# Introduction to the Electromagnetic Theory

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Part 1.

# Introduction:

# Maxwell's Equations

# Motivation: control of charged particle beams

To control a charged particle beam we use electromagnetic fields. Recall the Lorentz force:

$$\boxed{\vec{F} = q \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right)}$$

where, in high energy machines, $|\vec{v}| \approx c \approx 3 \cdot 10^8$ m/s. In particle accelerators, transverse deflection is usually given by magnetic fields, whereas acceleration can only be given by electric fields.

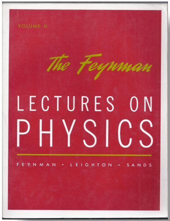
Comparison of electric and magnetic force:

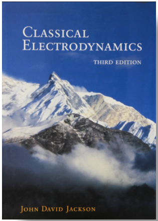
$$\begin{aligned} \left|\vec{E}\right| &= 1 \quad \text{MV/m} \\ \left|\vec{B}\right| &= 1 \quad \text{T} \\ \frac{F_{\text{magnetic}}}{F_{\text{electric}}} &= \frac{evB}{eE} = \frac{\beta c B}{E} \simeq \beta \frac{3 \cdot 10^8}{10^6} = 300\,\beta \end{aligned}$$

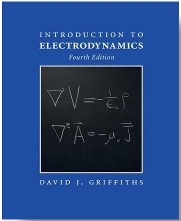
$\Rightarrow$ the magnetic force is much stronger then the electric one: in an accelerator, use magnetic fields whenever possible.

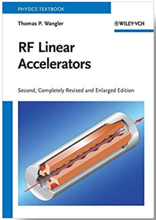
# Some references

1. Richard P. Feynman, Lectures on Physics, 1963, on-line
2. J. D. Jackson, Classical Electrodynamics, Wiley, 1998
3. David J. Griffiths, Introduction to Electrodynamics, Cambridge University Press, 2017
4. Thomas P. Wangler, RF Linear Accelerators, Wiley, 2008



# Variables and units

| | | |
|---|---|---|
| $\mathbf{E}$ | | electric field [V/m] |
| $\mathbf{B}$ | | magnetic field [T] |
| $\mathbf{D}$ | | electric displacement [$C/m^2$] |
| $\mathbf{H}$ | | magnetizing field [A/m] |
| $q$ | | electric charge [C] |
| $\rho$ | | electric charge density [$C/m^3$] |
| $\mathbf{j}$ | $= \rho \mathbf{v}$ | current density [$A/m^2$] |
| $\epsilon_0$ | | permittivity of vacuum, $8.854 \cdot 10^{-12}$ [F/m] |
| $\mu_0$ | $= \frac{1}{\epsilon_0 c^2}$ | permeability of vacuum, $4\pi \cdot 10^{-7}$ [H/m or $N/A^2$] |
| $c$ | | speed of light in vacuum, $2.99792458 \cdot 10^8$ [m/s] |

# Differentiation with vectors

- We define the operator "nabla":

$$\nabla \stackrel{\text{def}}{=} \left( \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z} \right)$$

  which we treat as a special vector.

- Examples:

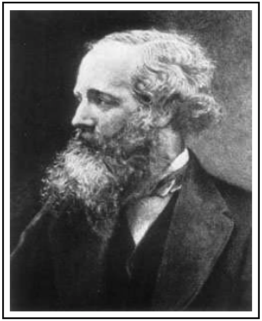
$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \qquad \text{divergence}$$

$$\nabla \times \mathbf{F} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \quad \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \quad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \qquad \text{curl}$$

$$\nabla \phi = \left( \frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial z} \right) \qquad \text{gradient}$$

# Maxwell's equations: integral form

1. Maxwell's equations can be written in **integral** or in differential form (SI units convention):



$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\int_A \vec{B} \cdot d\vec{A} = 0$$

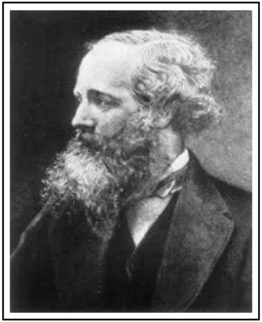
$$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt}\right) \cdot d\vec{A}$$

$$\oint_C \vec{B} \cdot d\vec{r} = \int_A \left(\mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}\right) \cdot d\vec{A}$$

*(1) Gauss' law;*
*(2) Gauss' law for magnetism;*
*(3) Maxwell–Faraday equation (Faraday's law of induction);*
*(4) Ampère's circuital law*

# Maxwell's equations: differential form

1. Maxwell's equations can be written in integral or in **differential** form (SI units convention):



$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} \; + \; \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

*(1) Gauss' law;*
*(2) Gauss' law for magnetism;*
*(3) Maxwell–Faraday equation (Faraday's law of induction);*
*(4) Ampère's circuital law*

Part 2.

# Electromagnetism:

# Static case

# Static case

- We will consider relatively simple situations.
- The easiest circumstance is one in which nothing depends on the time—this is called the static case:
  - All charges are permanently fixed in space, or if they do move, they move as a steady flow in a circuit (so $\rho$ and $\mathbf{j}$ are constant in time).
- In these circumstances, all of the terms in the Maxwell equations which are time derivatives of the field are zero. In this case, the Maxwell equations become:

***Electrostatics:***

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0},$$

$$\nabla \times \boldsymbol{E} = 0.$$

***Magnetostatics:***

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}}{\epsilon_0 c^2},$$

$$\nabla \cdot \boldsymbol{B} = 0.$$

# Electrostatics: principle of superposition

- Coulomb's Law: Electric field due to a stationary point charge $q$, located in $\mathbf{r}_1$:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}_1}{\left|\mathbf{r}-\mathbf{r}_1\right|^3}$$

- Principle of superposition, tells that a distribution of charges $q_i$ generates an electric field:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum q_i \frac{\mathbf{r}-\mathbf{r}_i}{\left|\mathbf{r}-\mathbf{r}_i\right|^3}$$

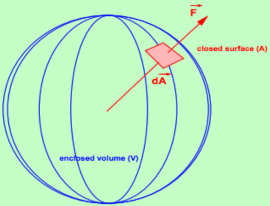
- Continuous distribution of charges, $\rho(\mathbf{r})$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\iiint_V \rho(\mathbf{r}')\,\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d\mathbf{r}$$

with $Q = \iiint_V \rho(\mathbf{r}')\,d\mathbf{r}$ as the total charge, and where $\rho$ is the charge density.

# Recall: Gauss' theorem

**Used in the following: Gauss' theorem to evaluate flux integral:**




$$\iint_A \vec{E} \cdot d\vec{A} = \iiint_V \nabla \cdot \vec{E} \cdot dV \quad \text{or}$$

$$\iint_A \vec{E} \cdot d\vec{A} = \iiint_V \text{div } \vec{E} \cdot dV$$

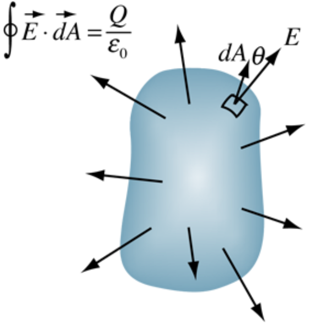
Integral through closed surface (flux) is integral of divergence in the enclosed volume

**Surface integral related to the divergence from the enclosed volume**

# Electrostatics: Gauss' law

Gauss' law states that the flux of $\vec{E}$ is:

$$\iint_A \vec{E}\cdot \mathrm{d}\vec{A} = \int_{\substack{\text{any closed}\\ \text{surface A}}} E_n\,\mathrm{da} = \frac{\text{sum of charges inside } A}{\epsilon_0}$$




We know that

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\iiint_V \rho(\mathbf{r}') \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} d\mathbf{r}$$

In differential form, using the Gauss' theorem (divergence theorem):

$$\iint \vec{E}\cdot d\vec{A} = \iiint \nabla\cdot\vec{E}\, d\mathbf{r}$$

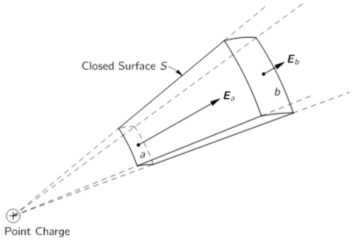
which gives the first Maxwell's equation in differential form:

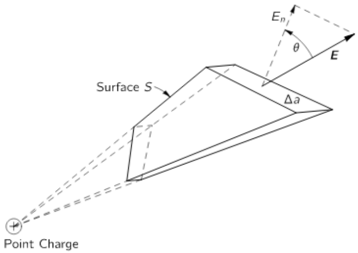
$$\boxed{\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}}$$

Example: case of a single point charge

$$\iint \vec{E}\cdot d\vec{A} = \begin{cases} \frac{q}{\epsilon_0} & \text{if } q \text{ lies inside A} \\ 0 & \text{if } q \text{ lies outside A} \end{cases}$$

# Electrostatics: Gauss' law




The flux of **E** out of the surface S is zero.

# Electrostatics: scalar potential and Poisson equation

The equations for electrostatics are:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \times \vec{E} = 0$$

The two can be combined into a single equation:

$$\boxed{\vec{E}=-\nabla\phi}$$

which leads to the Poisson's equation:

$$\nabla \cdot \nabla\phi = \boxed{\nabla^2\phi=-\frac{\rho}{\epsilon_0}}$$

Where the operator $\nabla^2$ is called Laplacian:

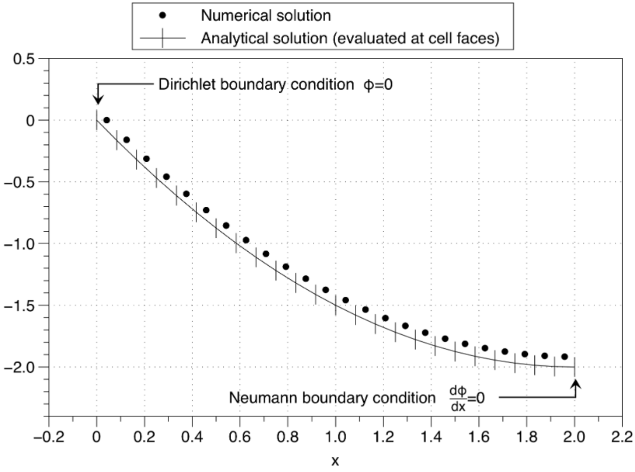
$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The Poisson's equation allows to compute the electric field generated by arbitrary charge distributions.

# Electrostatics: Poisson's equation

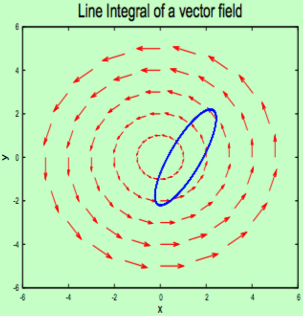
$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = -\frac{\rho}{\epsilon_0}$$



# Recall: Stokes' theorem

**Used in the following: Stoke's theorem**




$$\oint_C \vec{F} \cdot d\vec{r} \;=\; \iint_A \nabla \times \vec{F} \cdot d\vec{A} \quad \text{or}$$

$$\oint_C \vec{F} \cdot d\vec{r} \;=\; \iint_A \text{curl } \vec{F} \cdot d\vec{A}$$

$$\text{obviously : div } \vec{F} \;=\; 0$$

# Magnetostatics: Ampère's and Biot-Savart laws

The equations for electrostatics are:

$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{B} = \frac{\vec{j}}{\epsilon_0 c^2}$$

The Stokes' theorem tells that:

$$\oint_C \vec{B} \cdot d\vec{r} = \iint_A \left( \nabla \times \vec{B} \right) \cdot d\vec{A}$$

This equation gives the Ampère's law:

$$\oint_C \vec{B} \cdot d\vec{r} = \frac{1}{\epsilon_0 c^2} \iint_A \vec{j} \cdot \vec{n} \, dA$$

From which one can derive the Biot-Savart law, stating that, along a current $j$:

$$\vec{B}\left(\vec{r}\right) = \frac{1}{4\pi\epsilon_0 c^2} = \oint_C \frac{j \, d\vec{r}' \times (\vec{r} - \vec{r}')}{\left|\vec{r} - \vec{r}'\right|^3}$$

This provides a practical way to compute $\vec{B}$ from current distributions.

# Magnetostatics: vector potential

The equations for electrostatics are:

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \frac{\vec{j}}{\epsilon_0 c^2}$$

They can be unified into one, introducing the vector potential $\vec{A}$:

$$\vec{B} = \nabla \times \vec{A}$$

Using the Stokes' theorem

$$\vec{B}\,(\vec{r}) = \frac{1}{4\pi\epsilon_0 c^2} = \oint_C \frac{j\,d\vec{r}\,' \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$$

one can derive the expression of the vector potential $\vec{A}$ from of the current $\vec{j}$:

$$\vec{A}\,(r) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{j}\,(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3\vec{r}\,'$$

# Summary of electro- and magneto- statics

One can compute the electric and the magnetic fields from the scalar and the vector potentials

$$\vec{E} = -\nabla\phi$$
$$\vec{B} = \nabla \times \vec{A}$$

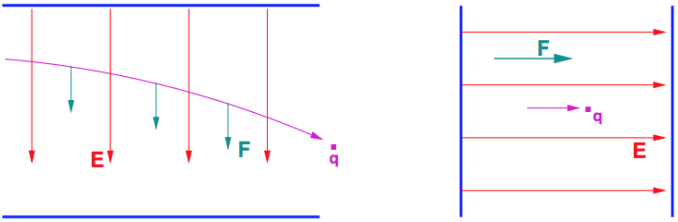
with

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}'$$
$$\vec{A}(r) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}'$$

being $\rho$ the charge density, and $\vec{j}$ the current density.

# Motion of a charged particle in an electric field

$$\vec{F} = q \cdot \left( \vec{E} + \vec{v} \times \not{\vec{B}} \right)$$




$\vec{v} \perp \vec{E}$ $\vec{v} \parallel \vec{E}$

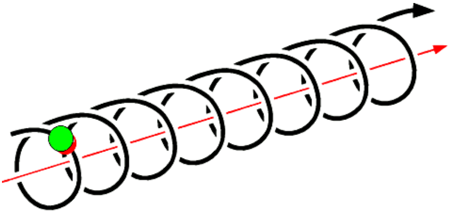
**Assume no magnetic field:**

$$\frac{d}{dt}(m\vec{v}) = \vec{f} = q \cdot \vec{E}$$

**Force always in direction of field $\vec{E}$, also for particles at rest.**

# Motion of a charged particle in a magnetic field

$$\vec{F} = q \cdot \left(\cancel{\vec{E}} + \vec{v} \times \vec{B}\right)$$



Without electric field : $\quad \frac{d}{dt}(m\vec{v}) = \vec{f} = q \cdot \vec{v} \times \vec{B}$

**Force is perpendicular to both, $\vec{v}$ and $\vec{B}$**

Part 3.

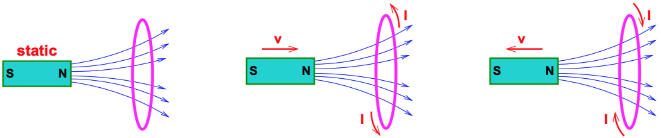
# Electromagnetism:

# Non-static case

# Magnetostatics: Faraday's law of induction

"The electromotive force around a closed path is equal to the negative of the time rate of change of the magnetic flux enclosed by the path."

$$\text{static flux}: \quad \Omega = \int_A \vec{B} \cdot \mathrm{d}\vec{A} \qquad\qquad \text{changing flux}: \quad \frac{\partial \Omega}{\partial t} = \int_A \frac{\partial(\vec{B})}{\partial t} \cdot \mathrm{d}\vec{A}$$



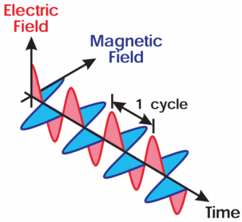

**Moving the magnet changes the flux (density or number of lines) through the area** $\Longrightarrow$

**Induces a <u>circulating</u> (curling) electric field $\vec{E}$ in the coil which "pushes" charges around the coil** $\Longrightarrow$

# Non-static case: electromagnetic waves

Electromagnetic wave equation:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i\left(\omega t - \vec{k}\cdot\vec{r}\right)}$$
$$\vec{B}(\vec{r}, t) = \vec{B}_0 e^{i\left(\omega t - \vec{k}\cdot\vec{r}\right)}$$




Important quantities:

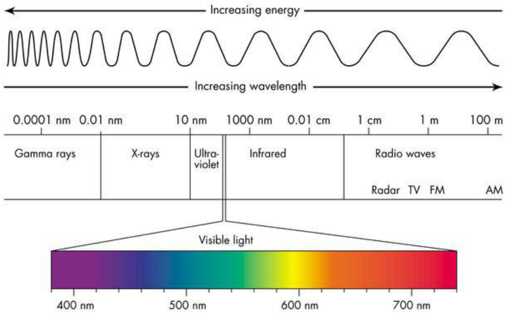
| | |
|---|---|
| $\left\vert\vec{k}\right\vert = \frac{2\pi}{\lambda} = \frac{\omega}{c}$ | wave-number vector |
| $\lambda = \frac{c}{f}$ | wave length |
| $f$ | frequency |
| $\omega = 2\pi f$ | angular frequency |

Magnetic and electric fields are transverse to direction of propagation:

$$\vec{E} \perp \vec{B} \perp \vec{k}$$

Short wave length $\rightarrow$high frequency $\rightarrow$ high energy

# Spectrum of electromagnetic waves




Examples:

- yellow light $\approx 5 \cdot 10^{14}$ Hz (i.e. $\approx 2$ eV !)
- LEP (SR) $\lesssim 2 \cdot 10^{20}$ Hz (i.e. $\approx 0.8$ MeV !)
- gamma rays $\lesssim 3 \cdot 10^{21}$ Hz (i.e. $\lesssim 12$ MeV !)

(For estimates using temperature: 3 K $\approx 0.00025$ eV )

# Electromagnetic waves impacting highly conductive materials

Highly conductive materials: RF cavities, wave guides.

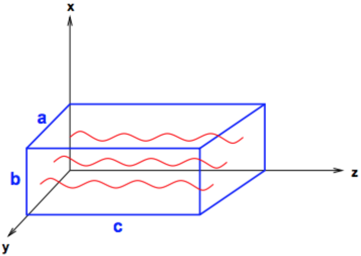
- In an ideal conductor:

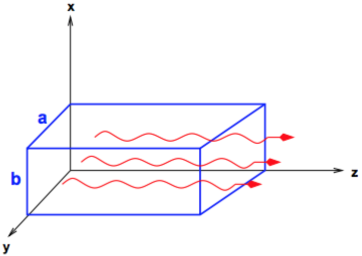
$$\vec{E}_{\parallel} = 0, \qquad \vec{B}_{\perp} = 0$$

- This implies:
  - All energy of an electromagnetic wave is reflected from the surface of an ideal conductor.
  - Fields at any point in the ideal conductor are zero.
  - Only some field patterns are allowed in waveguides and RF cavities.

# Example: RF cavities and wave guides

**Rectangular, conducting cavities and wave guides (schematic) with dimensions $a \times b \times c$ and $a \times b$:**




- **RF cavity, fields can persist and be stored (reflection !)**
- **Plane waves can propagate along wave guides, here in z-direction**

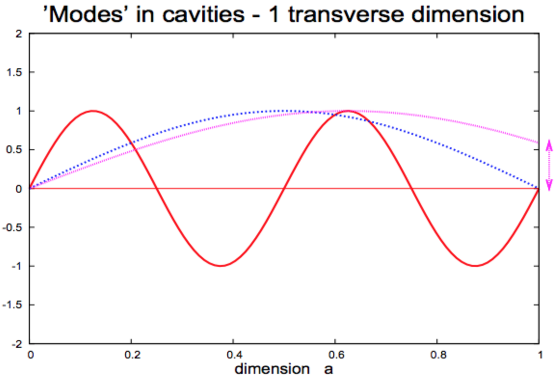
# Example: Fields in RF cavities

**Assume a rectangular RF cavity ($a$, $b$, $c$), ideal conductor.**

**Without derivations, the components of the fields are:**

$$E_x = E_{x0} \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$
$$E_y = E_{y0} \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$
$$E_z = E_{z0} \cdot \sin(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$

$$B_x = \frac{i}{\omega}(E_{y0} k_z - E_{z0} k_y) \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$
$$B_y = \frac{i}{\omega}(E_{z0} k_x - E_{x0} k_z) \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$
$$B_z = \frac{i}{\omega}(E_{x0} k_y - E_{y0} k_x) \cdot \cos(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$




**No electric field at boundaries, wave must have "nodes" = zero fields at the boundaries**

**Only modes which 'fit' into the cavity are allowed**

**In the example:** $\frac{\lambda}{2} = \frac{a}{4}$, $\frac{\lambda}{2} = \frac{a}{1}$, $\frac{\lambda}{2} = \frac{a}{0.8}$

**(then either "sin" or "cos" is 0)**

# Example: Consequences for RF cavities

**Field must be zero at conductor boundary, only possible if:**

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

**and for $k_x, k_y, k_z$ we can write, (then they all fit):**

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \frac{m_z \pi}{c},$$

**The integer numbers $m_x, m_y, m_z$ are called mode numbers, important for design of cavity !**

**$\rightarrow$ half wave length $\lambda/2$ must always fit exactly the size of the cavity.**

**(For cylindrical cavities: use cylindrical coordinates )**

# Example: Consequences for wave guides

**Similar considerations as for cavities, no field at boundary.**
**We must satisfy again the condition:**

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

**This leads to modes like (no boundaries in direction of propagation $z$):**

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b},$$

**The numbers $m_x, m_y$ are called mode numbers for planar waves in wave guides !**

**In z direction: No Boundary - No Boundary Condition ...**

**Re-writing the condition as:**

$$k_z^2 = \frac{\omega^2}{c^2} - k_x^2 - k_y^2 \quad \rightarrow \quad k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

**Propagation without losses requires $k_z$ to be real, i.e.:**

$$\frac{\omega^2}{c^2} > k_x^2 + k_y^2 = (\frac{m_x \pi}{a})^2 + (\frac{m_y \pi}{b})^2$$

**which defines a cut-off frequency $\omega_c$. For lowest order mode:**

$$\omega_c = \frac{\pi \cdot c}{a}$$

- **Above cut-off frequency: propagation without loss**
- **At cut-off frequency: standing wave**
- **Below cut-off frequency: attenuated wave (it does not "fit in").**

**There is a very easy way to show that very high frequencies easily propagate**

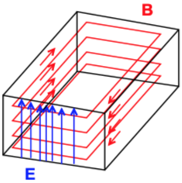
# Classification of modes

**Transverse electric modes (TE):** $E_z = 0 \quad H_z \neq 0$

**Transverse magnetic modes (TM):** $E_z \neq 0 \quad H_z = 0$

**Transverse electric-magnetic modes (TEM):** $E_z = 0 \quad H_z = 0$

**(Not all of them can be used for acceleration ... !)**

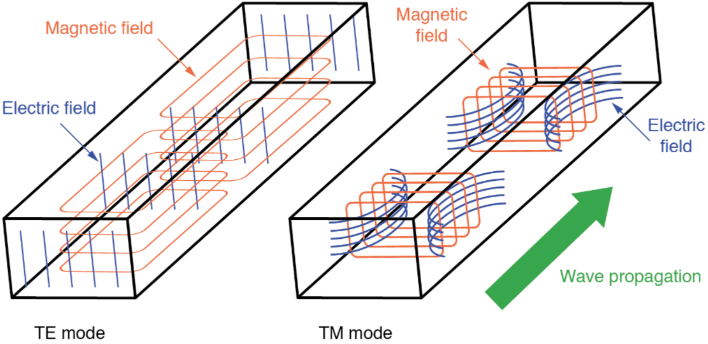



Note (here a TE mode) :

Electric field lines end at boundaries

Magnetic field lines appear as "loops"

# Classification of modes




*Magnetic flux lines appear as continuous loops*
*Electric flux lines appear with beginning and end points*

# ...The End!

# Thank you

# for your attention!

Special thanks to Werner Herr, for the pictures I took from his slides.