QCD and precision calculations
Lecture 2: NLO

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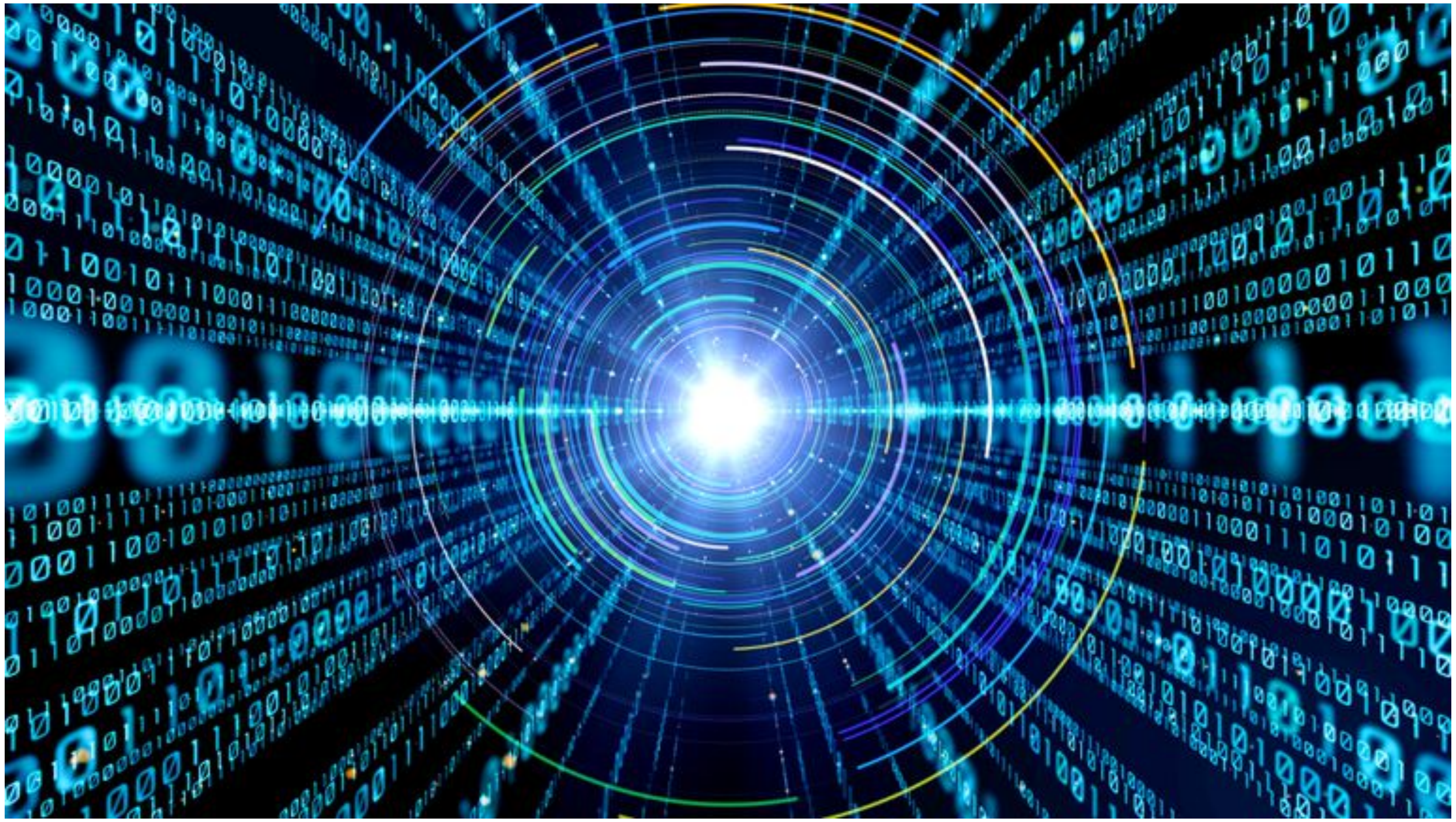

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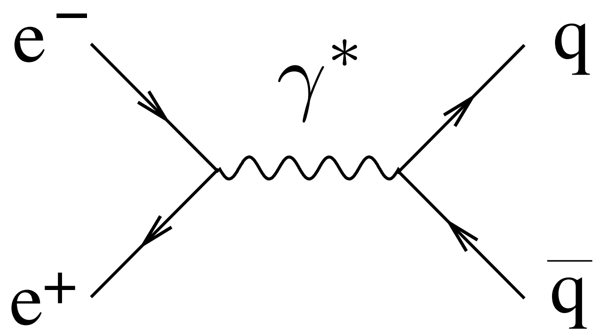
Contents

- building blocks of NLO cross sections
- infrared singularities and their cancellation
- dimensional regularisation
- hadronic initial states
- parton distribution functions
- one-loop integrals
- regularisation schemes

NLO basics

start with simple example: e+e- annihilation

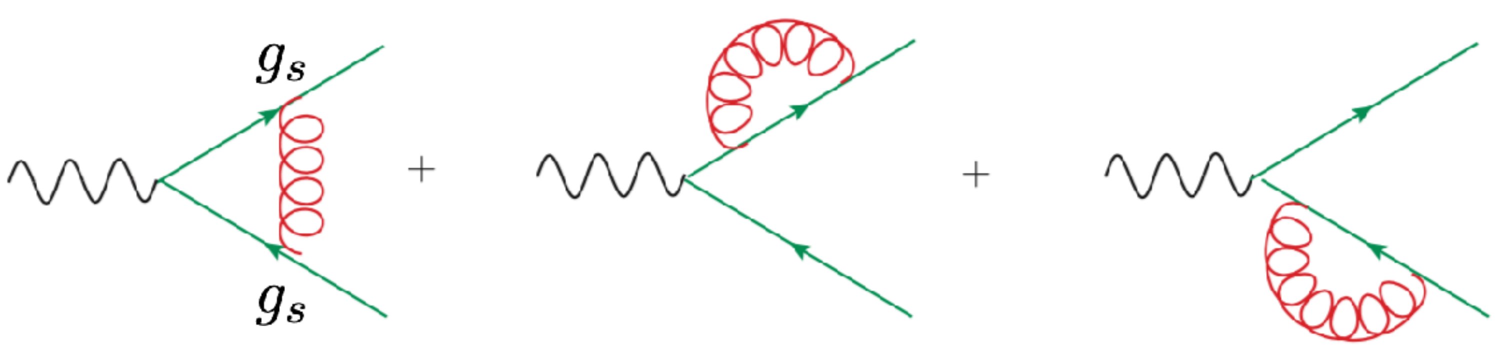


at leading order: $\sigma^{LO} = \frac{4\pi\alpha^2}{3s} e_q^2 N_c$

(Z exchange not considered)

split off leptonic part and consider $\gamma^* \to q\bar{q}$

NLO: order α_s corrections at cross section level

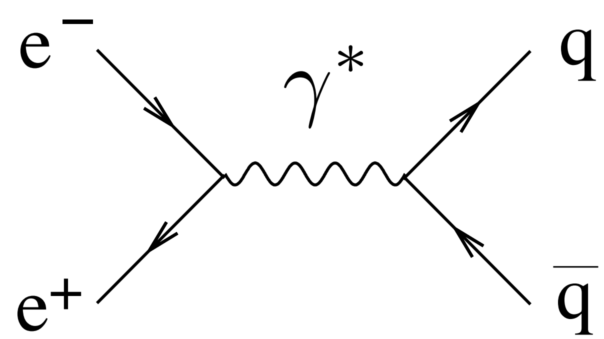


virtual

real

NLO basics

start with simple example: e+e- annihilation



at leading order: $$\sigma^{LO} = \frac{4\pi\alpha^2}{3s}\, e_q^2\, N_c$$

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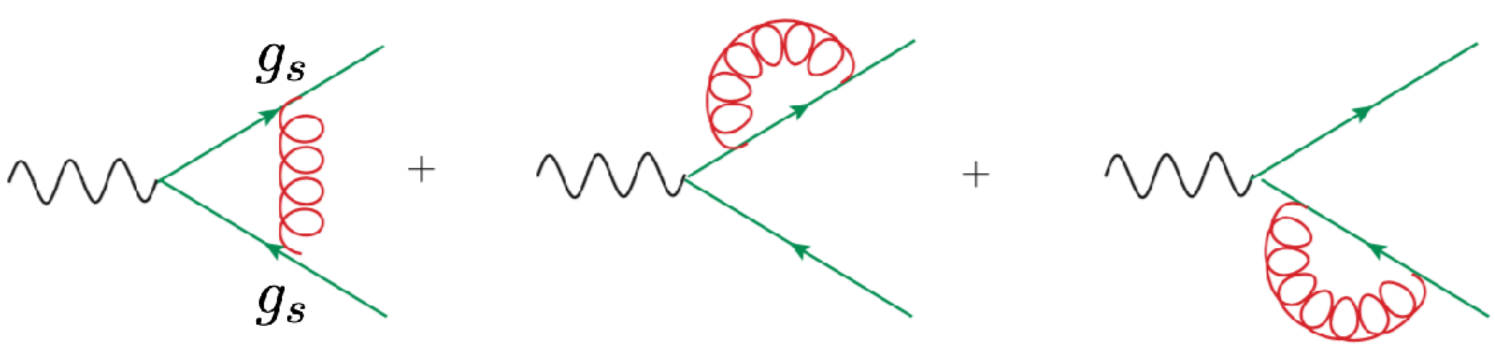
split off leptonic part and consider $\gamma^* \to q\bar{q}$

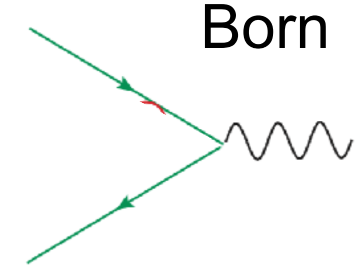
NLO: order α_s corrections at cross section level

will be interfered with



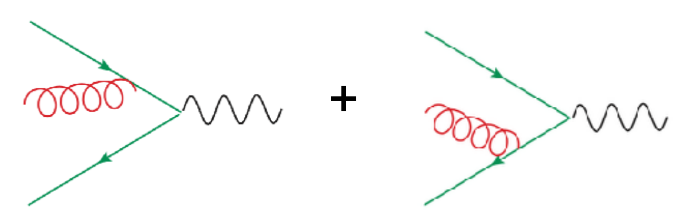
virtual

Born



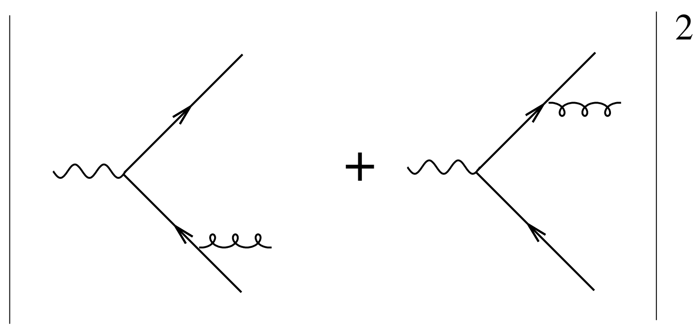
real

itself

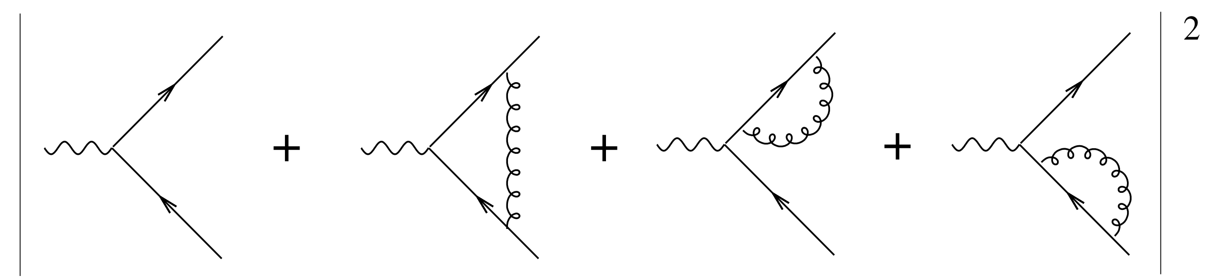


NLO basics

$$\sigma^{NLO} = \underbrace{\int \mathrm{d}\phi_2 \, |\mathcal{M}_0|^2}_{\sigma^{LO}} + \int_R \mathrm{d}\phi_3 \, |\mathcal{M}_{\text{real}}|^2 + \int_V \mathrm{d}\phi_2 \, 2Re\left(\mathcal{M}_{\text{virt}} \mathcal{M}_0^*\right)$$

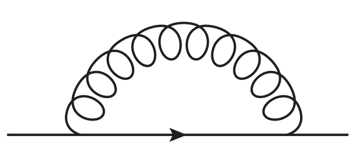


real radiation



virtual corrections

- virtual corrections contain UV divergences, but they cancel here due to Ward Identity

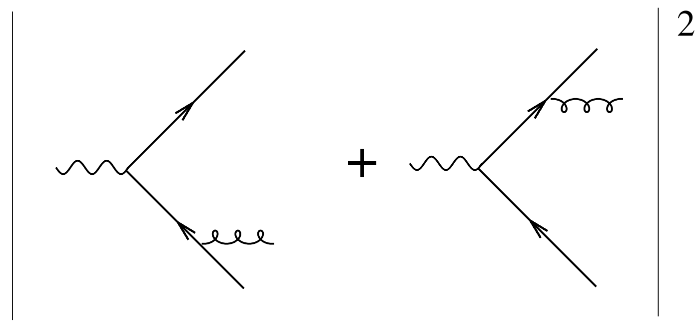


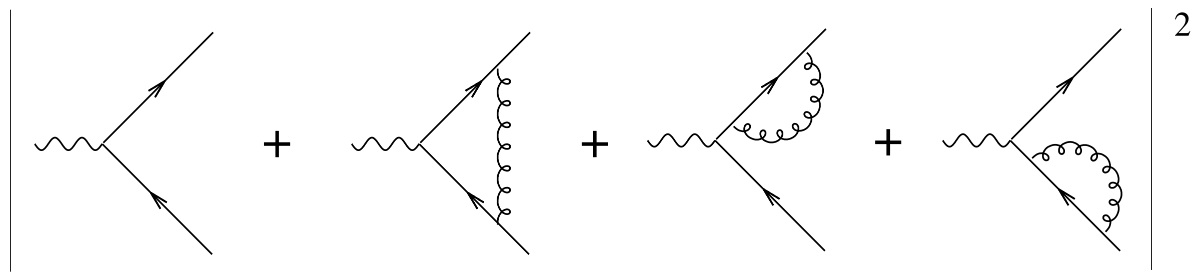
is zero for massless quarks (scaleless integral)

(see later, dimensional regularisation)

in fact it is something like $\dfrac{1}{\epsilon_{UV}} - \dfrac{1}{\epsilon_{IR}}$

NLO basics

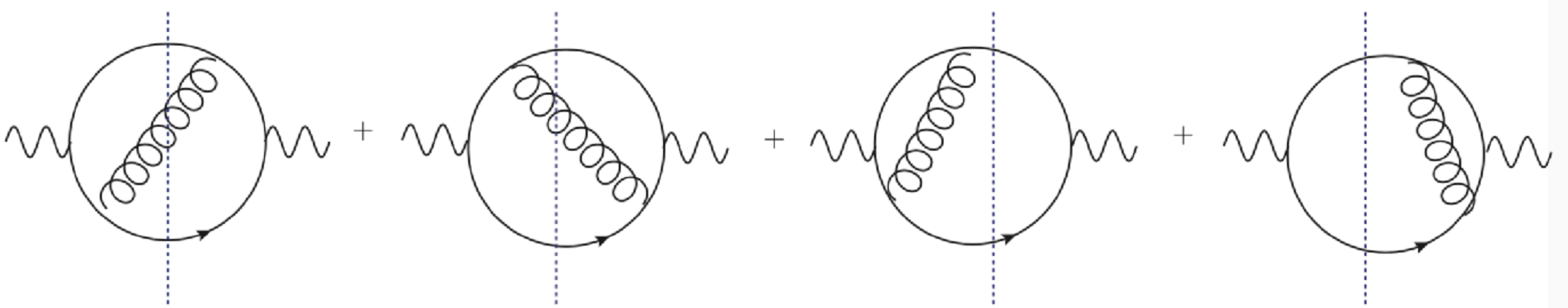




real radiation

virtual corrections

$|\mathcal{M}|^2$ pictorially: $\mathcal{M}$ left of the cut, $\mathcal{M}^*$ right of the cut



claim: sum over all cuts above is finite

individual diagrams contain infrared singularities

must be so due to KLN-Theorem

cancellation of IR singularities

KLN Theorem

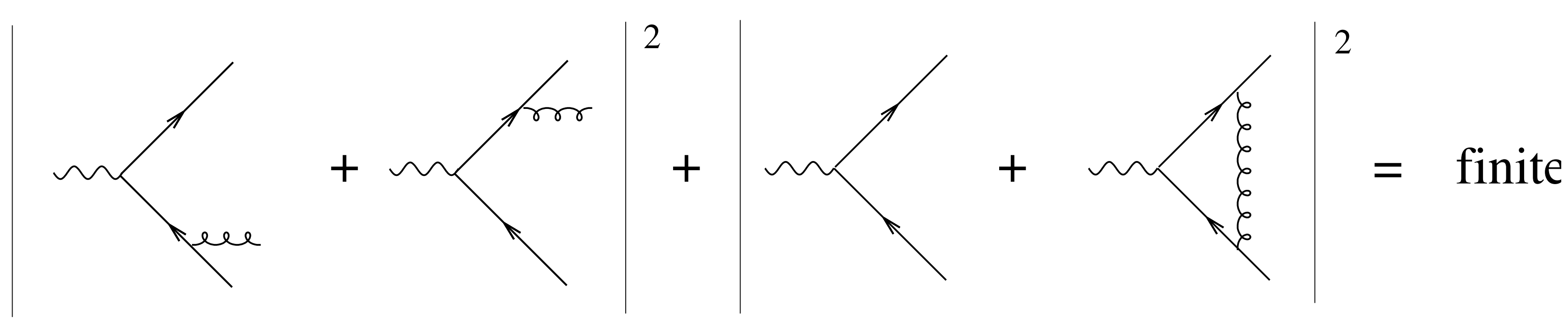
Kinoshita, Lee, Nauenberg, 1960's

Soft and collinear singularities cancel in the sum over degenerate states

what are degenerate states ?

- a quark emitting a soft gluon, or a collinear quark-gluon system cannot be distinguished from simply a quark
- virtual corrections are not directly observable

$\Rightarrow$ in the considered inclusive cross section,
singularities cancel between real and virtual corrections



cancellation of IR singularities

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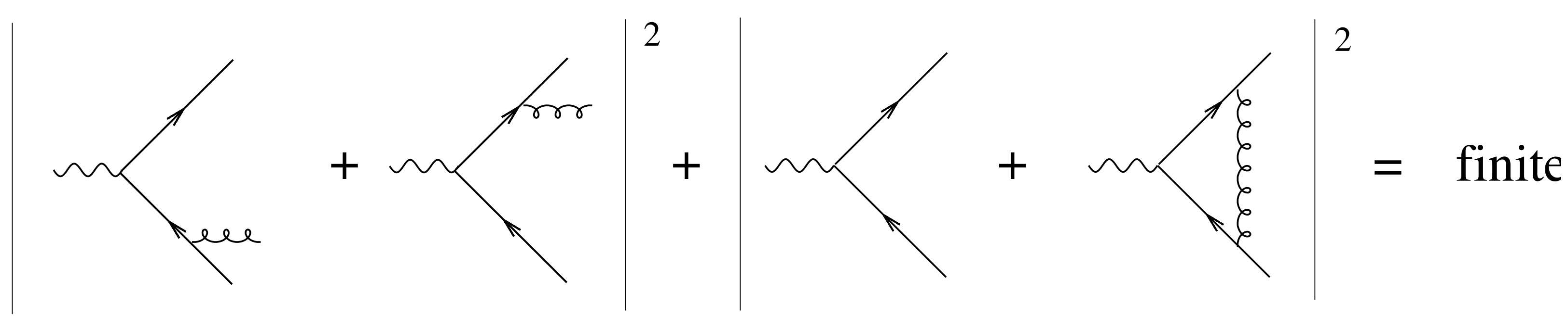


warning:

does not hold for initial state radiation in hadronic collisions

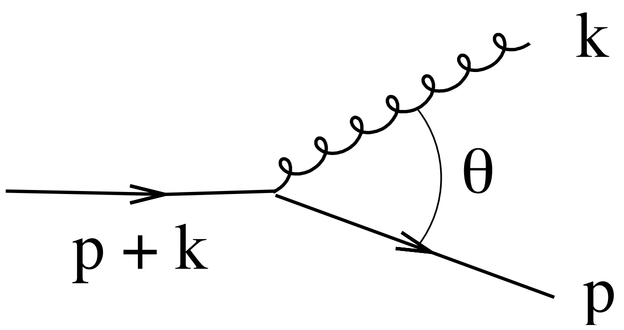
reason: cannot sum over degenerate states for partons in the proton

(see later)

IR singularities

two types: (a) soft, (b) collinear

Consider the emission of a gluon from a hard quark:



$$\begin{aligned} p &= E\,(1,0,0,1) \\ k &= \omega\,(1,0,\sin\theta,\cos\theta) \end{aligned}$$

$$\boxed{(p+k)^2 = 2E\,\omega\,(1-\cos\theta)}$$

will go to zero if the gluon becomes soft $(\omega \to 0)$

or if quark and gluon become collinear $(\theta \to 0)$

note: collinear singularity will be absent for massive quarks $(p^2 = m^2)$

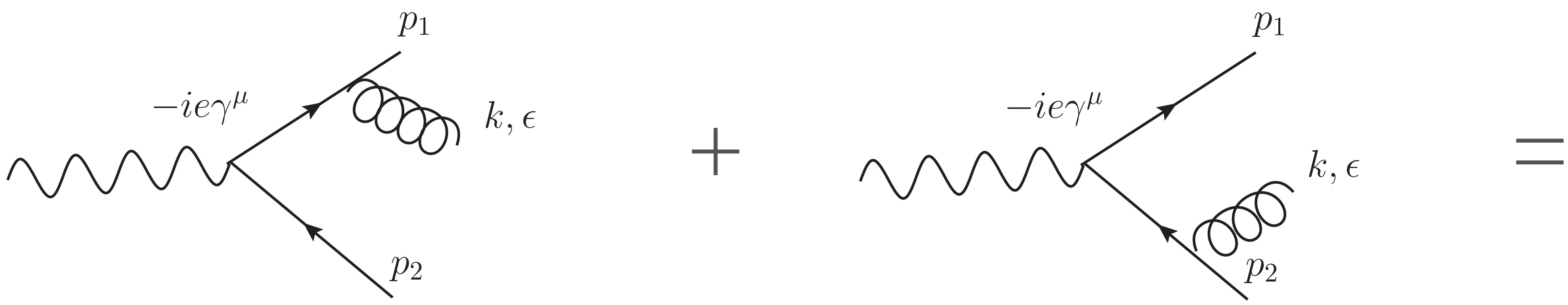
1/propagator ~ $(p+k)^2 - m^2 = 2E\omega\,(1-\beta\cos\theta)\,,\ \beta = \sqrt{1-m^2/E^2}$

nonzero for $\theta \to 0$, but soft singularity still present

therefore collinear singularities are sometimes called *mass singularities*

soft singularities

Consider real emission diagrams in more detail:



$$\begin{aligned} \mathcal{M}^\mu_{q\bar{q}g} = \quad & \bar{u}(p_1)\,(-igt^A \not\epsilon)\,\frac{i(\not p_1 + \not k)}{(p_1+k)^2}\,(-ie\gamma^\mu)\,v(p_2) \\ + \quad & \bar{u}(p_1)\,(-ie\gamma^\mu)\,\frac{-i(\not p_2 + \not k)}{(p_2+k)^2}\,(-igt^A \not\epsilon)\,v(p_2) \end{aligned}$$

If gluon becomes soft: neglect *k* except for linear terms in denominator:

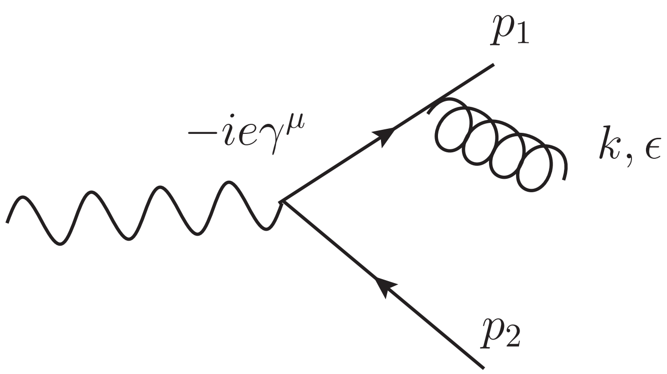
$$\mathcal{M}^\mu_{q\bar{q}g} \overset{soft}{=} -iegt^A\,\bar{u}(p_1)\,\gamma^\mu \left(\frac{\not\epsilon \not p_1}{2p_1 k} - \frac{\not p_2 \not\epsilon}{2p_2 k} \right) v(p_2)$$

$$|\mathcal{M}_{q\bar{q}g}|^2 \overset{soft}{\to} |\mathcal{M}_{q\bar{q}}|^2\, g^2 C_F\, \frac{p_1 p_2}{(p_1 k)(p_2 k)}$$

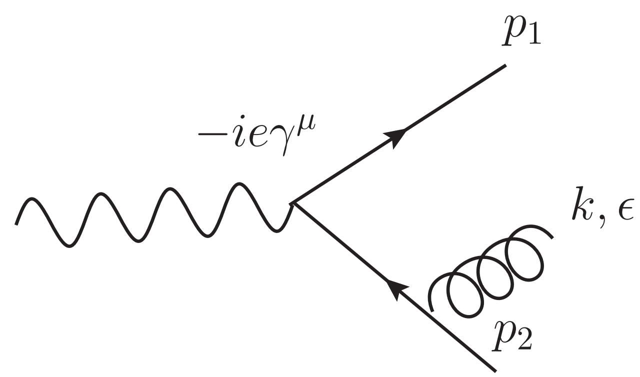
Factorisation into Born matrix element and Eikonal factor

soft singularities

Consider real emission diagrams in more detail:



$+$



$=$

$$\mathcal{M}^\mu_{q\bar{q}g} = \bar{u}(p_1)\,(-igt^A\not{\epsilon})\,\frac{i(\not{p}_1+\not{k})}{(p_1+k)^2}\,(-ie\gamma^\mu)\,v(p_2)$$
$$+ \quad \bar{u}(p_1)\,(-ie\gamma^\mu)\,\frac{-i(\not{p}_2+\not{k})}{(p_2+k)^2}\,(-igt^A\not{\epsilon})\,v(p_2)$$

If gluon becomes soft: neglect *k* except for linear terms in denominator:

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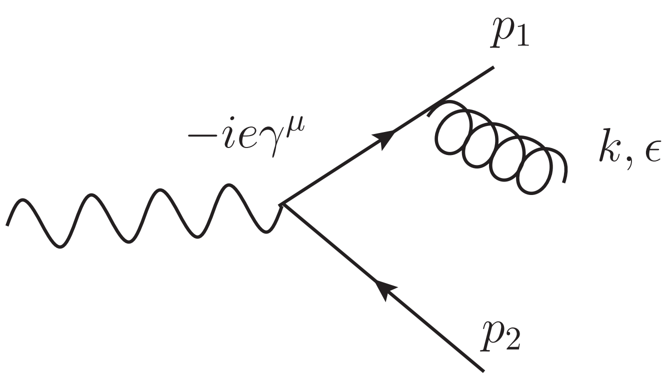
$$|\mathcal{M}_{q\bar{q}g}|^2 \stackrel{soft}{\rightarrow} |\mathcal{M}_{q\bar{q}}|^2\, g^2 C_F\, \frac{p_1p_2}{(p_1k)(p_2k)}$$

Note: colour will in general **not** factorise in the soft limit

Factorisation into Born matrix element and Eikonal factor

collinear singularities



$$(p_1 + k)^2 = 2E\,\omega\,(1 - \cos\theta) \to 0 \quad \text{for} \quad \theta \to 0$$

convenient parametrisation of momenta:

"Sudakov parametrisation"

$$\begin{aligned} p_1 &= z\,p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z}\,\frac{n^\mu}{2p_1 n} \\ k &= (1-z)\,p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z}\,\frac{n^\mu}{2p_1 n} \\ \Rightarrow 2p_1 k &= -\frac{k_\perp^2}{z(1-z)} \end{aligned}$$

p^μ collinear direction

n^μ light-like auxiliary vector

$k_\perp p = k_\perp n = 0$

$z = \frac{E_1}{E_1 + E_g}$

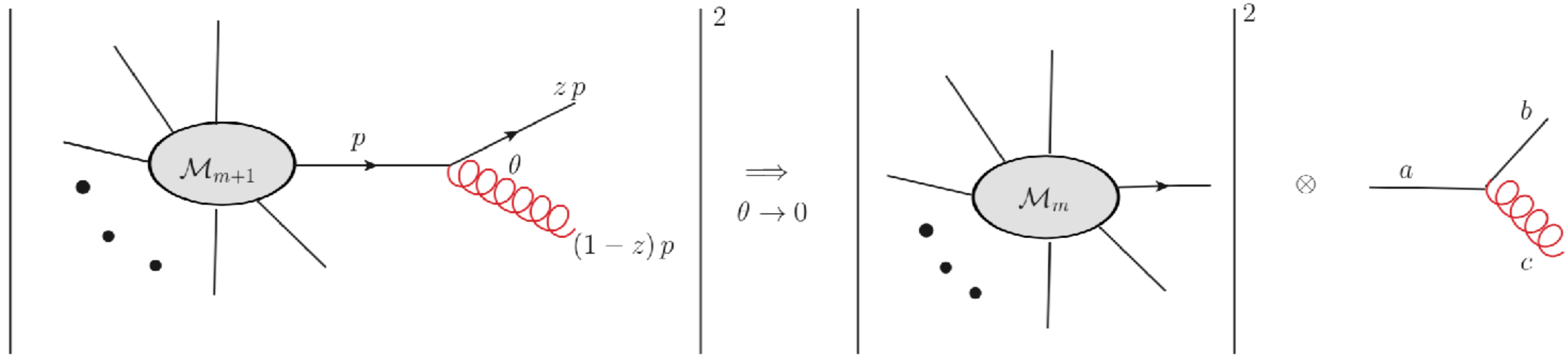
collinear limit in this parametrisation: $k_\perp \to 0$

$$|\mathcal{M}_1(p_1, k, p_2)|^2 \quad \stackrel{coll}{\to} \quad g^2\,\frac{1}{p_1 \cdot k}\,P_{qq}(z)\;|\mathcal{M}_0(p_1 + k, p_2)|^2$$

$P_{qq}(z)$: splitting functions

collinear singularities

factorisation property of amplitudes in the collinear limit:



$$|\mathcal{M}_{m+1}|^2\,\mathrm{d}\Phi_{m+1} \to |\mathcal{M}_m|^2\mathrm{d}\Phi_m\,\frac{\alpha_s}{2\pi}\,\frac{\mathrm{d}k_\perp^2}{k_\perp^2}\,\frac{\mathrm{d}\phi}{2\pi}\,\mathrm{d}z\,P_{a\to bc}(z)$$

note that the phase space also can be factorised in this limit

$$\mathrm{d}\Phi_{m+1} \to \mathrm{d}\Phi_m \otimes \mathrm{d}\Phi_k \qquad (k^+ = k\cdot n = (1-z)\,p\cdot n\,)$$

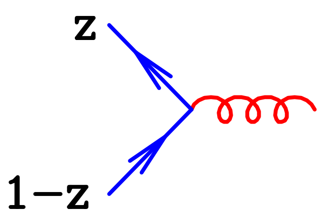
$$\mathrm{d}\Phi_k \equiv \frac{\mathrm{d}^4k}{(2\pi)^3}\delta(k^2) = \frac{1}{8\pi^2}\,\frac{\mathrm{d}\phi}{2\pi}\,\frac{\mathrm{d}k^+}{2k^+}\,\mathrm{d}k_\perp^2 = \frac{1}{16\pi^2}\,\frac{\mathrm{d}z}{(1-z)}\,\mathrm{d}k_\perp^2$$

this factorisation does not depend on the details of $\mathcal{M}_m$

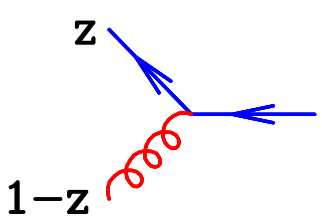
splitting functions

Dokshitzer, Gribov, Lipatov, Altarerlli, Parisi

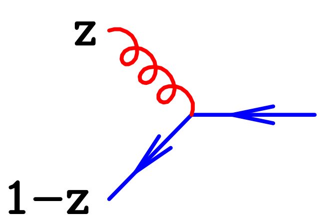
it only depends on the types of splitting partons



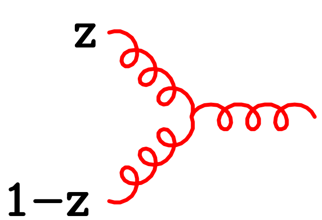
$$\hat{P}_{qg}(z) = T_R \left[z^2 + (1-z)^2 \right], \quad T_R = \frac{1}{2} \, ,$$



$$\hat{P}_{qq}(z) = C_F \left[\frac{1+z^2}{(1-z)} \right] \, ,$$



$$\hat{P}_{gq}(z) = C_F \left[\frac{1+(1-z)^2}{z} \right] \, ,$$

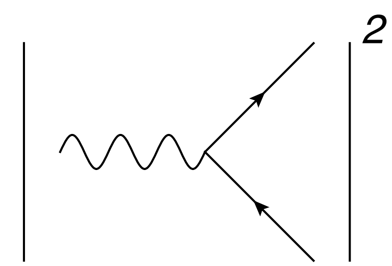


$$\hat{P}_{gg}(z) = C_A \left[\frac{z}{(1-z)} + \frac{1-z}{z} + z\,(1-z) \right]$$

real radiation matrix element

remember $\quad |\overline{\mathcal{M}}|^2 \;\to\; \overline{\sum}_{\lambda,c} |\mathcal{M}_{\lambda,c}|^2 = \frac{1}{\prod_{\text{initial}} N_{\text{pol}} N_{\text{col}}} \sum_{\text{final pol,col}} |\mathcal{M}_{\lambda,c}|^2$

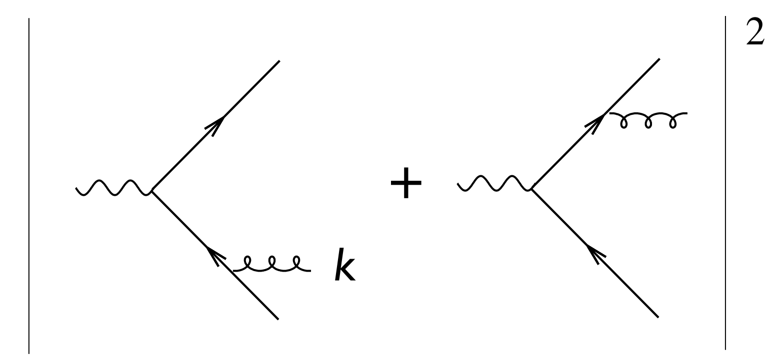
at LO: $\quad |\overline{\mathcal{M}}_0|^2 \;=\; \frac{1}{3}\, 4e^2 Q_q^2 N_c\, s$



with extra gluon radiation:

$$
\begin{aligned}
p^\gamma &= \sqrt{s}\,(1,0,0,0) \\
p_1 &= E_1\,(1,0,0,1) \\
p_2 &= E_2\,(1,0,\sin\theta,\cos\theta) \\
k &\equiv p_3 = p^\gamma - p_1 - p_2
\end{aligned}
$$

$s_{ij} = (p_i + p_j)^2$



$$
|\overline{\mathcal{M}}_1|^2 \;=\; |\overline{\mathcal{M}}_0|^2\, \frac{2g^2\, C_F}{s} \left(\frac{s_{13}}{s_{23}} + \frac{s_{23}}{s_{13}} + 2s\, \frac{s_{12}}{s_{13}s_{23}} \right)
$$

define $\quad x_1 = 2E_1/\sqrt{s},\, x_2 = 2E_2/\sqrt{s}$

$$
\Rightarrow |\overline{\mathcal{M}}_1|^2 \;=\; |\overline{\mathcal{M}}_0|^2\, \frac{2g^2\, C_F}{s} \left(\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \right)
$$

gluon energy:

$E_g = \sqrt{s}\,(1 - x_1 - x_2)$

singularity structure

$$|\overline{\mathcal{M}}_1|^2 = |\overline{\mathcal{M}}_0|^2 \frac{2g^2 C_F}{s} \left(\frac{s_{13}}{s_{23}} + \frac{s_{23}}{s_{13}} + 2s \frac{s_{12}}{s_{13}s_{23}} \right)$$

$$= |\overline{\mathcal{M}}_0|^2 \frac{2g^2 C_F}{s} \left(\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \right) \qquad x_1 = 2E_1/\sqrt{s}, x_2 = 2E_2/\sqrt{s}$$

$x_1 \to 1$: collinear singularity $p_1 \parallel p_3$, $x_2 \to 1$: collinear singularity $p_2 \parallel p_3$

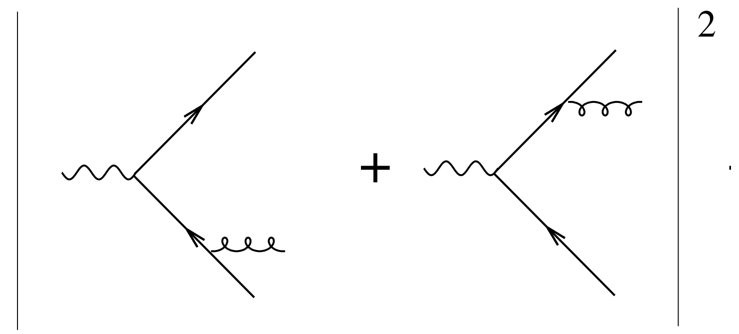
$x_1 \to 1 - x_2$: soft gluon $E_g = \sqrt{s}\,(1 - x_1 - x_2)$

in these limits the matrix element is singular

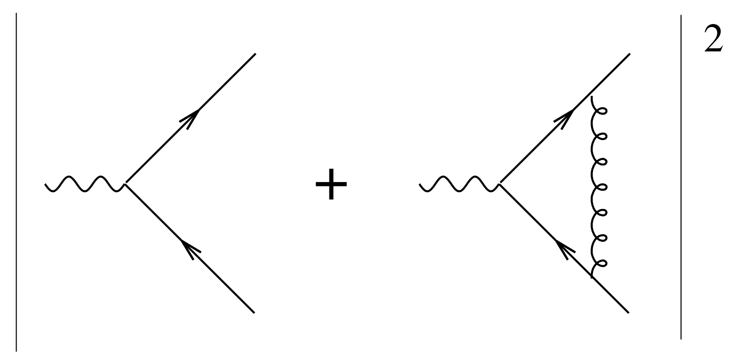
- we know that the singularities should cancel with the virtual corrections
- however we first have to isolate them to make the cancellation manifest

cancellation of singularities

real and virtual corrections live on different phase spaces



3-particle phase space



2-particle phase space

$$\sigma^{NLO} = \underbrace{\int \mathrm{d}\phi_2 \, |\mathcal{M}_0|^2}_{\sigma^{LO}} + \int_R \mathrm{d}\phi_3 \, |\mathcal{M}_{\mathrm{real}}|^2 + \int_V \mathrm{d}\phi_2 \, 2Re \left(\mathcal{M}_{\mathrm{virt}} \mathcal{M}_0^* \right)$$

cancellation of singularities

widely used procedure, for n-particle production:

$$\mathcal{B}_n = \int d\phi_n \left|\mathcal{M}_0\right|^2 = \int d\phi_n B_n$$

$$\mathcal{V}_n = \int d\phi_n \, 2Re\left(\mathcal{M}_{\rm virt}\mathcal{M}_0^*\right) = \int d\phi_n \frac{V_n}{\epsilon}$$ + finite

$$\mathcal{R}_n = \int d\phi_{n+1} \left|\mathcal{M}_{\rm real}\right|^2 = \int d\phi_n \int_0^1 dx\, x^{-1-\epsilon}\, R_n(x)$$ + finite'

$$\sigma^{NLO} = \int d\phi_n \left\{ \left(B_n + \frac{V_n}{\epsilon}\right) J(p_1 \dots p_n, 0) + \int_0^1 dx\, x^{-1-\epsilon}\, R_n(x)\, J(p_1 \dots p_n, x) \right\}$$

+ Finite

with $\lim_{x\to 0} J(p_1 \dots p_n, x) = J(p_1 \dots p_n, 0) \quad (*)$

J is called *measurement function* and defines the observable,

the property $(*)$ is called *infrared safety*

cancellation of singularities

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$$\mathcal{R}_n = \int \mathrm{d}\phi_{n+1} \left|\mathcal{M}_{\mathrm{real}}\right|^2 = \int \mathrm{d}\phi_n \int_0^1 \mathrm{d}x \, x^{-1-\epsilon} R_n(x) \quad \text{+ finite'}$$

$$\sigma^{NLO} = \int \mathrm{d}\phi_n \left\{ \left(B_n + \frac{V_n}{\epsilon} \right) J(p_1 \ldots p_n, 0) + \int_0^1 \mathrm{d}x \, x^{-1-\epsilon} R_n(x) \, J(p_1 \ldots p_n, x) \right\}$$

+ Finite

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but what is ϵ ?

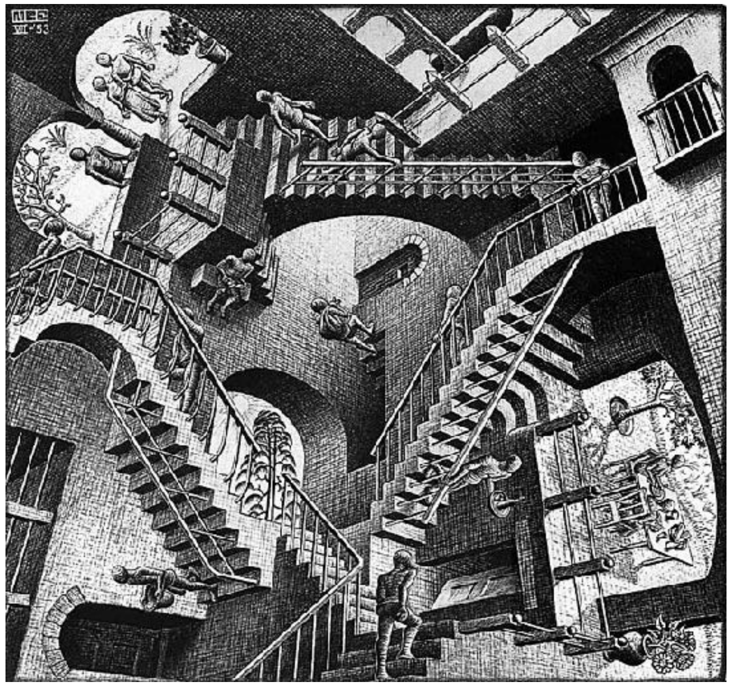
dimensional regularisation

't Hooft, Veltman '72; Bollini, Gambiagi '72

A convenient way to isolate singularities:

continue space-time from 4 to $\boxed{D = 4 - 2\epsilon}$ dimensions

- regulates both UV and IR divergences
  formally UV: $\epsilon > 0$, IR: $\epsilon < 0$
- does not violate gauge invariance
- poles can be isolated in terms of $1/\epsilon^b$



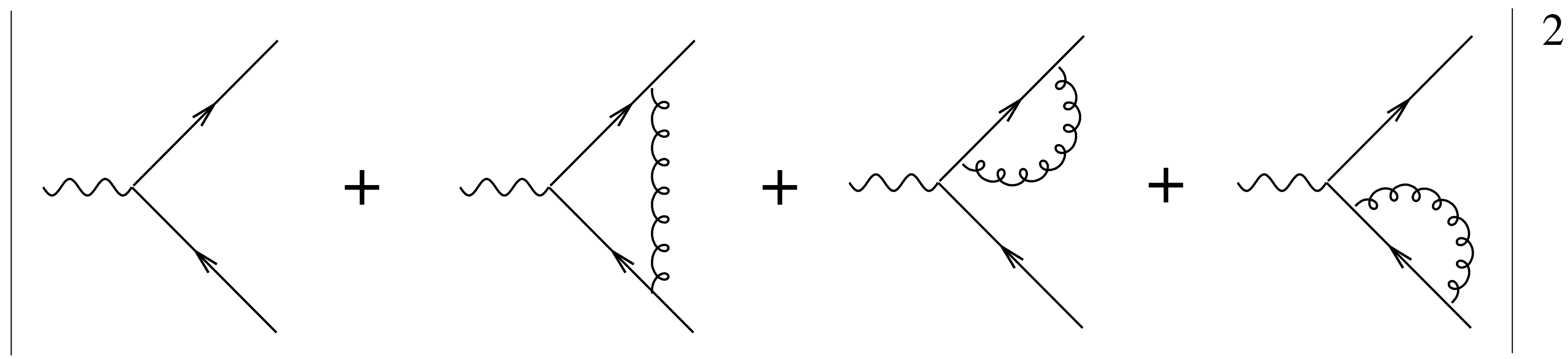
➡ need phase space integrals in D dimensions

➡ need integration over virtual loop momenta in D dimensions

$$g^2 \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \longrightarrow g^2 \mu^{2\epsilon} \int_{-\infty}^{\infty} \frac{d^Dk}{(2\pi)^D} \quad , \quad \mu^{2\epsilon}$$

is introduced to keep coupling (mass-)dimensionless in D dim.

virtual corrections



we will not go through the calculation but only quote the result:

$$R^{\text{virt}} = R^{LO} \times \frac{\alpha_s}{2\pi} C_F \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{-s}{4\pi\mu^2}\right)^{-\epsilon} \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon) \right\}$$

phase space in D dimensions

1 to N particle phase space:

$$Q \to p_1 + \ldots + p_N$$

$$\int d\Phi_N^D \;=\; (2\pi)^{N-D(N-1)} \int \prod_{j=1}^{N} d^D p_j \, \delta^+(p_j^2 - m_j^2) \delta^{(D)}\Big(Q - \sum_{i=1}^{N} p_i\Big)$$

In the following consider massless case $p_j^2 = 0$. Use for $i = 1, \ldots, N-1$

$$\begin{aligned} \int d^D p_i \delta^+(p_i^2) \;&\equiv\; \int d^D p_i \delta(p_i^2)\theta(E_i) = \int d^{D-1}\vec{p}_i \, dE_i \, \delta(E_i^2 - \vec{p}_i^{\,2})\theta(E_i) \\ &= \frac{1}{2E_i} \int d^{D-1}\vec{p}_i \Big|_{E_i = |\vec{p}_i|} \end{aligned}$$

and eliminate p_N by momentum conservation

$$\Rightarrow \quad \int d\Phi_N^D = (2\pi)^{N-D(N-1)} \, 2^{1-N} \int \prod_{j=1}^{N-1} d^{D-1}\vec{p}_j \, \frac{\Theta(E_j)}{E_j} \, \delta^+([Q - \sum_{i=1}^{N-1} p_i]^2)\Big|_{E_j = |\vec{p}_j|}$$

for polar coordinates need phase space volume of unit sphere in D dimensions

$$\int d\Omega_{D-1} \;=\; V(D) = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \qquad V(D) \;=\; \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \sin\theta_2 \ldots \int_0^{\pi} d\theta_{D-1} (\sin\theta_{D-1})^{D-2}$$

real radiation in D dimensions

polar coord. $$\frac{\mathrm{d}^{D-1}\vec{p}}{|\vec{p}|} f(|\vec{p}|) = \mathrm{d}\Omega_{D-2}\,\mathrm{d}|\vec{p}|\,|\vec{p}|^{D-3} f(|\vec{p}|)$$, use $|\vec{p}_j| = E_j$

(massless case)

1 to 3 particle phase space:

$$\begin{aligned} p^{\gamma} &= (\sqrt{s}, \vec{0}^{(D-1)}) \\ p_1 &= E_1\,(1, \vec{0}^{(D-2)}, 1) \\ p_2 &= E_2\,(1, \vec{0}^{(D-3)}, \sin\theta, \cos\theta) \\ p_3 &= p^{\gamma} - p_2 - p_1 \end{aligned}$$

$$x_i = \frac{2p_i \cdot p^{\gamma}}{s}$$

$$d\Phi_{1\to 3} = \frac{1}{4}(2\pi)^{3-2\,D}\; dE_1 dE_2 d\theta\,[E_1 E_2 \sin\theta]^{D-3} d\Omega_{D-2}\; d\Omega_{D-3}$$

$$= (2\pi)^{3-2\,D}\,\frac{2^{4-D}}{32}\; s^{D-3}\; d\Omega_{D-2}\; d\Omega_{D-3}\,[(1-x_1)\,(1-x_2)\,(1-x_3)]^{D/2-2}$$

$-\epsilon$

$$dx_1\,dx_2\,dx_2\,\Theta(1-x_1)\,\Theta(1-x_2)\,\Theta(1-x_3)\delta(2-x_1-x_2-x_3)$$

$$|\overline{\mathcal{M}}_1|^2 = |\overline{\mathcal{M}}_0^{(D)}|^2\,\frac{2g^2\,C_F}{s}\left(\frac{(x_1^2+x_2^2)(1-\epsilon)+2\epsilon(1-x_3)}{(1-x_1)(1-x_2)} - 2\epsilon\right)$$

combine real and virtual

$$R^{\mathrm{real}} = R^{LO} \times \frac{\alpha_s}{2\pi} C_F \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-3\epsilon)} \left(\frac{s}{4\pi\mu^2}\right)^{-\epsilon} \left\{\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} + \mathcal{O}(\epsilon)\right\}$$

gluon both soft and collinear

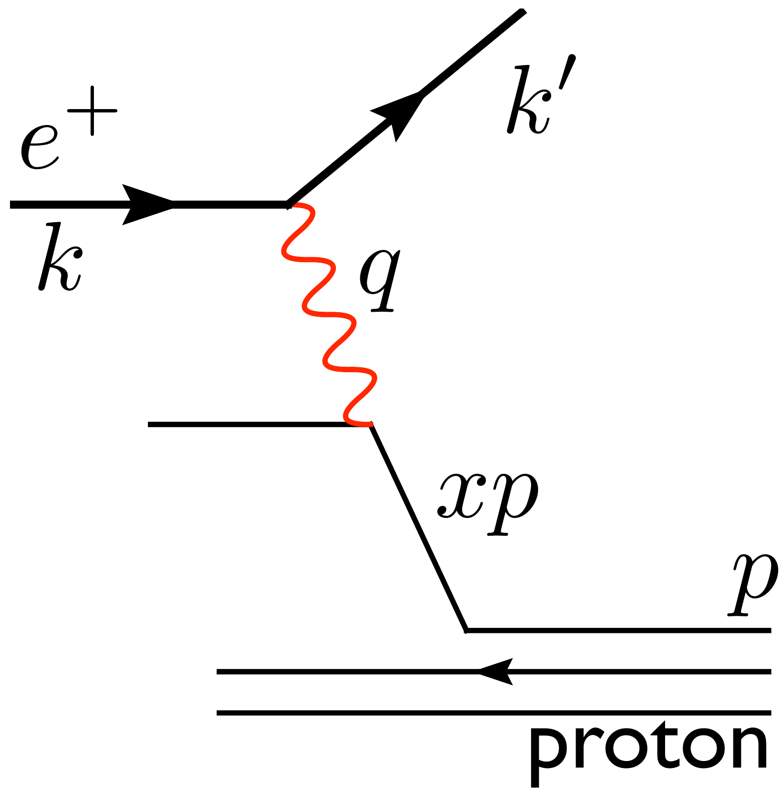
$$R^{\mathrm{virt}} = R^{LO} \times \frac{\alpha_s}{2\pi} C_F \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{-s}{4\pi\mu^2}\right)^{-\epsilon} \left\{-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon)\right\}$$

KLN theorem at work!

$$R = R^{LO} \times \left\{1 + \frac{3}{4} C_F \frac{\alpha_s(\mu)}{\pi} + \mathcal{O}(\alpha_s^2)\right\}$$

hadrons in the initial state

deeply inelastic scattering (DIS) $e(k) + p(P) \to e(k') + X$



$$s = (P+k)^2 \quad [\text{cms energy}]^2$$

$$q^\mu = k^\mu - k'^\mu \quad [\text{momentum transfer}]$$

$$Q^2 = -q^2 = 2MExy \qquad (Q^2 \gg 1\,\text{GeV}^2)$$

$$x = \frac{Q^2}{2P\cdot q} \quad [\text{scaling variable}]$$

$$y = \frac{P\cdot q}{P\cdot k} = 1 - \frac{E'}{E} \quad [\text{relative energy loss}]$$

$$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\alpha^2}{y\,Q^2}\left[\left(1+(1-y)^2\right)F_1 + \frac{1-y}{x}\left(F_2 - 2xF_1\right)\right]$$

F_1, F_2 : structure functions

Deep-inelastic scattering

in the scaling limit $Q^2 \to \infty$ with x fixed:

$2xF_1 \to F_2$ (Callan-Gross relation) and $F_2(x, Q^2) \to F_2(x)$

characteristic for elastic scattering at spin-1/2 particles

$\longrightarrow$ confirmation of the parton model, since it predicts

$$F_2(x) = \sum_i \int_0^1 d\xi \, f_i(\xi) \, x \, e_{q_i}^2 \, \delta(x - \xi) = x \sum_i e_{q_i}^2 \, f_i(x)$$

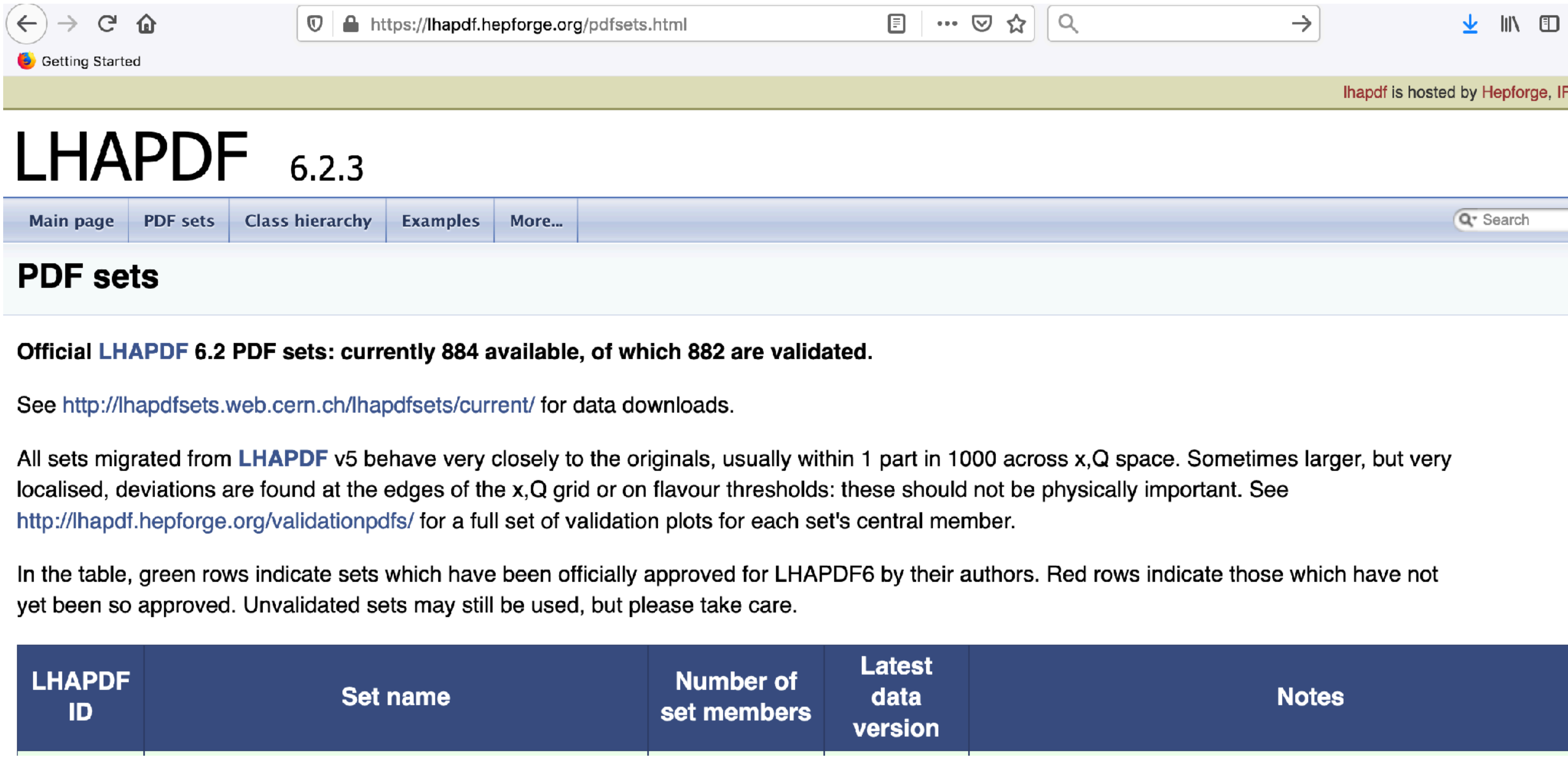
$f_i(\xi)$ denotes the probability that a parton ($q, \bar{q}, g$) with flavour *i*

carries a momentum fraction of the proton between ξ and $\xi + d\xi$

$f_i(\xi)$: parton distribution functions (PDFs)

are fitted from data, but their energy scale dependence is calculable in perturbation theory

PDF sets



https://lhapdf.hepforge.org/pdfsets.html

Getting Started

lhapdf is hosted by Hepforge, IF

LHAPDF 6.2.3

Main page | PDF sets | Class hierarchy | Examples | More...

Search

PDF sets

Official LHAPDF 6.2 PDF sets: currently 884 available, of which 882 are validated.

See http://lhapdfsets.web.cern.ch/lhapdfsets/current/ for data downloads.

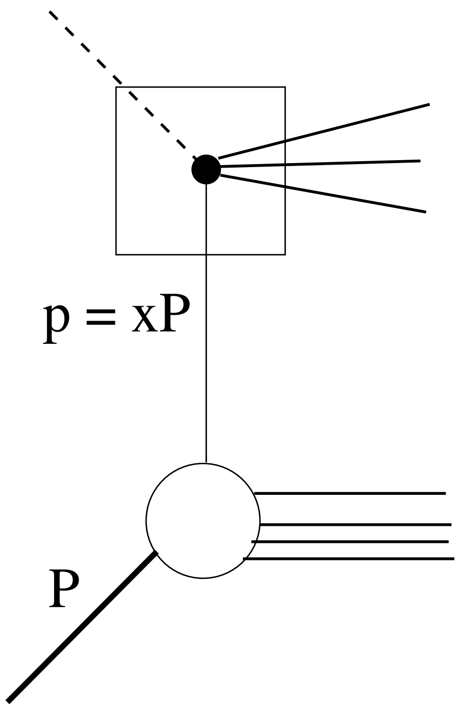
All sets migrated from **LHAPDF** v5 behave very closely to the originals, usually within 1 part in 1000 across x,Q space. Sometimes larger, but very localised, deviations are found at the edges of the x,Q grid or on flavour thresholds: these should not be physically important. See http://lhapdf.hepforge.org/validationpdfs/ for a full set of validation plots for each set's central member.

In the table, green rows indicate sets which have been officially approved for LHAPDF6 by their authors. Red rows indicate those which have not yet been so approved. Unvalidated sets may still be used, but please take care.

LHAPDF ID	Set name	Number of set members	Latest data version	Notes

hadronic initial states

in general:



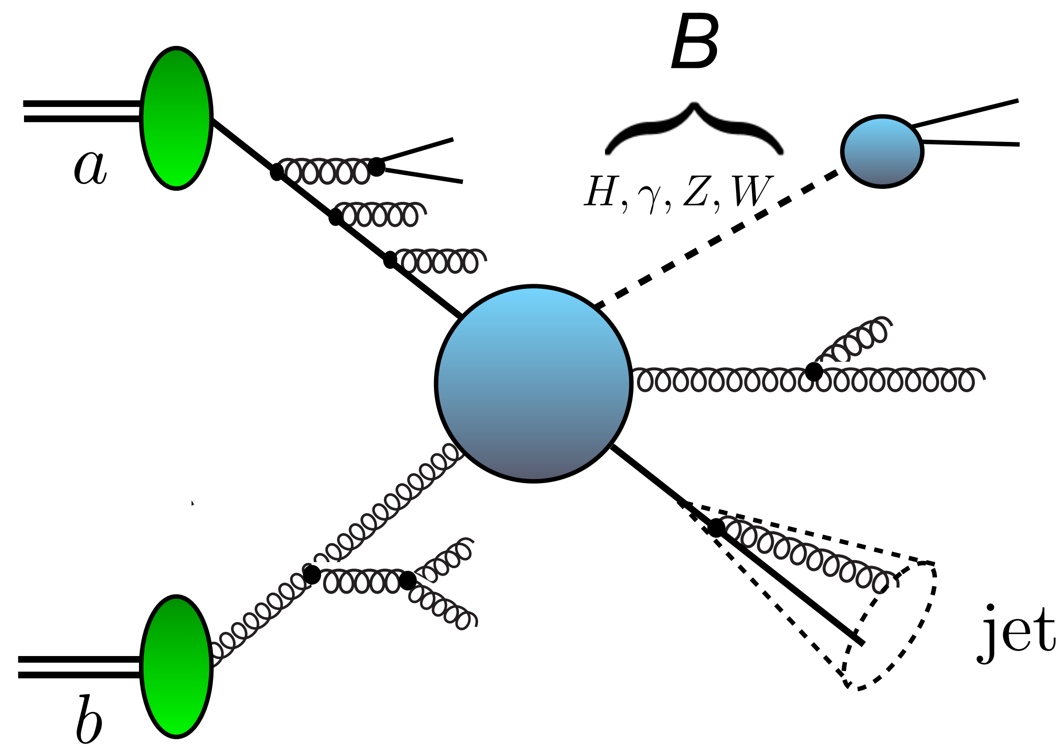
- factorisation allows to separate short-distance from long-distance effects

- hadronic cross section is written as a convolution of the partonic cross section $\hat{\sigma}_i$ with the corresponding PDF $f_{i/H}$

$$\sigma_H(P) = \sum_i \int_0^1 dx\, f_{i/H}(x)\, \hat{\sigma}_i(xP)$$

in principle the same for two hadrons in the initial state

hadron-hadron collisions



$$d\sigma_{pp\to B+X} = \sum_{i,j} \int_0^1 dx_1\, f_{i/p_a}(x_1, \alpha_s, \mu_f) \int_0^1 dx_2\, f_{j/p_b}(x_2, \alpha_s, \mu_f)$$

$$\times d\hat{\sigma}_{ij\to B+X}(\{p\}, x_1, x_2, \alpha_s(\mu_r), \mu_r, \mu_f)\, J(\{p\}) \;+\; \mathcal{O}\left(\frac{\Lambda}{Q}\right)^p$$

measurement function

partonic momenta

factorisation scale μ_f

renormalisation scale μ_r , $\alpha_s(\mu_r)$

back to DIS

$$F_2(x) = x \sum_{i=u,d,s,\ldots} e_i^2 \; [q_i(x) + \bar{q}_i(x)]$$

corresponds to the naive parton model

There are perturbative corrections from the “splitting” of partons as well as non-perturbative effects

For example $\sum_i \int_0^1 dx\, x\, [q_i(x) + \bar{q}_i(x)] \simeq 0.5$

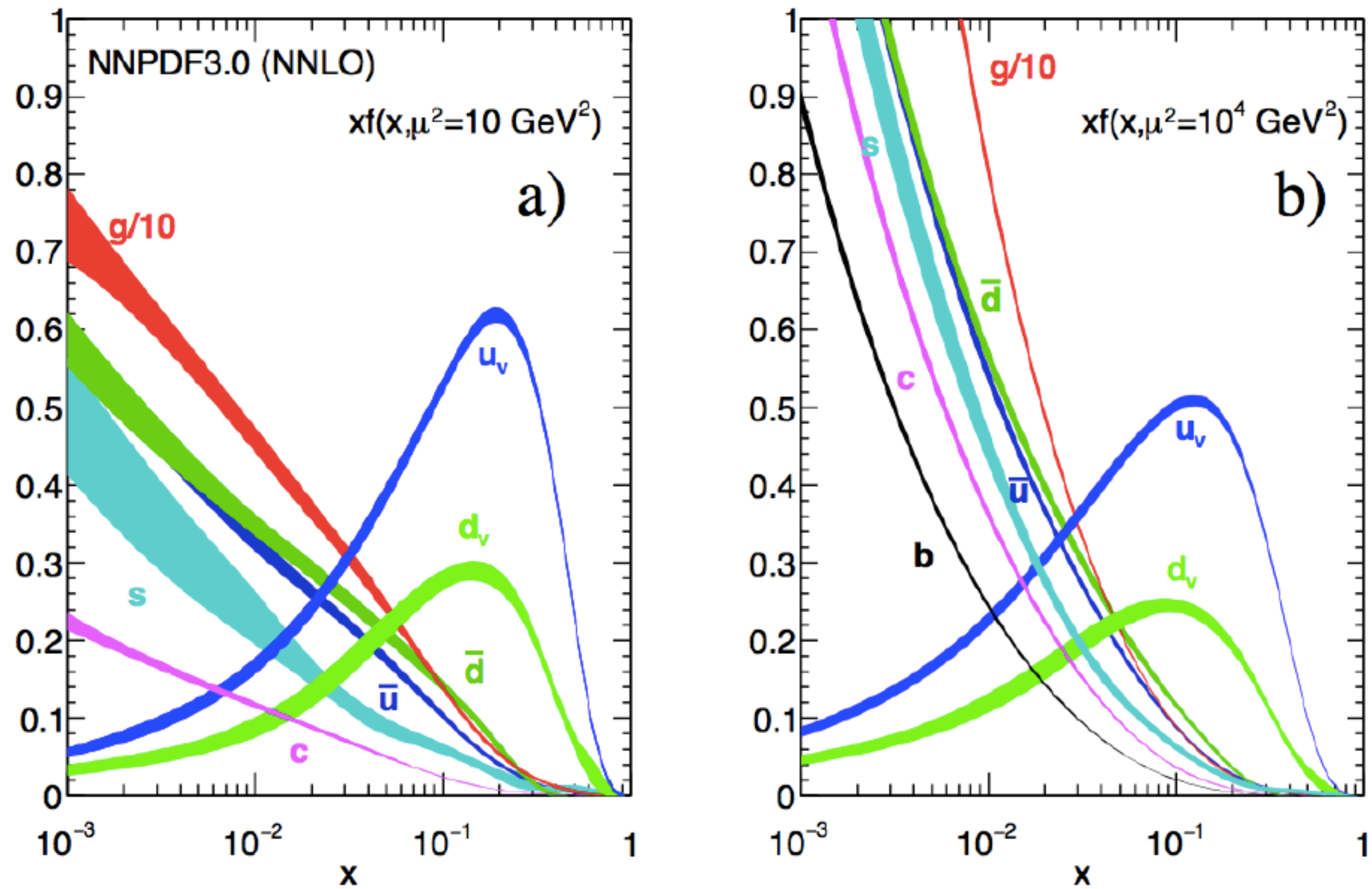
So quarks carry only about half of the proton momentum,

the rest is carried by gluons

PDFs

sea quarks and gluons play a larger role than valence quarks at

- low x
- large Q^2



source: Particle Data Group

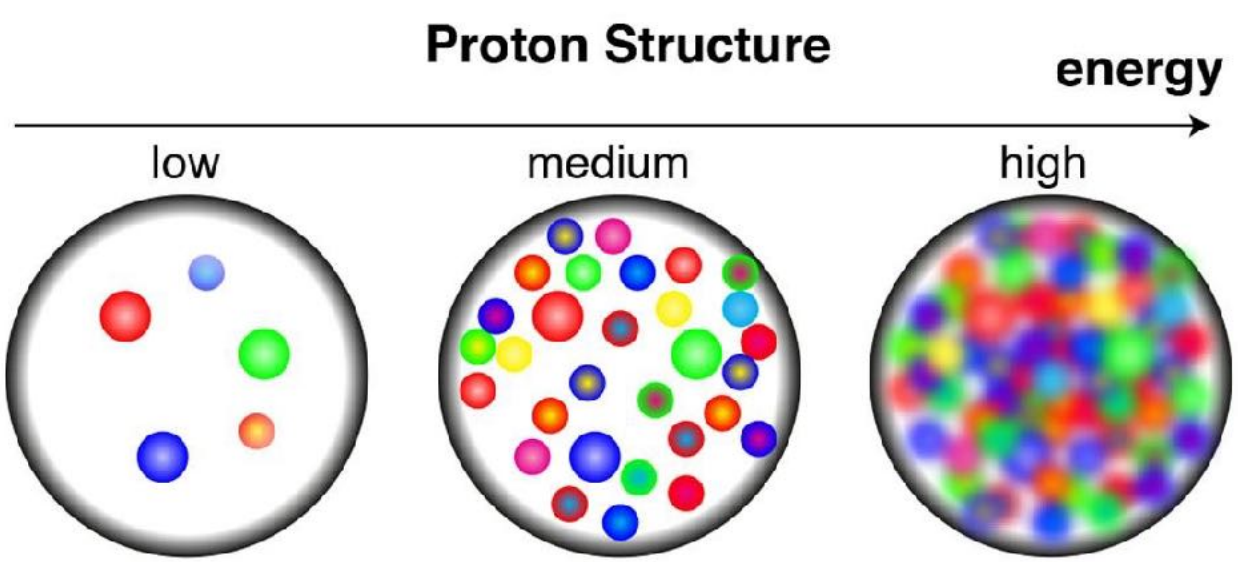
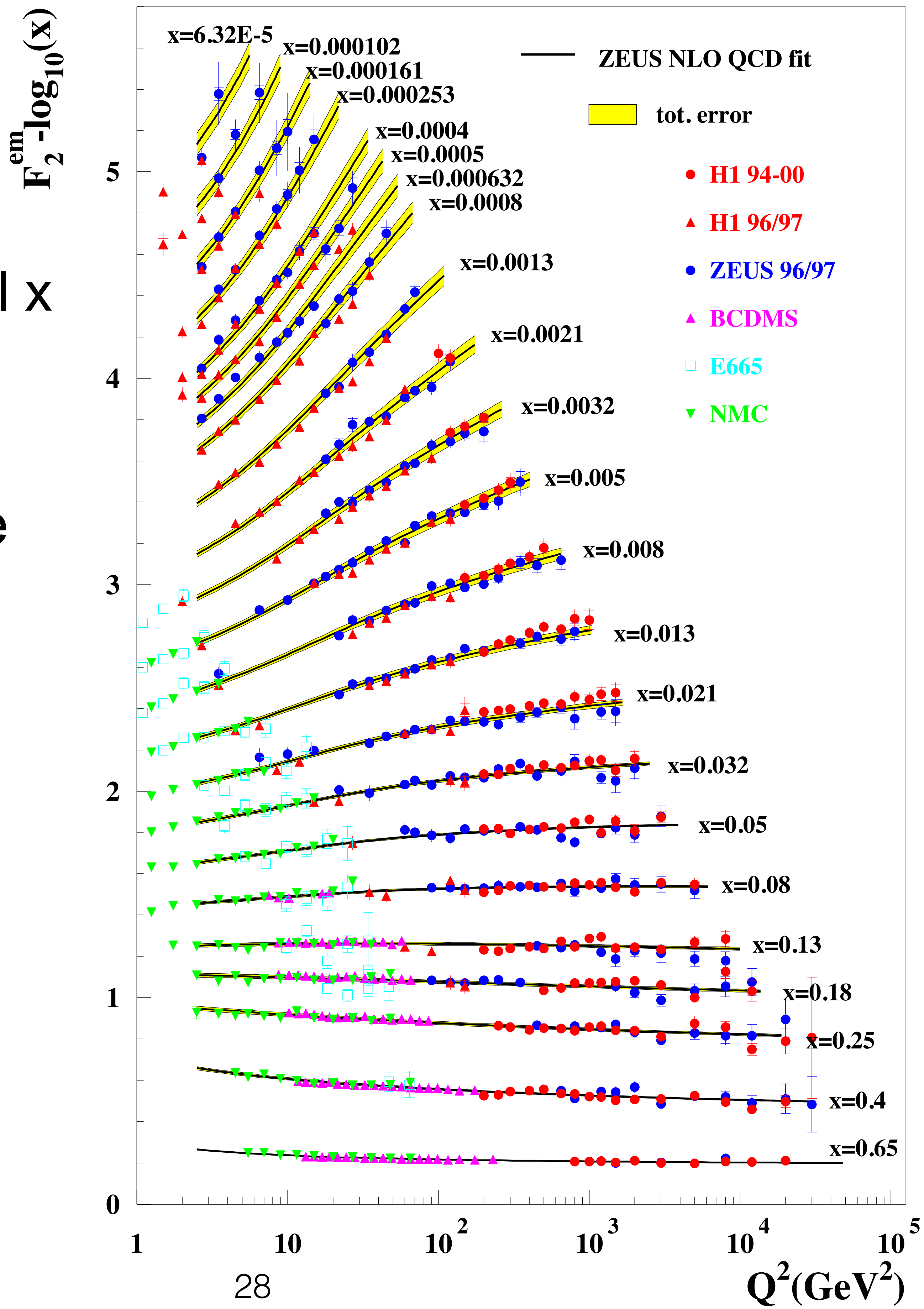


image source: Utrecht University

(almost) Scaling

- scaling is violated for small x

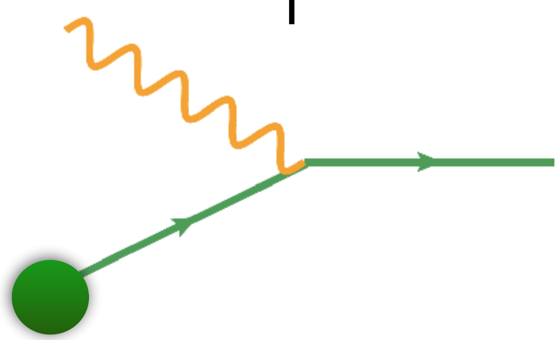
- can be understood from higher order perturbative corrections in α_s



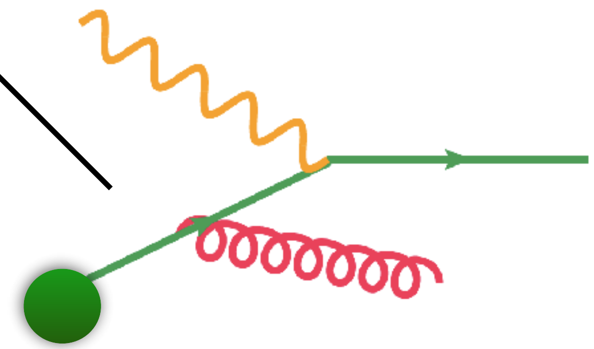
hep-ex/0212008

beyond the parton model

$$\hat{F}_{2,q}(x) = e_q^2 x \left[\delta(1-x) + \frac{\alpha_\mathrm{s}}{4\pi}\left(-\left(\frac{Q^2}{\mu^2}\right)^{-\epsilon}\frac{1}{\epsilon}P_{q\to qg}(x) + C_2^q(x)\right)\right]$$



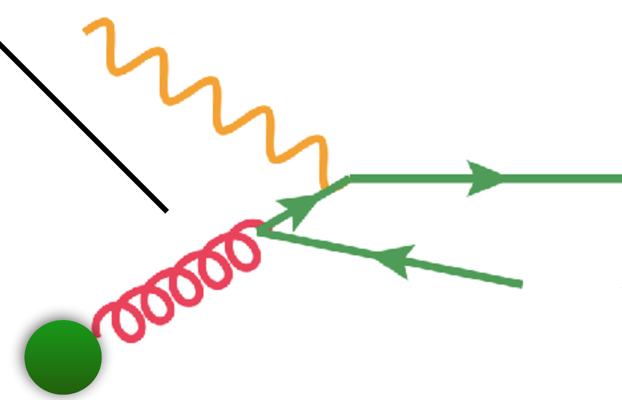
parton model



gluon emission

decreases parton momentum

$$\hat{F}_{2,g}(x) = \sum_q e_q^2 x \left[0 + \frac{\alpha_s}{4\pi}\left(-\left(\frac{Q^2}{\mu^2}\right)^{-\epsilon}\frac{1}{\epsilon}P_{g\to q\bar{q}}(x) + C_2^g(x)\right)\right]$$

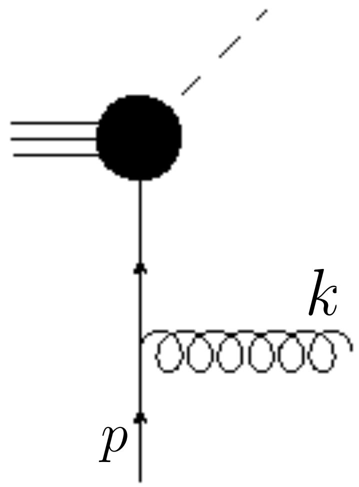


splitting of a gluon into a quark-antiquark pair

PDFs and DGLAP evolution

consider the emission of one gluon in the initial state

(we have encountered this already for final state emission)



phase space factor for one gluon emission:

$$d\Phi \sim \frac{d^{D-1}k}{2k_0} \sim dz\,(1-z)^{-1-\epsilon} dk_\perp^2 (k_\perp^2)^{-\epsilon}$$

In the collinear limit $k_\perp^2 \to 0$

$$d\Phi \left|\bar{M}_1^{\mathrm{real}}(p,k)\right|^2 \sim \frac{\alpha_s}{2\pi} \frac{dk_\perp^2}{(k_\perp^2)^{1+\epsilon}} dz\,(1-z)^{-\epsilon} P_{qq}(z,\epsilon) \left|\bar{M}_0(zp)\right|^2$$

$$P_{qq}(z,\epsilon) = C_F \frac{1+z^2}{1-z} - \epsilon\,(1-z)$$

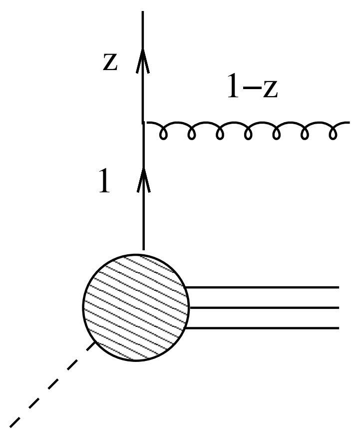
virtual corrections in IR limit: $\sim |\bar{M}_0(p)|^2$

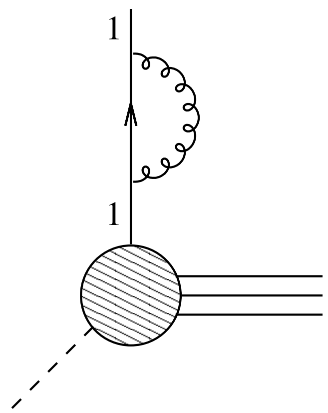
note that soft limit is z $\longrightarrow$ 1 $\Rightarrow$ cancellation in soft limit but not in collinear limit

PDFs and DGLAP evolution

Recap:

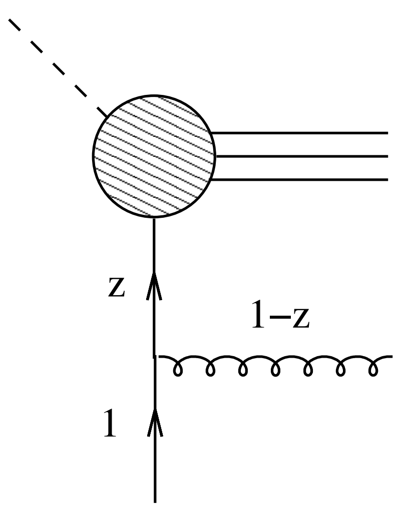
gluon emission in final state:

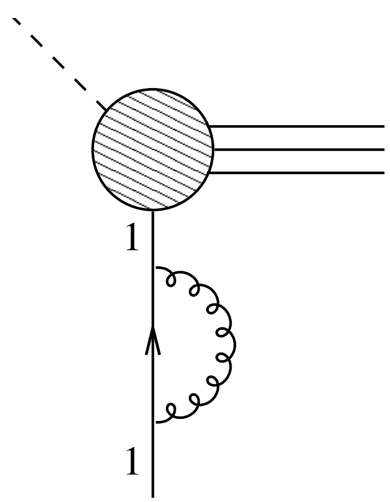




both soft and collinear singularities
cancel between real and virtual
corrections

gluon emission in initial state:





only soft singularities
cancel between real and
virtual corrections

PDFs and DGLAP evolution

Absorb initial state singularities at factorisation scale μ
into "bare PDFs" to obtain the measured PDFs

$$f_i(x,\mu_f^2) = f_i^{(0)}(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{\mathrm{d}\xi}{\xi}\left\{ f_i^{(0)}(\xi)\left[-\frac{1}{\epsilon}\left(\frac{\mu_f^2}{\mu^2}\right)^{-\epsilon} P_{q\to qg}\left(\frac{x}{\xi}\right) + K_{qq}\right]\right\}$$

evolution with μ^2 can be predicted within perturbative QCD

$$\mu^2 \frac{\partial f_{i/H}(x,\mu)}{\partial \mu^2} = \frac{\alpha_s}{2\pi}\sum_j \int_x^1 \frac{dz}{z}\left[P_{ij}(z)\right]_+ f_{j/H}\left(\frac{x}{z},\mu\right)$$

DGLAP evolution equation

(Dokshitzer, Gribov, Lipatov, Altarelli, Parisi)

can be extended to higher orders in α_s

$$\mu^2 \frac{\partial f_{i/H}(x,\mu)}{\partial \mu^2} = \sum_j \int_x^1 \frac{dz}{z}\left[\mathcal{P}_{ij}(\alpha_s(\mu),z)\right]_+ f_{j/H}\left(\frac{x}{z},\mu\right)$$

$$\mathcal{P}_{ij}(\alpha_s(\mu),z) = P_{ij}^{(0)}(z) + \frac{\alpha_s(\mu)}{2\pi}P_{ij}^{(1)}(z) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 P_{ij}^{(2)}(z) + \ldots$$

LO (1974) NLO (1980) NNLO (2004, Moch, Vermaseren Vogt)

DGLAP evolution

(flavour) singlet evolution equations: $\Sigma(x,Q^2) \equiv \sum_{i=1}^{n_f} \left(q_i(x,Q^2) + \bar{q}_i(x,Q^2)\right)$

$$\frac{\partial}{\partial \ln Q^2} \begin{pmatrix} \Sigma(x,Q^2) \\ g(x,Q^2) \end{pmatrix} = \int_x^1 \frac{\mathrm{d}y}{y} \begin{pmatrix} P_{qq}^S\left(\frac{x}{y}, \alpha_S(Q^2)\right) & 2n_f P_{qg}^S\left(\frac{x}{y}, \alpha_S(Q^2)\right) \\ P_{gq}^S\left(\frac{x}{y}, \alpha_S(Q^2)\right) & P_{gg}^S\left(\frac{x}{y}, \alpha_S(Q^2)\right) \end{pmatrix} \begin{pmatrix} \Sigma(y,Q^2) \\ g(y,Q^2) \end{pmatrix}$$

non-singlet: $q_{ij}^{\mathrm{NS}}(x,Q^2) = q_i(x,Q^2) - q_j(x,Q^2)$

$$\frac{\partial}{\partial \ln Q^2} q_{ij}^{\mathrm{NS}}(x,Q^2) = \int_x^1 \frac{\mathrm{d}y}{y} P_{ij}^{\mathrm{NS}}\left(\frac{x}{y}, \alpha_S(Q^2)\right) q_{ij}^{\mathrm{NS}}(y,Q^2)$$

constraints: $\int_0^1 \mathrm{d}x\, x \left[\sum_{i=1}^{n_f} \left(q_i(x,Q^2) + \bar{q}_i(x,Q^2)\right) + g(x,Q^2)\right] = 1.$ (total momentum of the proton is carried by its constituents)

$\int_0^1 \mathrm{d}x \left(q_i(x,Q^2) - \bar{q}_i(x,Q^2)\right) = n_i \qquad (n_u = 2, n_d = 1, n_{s,c,b,t} = 0)$ (baryon number conservation)

number of valence quarks

recent developments

from PDF determination “wishlist” 2013 [S.Forte, G.Watt, **1301.6754**]

- The parametrisation should be sufficiently general and unbiased

  e.g. new approach based on deep learning [S.Carrazza et al. ’19]

- The experimental uncertainties should be understood and carefully propagated

  LHAPDF6: metadata `ErrorType`, `ErrorConfLevel` [A.Buckley et al. ’14]

- PDFs including electroweak corrections will have to be constructed

  QED corrections done (see next slide)

- The treatment of heavy quarks will have to include mass-suppressed terms

  in progress, see e.g. Blümlein, Moch et al.

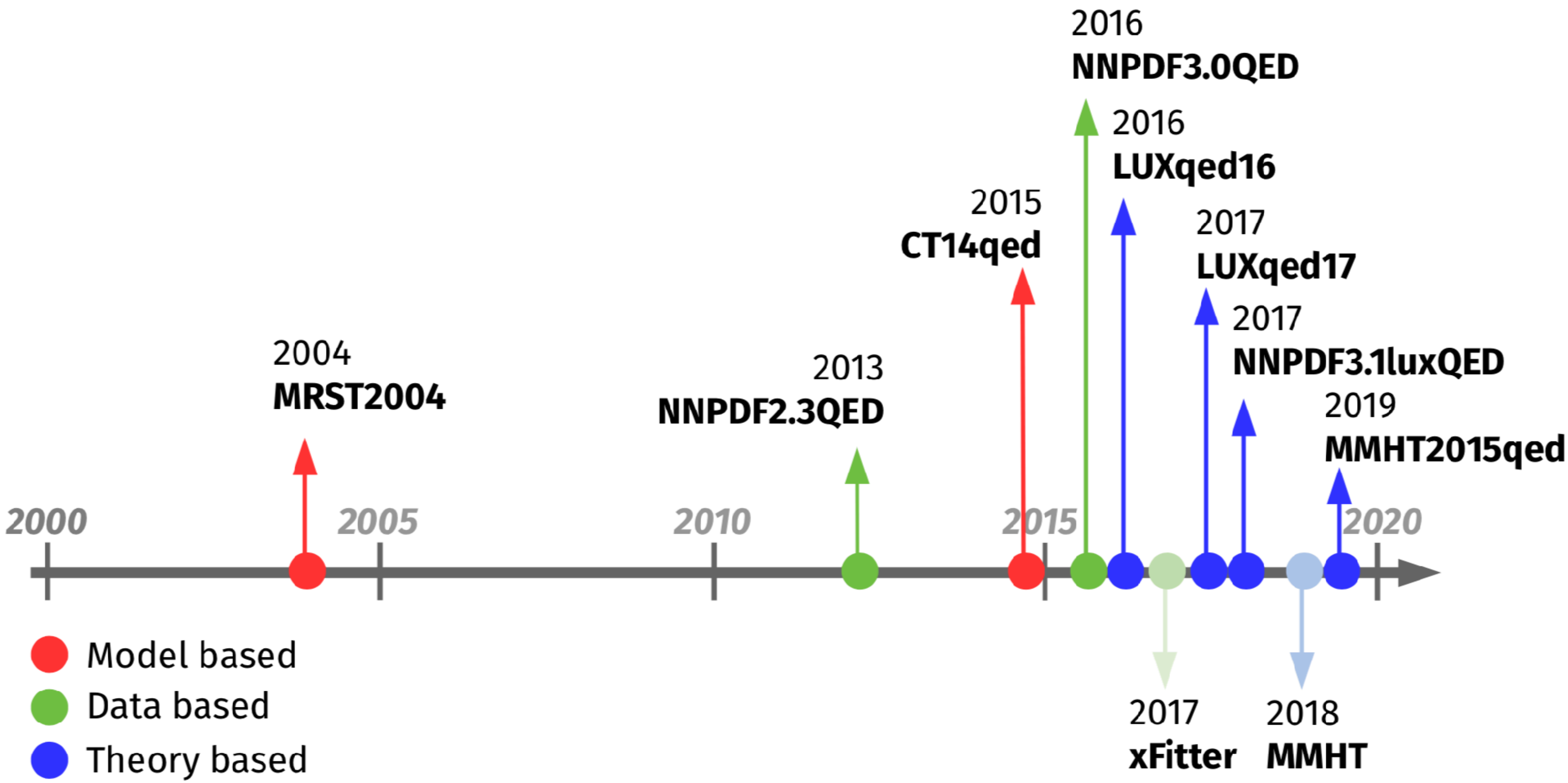
- The strong coupling, in addition to being determined simultaneously with PDFs, should also be decoupled from the PDF determination,

  available, see e.g. **PDF4LHC15** J. Butterworth et al. ‘15

- An estimate of theoretical uncertainties should be performed together with PDF sets

  depends …

PDFs with QED corrections



S.Carrazza, E.Villa et al, 1909.10547

One-loop integrals

simple example:

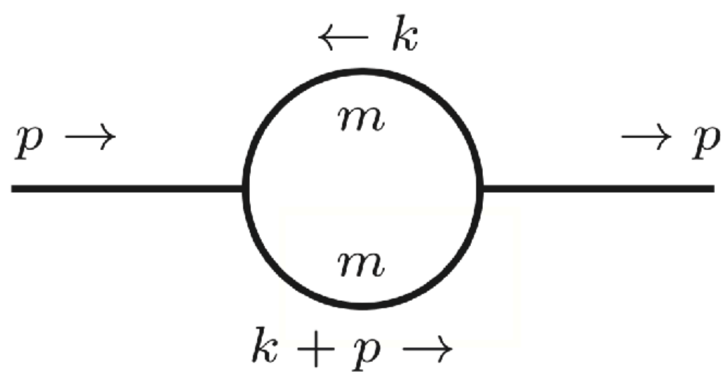


figure: Stephan Jahn

$$I_2 = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2 + i\delta][(k+p)^2 - m^2 + i\delta]}$$

for $|k| \to 0$ denominator cannot vanish if $m \neq 0$

for $|k| \to \infty$: spherical coordinates:

$$I_2 \sim \int \mathrm{d}\Omega_3 \int_{|k|_{\min}}^{\infty} \mathrm{d}|k| \frac{|k|^3}{|k|^4} \sim \lim_{\Lambda \to \infty} \int_{|k|_{\min}}^{\Lambda} \frac{\mathrm{d}|k|}{|k|}$$

divergent for $|k| \to \infty$ (UV)

one-loop integrals

we can isolate the divergence in terms of $\log \Lambda$

however a regulator that preserves Lorentz covariance is much more convenient (gauge invariance, renormalisation, …)

dimensional regularisation: (see previous lecture)

work in $D = 4 - 2\epsilon$ dimensions

$$g^2 \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \longrightarrow g^2 \mu^{2\epsilon} \int_{-\infty}^{\infty} \frac{d^Dk}{(2\pi)^D}$$

decreasing the dimension will help the UV problem (less powers of $|k|$ in the numerator)

so to regulate UV divergences, formally $\epsilon > 0$

(however it is an analytic continuation of the integral where the sign does not need to be specified)

dimensional regularisation

to cure *IR divergences*, it helps to *increase* the dimension $(\epsilon < 0)$

how can we use both signs at the same time?

formally:

- first calculate amplitude assuming IR divergences are regulated (off-shell, mass)
- then all 1/eps poles will be of UV nature $\longrightarrow$ perform UV renormalisation
- for UV finite amplitude, analytically continue to $Re(D) > 4$
- remove auxiliary IR regulator $\longrightarrow$ IR poles will manifest as 1/eps poles

in practice, we just use *D,* both UV and IR poles appear as powers of $1/\epsilon$

note: other methods than dim. reg. exist and are appealing, making pole cancellations manifest at integrand level; however this is not straightforward

regularisation schemes

Clifford algebra needs to be extended to D dimensions:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu} \quad \text{with} \quad g^\mu_\mu = D}$$

leads for example to $\gamma_\mu \not{p} \gamma^\mu = (2-D)\not{p}$

problem: $$\boxed{\gamma_5 \equiv i\,\gamma_0\gamma_1\gamma_2\gamma_3 = \frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma}$$

$\varepsilon^{\mu\nu\rho\sigma}$ totally antisymmetric tensor

is an intrinsically 4-dim. quantity

in 4-dim:

$$\gamma_5^2 = \mathbf{1}\ ,\ \{\gamma_\mu, \gamma_5\} = 0\ ,\ \mathrm{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5) = 4i\varepsilon_{\mu\nu\rho\sigma}$$

in D-dim. these conditions cannot hold simultaneously!

regularisation schemes

proof: consider the expression $\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(\gamma_\tau\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma^\tau\gamma_5\right)$

and use $\{\gamma_\mu,\gamma_5\}=0$ and the cyclicity of the trace or $\{\gamma^\mu,\gamma^\nu\}=2\,g^{\mu\nu}$

to contract $\gamma_\tau\gamma^\tau=D$

leads to $(D-4)\;\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right)=0$

different prescriptions are available in the literature to remedy this, e.g.
[‘tHooft,Veltman ’72; Breitenlohner, Maison ’77; Larin ’93]

$$\gamma_\mu=\bar\gamma_\mu+\tilde\gamma_\mu$$

$$\tilde\gamma_\mu:(D-4)-\mathrm{dim.}$$

$$\{\gamma^\mu,\gamma_5\}=\begin{cases}0 & \mu\in\{0,1,2,3\}\\ 2\tilde\gamma^\mu\gamma_5 & \text{otherwise.}\end{cases}$$

breaks axial Ward Identities, fix by “finite renormalisation”

or give up cyclicity of the trace, but keep $\{\gamma_\mu^{(D)},\gamma_5\}=0$ [Kreimer, Körner, Schilcher ’92]

see also recent paper by N.Zerf https://arxiv.org/abs/1911.06345

regularisation schemes

even without γ_5 the extension to D dimensions is not unique

in principle only the unobserved momenta need to be D-dim.

some possibilities: (see also Signer, Stöckinger 0807.4424)

- **CDR:** "conventional dim. reg."
  internal and external gluons (and other vector fields) are treated as D-dim.
- **HV:** "'t Hooft-Veltman" internal: D-dim., external: 4-dim.
- **DR:** "dimensional reduction" only loop momenta D-dim., otherwise (quasi-) 4-dim.
- **FDH:** "four-dimensional helicity scheme" as DR, but external states strictly 4-dim.

- at one loop, CDR and HV are equivalent,
  similarly DR and FDH are equivalent,
  as terms of order epsilon in external momenta do not play a role
- different beyond one loop!

more about schemes when we discuss UV renormalisation …

addendum to regularisation schemes

distinguish

	CDR	HV	DR	FDH
internal gluon	$\hat{g}^{\mu\nu}$	$\hat{g}^{\mu\nu}$	$g^{\mu\nu}$	$g^{\mu\nu}$
external gluon	$\hat{g}^{\mu\nu}$	$\bar{g}^{\mu\nu}$	$g^{\mu\nu}$	$\bar{g}^{\mu\nu}$

$g^{\mu\nu}$ quasi-4-dim.

$\hat{g}^{\mu\nu}$ D-dim. (subspace of above)

$\bar{g}^{\mu\nu}$ strictly 4-dim.

$$g^{\mu\nu}g_{\mu\nu} = 4,\ \hat{g}^{\mu\nu}\hat{g}_{\mu\nu} = D = 4 - 2\epsilon,\ \bar{g}^{\mu\nu}\bar{g}_{\mu\nu} = 4$$

in projections dimensionality matters!

$$g^{\mu\nu}\hat{g}_{\nu}{}^{\rho} = \hat{g}^{\mu\rho},\ g^{\mu\nu}\bar{g}_{\nu}{}^{\rho} = \bar{g}^{\mu\rho},\ \hat{g}^{\mu\nu}\bar{g}_{\nu}{}^{\rho} = \bar{g}^{\mu\rho}$$

Summary

- We know the building blocks of NLO cross sections
- We have seen how IR singularities arise
- We have seen how they cancel in inclusive quantities
- We know the origin and evolution of parton distribution functions to deal with hadronic initial states
- Dimensional regularisation: we know about different regularisation schemes