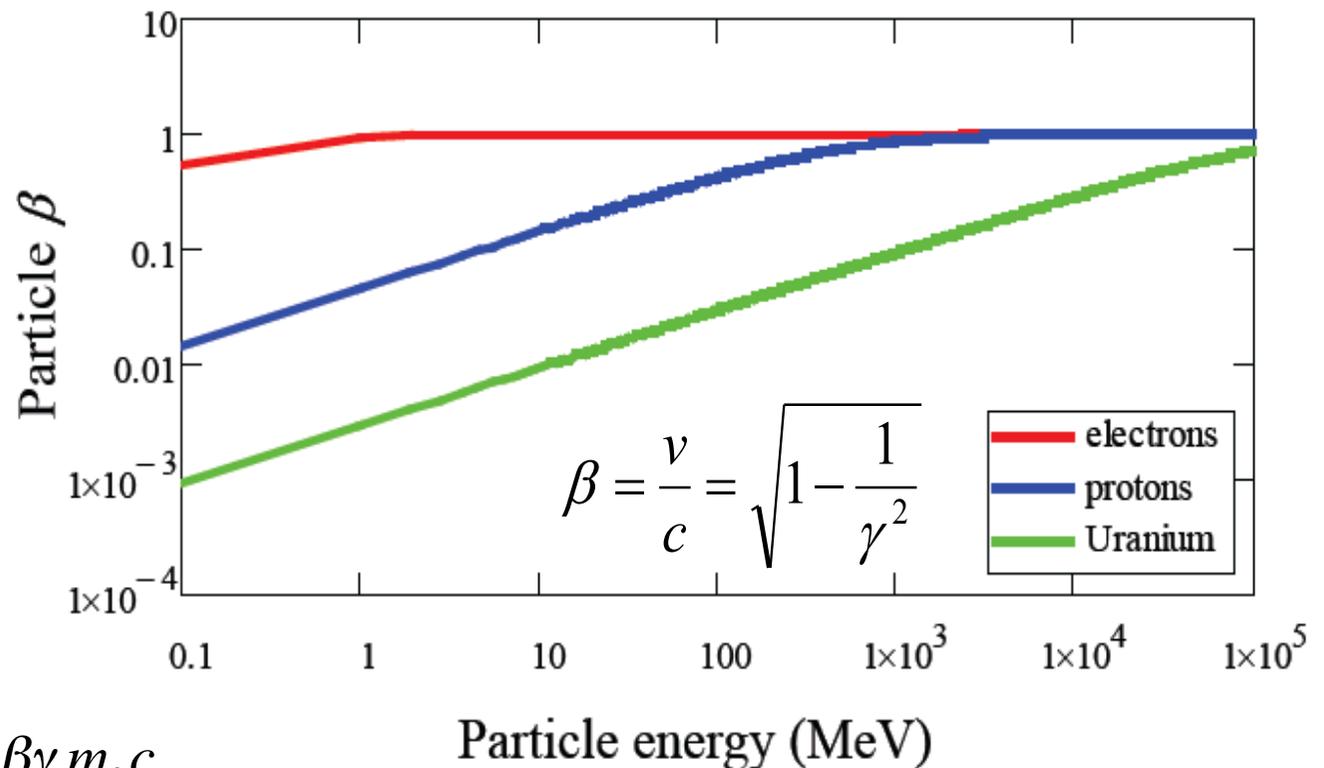^{239}U   ~220000 MeV

Total Energy: $E = \gamma m_0 c^2$

Relativistic gamma factor:

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$$

Momentum:

$$p = mv = \frac{E}{c^2}\beta c = \beta \frac{E}{c} = \beta\gamma\, m_0 c$$

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$

		electrons
		protons
		Uranium

Particle β vs Particle energy (MeV)

Revolution frequency variation

The revolution and RF frequency will be changing during acceleration
Much more important for lower energies (values are kinetic energy - protons).

PS Booster: 50 MeV (β= 0.314) -> 1.4 GeV (β=0.915)
(pre LS2)   602 kHz -> 1746 kHz => **190% frequency increase**
(post LS2):   160 MeV (β= 0.520) -> 2 GeV (β=0.948) => **95% increase**

PS: 1.4 GeV (β=0.915) -> 25.4 GeV (β =0.9994)
                437 KHz -> 477 kHz => **9% increase**
(post LS2):   2 GeV (β=0.948) -> 25.4 GeV (β =0.9994) => **5% increase**

SPS: 25.4 GeV -> 450 GeV (β=0.999998)
                => **0.06% frequency increase**

LHC: 450 GeV -> 7 TeV (β= 0.999999991)
                => only **2 10^{-6} increase**

RF system needs more flexibility in lower energy accelerators.

Acceleration: May the force be with you

To accelerate, we need a **force in** the **direction of motion**!

Newton-Lorentz Force on a charged particle with charge e (electron, proton)

$$\vec{F} = \frac{d\vec{p}}{dt} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

2nd term always perpendicular to motion => no acceleration

Hence, it is necessary to have an electric field E (preferably) along the direction of the initial momentum (z), which changes the momentum of the particle.

$$\frac{dp}{dt} = eE_z$$

The 2nd term - larger at high velocities – is used for:

- BENDING: generated by a magnetic field perpendicular to the plane of the particle trajectory. The bending radius ρ obeys to the relation :

$$\frac{p}{e} = B\rho$$

in practical units: $B\,\rho\,[\text{Tm}] \approx \dfrac{p\,[\text{GeV/c}]}{0.3}$

- FOCUSING: the bending effect is used to bring the particles trajectory closer to the axis, hence to increase the beam density.

Energy Gain

The acceleration increases the **momentum**, providing **kinetic energy** to the charged particles.

In relativistic dynamics, total energy E and momentum p are linked by

$$E^2 = E_0^2 + p^2 c^2$$

$$(E = E_0 + W)$$

W kinetic energy
E_0 rest energy

Hence:

$$dE = v\,dp \qquad \left(2E\,dE = 2c^2 p\,dp \Leftrightarrow dE = c^2 mv / E\ dp = v\,dp\right)$$

The rate of energy gain per unit length of acceleration (along z) is then:

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

and the kinetic energy gained from the field along the z path is:

$$dW = dE = eE_z\,dz \qquad \rightarrow \qquad W = e\int E_z\,dz = eV$$

where V is just a potential.

Unit of Energy

Today's accelerators and future projects work/aim at the TeV energy range.

        LHC: 7 TeV -> 14 TeV

        CLIC: 3 TeV

        HE/VHE-LHC: 33/100 TeV

In fact, this energy unit comes from acceleration:

> **1 eV (electron Volt)** is the energy that 1 elementary charge e (like one electron or proton) gains when it is accelerated in a potential (voltage) difference of 1 Volt.
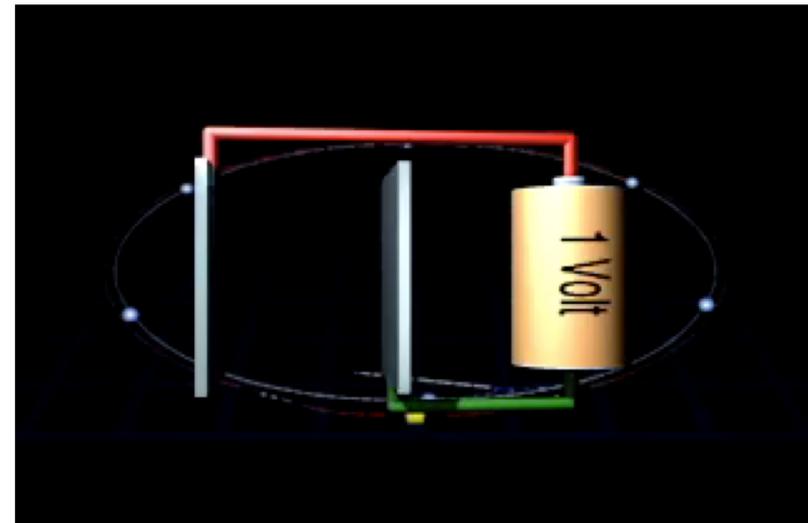
Basic Unit: **eV (electron Volt)**

        $keV = 1000\ eV = 10^3\ eV$

        $MeV = 10^6\ eV$

        $GeV = 10^9\ eV$

        $TeV = 10^{12}\ eV$

LHC = ~450 Million km of batteries!!!
        3x distance Earth-Sun

1 Volt

Electrostatic Acceleration

source
$\vec{E}$
vacuum envelope
ΔV
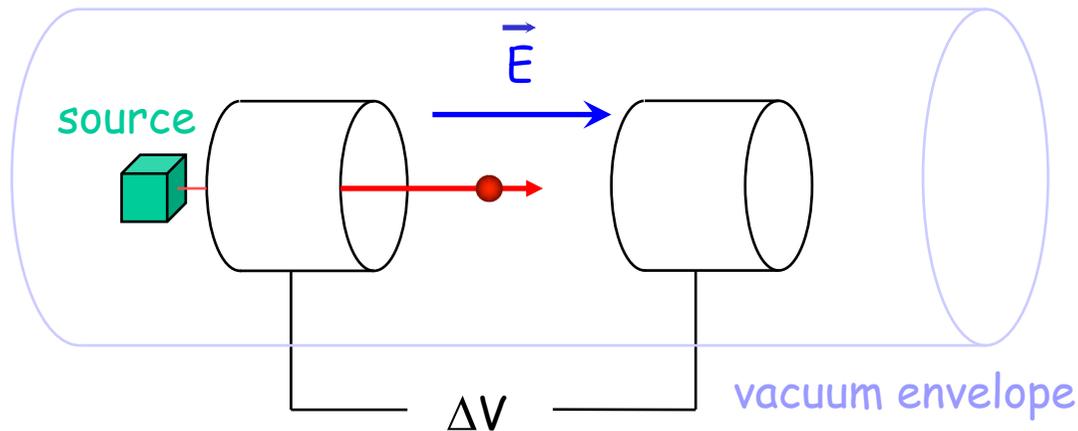
Electrostatic Field:

Force: $\vec{F} = \dfrac{d\vec{p}}{dt} = e\,\vec{E}$

Energy gain: $W = e\,\Delta V$

used for first stage of acceleration:
particle sources, electron guns,
x-ray tubes

Limitation: insulation problems
   maximum high voltage (~ 10 MV)
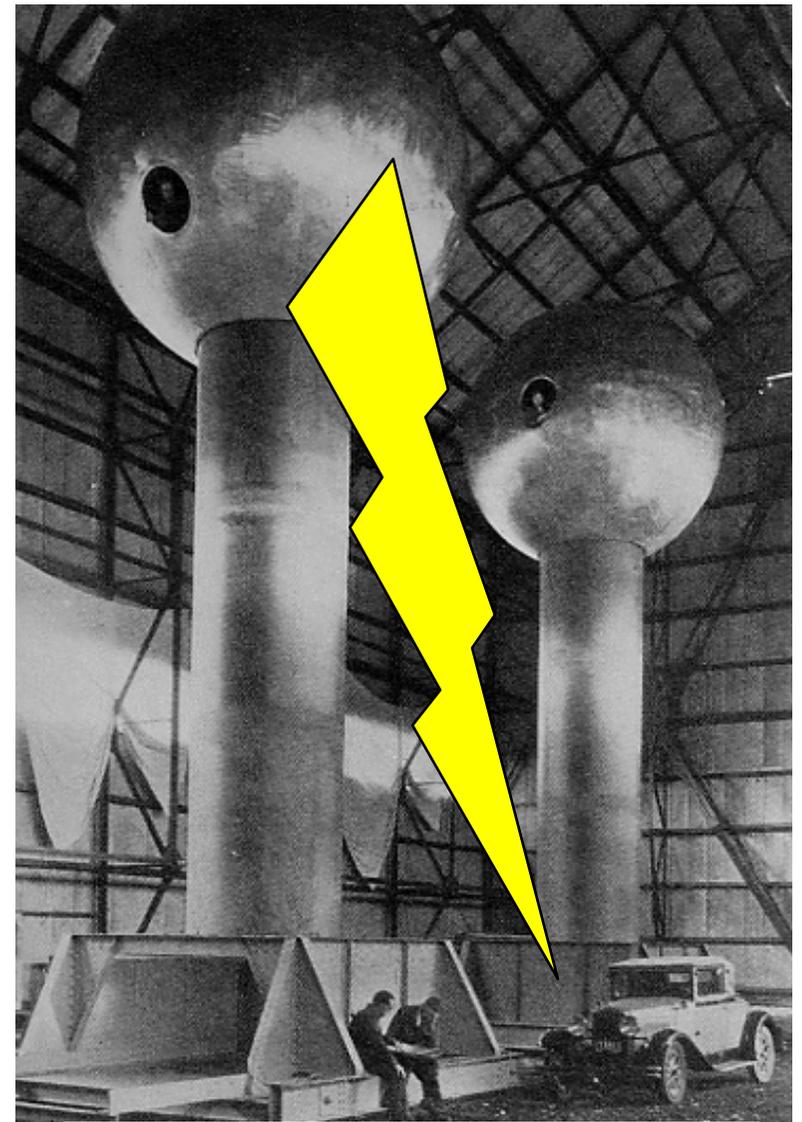
Van-de-Graaf generator at MIT

Radio-Frequency (RF) Acceleration
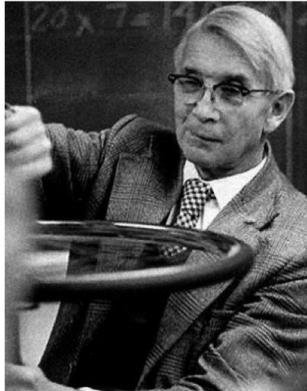
Electrostatic acceleration limited by insulation possibilities => use time-varying fields

1924: Ising suggests drift-tubes with time-varying fields

1928: Widerøe builds first demonstration linac

G.Ising

R.Widerøe

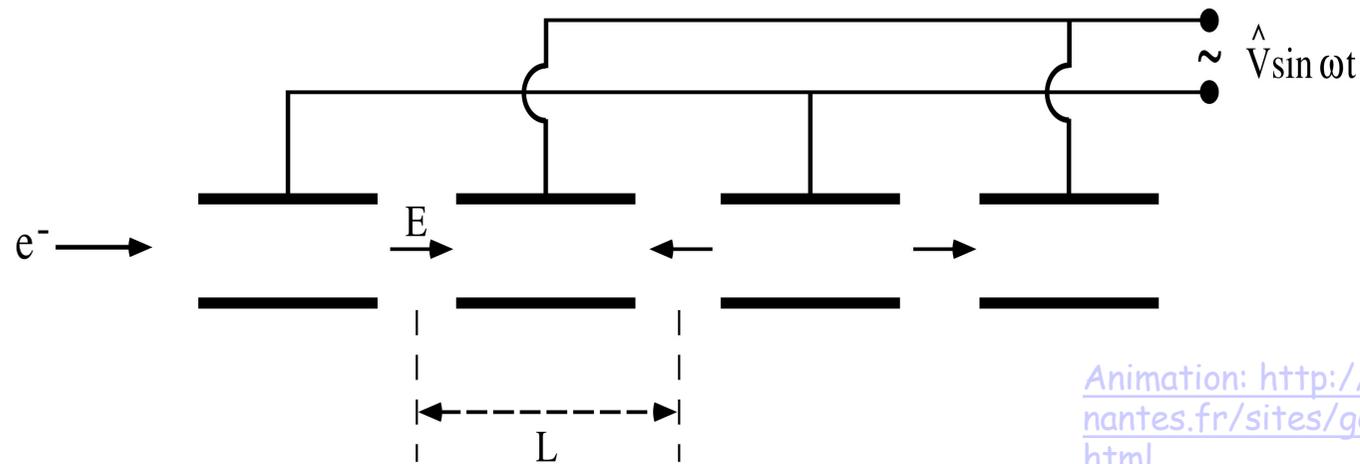Prinzip einer Methode zur Herstellung von Kanalstrahlen hoher Voltzahl.

Von

GUSTAF ISING.

Mit 2 Figuren im Texte.

Mitgeteilt am 12. März 1924 durch C. W. OSEEN und M. SIEGBAHN.
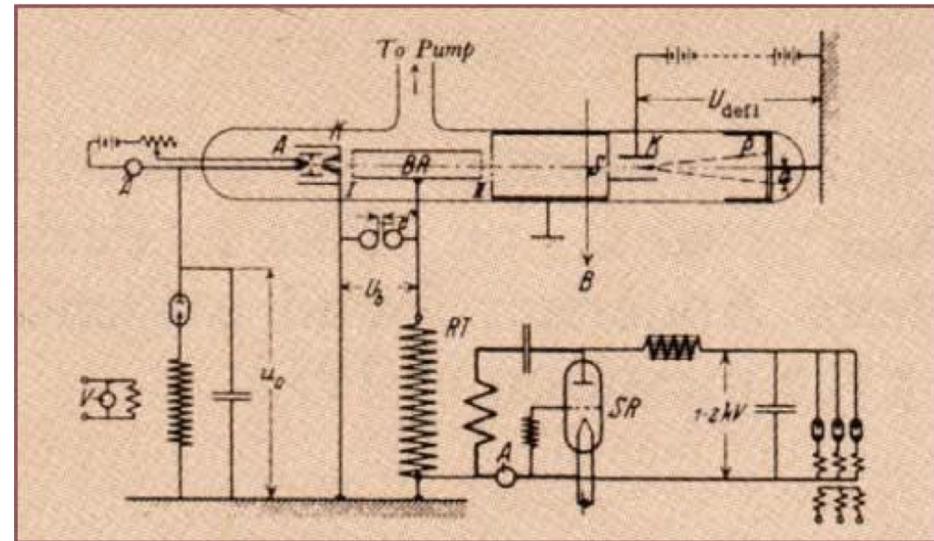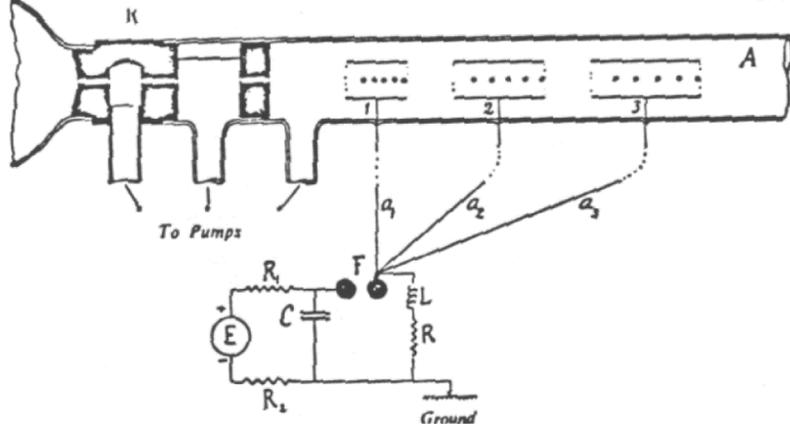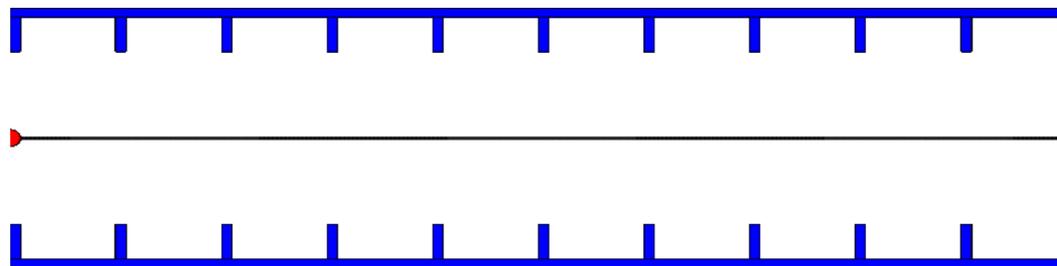
P.Lebrun

Radio-Frequency (RF) Acceleration

$\hat{V}\sin\omega t$

e$^-$

E

L

Widerøe-type structure

Animation: http://www.sciences.univ-nantes.fr/sites/genevieve_tulloue/Meca/Charges/linac.html

Cylindrical electrodes (drift tubes) separated by gaps and fed by a RF generator, as shown above, lead to an alternating electric field polarity

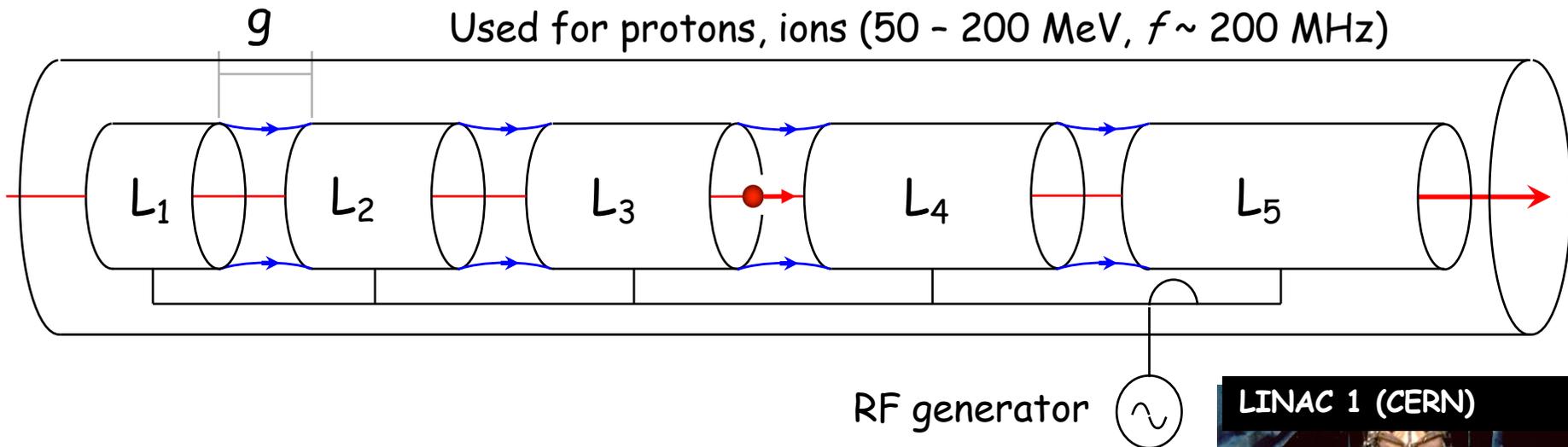Synchronism condition $\longrightarrow$ L = v T/2

v = **particle velocity**
T = **RF period**

Consequence: We can only accelerate bunched beam!

Similar for standing wave cavity as shown (with v≈c)

D.Schulte

RF acceleration: Alvarez Structure
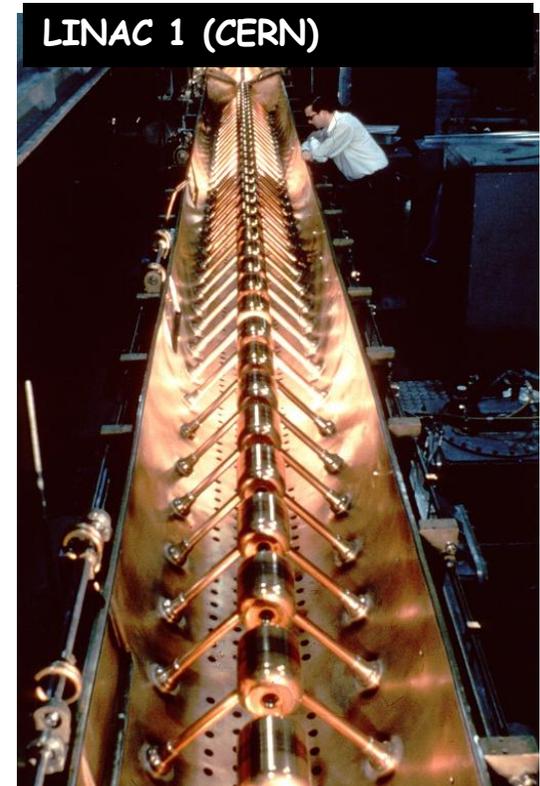
g

Used for protons, ions (50 – 200 MeV, $f \sim$ 200 MHz)

L$_1$ L$_2$ L$_3$ L$_4$ L$_5$

RF generator

LINAC 1 (CERN)

Synchronism condition $\left(g << L \right)$

➡️

$$L = v_s \, T_{RF} = \beta_s \, \lambda_{RF}$$

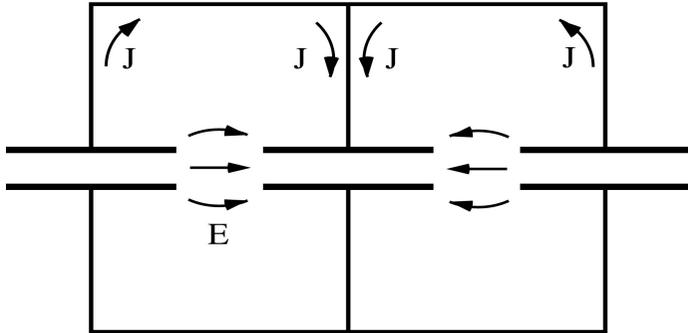$$\omega_{RF} = 2\pi f_{RF} = 2\pi \frac{v_s}{L}$$

Resonant RF Cavities

- Considering RF acceleration, it is obvious that when particles get high velocities the drift spaces get longer and one looses on the efficiency.
  => The solution consists of using a higher operating frequency.

- The power lost by radiation, due to circulating currents on the electrodes, is proportional to the RF frequency.
  => The solution consists of enclosing the system in a cavity which resonant frequency matches the RF generator frequency.

$\vec{H}$ ou $\vec{J}$
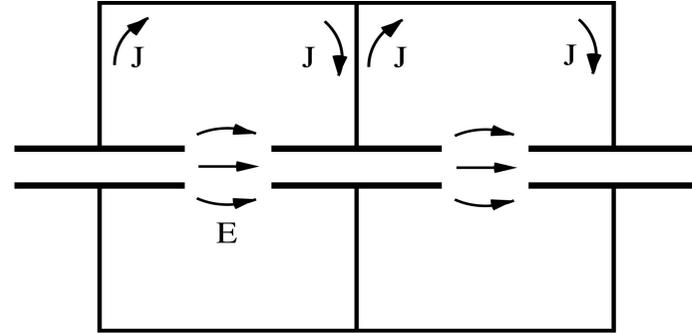
$\vec{E}_z$

ω_{RF}
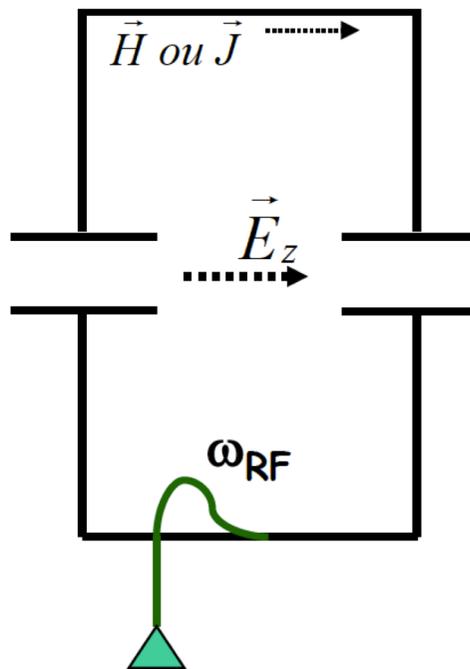
- The electromagnetic power is now constrained in the resonant volume

- Each such cavity can be independently powered from the RF generator

- Note however that joule losses will occur in the cavity walls (unless made of superconducting materials)
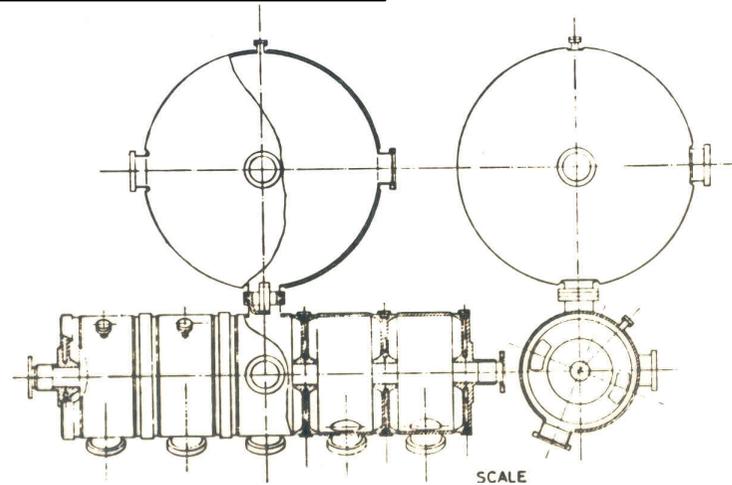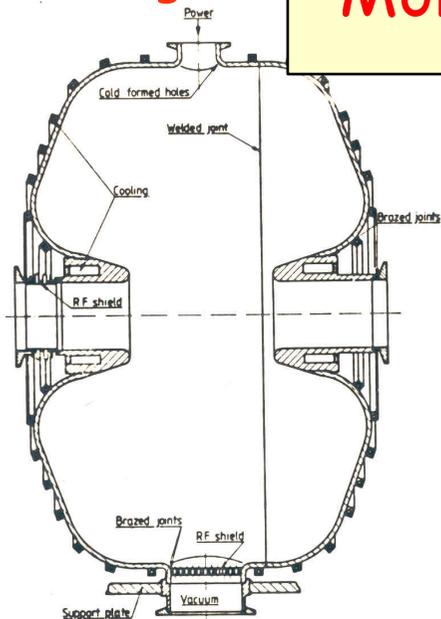
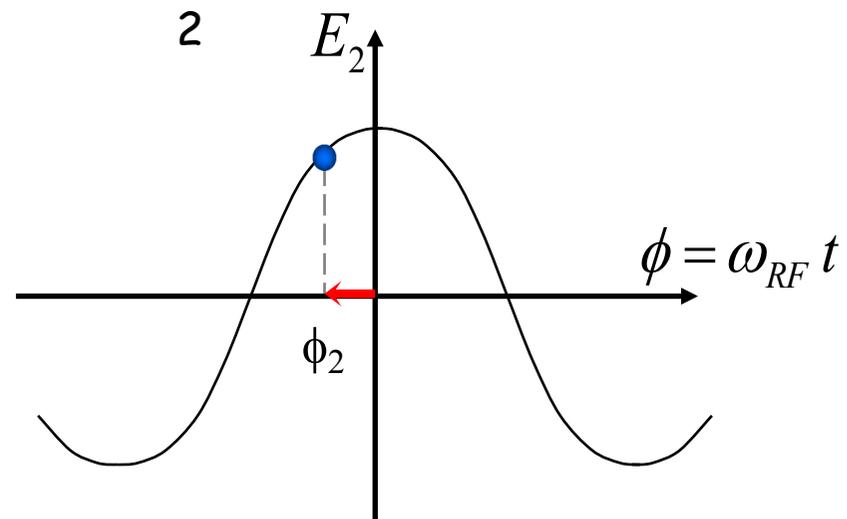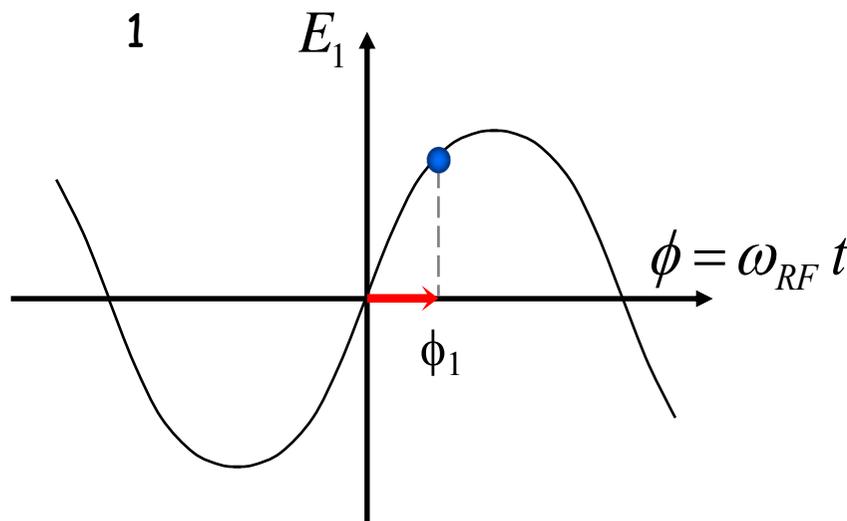# Some RF Cavity Examples

L = vT/2 (π mode)

L = vT (2π mode)

J J J J

E

J J J J

E

Single G... ...p

More in RF Systems

Power
Cold formed holes
Welded joint
Cooling
Brazed joints
RF shield
Brazed joints
RF shield
Support plate
Vacuum

SCALE

Common Phase Conventions

1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope

2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

Time t= 0 chosen such that:

1

E_1

$\phi = \omega_{RF}\, t$

ϕ_1

$$E_1(t) = E_0 \sin\left(\omega_{RF}\, t\right)$$

2

E_2

$\phi = \omega_{RF}\, t$

ϕ_2

$$E_2(t) = E_0 \cos\left(\omega_{RF}\, t\right)$$

3. I will stick to **convention 1** in the following to avoid confusion

Principle of Phase Stability (Linac)

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φs .

$$eV_S = e\hat{V} \sin \Phi_S$$

is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1 , P_2, …… are fixed points.

For a 2π mode, the electric field is the same in all gaps at any given time.
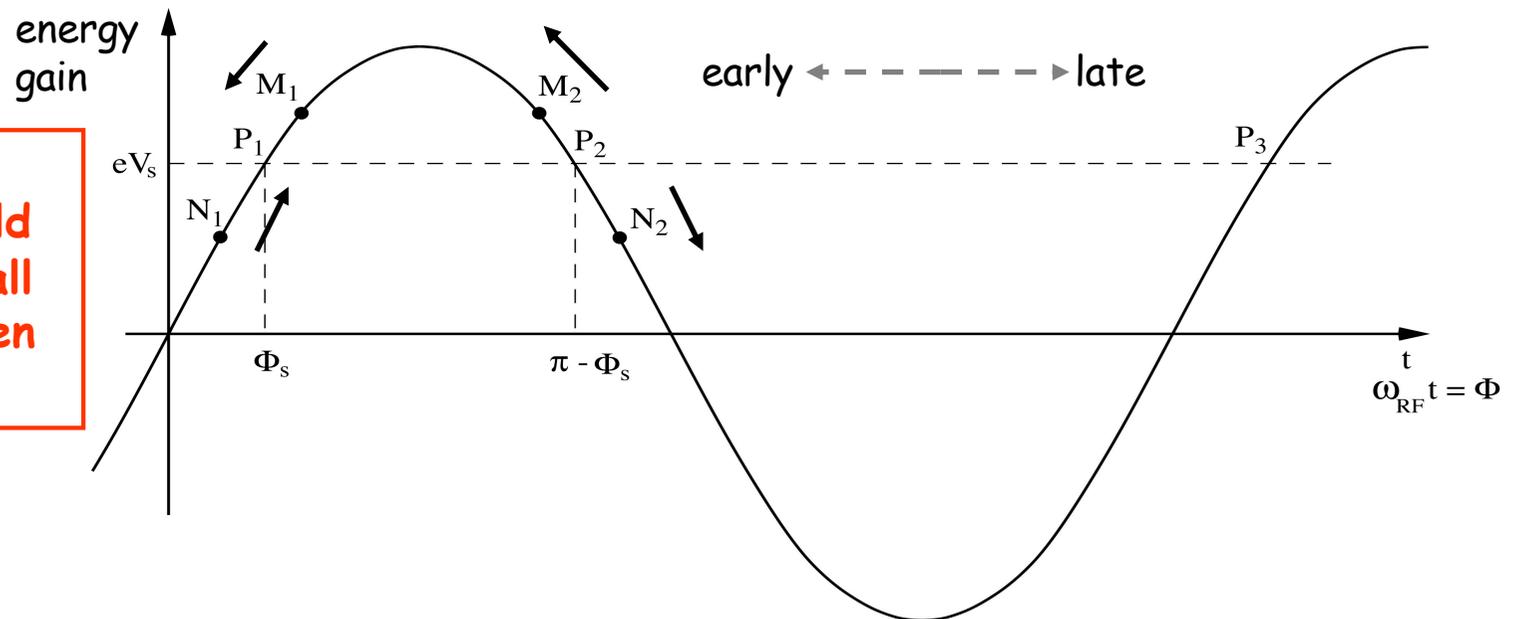
If an energy increase is transferred into a velocity increase =>
     M$_1$ & N$_1$ will move towards P$_1$ => stable
     M$_2$ & N$_2$ will go away from P$_2$ => unstable
(Highly relativistic particles have no significant velocity change)

A Consequence of Phase Stability
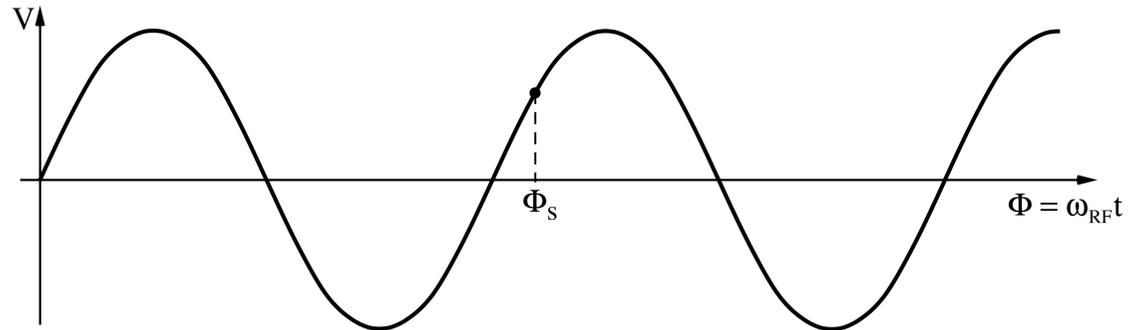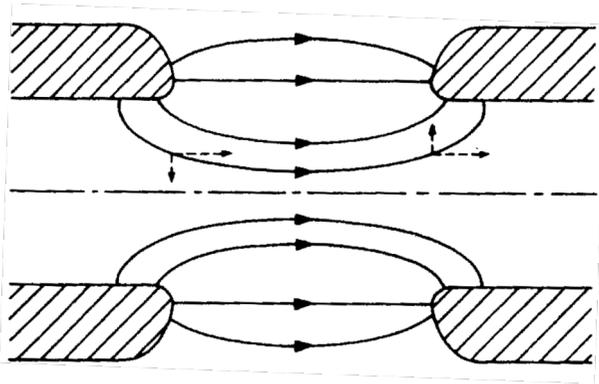
The divergence of the field is zero according to Maxwell :

$$\nabla \vec{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} = -\frac{\partial E_z}{\partial z}$$

Transverse fields
* focusing at the entrance and
* defocusing at the exit of the cavity.

Electrostatic case: Energy gain inside the cavity leads to focusing
RF case: Field increases during passage => transverse defocusing!

External focusing (solenoid, quadrupole) is then necessary

Energy-phase Oscillations (Small Amplitude) (1)

- Rate of **energy gain** for the **synchronous particle**:

$$\frac{dE_s}{dz} = \frac{dp_s}{dt} = eE_0 \sin\phi_s$$

- Use **reduced variables** with respect to **synchronous particle**

$$w = W - W_s = E - E_s \qquad\qquad \varphi = \phi - \phi_s$$

Energy gain: $\dfrac{dw}{dz} = eE_0\left[\sin(\phi_s + \varphi) - \sin\phi_s\right] \approx eE_0 \cos\phi_s . \varphi \quad (small\ \varphi)$

- Rate of **phase change** with respect to the synchronous one:

$$\frac{d\varphi}{dz} = \omega_{RF}\left(\frac{dt}{dz} - \left(\frac{dt}{dz}\right)_s\right) = \omega_{RF}\left(\frac{1}{v} - \frac{1}{v_s}\right) \cong -\frac{\omega_{RF}}{v_s^2}(v - v_s)$$

Leads finally to: $\dfrac{d\varphi}{dz} = -\dfrac{\omega_{RF}}{m_0 v_s^3 \gamma_s^3} w$

Energy-phase Oscillations (Small Amplitude) (2)

Combining the two 1st order equations into a 2nd order equation gives the equation of a **harmonic oscillator**:

$$\frac{d^2\varphi}{dz^2} + \Omega_s^2 \varphi = 0 \qquad \text{with} \qquad \Omega_s^2 = \frac{eE_0\omega_{RF}\cos\phi_s}{m_0 v_s^3 \gamma_s^3}$$

Slower for higher energy!

Stable harmonic oscillations imply:

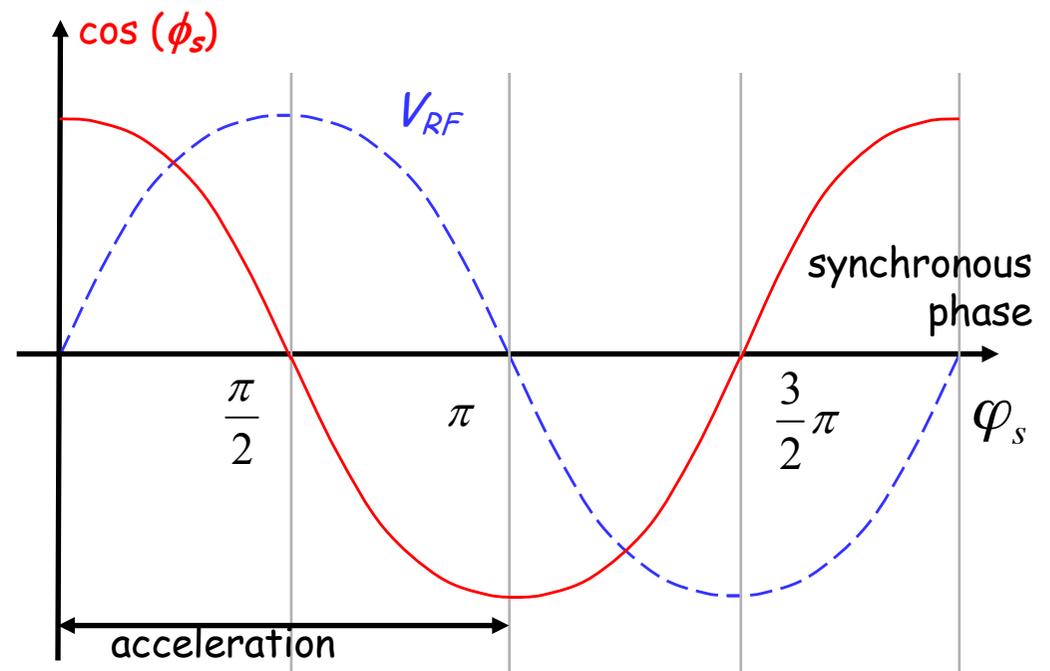$$\Omega_s^2 > 0 \quad \text{and real}$$

hence: $\cos\phi_s > 0$

And since acceleration also means:

$$\sin\phi_s > 0$$

You finally get the result for the **stable phase range**:

$$0 < \phi_s < \frac{\pi}{2}$$

Positive rising RF slope!

Summary up to here…

- **Acceleration by electric fields**, static fields limited
  => time-varying fields

- **Synchronous condition** needs to be fulfilled for acceleration

- Particles perform **oscillation** around synchronous phase

- Stable acceleration on the rising slope in a linac.

- Electrons are quickly relativistic, speed does not change

- Protons and ions need changing structure geometry and certain RF frequency range

Circular accelerators

Betatron

Cyclotron

Synchrotron

Acceleration by Induction: The Betatron
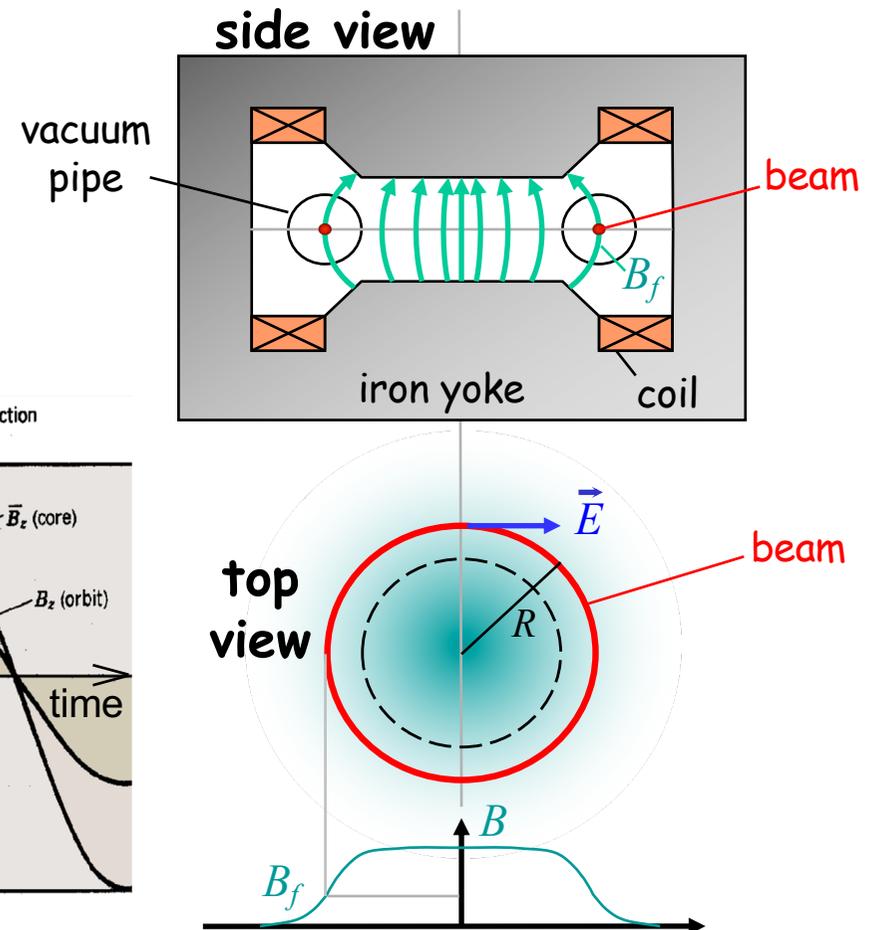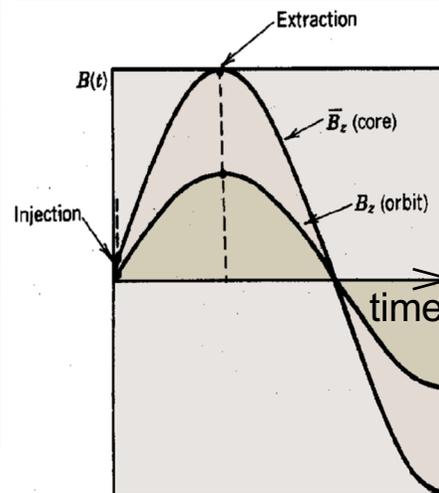
A ramping magnetic field
- Guides particles on a circular trajectory and
- Creates a tangential electric field that accelerates the particles

Limited by saturation in iron (~300 MeV e-)

Used in industry and medicine, as they are compact accelerators for electrons

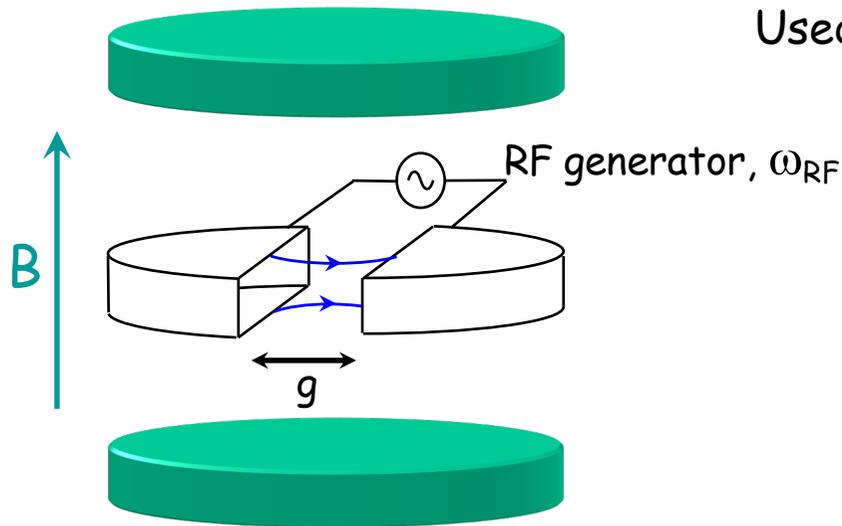Donald Kerst with the first betatron, invented at the University of Illinois in 1940

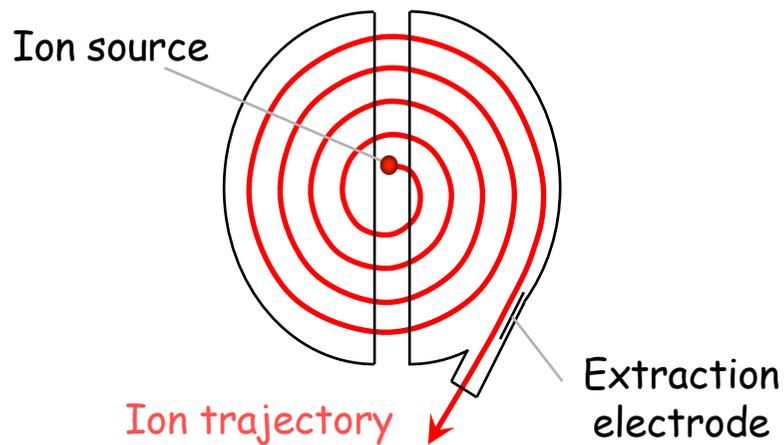## Circular accelerators: Cyclotron

Courtesy: EdukiteLearning, https://youtu.be/cNnNM2ZqIsc

Circular accelerators: Cyclotron

Used for protons, ions

B = constant

ω_{RF} = constant

RF generator, ω_{RF}

B

g

Synchronism condition

$$\omega_s = \omega_{RF}$$

$$2\pi \, \rho = v_s \, T_{RF}$$

Ion source

Cyclotron frequency $\qquad \omega = \dfrac{q \, B}{m_0 \, \gamma}$

1. γ increases with the energy
   $\Rightarrow$ no exact synchronism
2. if $\mathbf{v} \ll \mathbf{c} \Rightarrow \gamma \cong \mathbf{1}$

Extraction electrode

Ion trajectory

Cyclotron Animation

Animation: http://www.sciences.univ-nantes.fr/sites/genevieve_tulloue/Meca/Charges/cyclotron.html

Cyclotron / Synchrocyclotron

TRIUMF 520 MeV cyclotron   Vancouver - Canada

CERN 600 MeV synchrocyclotron

Synchrocyclotron: Same as cyclotron, except a modulation of ω_{RF}

B = constant

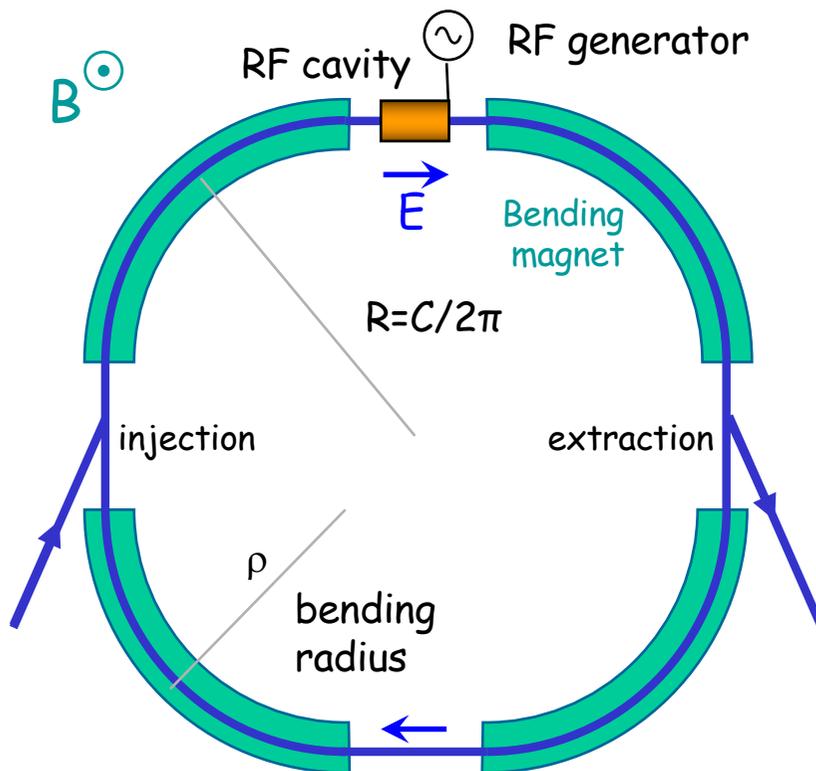$\gamma \, \omega_{RF}$ = constant ω_{RF} decreases with time

The condition:

$$\omega_s(t) = \omega_{RF}(t) = \frac{q\,B}{m_0\,\gamma(t)}$$

Allows to go beyond the non-relativistic energies

Empty Slide – Wait for the talk

Circular accelerators: The Synchrotron

1. **Constant orbit** during acceleration

2. To keep particles on the closed orbit, **B should increase** with time

3. ω and ω_{RF} **increase** with energy

RF frequency can be multiple of revolution frequency

$$\omega_{RF} = h\omega$$

Synchronism condition ➡

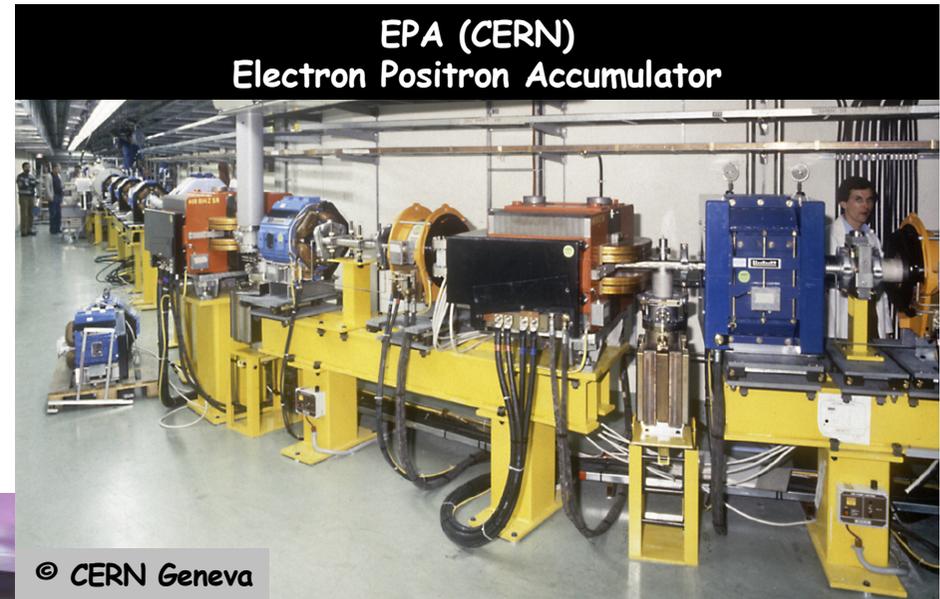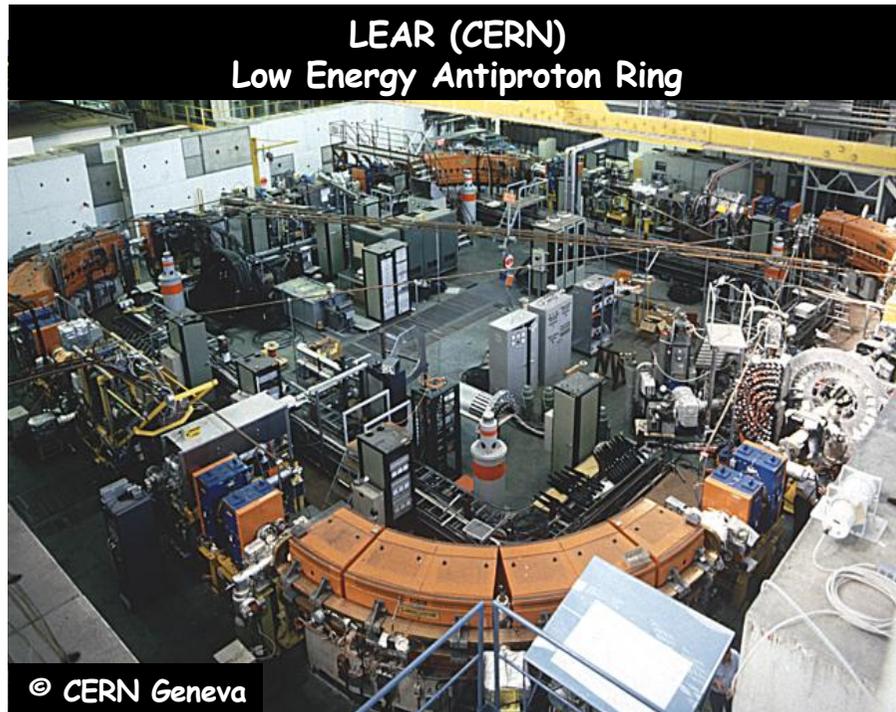$$T_s = h\,T_{RF}$$

$$\frac{2\pi R}{v_s} = h\,T_{RF}$$

h integer,
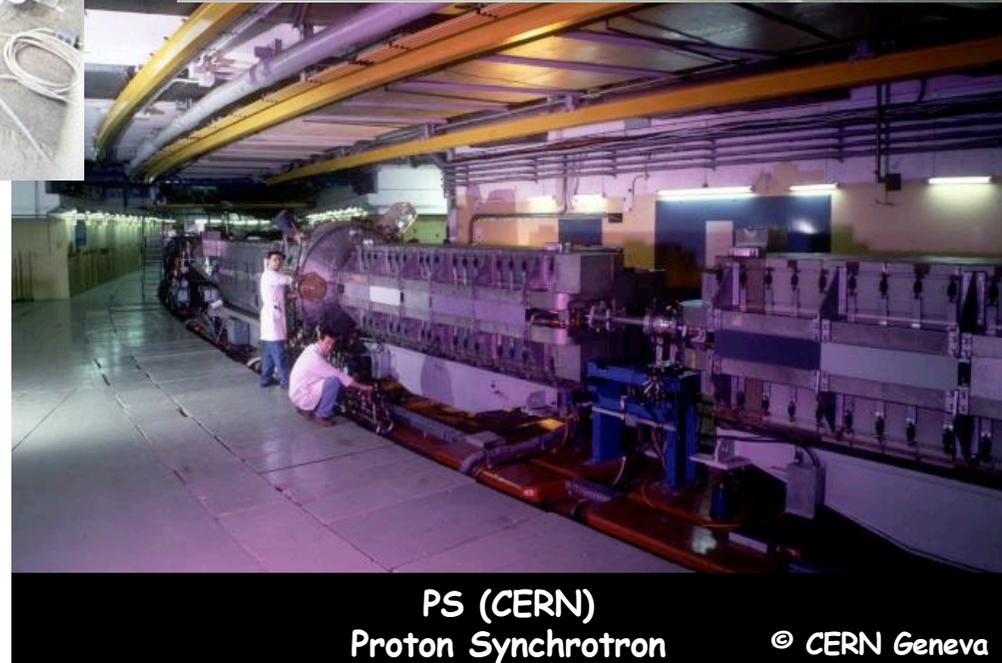**harmonic number:**
number of RF cycles per revolution

h is the **maximum number of bunches** in the synchrotron.
Normally less bunches due to gaps for kickers, collision constraints,…

Circular accelerators: The Synchrotron

LEAR (CERN)
Low Energy Antiproton Ring
© CERN Geneva

EPA (CERN)
Electron Positron Accumulator
© CERN Geneva

PS (CERN)
Proton Synchrotron © CERN Geneva

Examples of different proton and electron synchrotrons at CERN

+ LHC (of course!)

The Synchrotron

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:
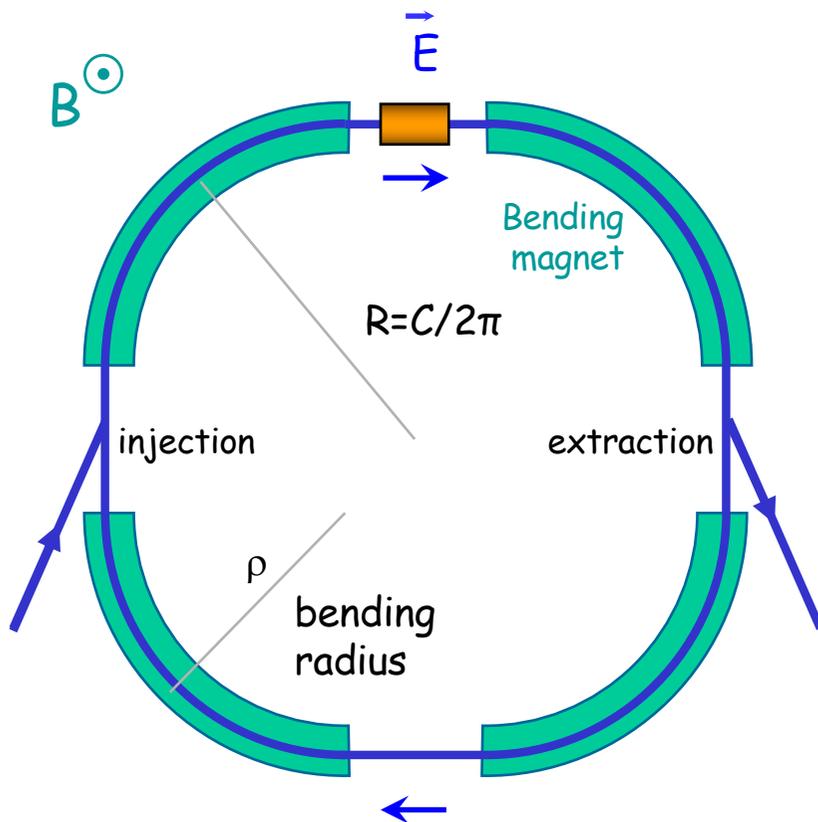
$e\hat{V}\sin\phi$ ⟶ **Energy gain per turn**

$\phi = \phi_s = cte$ ⟶ **Synchronous particle**

$\omega_{RF} = h\omega$ ⟶ **RF synchronism (h - harmonic number)**

$\rho = cte \quad R = cte$ ⟶ **Constant orbit**

$B\rho = P/e \Rightarrow B$ ⟶ **Variable magnetic field**

If v≈c, ω hence ω_RF remain constant (ultra-relativistic e⁻)

The Synchrotron – LHC Operation Cycle

The magnetic field (dipole current) is increased during the acceleration.
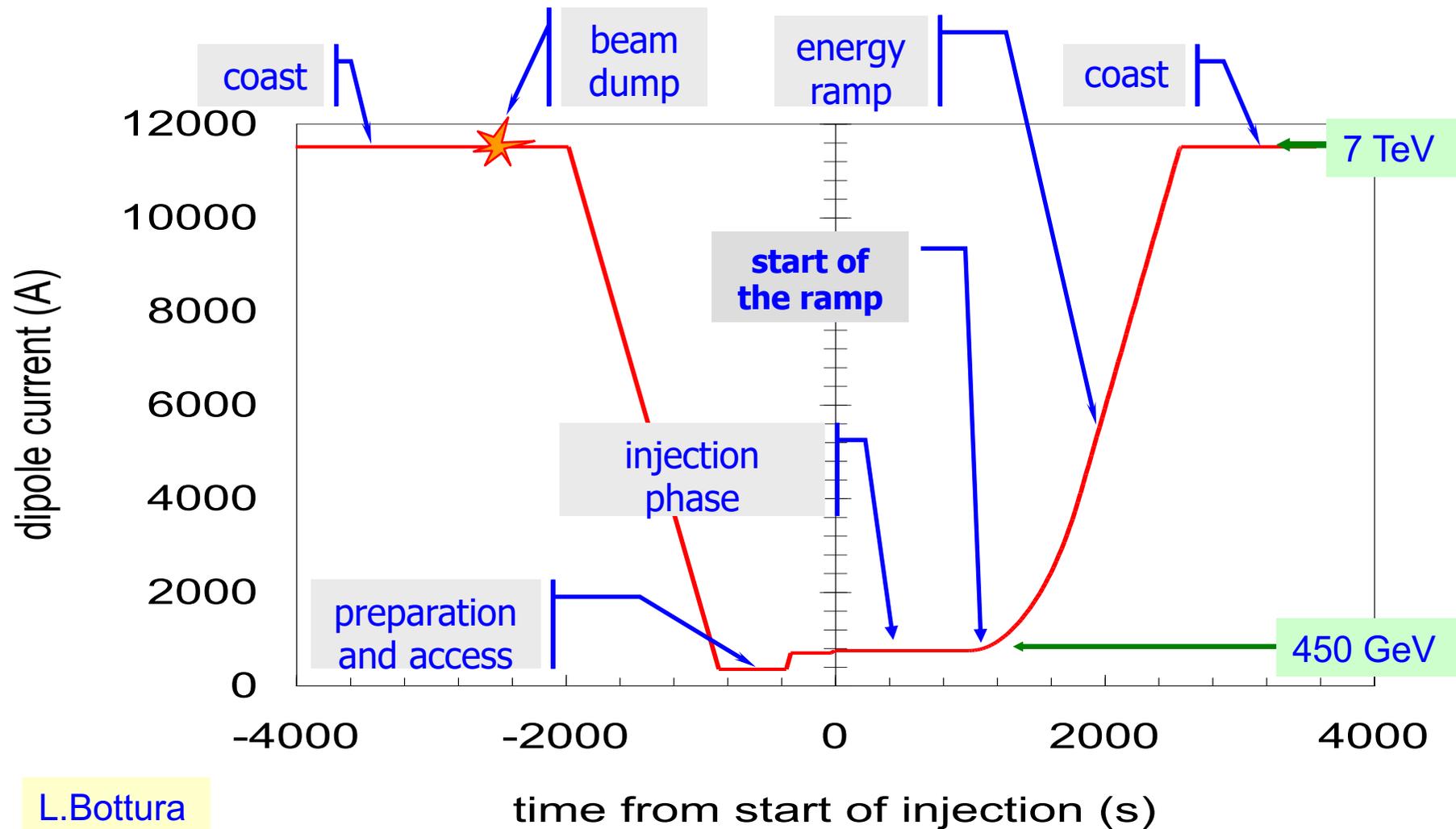
L.Bottura

The Synchrotron – Energy ramping

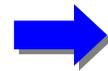Energy ramping by increasing the B field (frequency has to follow v):

$$p = eB\rho \underset{\rho \text{ const.}}{\Rightarrow} \frac{dp}{dt} = e\rho \dot{B} \Rightarrow (\Delta p)_{turn} = e\rho \dot{B} T_r = \frac{2\pi e\rho R \dot{B}}{v}$$

With

$$E^2 = E_0^2 + p^2 c^2 \Rightarrow \Delta E = v\Delta p \qquad \left(\Delta E\right)_{turn} = \left(\Delta W\right)_s = 2\pi e\rho R\dot{B} = e\hat{V}\sin\phi_s$$

Synchronous phase φ_s changes during energy ramping

$$\sin\phi_s = 2\pi\rho R\frac{\dot{B}}{\hat{V}_{RF}} \qquad \Longrightarrow \qquad \phi_s = \arcsin\left(2\pi\rho R\frac{\dot{B}}{\hat{V}_{RF}}\right)$$

- **The synchronous phase depends on**
  - **the change of the magnetic field**
  - **and the RF voltage**
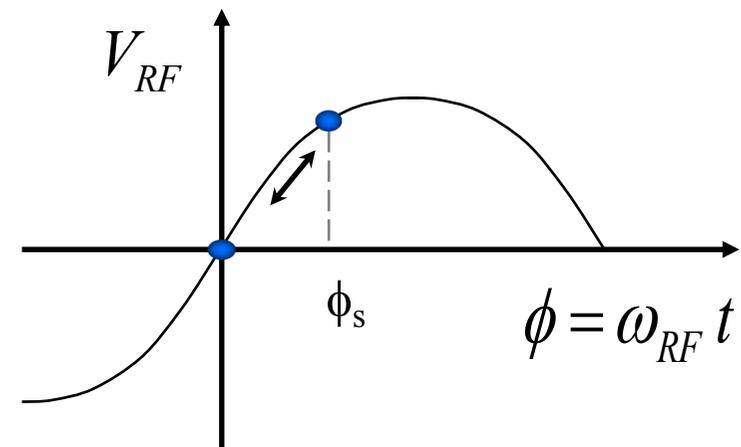
The Synchrotron – Frequency change

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency :

$$\omega = \frac{\omega_{RF}}{h} = \omega(B, R_s)$$

Hence: $\dfrac{f_{RF}(t)}{h} = \dfrac{v(t)}{2\pi R_s} = \dfrac{1}{2\pi}\dfrac{ec^2}{E_s(t)}\dfrac{\rho}{R_s}B(t)$ (using $p(t) = eB(t)\rho, \quad E = mc^2$)
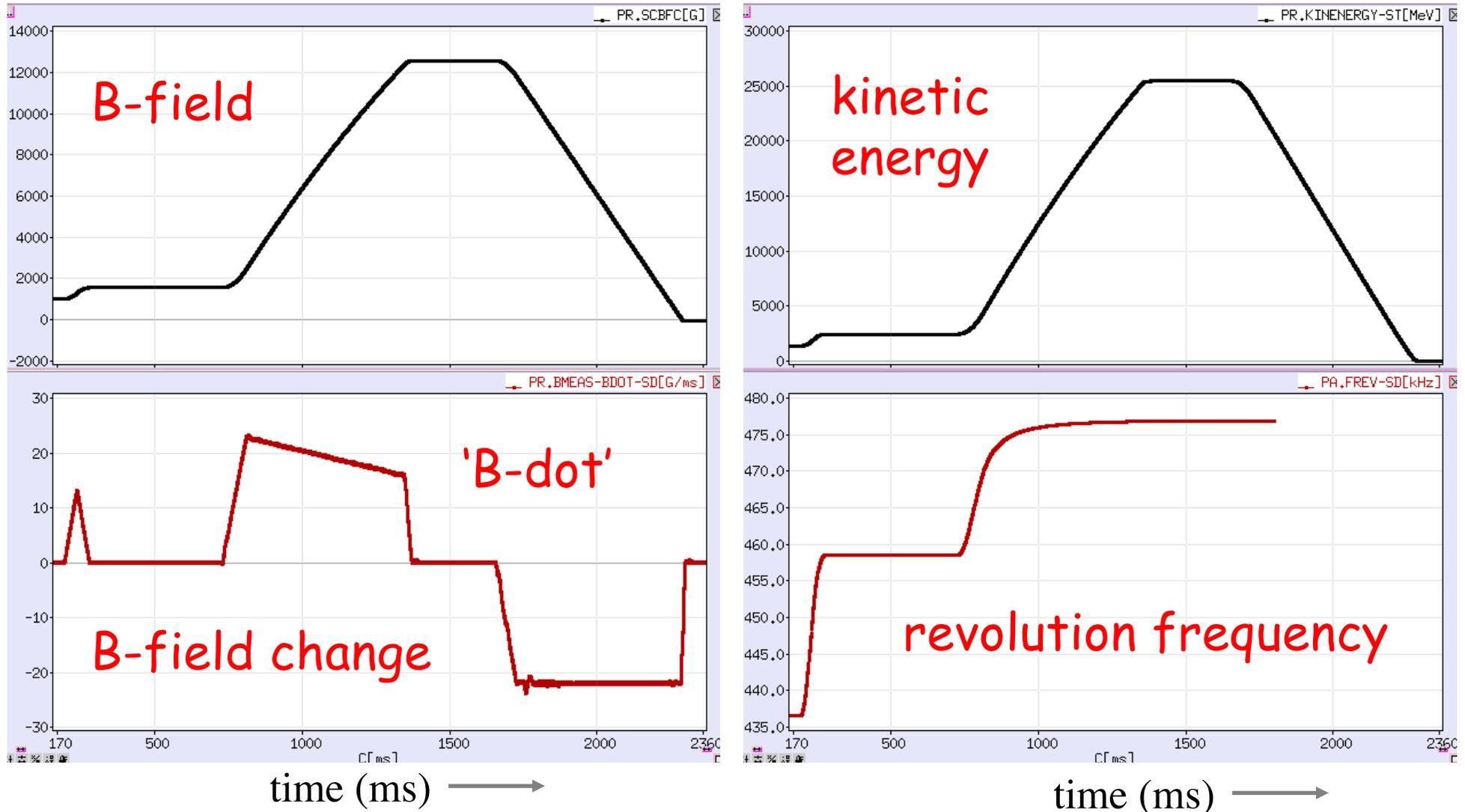
Since $E^2 = (m_0 c^2)^2 + p^2 c^2$ the RF frequency must follow the variation of the B field with the law

$$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s}\left\{\frac{B(t)^2}{(m_0 c^2 / ec\rho)^2 + B(t)^2}\right\}^{\frac{1}{2}}$$

RF frequency program during acceleration determined by B-field !

Example: PS - Field / Frequency change

During the energy ramping, the **B-field** and the **revolution frequency** increase

time (ms) →

time (ms) →

Overtaking in a roundabout

Finally a real-life problem: what is the fastest way through a roundabout?

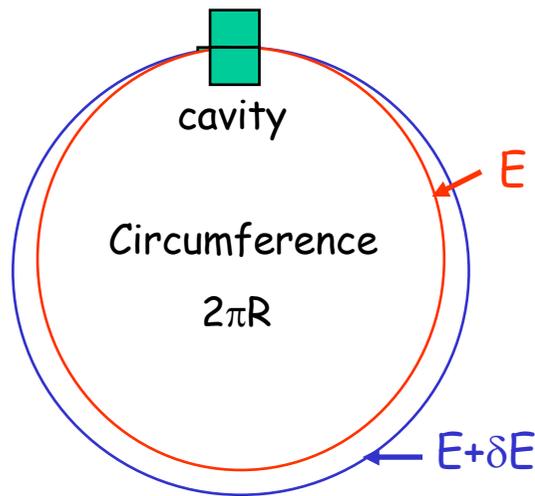Most CERN people encounter this near the French entrance to CERN.

Empty Slide – Wait for the talk

Empty Slide – Wait for the talk

Empty Slide – Wait for the talk

Empty Slide – Wait for the talk

Overtaking in a Synchrotron

cavity

E

Circumference

$2\pi R$

← E+δE

A particle slightly shifted in momentum will have a

- dispersion orbit and a **different orbit length** (higher momentum => less bent in magnet)
- a **different velocity**.

As a result of both effects the revolution frequency changes with a "slip factor η":

$$\eta = \frac{\mathrm{d}f_r/f_r}{\mathrm{d}p/p} \quad \Rightarrow \quad \eta = \frac{p}{f_r}\frac{df_r}{dp}$$

Note: you also find η defined with a minus sign!

p=particle momentum

R=synchrotron physical radius

f_r=revolution frequency

Effect from orbit defined by Momentum compaction factor:

$$\alpha_c = \frac{dL/L}{dp/p}$$

Property of the transverse beam optics:
(derivation in Appendix)

$$\alpha_c = \frac{1}{L}\int_C \frac{D_x(s)}{\rho(s)}\,ds_0$$

Dispersion Effects – Revolution Frequency

The two effects of the orbit length and the particle velocity change the revolution frequency as:

$$f_r = \frac{\beta c}{2\pi R} \quad \Rightarrow \quad \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R} = \frac{d\beta}{\beta} - \alpha_c \frac{dp}{p}$$

definition of momentum compaction factor
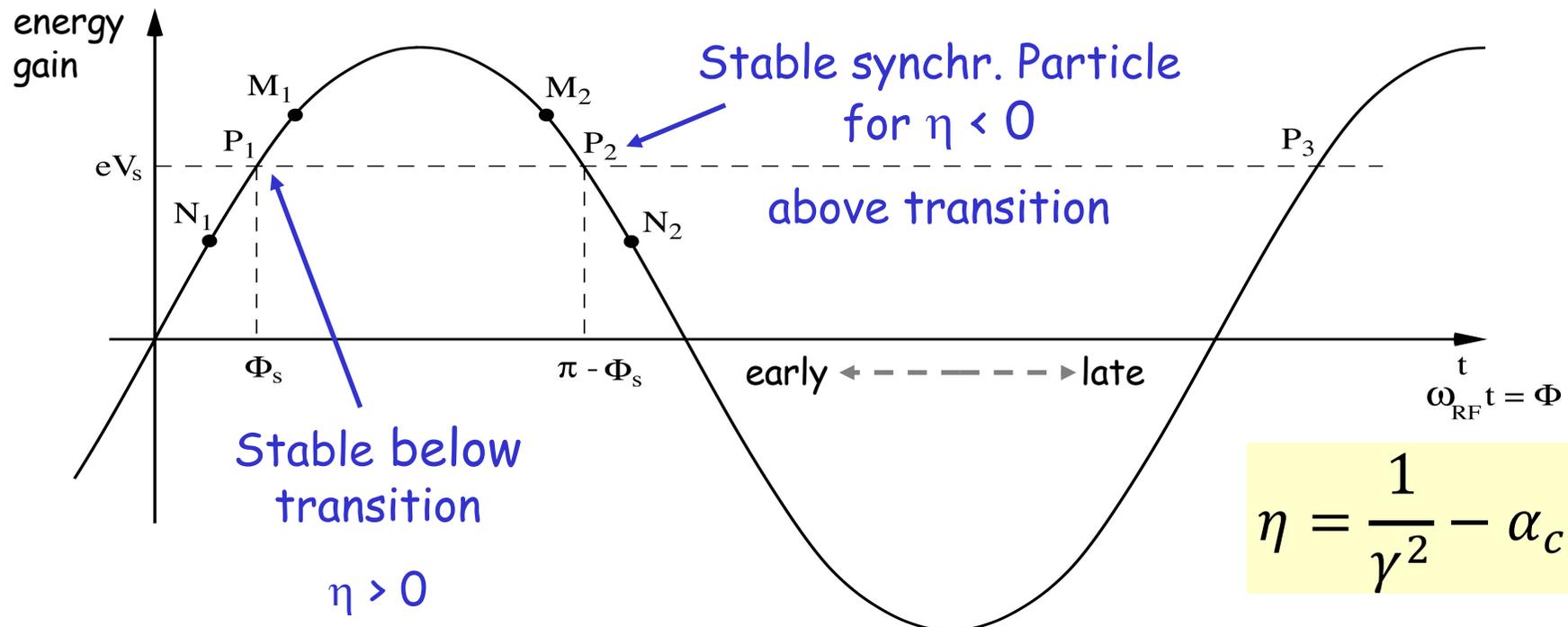
$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha_c \right) \frac{dp}{p}$$

$$p = mv = \beta\gamma \frac{E_0}{c} \quad \Rightarrow \quad \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-\frac{1}{2}}}{\left(1-\beta^2\right)^{-\frac{1}{2}}} = \underbrace{\left(1-\beta^2\right)^{-1}}_{\gamma^2} \frac{d\beta}{\beta}$$

Slip factor:

$$\eta = \frac{1}{\gamma^2} - \alpha_c$$

or

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_t^2}$$

with

$$\gamma_t = \frac{1}{\sqrt{\alpha_c}}$$

At transition energy, $\eta = 0$, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Phase Stability in a Synchrotron

From the definition of η it is clear that an increase in momentum gives
- below transition ($\eta > 0$) a higher revolution frequency
  (increase in velocity dominates) while

- above transition ($\eta < 0$) a lower revolution frequency ($v \approx c$ and longer path)
 where the momentum compaction (generally > 0) dominates.

energy gain

Stable synchr. Particle for $\eta < 0$

above transition

M_1 M_2

P_1 P_2 P_3

eV_s

N_1 N_2

Φ_s $\pi - \Phi_s$ early ← – – – – – → late   t $\omega_{RF} t = \Phi$

Stable below transition

$\eta > 0$

$$\eta = \frac{1}{\gamma^2} - \alpha_c$$

Crossing Transition

At transition, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a rapid change of the RF phase, a 'phase jump'.

$$\alpha_c \sim \frac{1}{Q_x^2} \qquad \gamma_t = \frac{1}{\sqrt{\alpha_c}} \sim Q_x$$
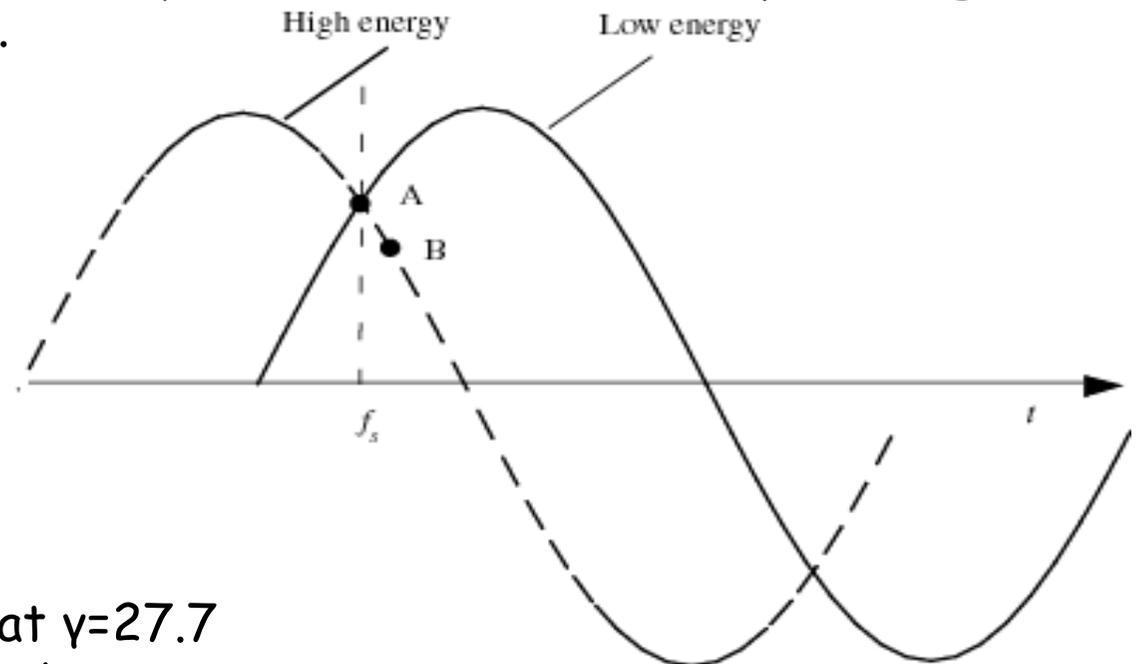
In the PS: γ_t is at ~6 GeV
In the SPS: γ_t = 22.8, injection at γ=27.7
        => no transition crossing!
In the LHC: γ_t is at ~55 GeV, also far below injection energy

Transition crossing not needed in leptons machines, why?

Dynamics: Synchrotron oscillations

Simple case (no accel.): B = const., below transition $\gamma < \gamma_t$

The phase of the synchronous particle must therefore be $\phi_0 = 0$.

Φ_1 — The particle B is accelerated
- Below transition, an energy increase means an increase in revolution frequency
- The particle arrives earlier – tends toward ϕ_0
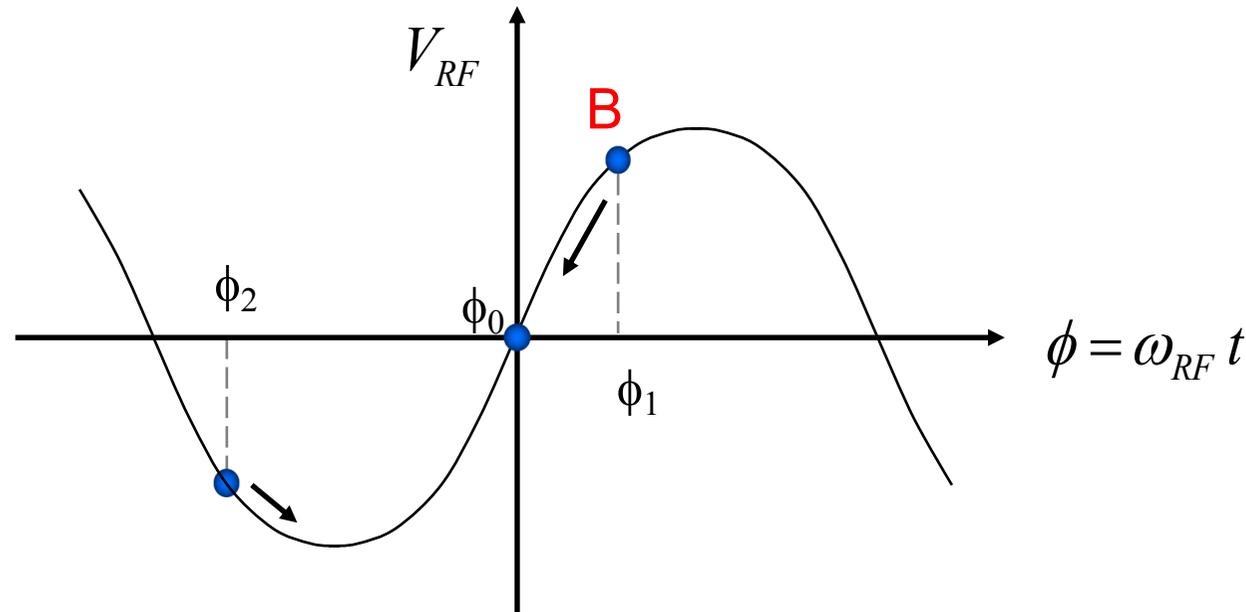
ϕ_2 — The particle is decelerated
- decrease in energy - decrease in revolution frequency
- The particle arrives later – tends toward ϕ_0

Synchrotron oscillations

800th revolution period
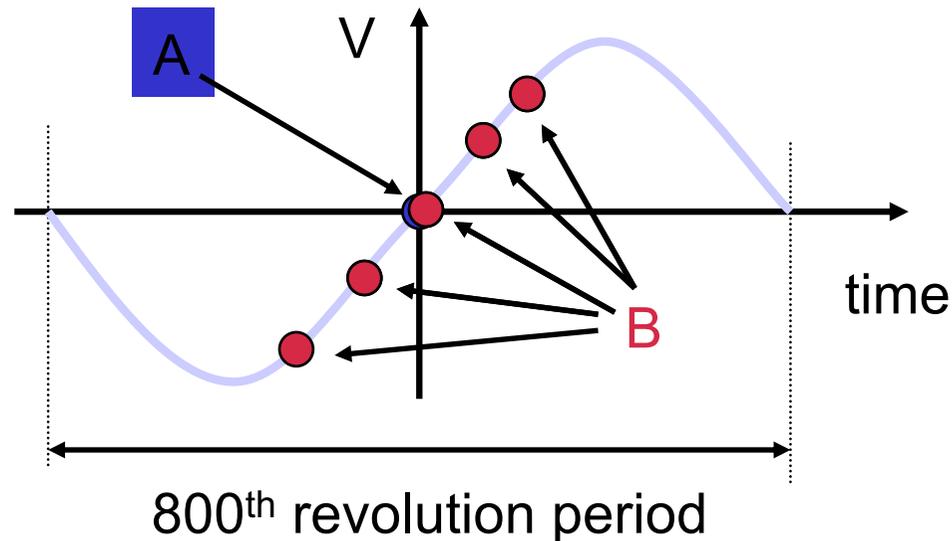
Particle B performs Synchrotron Oscillations around synchronous particle A.

The amplitude depends on the initial phase and energy.

The oscillation frequency is much slower than in the transverse plane. It takes a large number of revolutions for one complete oscillation. The restoring electric force is smaller than the magnetic force.

- proton synchrotrons of the order of 1000 turns

- electron storage rings of the order of ~10 turns

The Potential Well
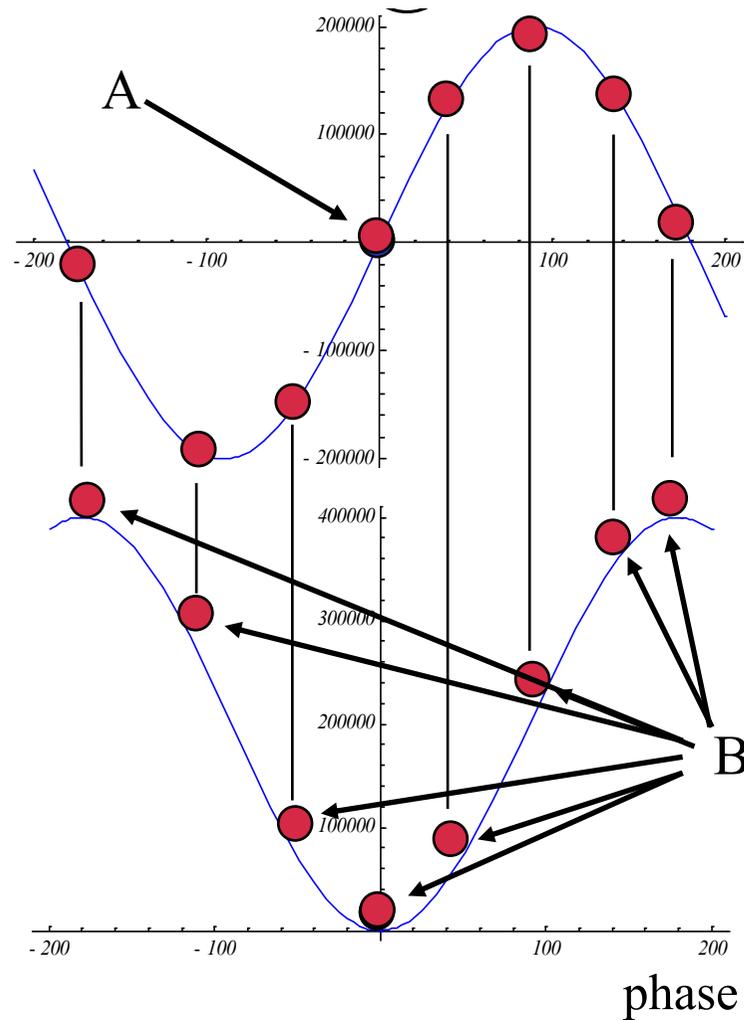
Cavity voltage

Potential well

Longitudinal phase space

The energy – phase oscillations can be drawn in phase space.
Similar to transverse, but here it's TIME and ENERGY!
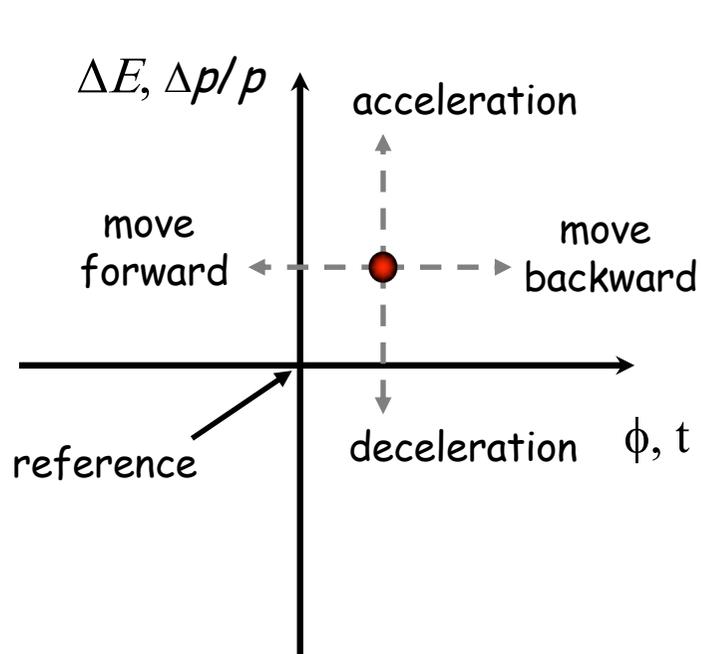
The particle trajectory in the phase space ($\Delta p/p$, ϕ) describes its longitudinal motion.
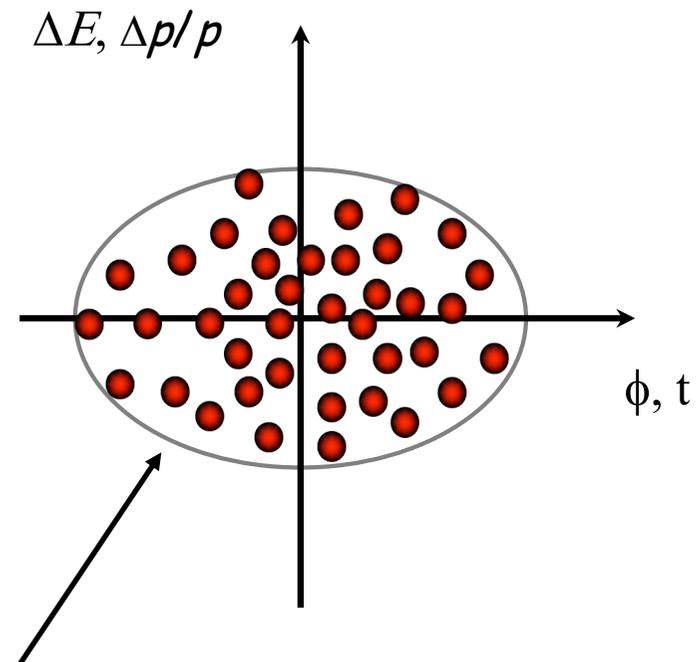
Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)
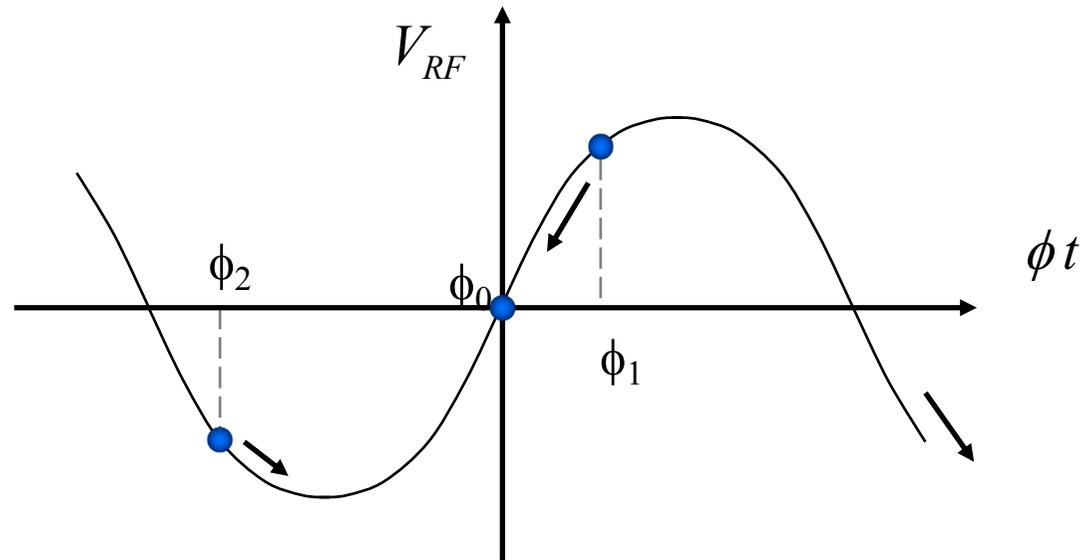
Longitudinal Phase Space Motion

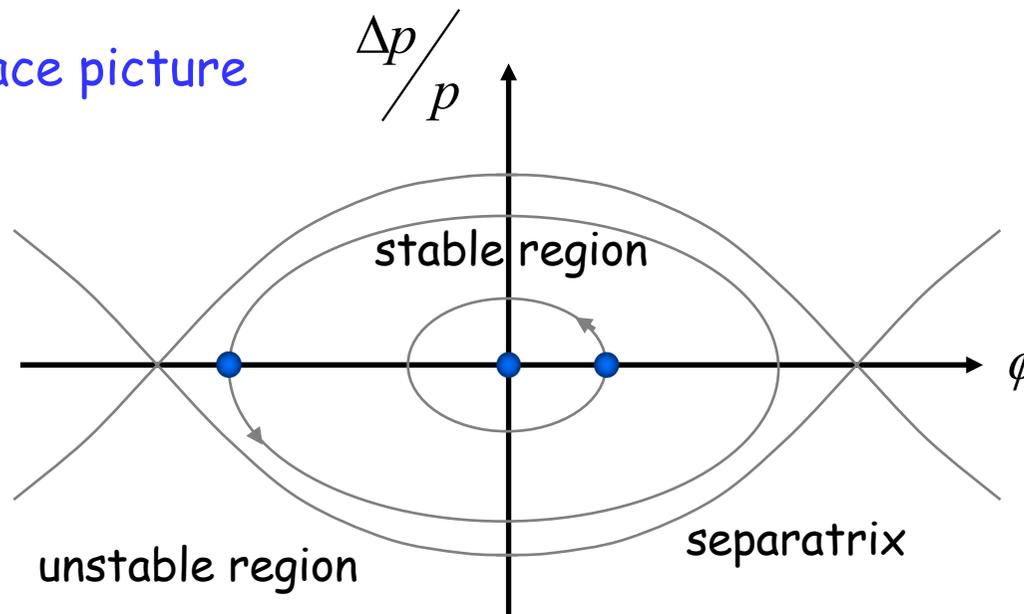Particle B oscillates around particle A in a synchrotron oscillation.

Plotting this motion in longitudinal phase space (time, energy) gives:
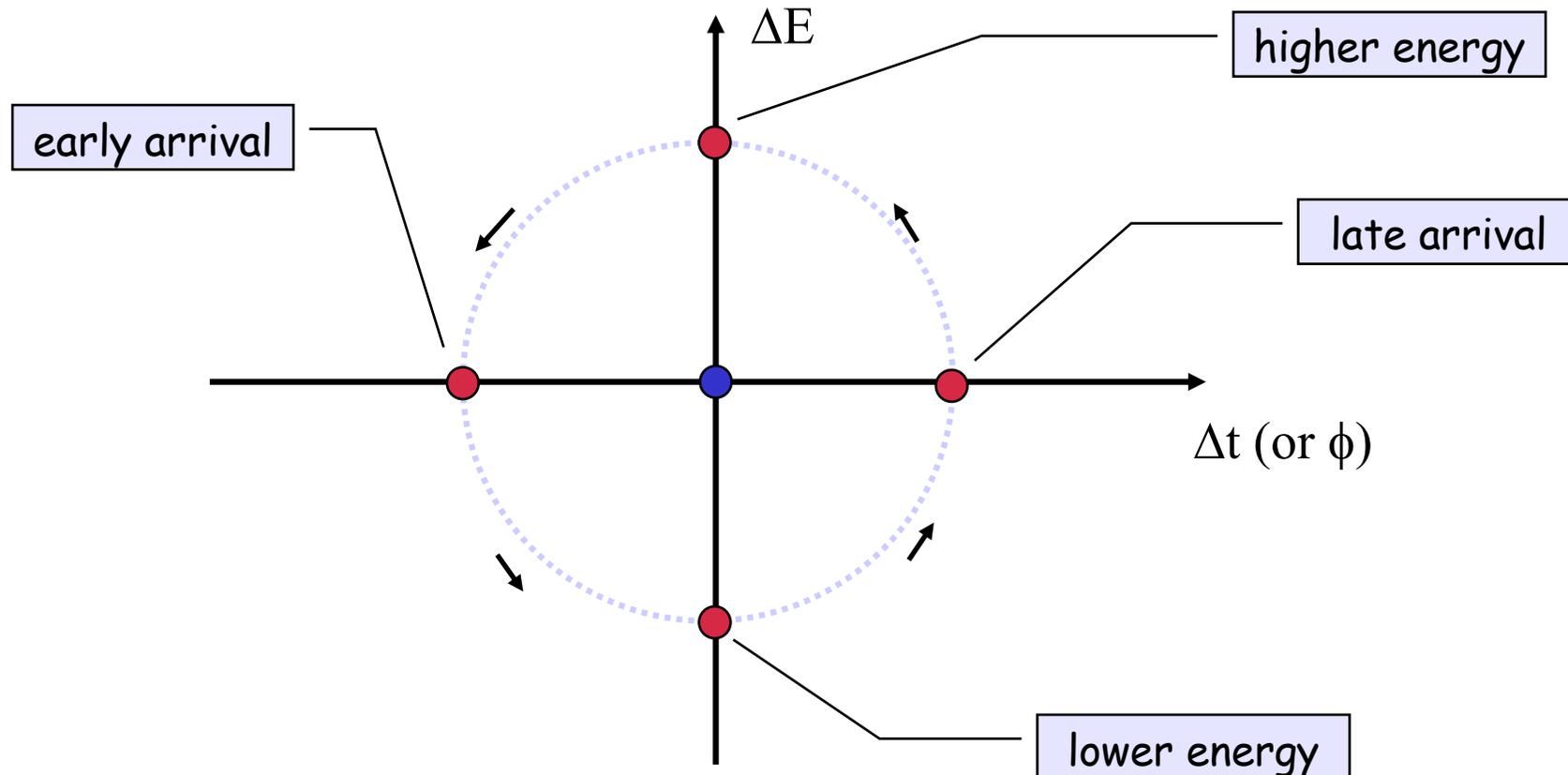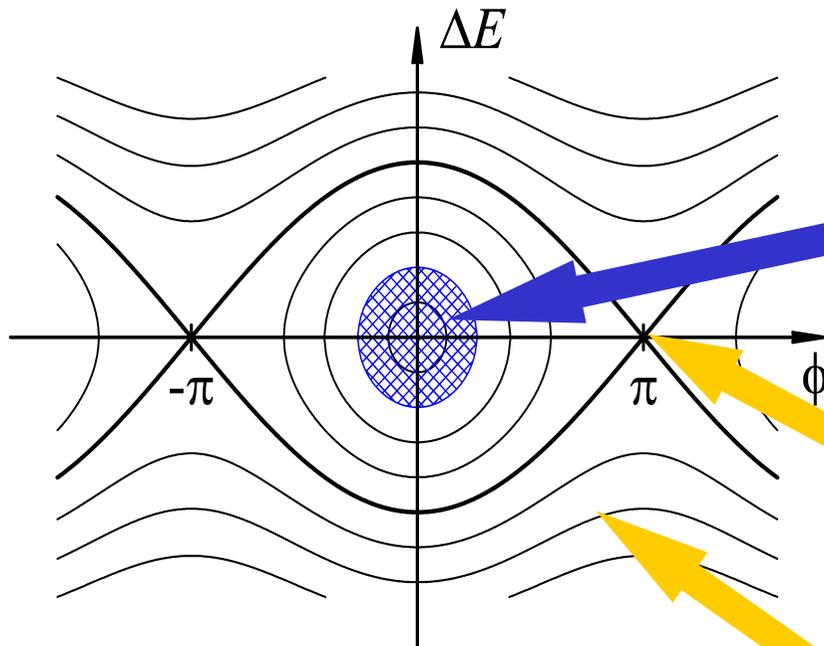
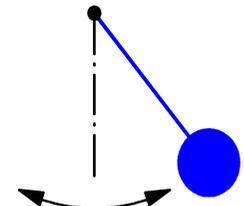# Synchrotron oscillations – No acceleration

Phase space picture

Synchrotron motion in phase space

ΔE-ϕ **phase space** of a stationary bucket
(when there is no acceleration)

Dynamics of a particle
Non-linear, conservative
oscillator $\rightarrow$ e.g. pendulum
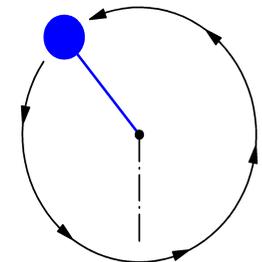

Particle **inside
the separatrix:**

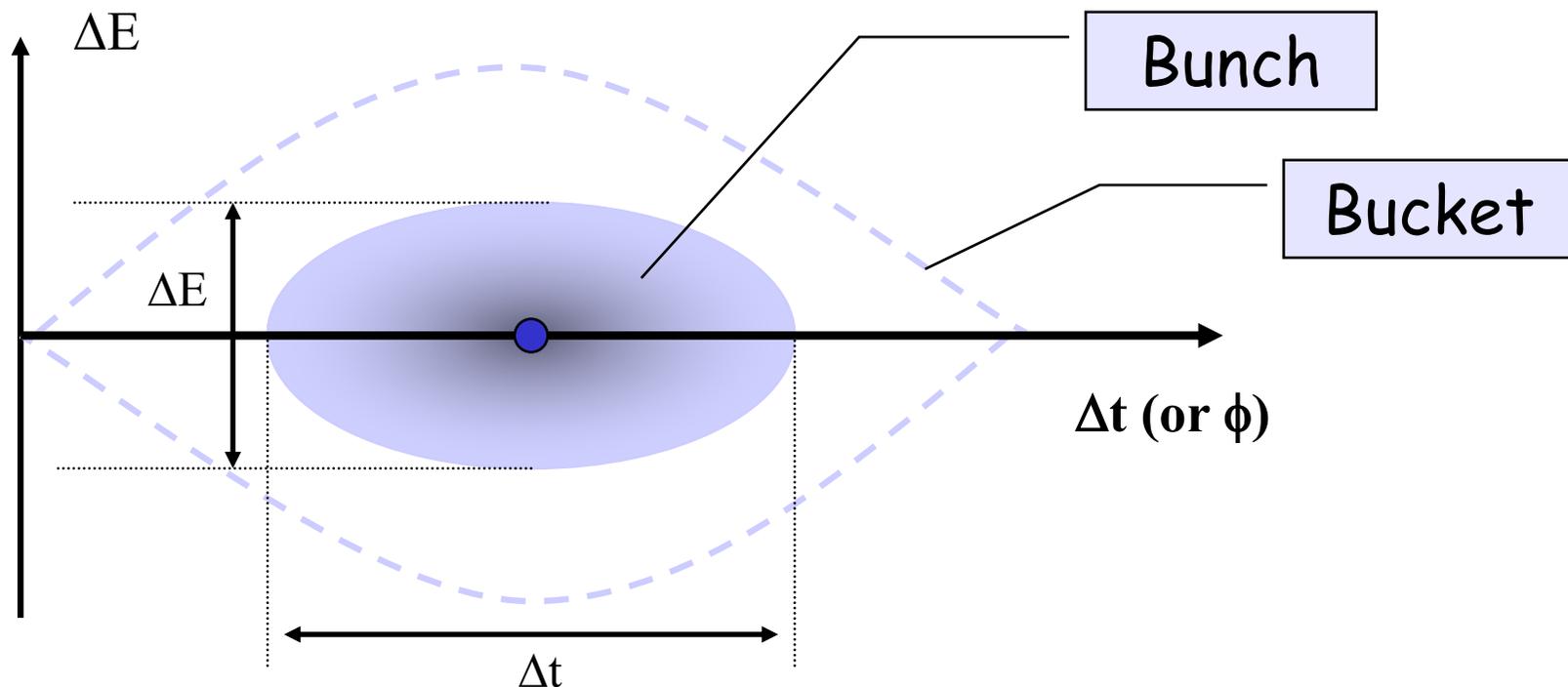Particle at the
unstable fix-point

Particle **outside
the separatrix:**

Bucket area: area enclosed
by the separatrix
The area covered by particles is
the longitudinal emittance

(Stationary) Bunch & Bucket

The **bunches** of the beam **fill** usually **a part of** the **bucket** area.

Bucket area = **longitudinal Acceptance** [eVs]

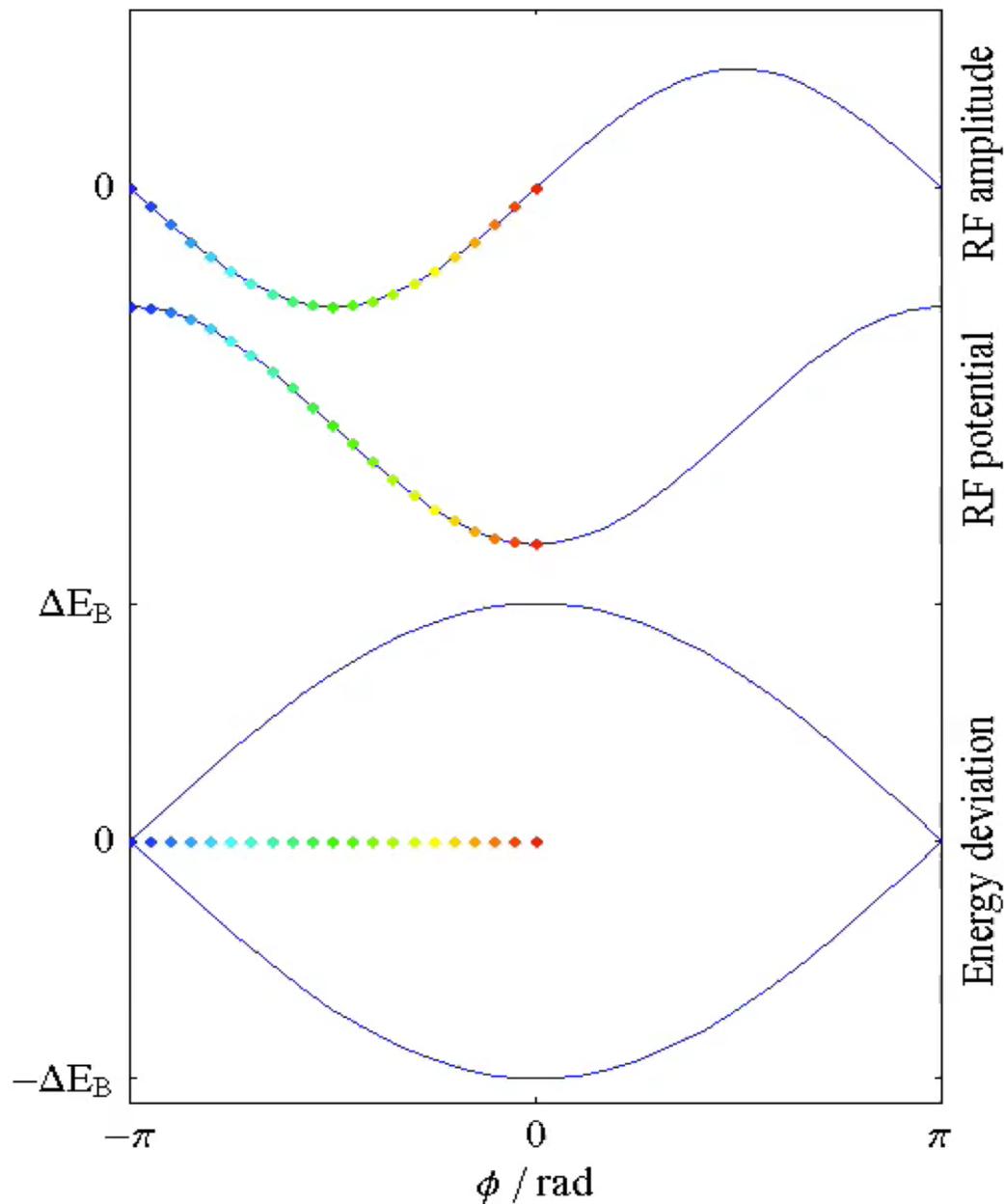Bunch area = **longitudinal beam emittance** = $4\pi\,\sigma_E\,\sigma_t$ **[eVs]**

Attention: Different definitions are used!

Synchrotron motion in phase space

The restoring force is non-linear.
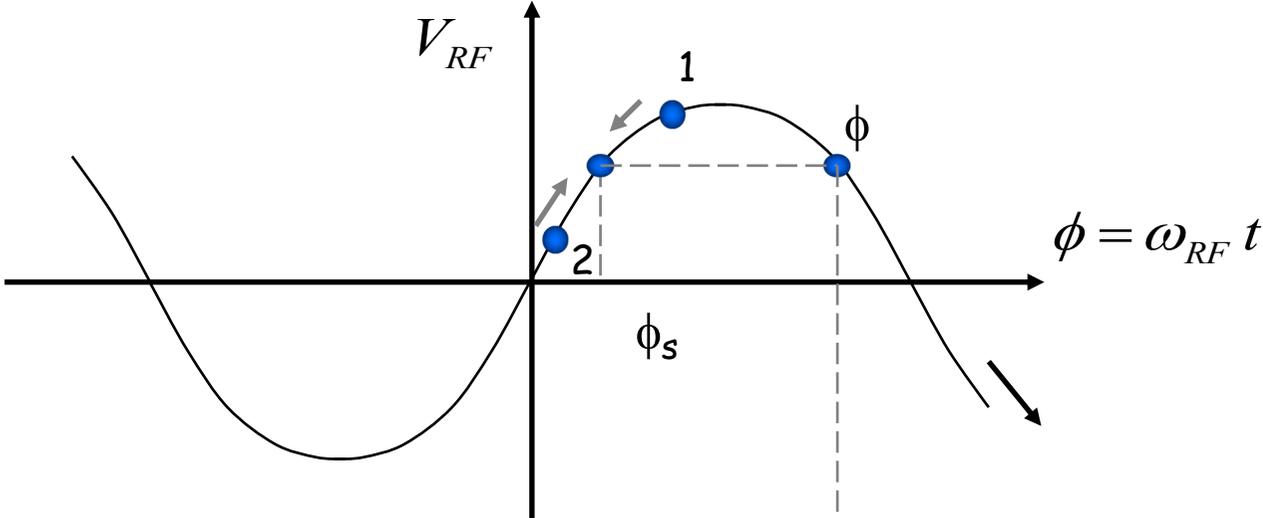⇒ speed of motion depends on position in phase-space

(here shown for a stationary bucket)

Synchrotron oscillations (with acceleration)
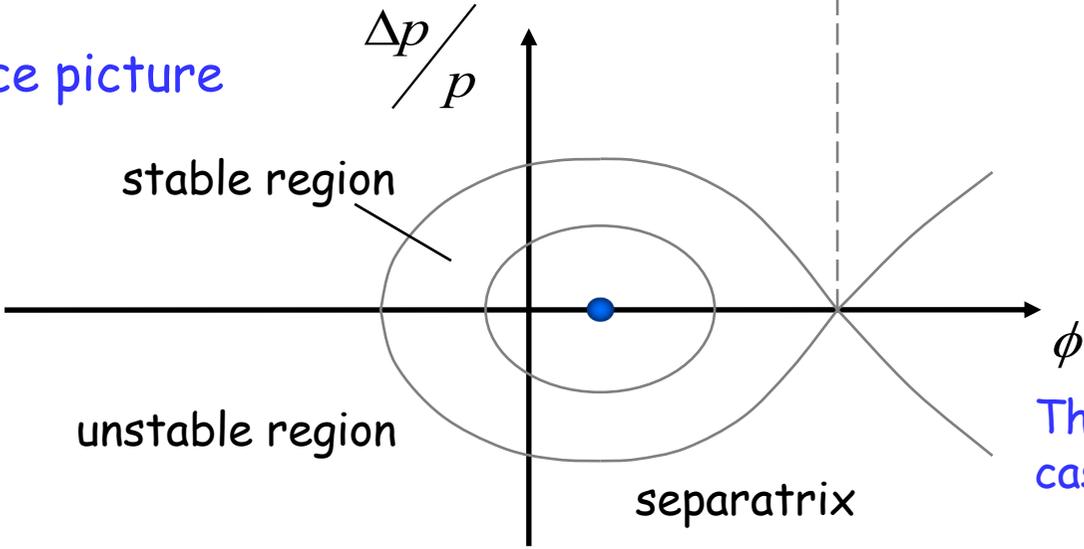
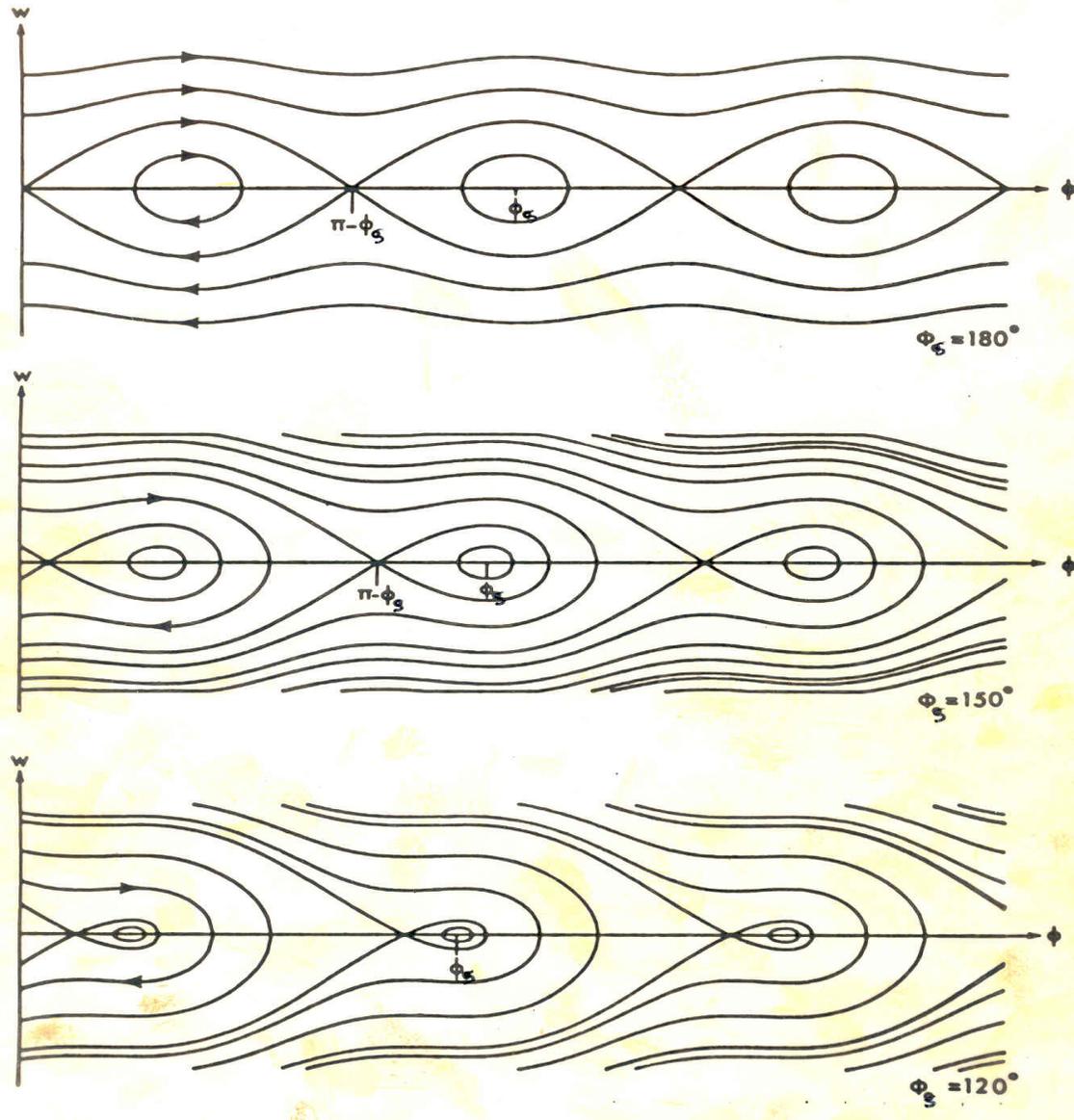Case with acceleration B increasing

$\gamma < \gamma_t$

$\phi = \omega_{RF}\, t$

$\phi_s < \phi < \pi - \phi_s$

Phase space picture

stable region

unstable region

separatrix

The symmetry of the case B = const. is lost

RF Acceptance versus Synchronous Phase

The areas of stable motion (closed trajectories) are called "BUCKET". The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which means no acceleration.

During acceleration, the buckets get smaller, both in length and energy acceptance.

=> Injection preferably without acceleration.

Longitudinal Dynamics in Synchrotrons

Now we will look more quantitatively at the "synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle.

Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient to follow other particles with respect to that particle.

So let's introduce the following **reduced variables**:

revolution frequency : $\Delta f_r = f_r - f_{rs}$

particle RF phase   : $\Delta\phi = \phi - \phi_s$

particle momentum   : $\Delta p = p - p_s$

particle energy   : $\Delta E = E - E_s$

azimuth angle   : $\Delta\theta = \theta - \theta_s$

Look at difference from synchronous particle

Equations of Longitudinal Motion

In these reduced variables, the equations of motion are (see Appendix):

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{R_s p_s}{h\eta\omega_{rs}}\frac{d\phi}{dt}\right] + \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.

We will simplify in the following…

Small Amplitude Oscillations

Let's assume constant parameters R_s, p_s, ω_s and η:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0 \qquad \text{with} \qquad \Omega_s^2 = \frac{h\eta\omega_{rs}e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Consider now small phase deviations from the reference particle:

$$\sin\phi - \sin\phi_s = \sin(\phi_s + \Delta\phi) - \sin\phi_s \cong \cos\phi_s\,\Delta\phi \qquad \text{(for small } \Delta\phi\text{)}$$

and the corresponding linearized motion reduces to a harmonic oscillation:

$$\ddot{\phi} + \Omega_s^2\,\Delta\phi = 0 \qquad \text{where } \Omega_s \text{ is the synchrotron angular frequency.}$$

The **synchrotron tune** ν_s is the number of synchrotron oscillations per revolution:

$$\nu_s = \Omega_s/\omega_s$$

Typical values are <<1, as it takes several 10 - 1000 turns per oscillation.
- proton synchrotrons of the order 10^{-3}
- electron storage rings of the order 10^{-1}

Stability condition for φ_s

$$\Omega_s^2 = \frac{e\hat{V}_{RF}\eta h \omega_s}{2\pi R_s p_s} \cos \phi_s \quad \Leftrightarrow \quad \Omega_s^2 = \omega_s^2 \frac{e\hat{V}_{RF}\eta h}{2\pi \beta^2 E} \cos \phi_s \quad \begin{array}{l} \text{with} \\ Rp = \dfrac{\beta^2 E}{\omega} \end{array}$$

Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\Omega_s^2 > 0$$

$\updownarrow$

$$\boxed{\eta \cos \phi_s > 0}$$

Stable in the region if

cos (φ_s) V_{RF}

$\dfrac{\pi}{2}$ π $\dfrac{3}{2}\pi$ ϕ

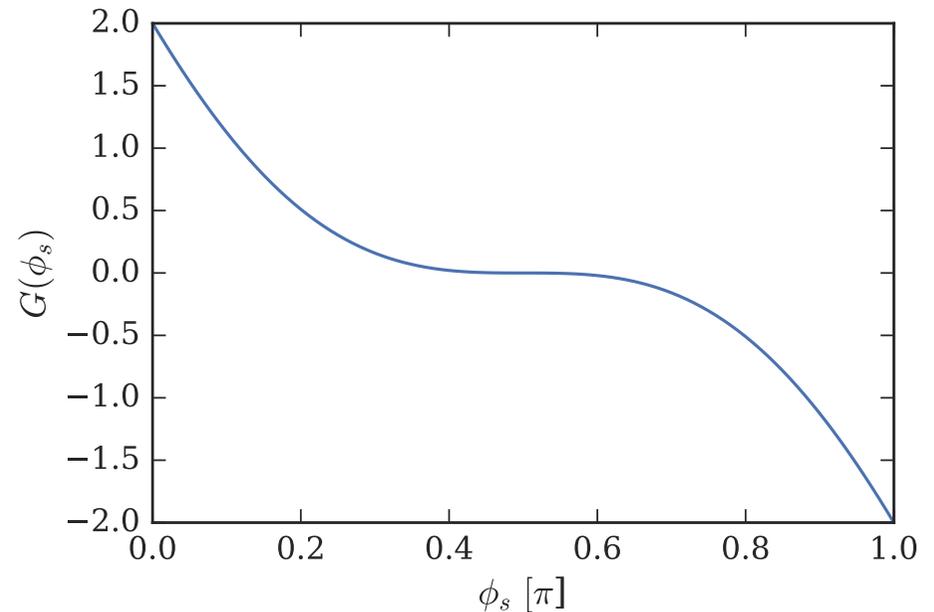| $\gamma < \gamma_{tr}$ | $\gamma > \gamma_{tr}$ | $\gamma > \gamma_{tr}$ | $\gamma < \gamma_{tr}$ |
| $\eta > 0$ | $\eta < 0$ | $\eta < 0$ | $\eta > 0$ |

acceleration   deceleration
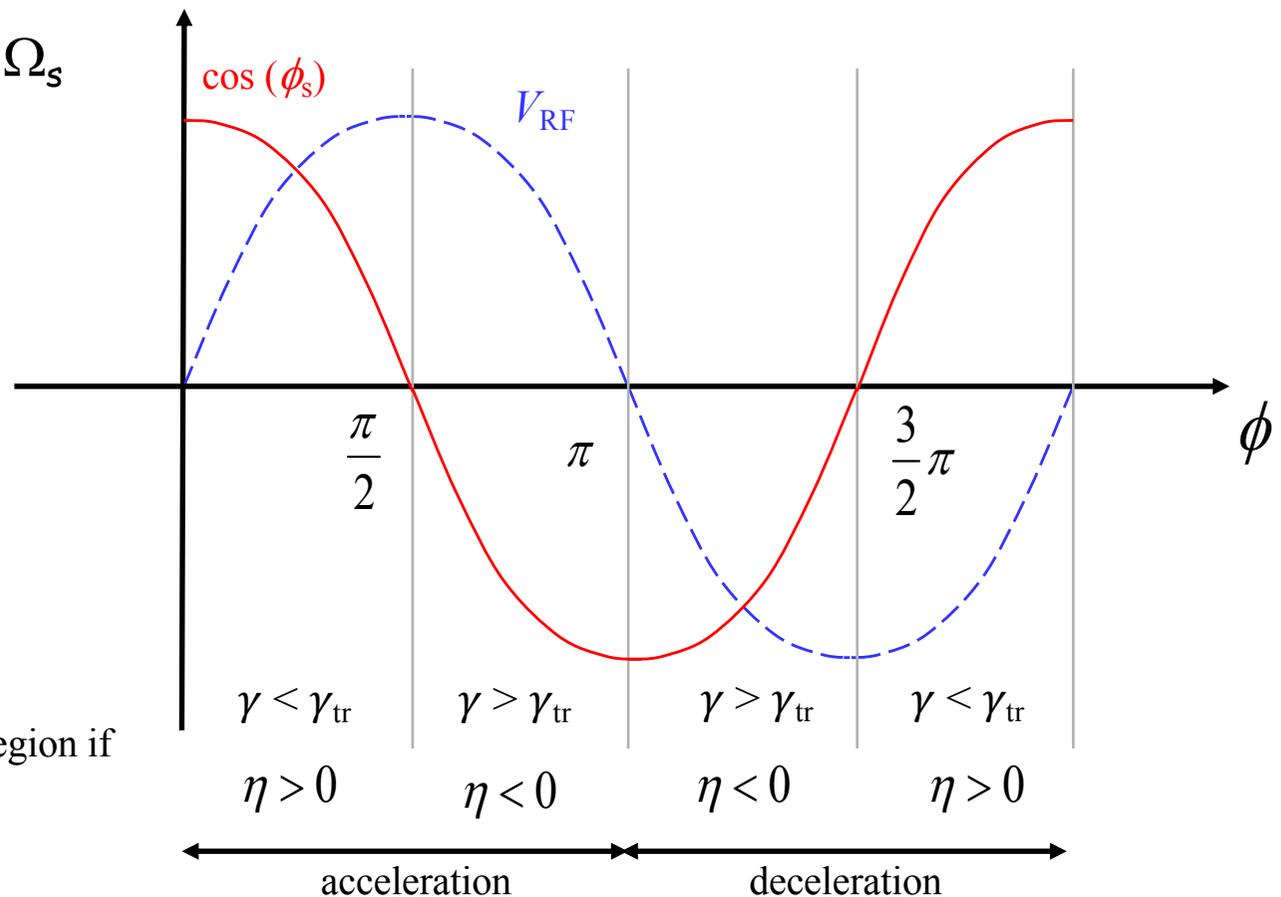
Energy Acceptance

From the equation of the separatrix, we can calculate (see appendix) the acceptance in energy:

$$\left(\frac{\Delta E}{E_s}\right)_{max} = \pm\beta\sqrt{\frac{e\hat{V}}{\pi h \eta E_s}\, G(\phi_s)}$$

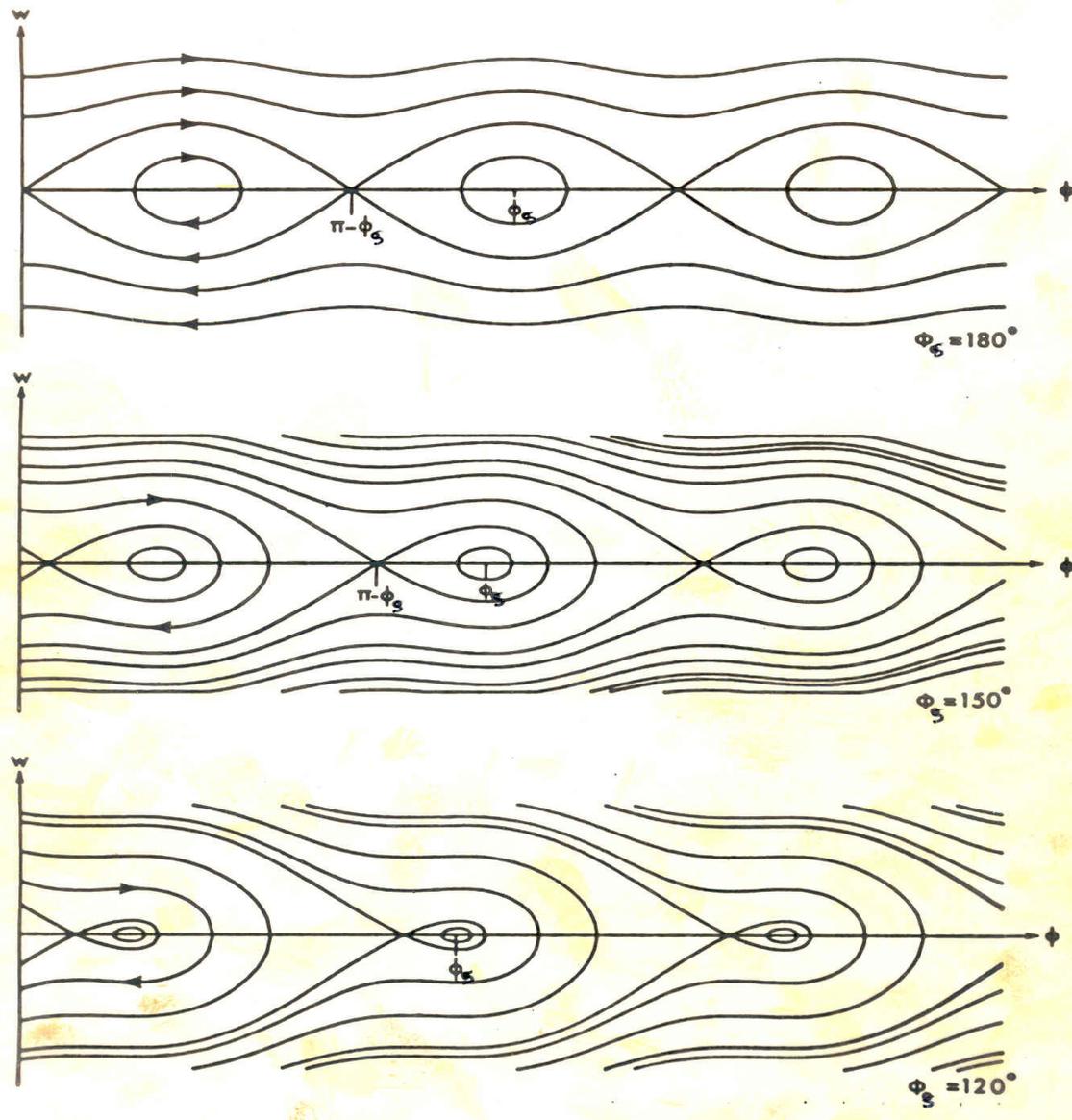$$G(\phi_s) = \left[2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s\right]$$

This "RF acceptance" depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

It's largest for $\phi_s=0$ and $\phi_s=\pi$ (no acceleration, depending on η).

It becomes smaller during acceleration, when ϕ_s is changing

Need a higher RF voltage for higher acceptance => need more $€ ☹

RF Acceptance versus Synchronous Phase

The areas of stable motion (closed trajectories) are called "BUCKET". The number of circulating buckets is equal to "h".

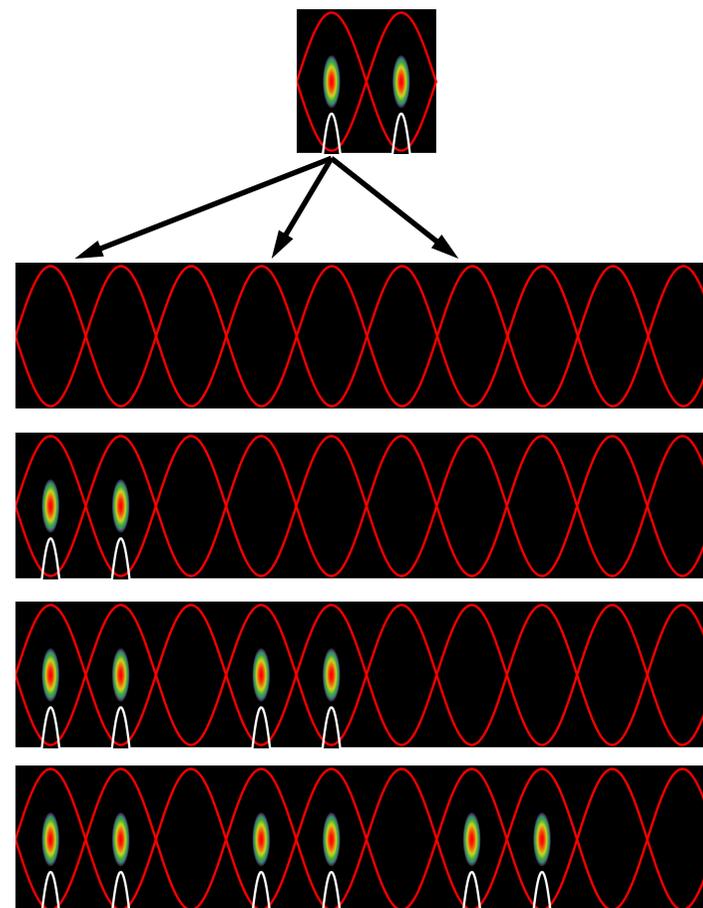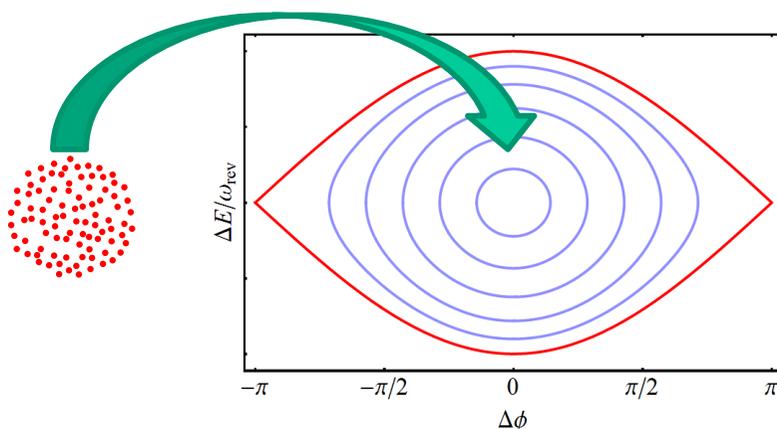The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which means no acceleration.

During acceleration, the buckets get smaller, both in length and energy acceptance.

=> Injection preferably without acceleration.

Injection: Bunch-to-bucket transfer

- Bunch from sending accelerator into the bucket of receiving


Advantages:

→ Particles always subject to longitudinal focusing

→ No need for RF capture of de-bunched beam in receiving accelerator

→ No particles at unstable fixed point

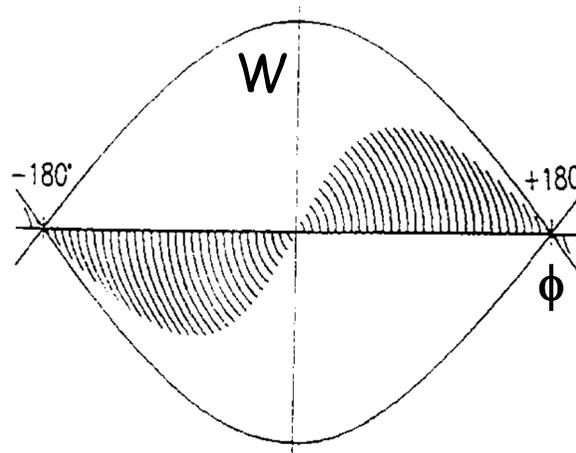→ Time structure of beam preserved during transfer

Effect of a Mismatch

Injected bunch: short length and large energy spread
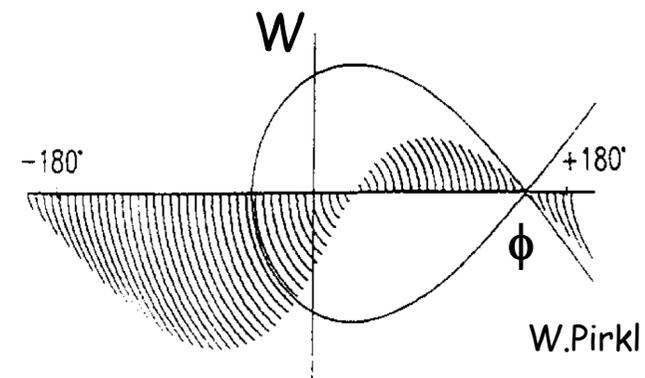after 1/4 synchrotron period: longer bunch with a smaller energy spread.
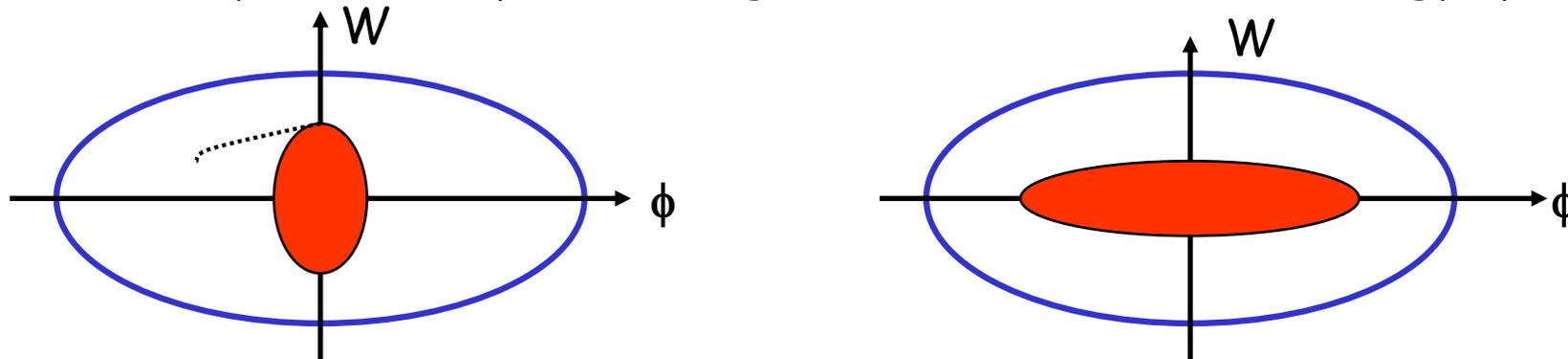
For larger amplitudes, the angular phase space motion is slower
(1/8 period shown below)   => can lead to filamentation and emittance growth
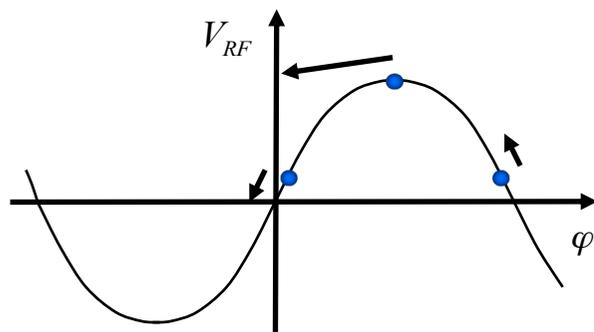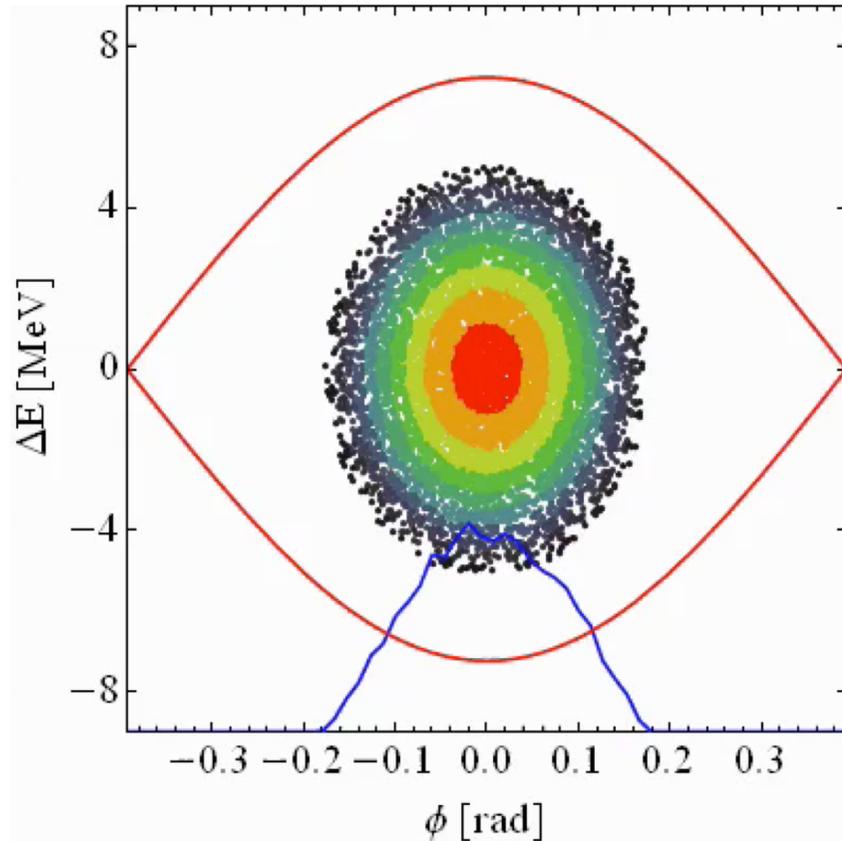

restoring force is
non-linear

stationary bucket

accelerating bucket

W.Pirkl

Effect of a Mismatch (2)
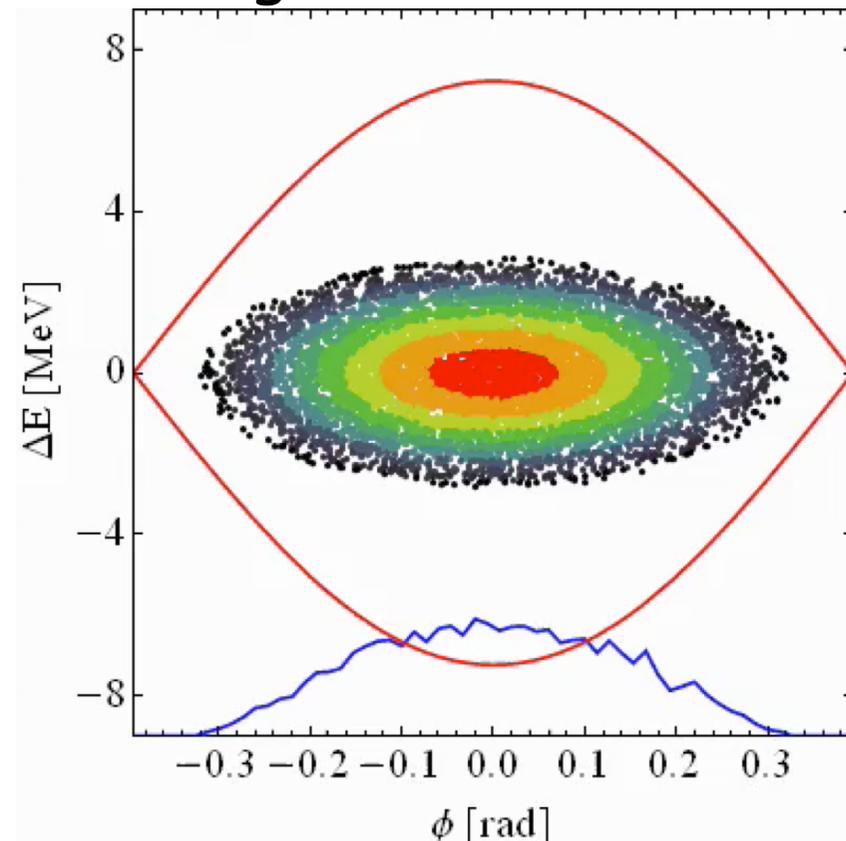
- Long. emittance is only preserved for correct RF voltage

Matched case

→ Bunch is fine, longitudinal emittance remains constant

Longitudinal mismatch

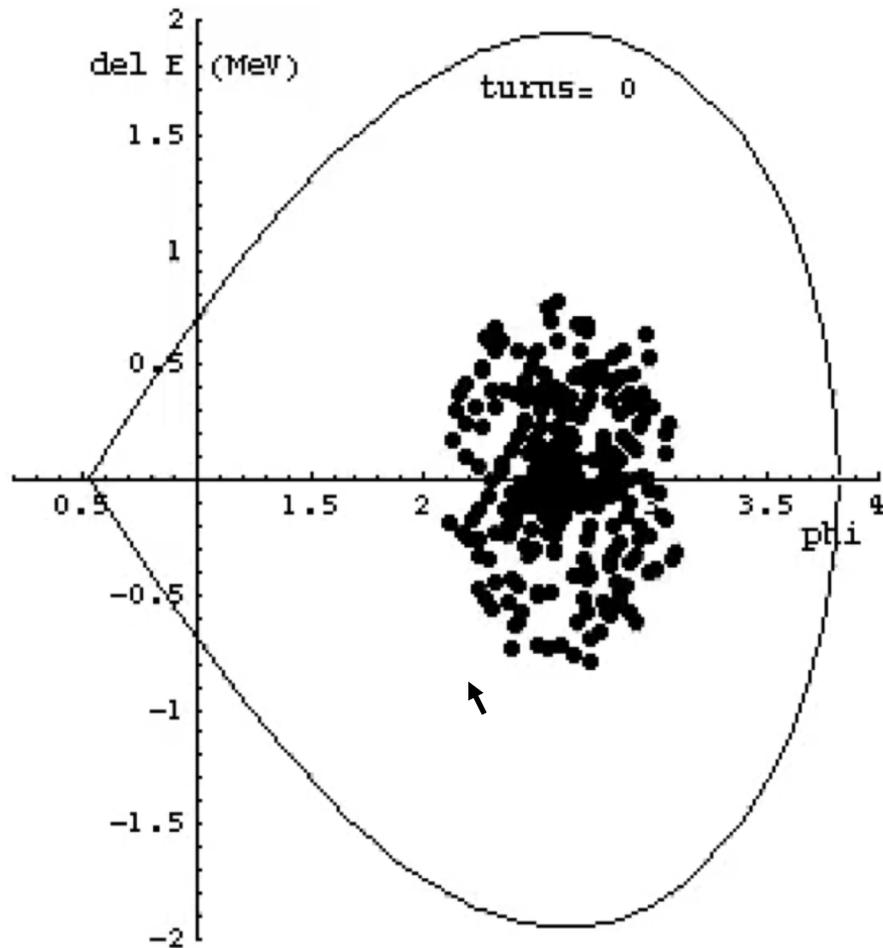→ Dilution of bunch results in increase of long. emittance

Effect of a Mismatch (3)

Evolution of an injected beam for the first 100 turns.

For a mismatched transfer, the emittance increases (right).
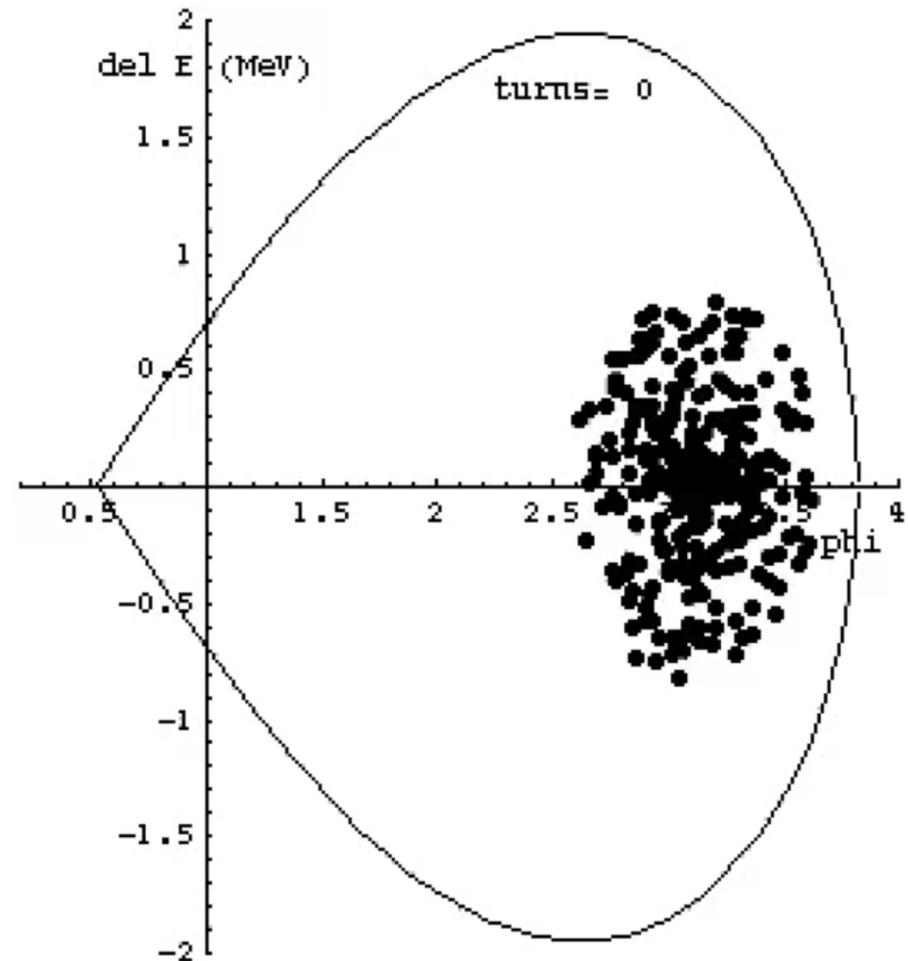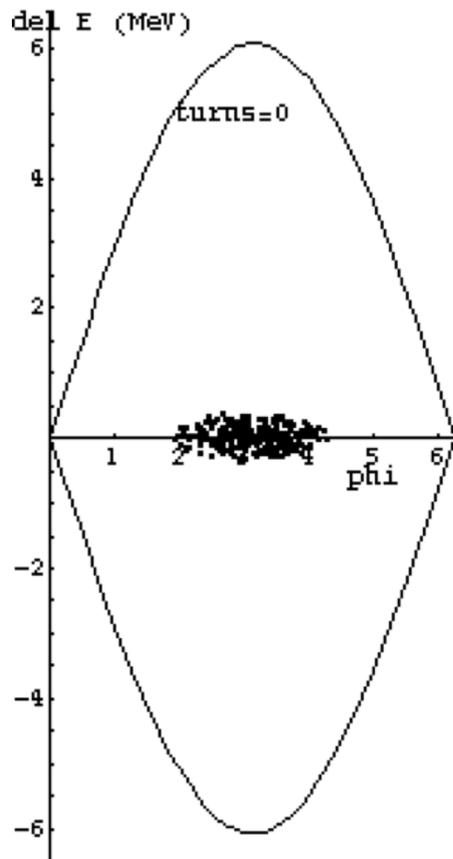

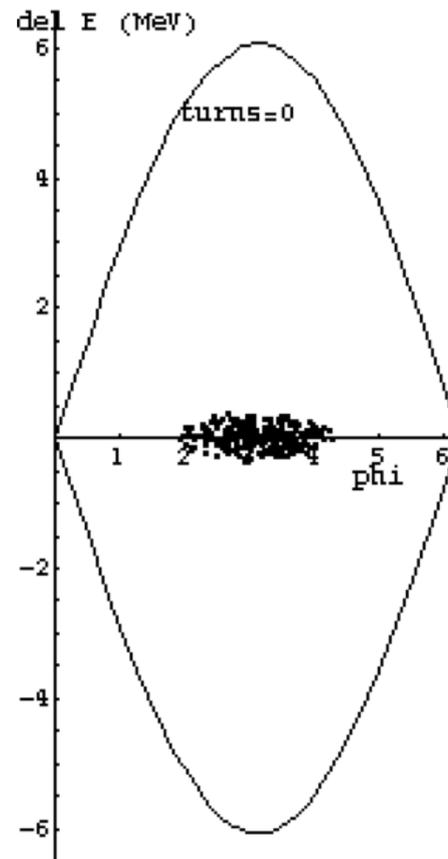matched beam   mismatched beam – phase error

Bunch Rotation

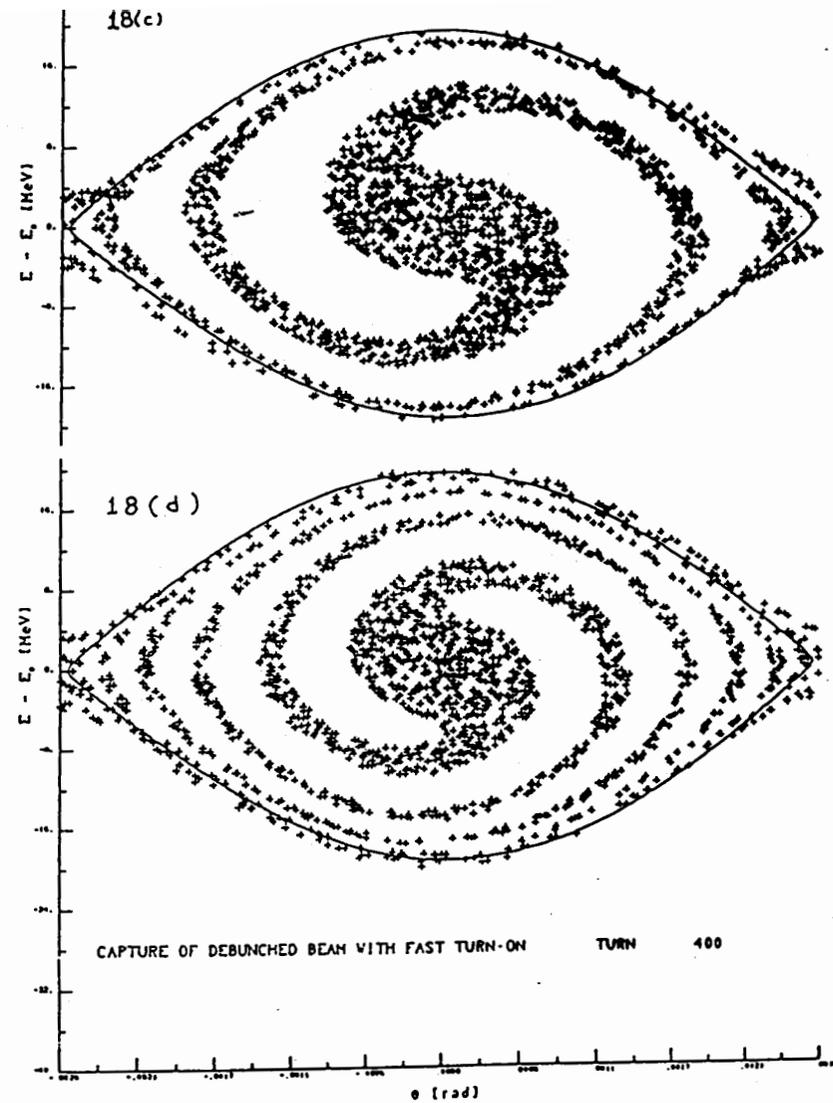Phase space motion can be used to make short bunches.

Start with a long bunch and extract or recapture when it's short.

initial beam

Capture of a Debunched Beam with Fast Turn-On
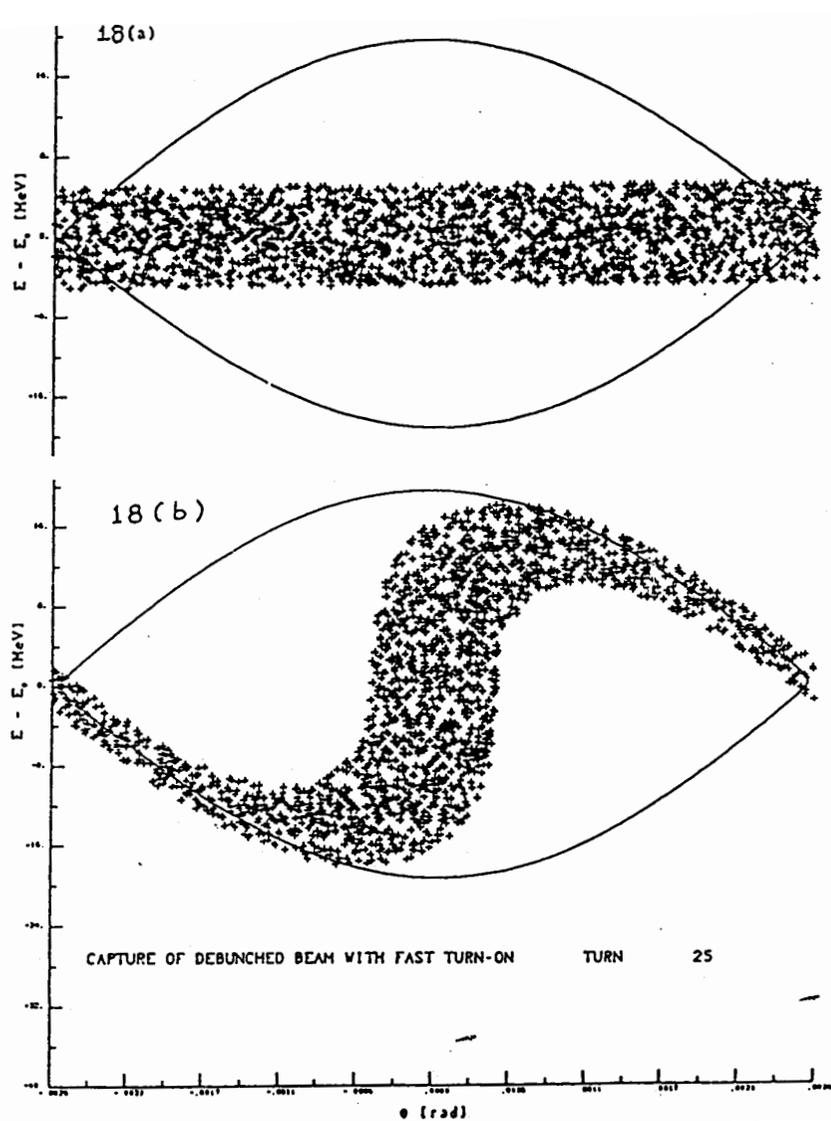
Capture of a Debunched Beam with Adiabatic Turn-On
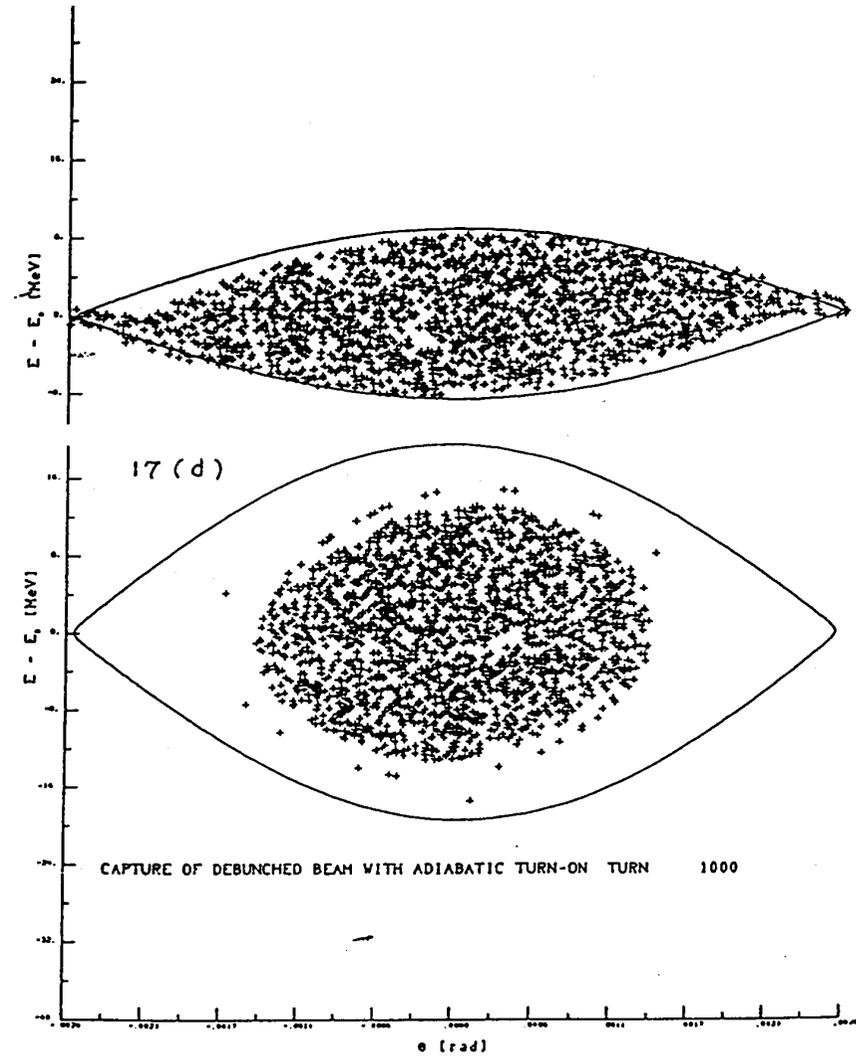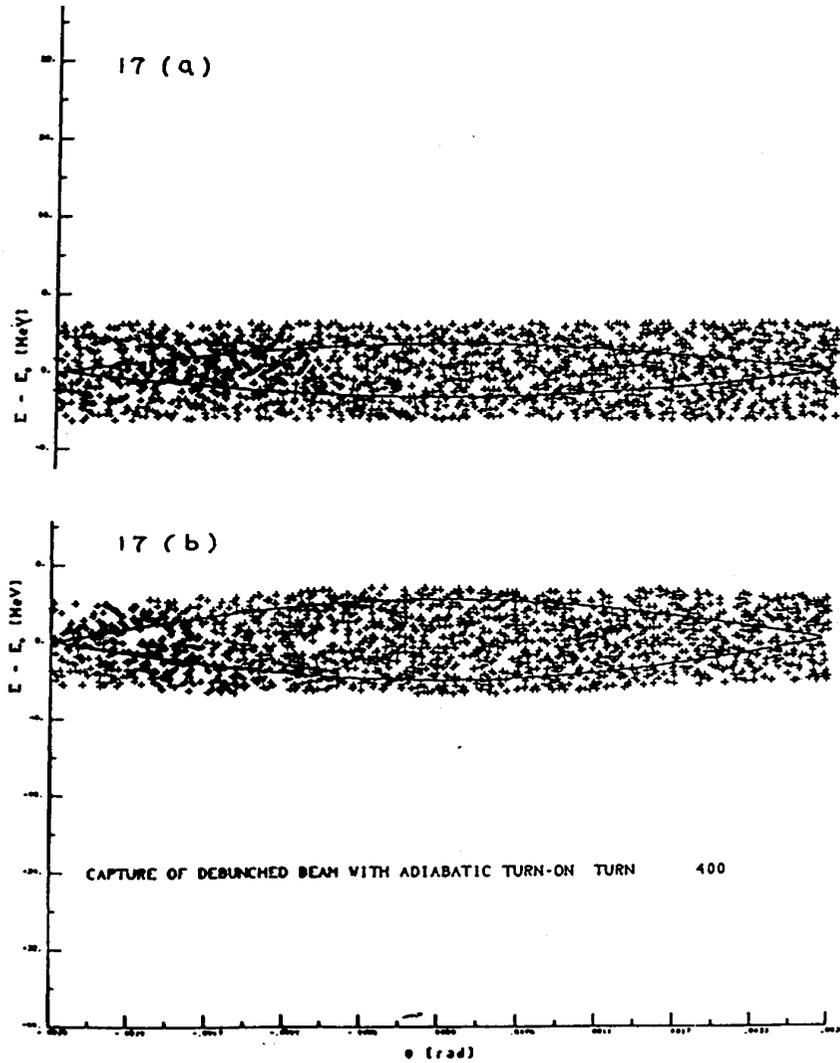
Generating a 25ns Bunch Train in the PS

- **Longitudinal bunch splitting (basic principle)**
  - Reduce voltage on principal RF harmonic and simultaneously rise voltage on multiple harmonics (adiabatically with correct phase, etc.)

PS ejection:
72 bunches
in 1 turn

320 ns beam gap
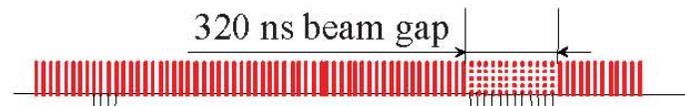
72 bunches on h=84

Quadruple splitting at 25 GeV

40 MHz fixed
RF = 40.0 MHz
1.1 × 10^{11} ppb
&
20 MHz fixed
RF = 20.0 MHz
2.2 × 10^{11} ppb

Acceleration to 25 GeV
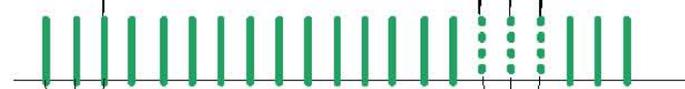
18 bunches on h=21

Triple splitting at 1.4 GeV

10 MHz system
RF = 9.18 MHz
4.4 × 10^{11} ppb

PS injection:
4+2 bunches
in 2 batches
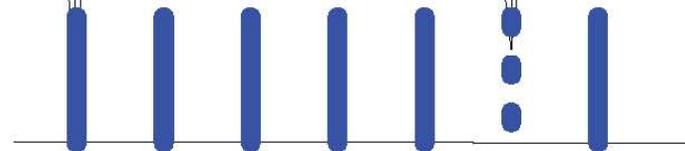
Empty bucket

6 bunches on h=7

10 MHz system
RF = 3.06 MHz
13.2 × 10^{11} ppb

R. Garoby

Use double splitting at 25 GeV to generate 50ns bunch trains instead

Production of the LHC 25 ns beam

1. Inject four bunches **~ 180 ns, 1.3 eVs**
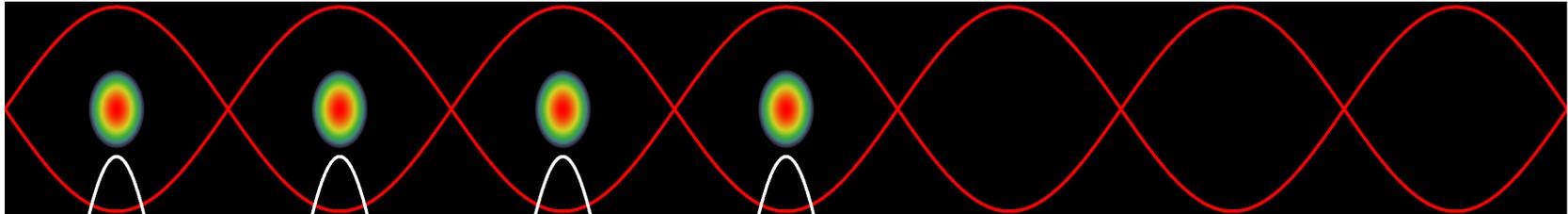
Wait 1.2 s for second injection

2. Inject two bunches
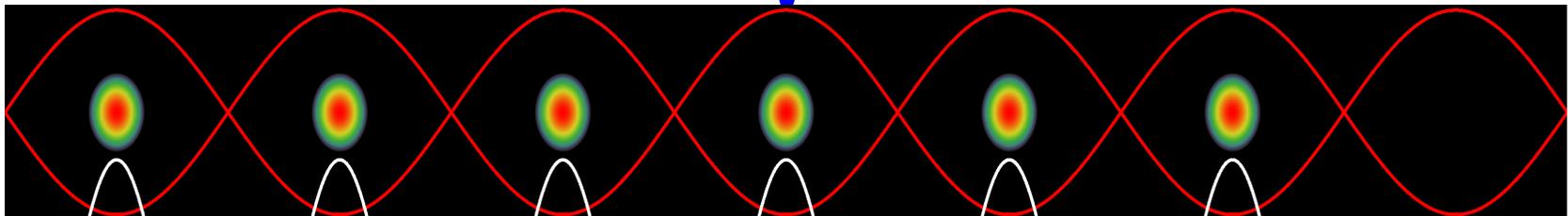
3. Triple split after second injection
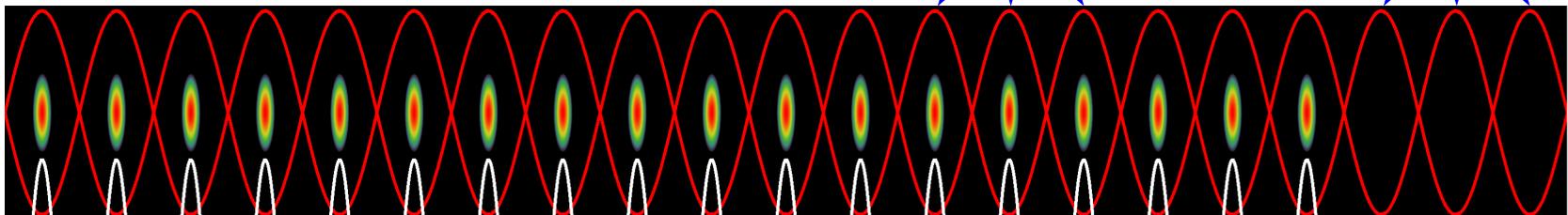
~ 0.7 eVs

4. Accelerate from 1.4 GeV (E_{kin}) to 26 GeV

Production of the LHC 25 ns beam

5. During acceleration: longitudinal emittance blow-up: 0.7 – 1.3 eVs
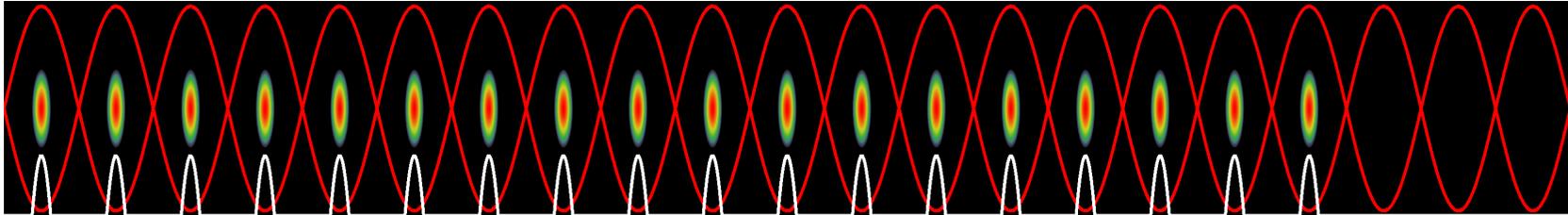
6. Double split (*h*21 → *h*42)
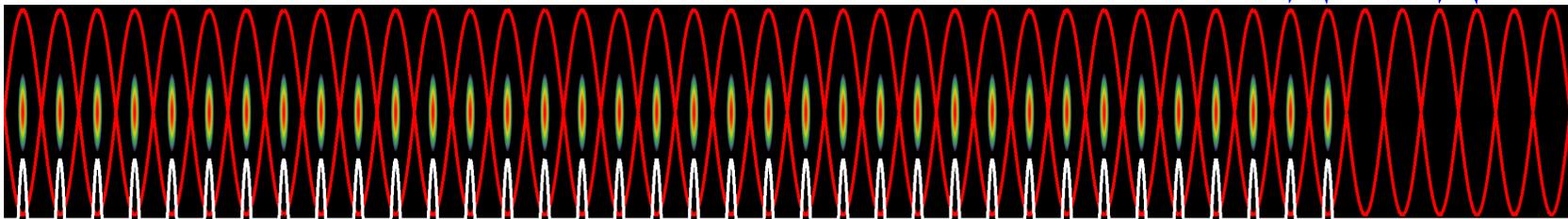
7. Double split (*h*42 → *h*84)   ~ 0.35 eVs, 4 ns
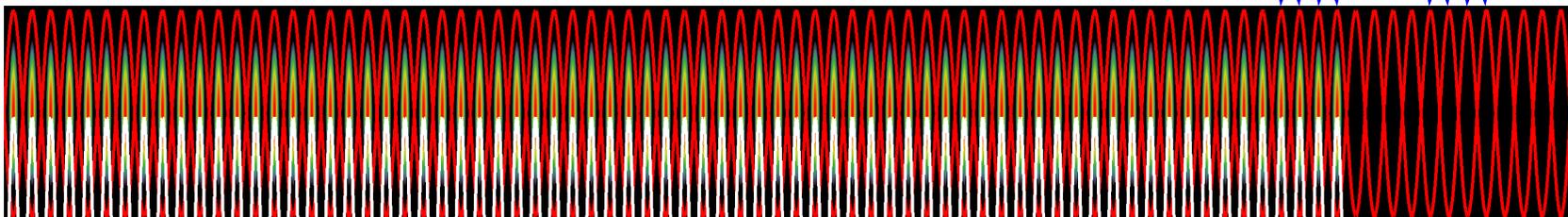
10. Fine synchronization, bunch rotation → Extraction!

The LHC25 (ns) cycle in the PS

Inject 4+2 bunches

$h = 7$

$h = 21$

γ_{tr}

Eject 72 bunches

$h = 84$

B [T]

Intensity [10^{13} ppp] (sketched)

Time [s]

Triple splitting after 2nd injection

Split in four at flat-top energy

2nd injection

Time [ms]

Time [μs]

1.4 GeV

Time [ms]

Time [μs]

26 GeV/c
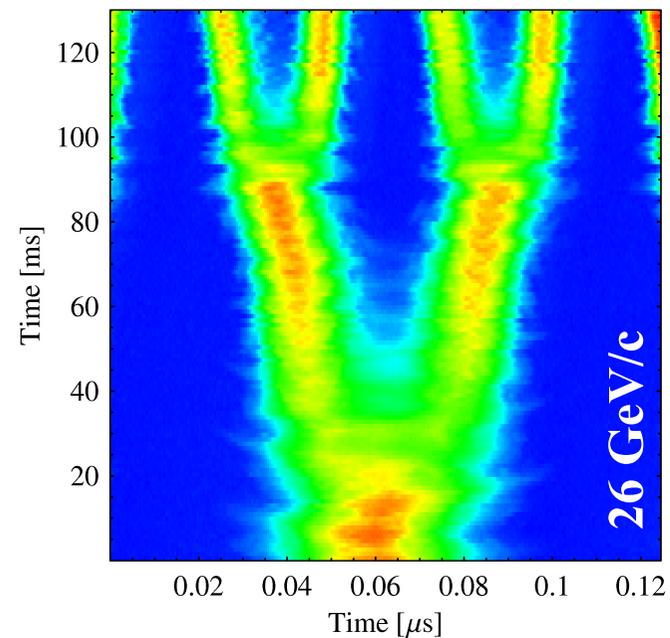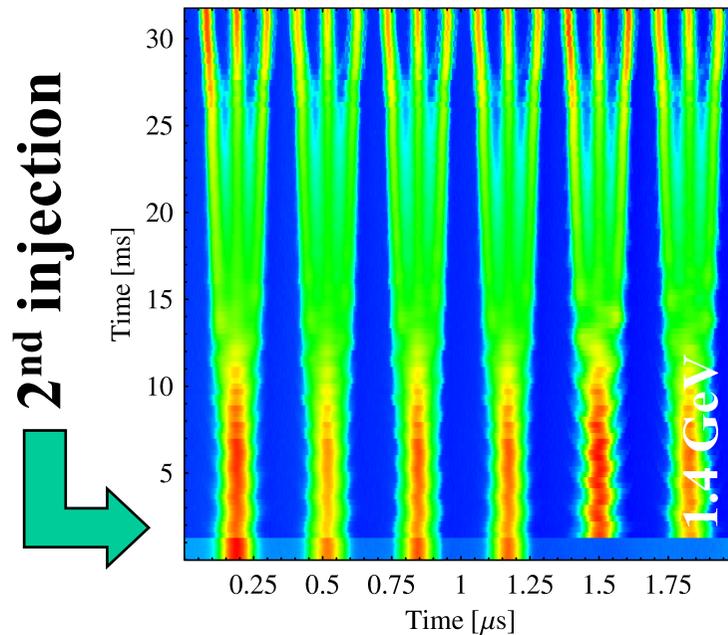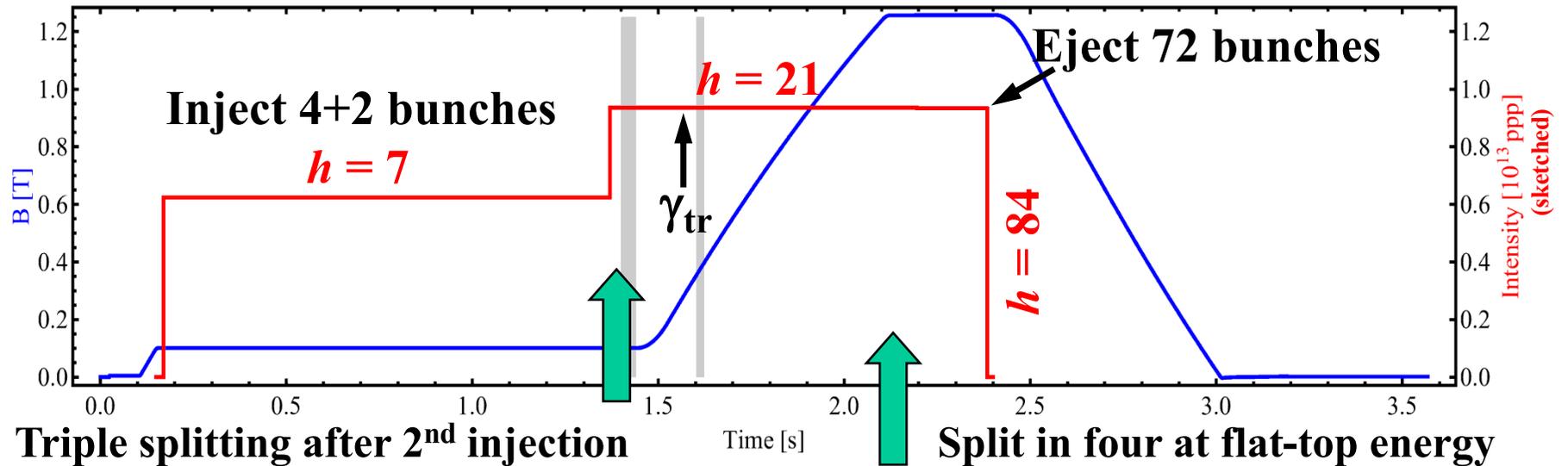
→ Each bunch from the Booster divided by 12 → 6 × 3 × 2 × 2 = 72

Triple splitting in the PS
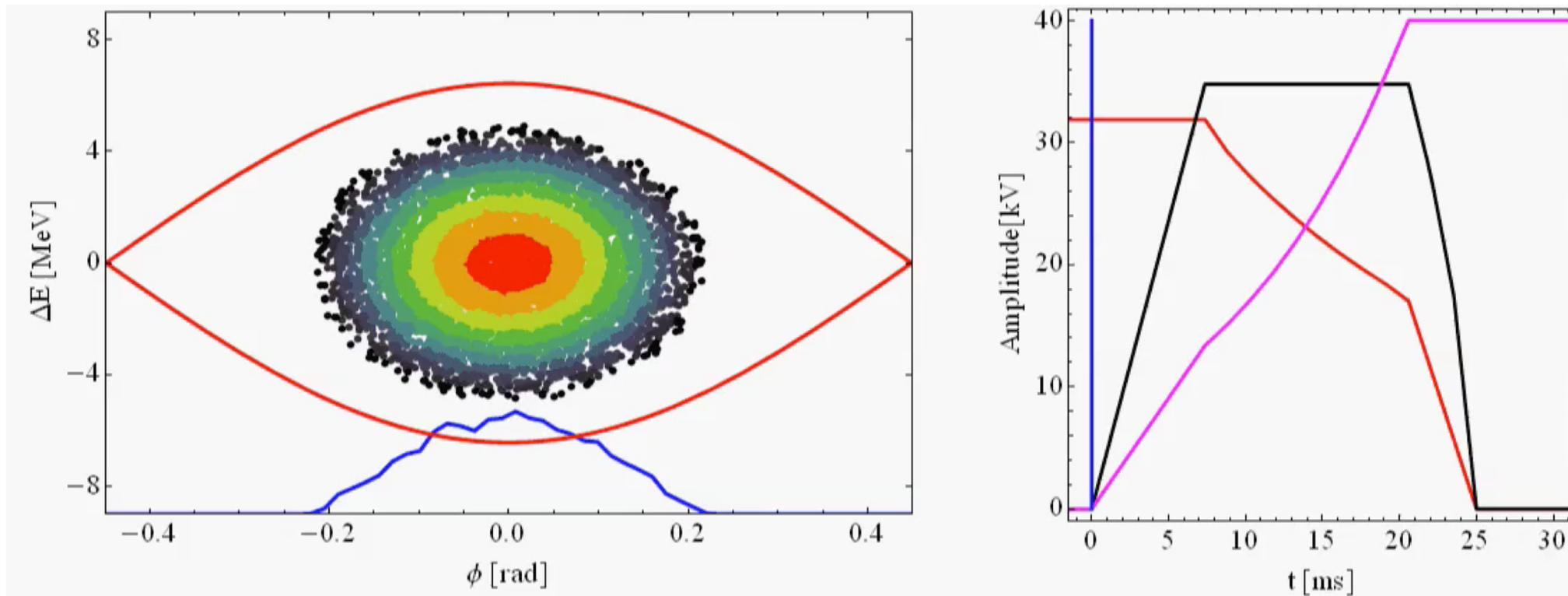
Two times double splitting in the PS

Two times double splitting and bunch rotation:
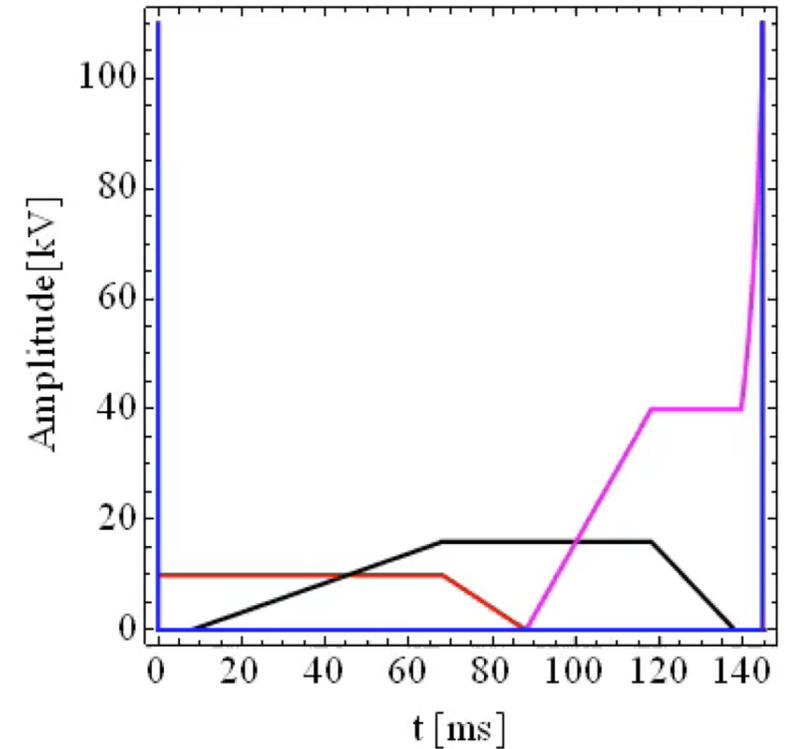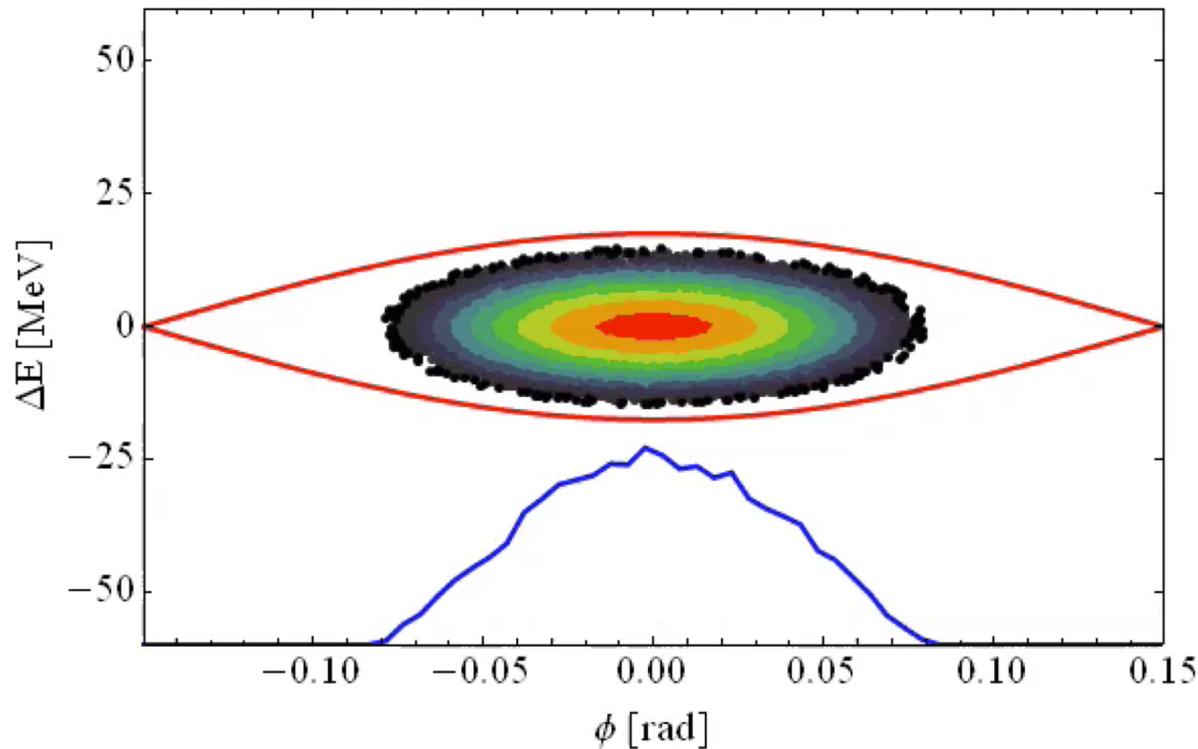
- Bunch is divided twice using RF systems at
  h = 21/42 (10/20 MHz) and h = 42/84 (20/40 MHz)

- Rotation: first part h84 only + h168 (80 MHz) for final part

Summary

- Cyclotrons/Synchrocylotrons for low energy

- Synchrotrons for high energies, constant orbit, rising field and frequency synchronously

- Particles with higher energy have a longer orbit (normally) but a higher velocity
  - at low energies (below transition) velocity increase dominates
  - at high energies (above transition) velocity almost constant

- Particles perform oscillations around synchronous phase
  - synchronous phase depending on acceleration
  - below or above transition

- Bucket is the stable region in phase space inside the separatrix

- Bunch is the area filled with beam

- Matching the shape of the bunch to the bucket is essential

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And CERN Accelerator Schools (CAS) Proceedings
In particular: CERN-2014-009
Advanced Accelerator Physics - CAS

Acknowledgements

I would like to thank everyone for the material that I have used.

In particular (hope I don't forget anyone):
- Joël Le Duff (from whom I inherited the course)
- Rende Steerenberg
- Gerald Dugan
- Heiko Damerau
- Werner Pirkl
- Genevieve Tulloue
- Mike Syphers
- Daniel Schulte
- Roberto Corsini
- Roland Garoby
- Luca Bottura
- Berkeley Lab
- Edukite Learning

Appendix

- Summary Relativity and Energy Gain

- Velocity, Energy, and Momentum

- Momentum compaction factor

- Synchrotron energy-phase oscillations

- Stability condition

- Separatrix stationary bucket

- Large amplitude oscillations

- Bunch matching into stationary bucket

Appendix: Relativity + Energy Gain

Newton-Lorentz Force $$\vec{F} = \frac{d\vec{p}}{dt} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

2nd term always perpendicular to motion => no acceleration

Relativistics Dynamics

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \qquad \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$$

$$p = mv = \frac{E}{c^2}\beta c = \beta\frac{E}{c} = \beta\gamma\, m_0 c$$

$$E^2 = E_0^2 + p^2 c^2 \quad\longrightarrow\quad dE = v\,dp$$

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

$$dE = dW = eE_z\,dz \quad\rightarrow\quad W = e\int E_z\,dz$$

RF Acceleration

$$E_z = \hat{E}_z \sin\omega_{RF}t = \hat{E}_z \sin\phi(t)$$

$$\int \hat{E}_z\,dz = \hat{V}$$

$$W = e\hat{V}\sin\phi$$

(neglecting transit time factor)

The field will change during the passage of the particle through the cavity
=> effective energy gain is lower

Appendix: Velocity, Energy and Momentum

normalized velocity $\quad \beta = \dfrac{v}{c} = \sqrt{1 - \dfrac{1}{\gamma^2}}$

=> electrons almost reach the speed of light very quickly (few MeV range)

total energy
—————————
rest energy

$$E = \gamma m_0 c^2$$

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}}$$

Momentum $\quad p = mv = \dfrac{E}{c^2}\beta c = \beta \dfrac{E}{c} = \beta \gamma m_0 c$

=> Magnetic field needs to follow the momentum increase

normalized velocity

electrons

protons

Beta — E_kinetic (MeV)

total energy
—————————
rest energy

electrons

protons

Gamma — E_kinetic (MeV)
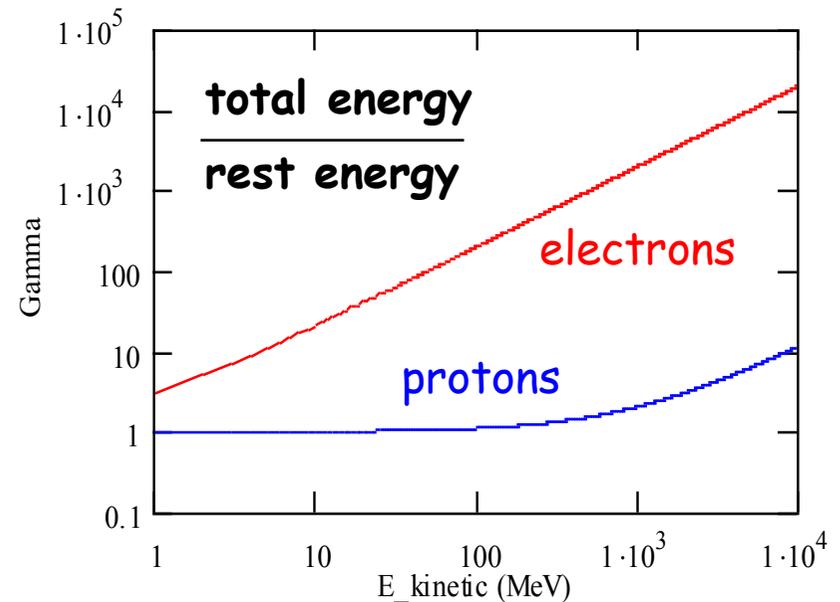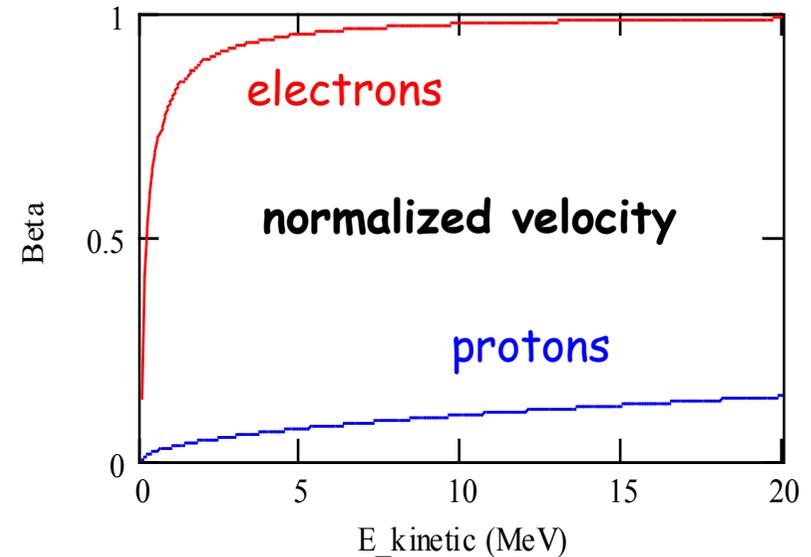
Appendix: Momentum Compaction Factor

$$\alpha_c = \frac{p}{L}\frac{dL}{dp}$$

$$ds_0 = \rho\, d\theta$$

$$ds = (\rho + x)\, d\theta$$
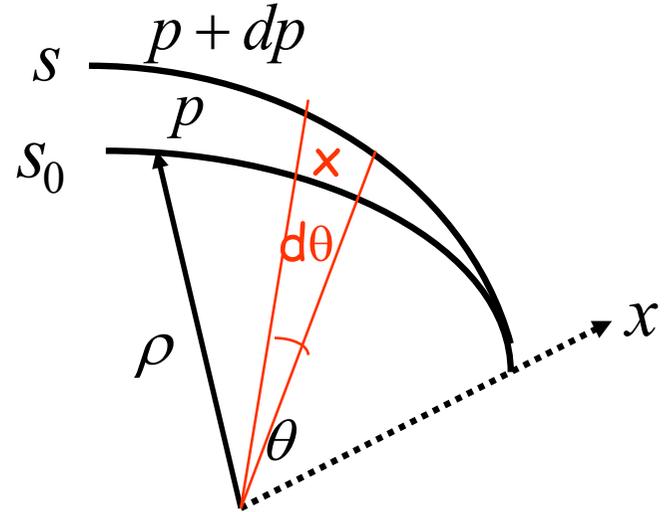
The elementary path difference from the two orbits is:

definition of dispersion D_x

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D_x}{\rho}\frac{dp}{p}$$

leading to the total change in the circumference:

$$dL = \int_C dl = \int \frac{x}{\rho}\, ds_0 = \int \frac{D_x}{\rho}\frac{dp}{p}\, ds_0$$

$$\alpha_c = \frac{1}{L}\int_C \frac{D_x(s)}{\rho(s)}\, ds_0$$

With $\rho=\infty$ in straight sections we get:

$$\alpha_c = \frac{\langle D_x \rangle_m}{R}$$

< >$_m$ means that the average is considered over the bending magnet only

Appendix: First Energy-Phase Equation
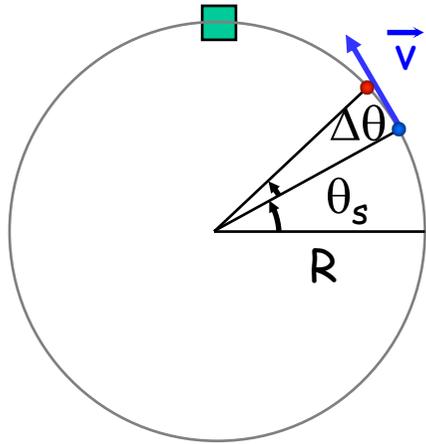
$$f_{RF} = h f_r \implies \Delta\phi = -h\,\Delta\theta \quad with \quad \theta = \int \omega\ dt$$

particle ahead arrives earlier
=> smaller RF phase

For a given particle with respect to the reference one:

$$\Delta\omega = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $\quad \eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega}{dp}\right)_s \quad$ and

$$E^2 = E_0^2 + p^2 c^2$$

$$\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$$

one gets: $\quad \dfrac{\Delta E}{\omega_{rs}} = -\dfrac{p_s R_s}{h\,\eta\,\omega_{rs}}\dfrac{d(\Delta\phi)}{dt} = -\dfrac{p_s R_s}{h\,\eta\,\omega_{rs}}\dot{\phi}$

Appendix: Second Energy-Phase Equation

The rate of energy gained by a particle is:

$$\frac{dE}{dt} = e\hat{V}\sin\phi\,\frac{\omega_r}{2\pi}$$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left-hand side to first order:

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}\Delta T_r + T_{rs}\Delta\dot{E} = \Delta E\,\dot{T}_r + T_{rs}\Delta\dot{E} = \frac{d}{dt}\left(T_{rs}\Delta E\right)$$

leads to the second energy-phase equation:

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}\left(\sin\phi - \sin\phi_s\right)$$

Appendix: Stability condition for ϕ_s

Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\Omega_s^2 = \frac{e\,\hat{V}_{RF}\,\eta\,h\,\omega_s}{2\pi\,R_s\,p_s}\cos\phi_s \quad \Rightarrow \quad \Omega_s^2 > 0 \quad \Leftrightarrow \quad \boxed{\eta\cos\phi_s > 0}$$

Stable in the region if

$\gamma < \gamma_{tr}$	$\gamma > \gamma_{tr}$	$\gamma > \gamma_{tr}$	$\gamma < \gamma_{tr}$
$\eta > 0$	$\eta < 0$	$\eta < 0$	$\eta > 0$

acceleration   deceleration
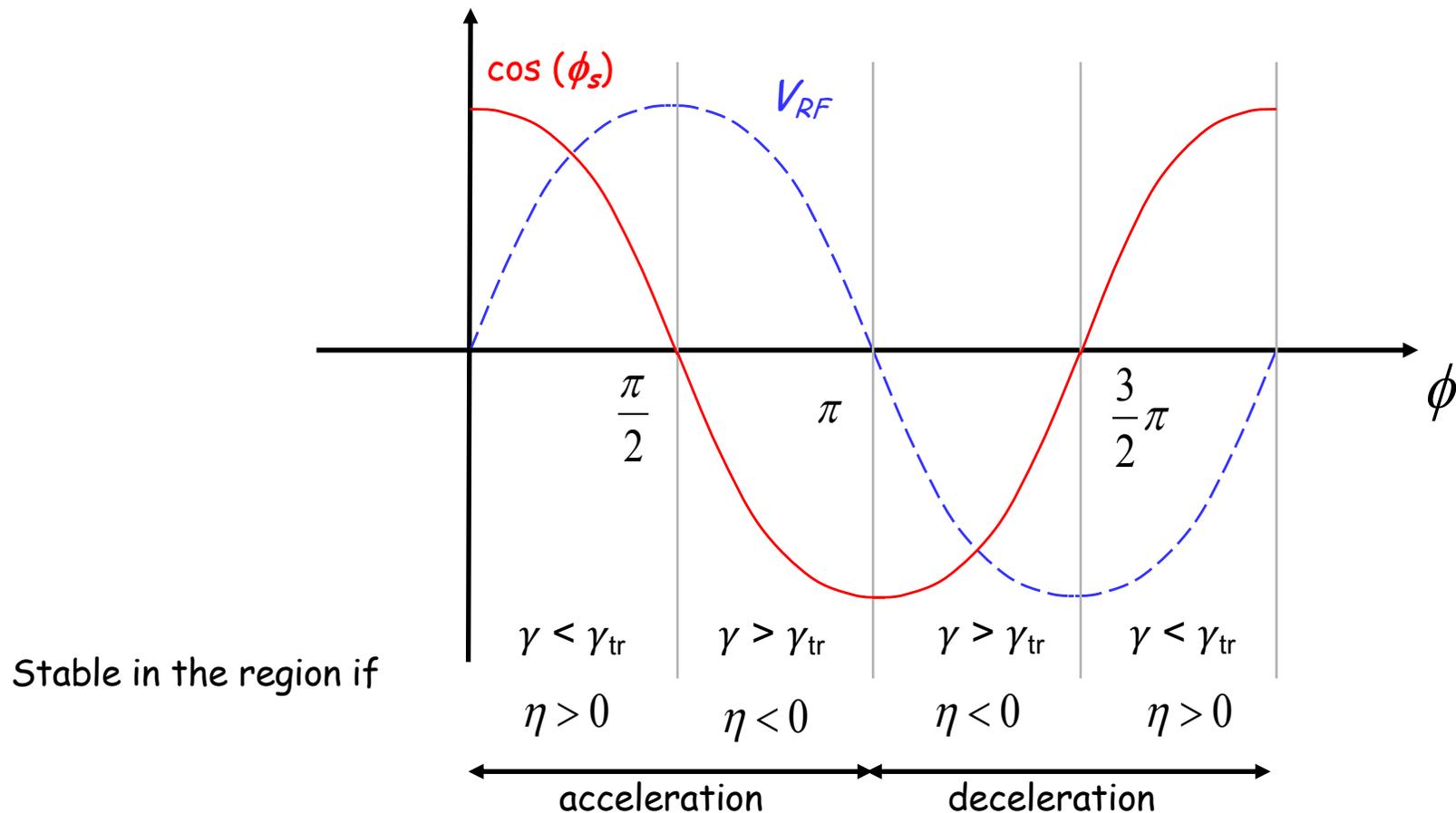
Appendix: Stationary Bucket - Separatrix

This is the case sinϕ_s=0 (no acceleration) which means ϕ_s=0 or π . The equation of the separatrix for ϕ_s= π (above transition) becomes:

$$\frac{\dot{\phi}^2}{2}+\Omega_s^2\cos\phi=\Omega_s^2$$

$$\frac{\dot{\phi}^2}{2}=2\Omega_s^2\sin^2\frac{\phi}{2}$$

Replacing the phase derivative by the (canonical) variable W:

W_{bk}

$$W=\frac{\Delta E}{\omega_{rf}}=-\frac{p_s R_s}{h\eta\omega_{rf}}\dot{\varphi}$$
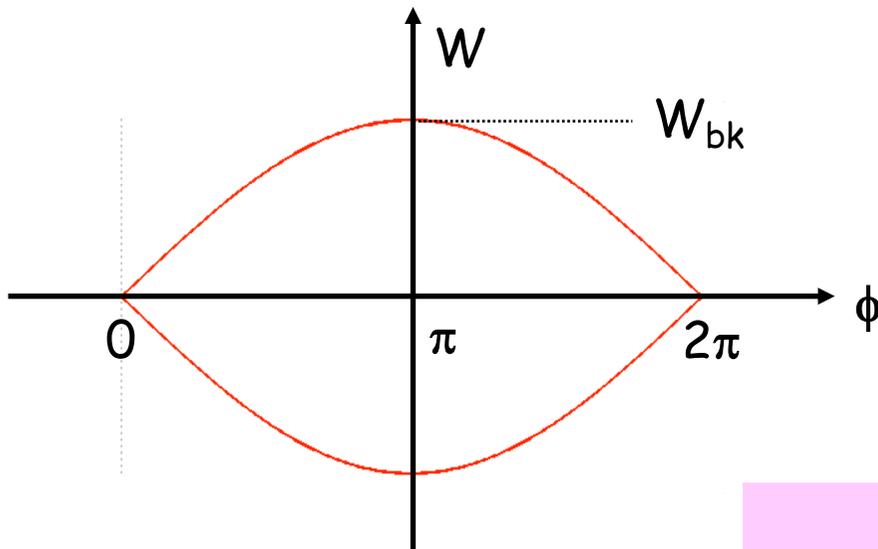
and introducing the expression for Ω_s leads to the following equation for the separatrix:

$$W=\pm\frac{C}{\pi hc}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}\sin\frac{\phi}{2} \ =\pm W_{bk}\sin\frac{\phi}{2}$$

with C=2πR$_s$

Stationary Bucket (2)

Setting $\phi=\pi$ in the previous equation gives the height of the bucket:

$$W_{bk} = \frac{C}{\pi h c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}$$

This results in the maximum energy acceptance:

$$\Delta E_{\max} = \omega_{rf}W_{bk} = \beta_s\sqrt{2\frac{-e\hat{V}_{RF}E_s}{\pi\eta h}}$$

The area of the bucket is: $\quad A_{bk}=2\int_0^{2\pi}W\,d\phi$

Since: $\quad \int_0^{2\pi}\sin\frac{\phi}{2}\,d\phi=4$

one gets: $\quad A_{bk}=8W_{bk}=8\frac{C}{\pi h c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}} \quad\longrightarrow\quad W_{bk}=\frac{A_{bk}}{8}$

Appendix: Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0 \qquad (\Omega_s \text{ as previously defined})$$

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$
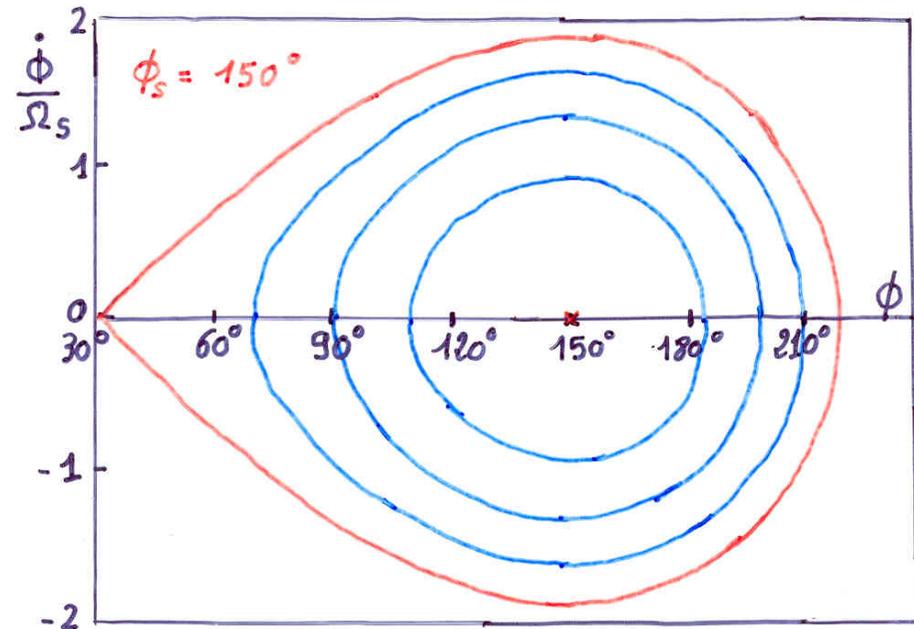
which for small amplitudes reduces to:

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2 \frac{(\Delta\phi)^2}{2} = I' \qquad (\text{the variable is } \Delta\phi, \text{ and } \phi_s \text{ is constant})$$

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches $\pi-\phi_s$ the force goes to zero and beyond it becomes non restoring.

Hence $\pi-\phi_s$ is an extreme amplitude for a stable motion which in the phase space($\dfrac{\dot\phi}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.

Equation of the separatrix:

$$\frac{\dot\phi^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s\right)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$

Energy Acceptance

From the equation of motion it is seen that $\dot\phi$ reaches an extreme when $\ddot\phi = 0$, hence corresponding to $\phi = \phi_s$.

Introducing this value into the equation of the separatrix gives:

$$\dot\phi_{\max}^2 = 2\Omega_s^2 \left\{ 2 + \left(2\phi_s - \pi \right) \tan\phi_s \right\}$$

That translates into an acceptance in energy:

$$\left(\frac{\Delta E}{E_s} \right)_{\max} = \pm\beta \sqrt{ \frac{e\hat{V}}{\pi h \eta E_s} G(\phi_s) }$$

$$G(\phi_s) = \left[2\cos\phi_s + \left(2\phi_s - \pi \right) \sin\phi_s \right]$$

This "RF acceptance" depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.
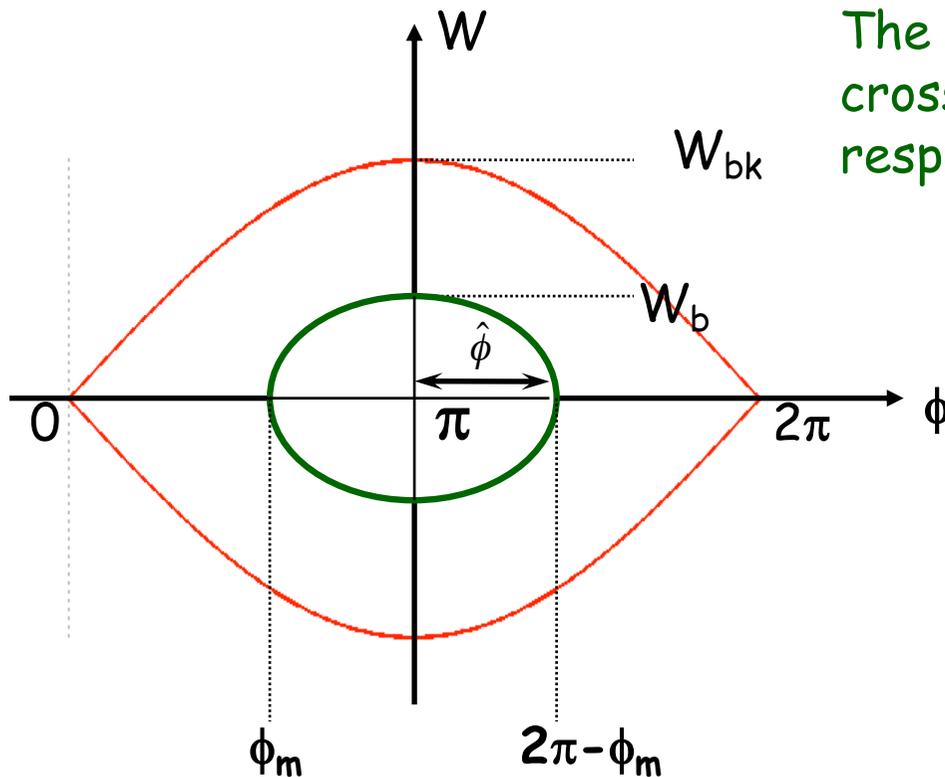
It's largest for $\phi_s=0$ and $\phi_s=\pi$ (no acceleration, depending on η).

Need a higher RF voltage for higher acceptance.

Bunch Matching into a Stationary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I \qquad \xrightarrow{\phi_s = \pi} \qquad \boxed{\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = I}$$

The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = \Omega_s^2\cos\phi_m$$

$$\dot{\phi} = \pm\Omega_s\sqrt{2\left(\cos\phi_m - \cos\phi\right)}$$

$$W = \pm W_{bk}\sqrt{\cos^2\frac{\varphi_m}{2} - \cos^2\frac{\varphi}{2}}$$

$$\cos(\phi) = 2\cos^2\frac{\phi}{2} - 1$$

Bunch Matching into a Stationary Bucket (2)

Setting $\phi = \pi$ in the previous formula allows to calculate the bunch height:

$$W_b = W_{bk} \cos\frac{\phi_m}{2} = W_{bk} \sin\frac{\hat{\phi}}{2}$$

or:

$$W_b = \frac{A_{bk}}{8} \cos\frac{\phi_m}{2}$$

$$\longrightarrow \quad \left(\frac{\Delta E}{E_s}\right)_b = \left(\frac{\Delta E}{E_s}\right)_{RF} \cos\frac{\phi_m}{2} = \left(\frac{\Delta E}{E_s}\right)_{RF} \sin\frac{\hat{\phi}}{2}$$

This formula shows that for a given bunch energy spread the proper matching of a shorter bunch (ϕ_m close to π, $\hat{\phi}$ small) will require a bigger RF acceptance, hence a higher voltage

For small oscillation amplitudes the equation of the ellipse reduces to:

$$W = \frac{A_{bk}}{16}\sqrt{\hat{\phi}^2 - (\Delta\phi)^2} \quad \longrightarrow \quad \left(\frac{16W}{A_{bk}\hat{\phi}}\right)^2 + \left(\frac{\Delta\phi}{\hat{\phi}}\right)^2 = 1$$

Ellipse area is called longitudinal emittance

$$A_b = \frac{\pi}{16} A_{bk} \hat{\phi}^2$$