Parton Distribution Functions Introductory Lectures: PDF1

Marco Guzzi

CTEQ School - 19 July 2019

University of Pittsburgh

Outline

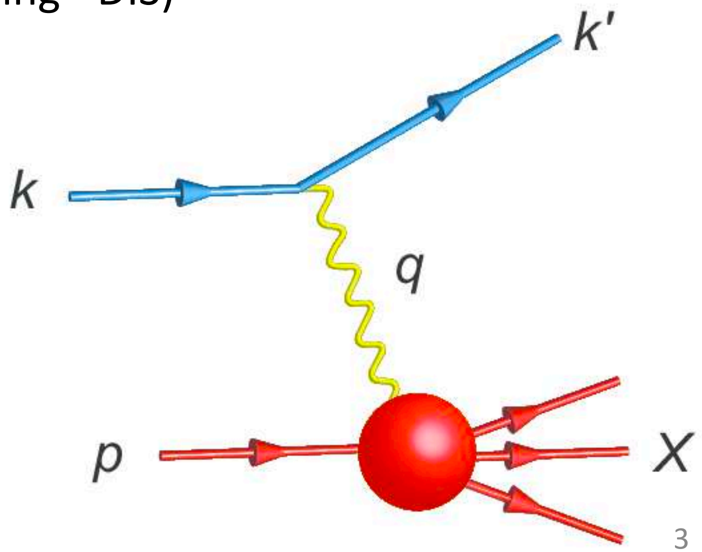
- Basics of Collisions
- Parton Model
- DIS
- Where are PDFs coming from?
- PDF’s evolution: DGLAP equations

Parton Model. Question:

How do we probe the inner structure of objects (hadrons, e.g. protons)?

?

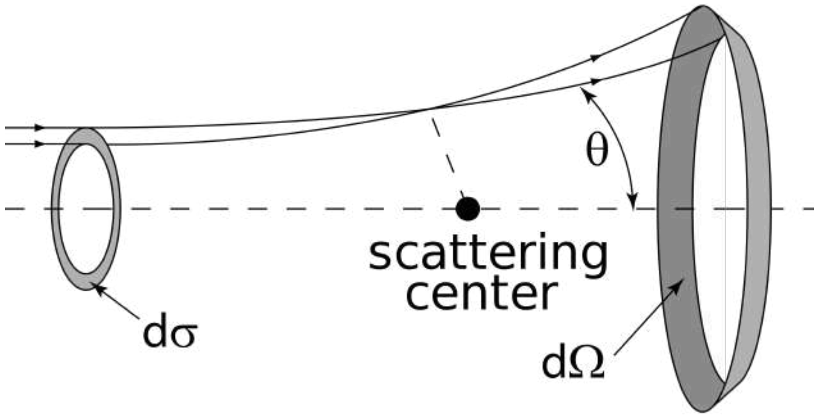
1) point-like projectile on the object (e.g. electron on a proton: Deep Inelastic Scattering - DIS)
2) smash the two objects (proton-(anti)proton collisions: e.g. Drell-Yan)



Partons: R. Feynman, 1969. A way to analyze high-energy hadron collisions.
Any hadron (for example, a proton) can be considered a composition of a number of point-like constituents.

The parton model was immediately applied to DIS by Bjorken and Paschos.

Basics of Collisions



- Rutherford's scattering

non-relativistic projectile with very heavy target (no recoil)

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 4E}\right)^2 \frac{1}{\sin^4(\theta/2)}$$

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}$$ e.m. coupling

For example, we can apply this to $e^-p^+ \rightarrow e^-p^+$ collisions

$$\frac{d\sigma}{d\Omega} = \frac{m_e^2 e^4}{4p^4 \sin^4 \frac{\theta}{2}}$$

Deflection of electrons from a central potential (of a ``heavy'' proton)

$$p = |\vec{p}_i| = |\vec{p}_f|$$ for elastic scattering

Basics of Collisions

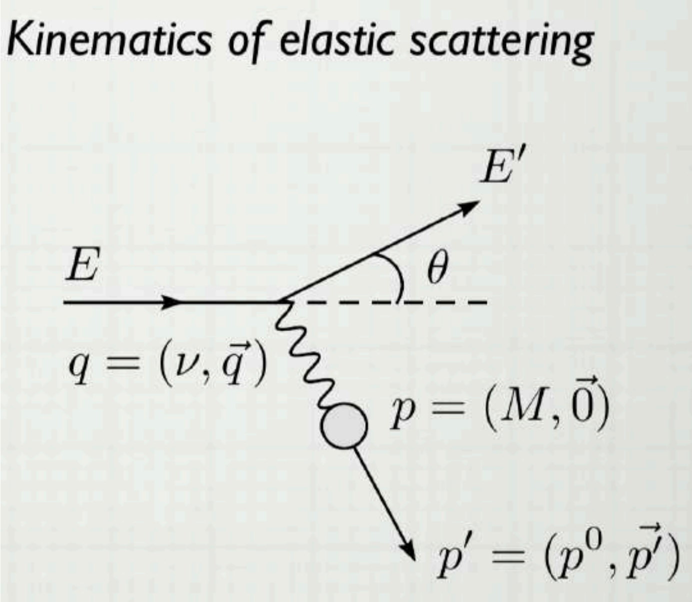
- Mott's scattering

(m_p>>m_e) relativistic correction with recoil of the target: if the proton were a point-charge

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Mott}} = \frac{(\alpha Z)^2}{4E^2 \sin^4 \theta/2} \frac{E'}{E} \cos^2 \theta/2$$

In the LAB frame.

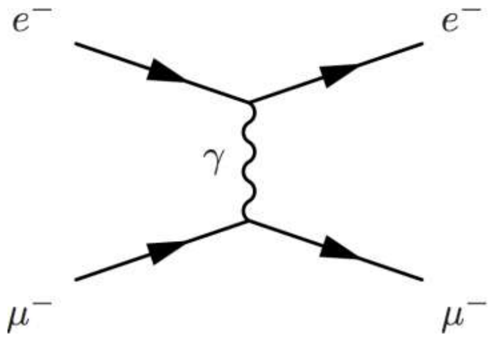
Here we have that E << m_p and m_p>>m_e



Basics of Collisions

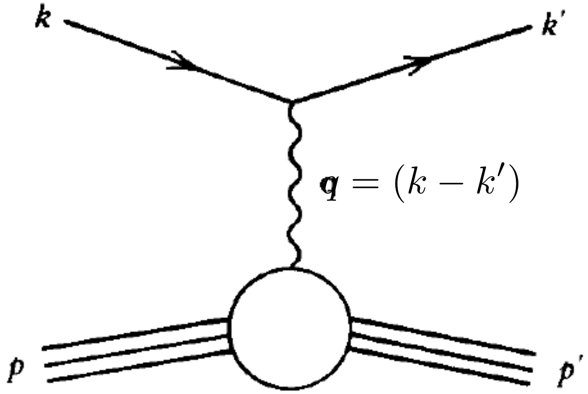
- ep → ep   vs   eμ → eμ

Similar to spin-1/2 relativistic scattering of electrons on muons (m_e= 0)



$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \theta/2} \frac{E'}{E} \left[\cos^2 \theta/2 - \frac{q^2}{2M^2} \sin^2 \theta/2\right]$$

- Variant of the Mott's formula with E >> m_e , $|\vec{k}| = E, \quad |\vec{k}'| = E'$
- Basically QED does not ``distinguish'' between a muon and a proton (up to the charge sign) in the elastic scattering limit



$$-\frac{q^2}{2M^2} = \frac{E - E'}{M}$$

Basics of Collisions

- …but the proton is not a point-charge: Rosenbluth's scattering

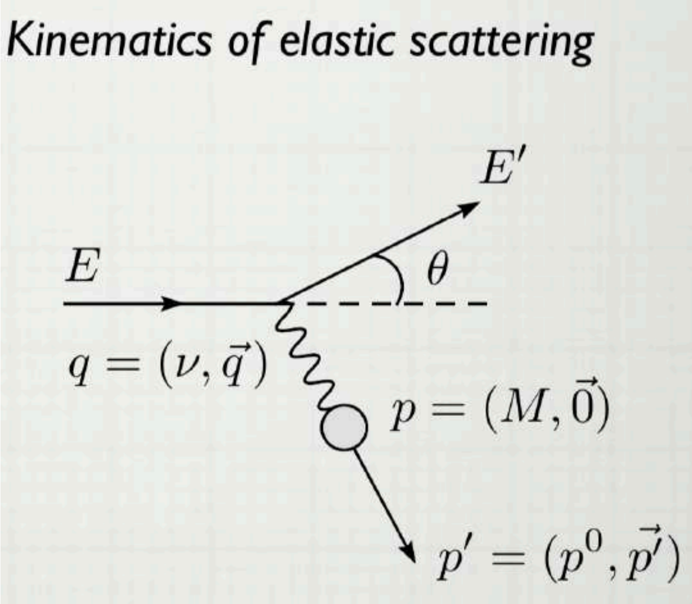
spin-1/2 relativistic projectile scattering on non-point like target with spin-1/2 (proton)

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left[G_M^2(Q^2)\frac{Q^2}{2M^2}\tan^2\frac{\theta}{2} + \frac{G_E^2(Q^2) + G_M^2(Q^2)Q^2/4M^2}{1+Q^2/4M^2} \right] \frac{E'}{E}$$

in terms of the magnetic and electric form factors.

In terms of F_1 and F_2 form factors it reads

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \frac{\alpha_e^2}{4E^2\sin^4\frac{\theta}{2}}\frac{E'}{E}\left\{\left(F_1^2 - \frac{q^2}{4m_p^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m_p^2}\left(F_1+F_2\right)^2\sin^2\frac{\theta}{2}\right\}$$



M is the proton mass

$$Q^2 = -q^2 = 4E'E\sin^2\theta/2$$ Momentum transferred to the proton

$$G_M(Q^2) = F_1(Q^2) + 2MF_2(Q^2)$$ Magnetic form factor

$$G_E(Q^2) = F_1(Q^2) - F_2(Q^2)Q^2/2M$$ Electric form factor

their Fourier transform gives info about the magnetic and charge spatial distribution of the proton.

$$\langle N, p'|j_\mu(0)|N, p\rangle = \bar{u}(p')\left[F_1(Q^2)\gamma_\mu + F_2(Q^2)i\sigma_{\mu\nu}q^\nu\right]u(p)$$ Electromagnetic current between proton states: spin-1/2 parametrization

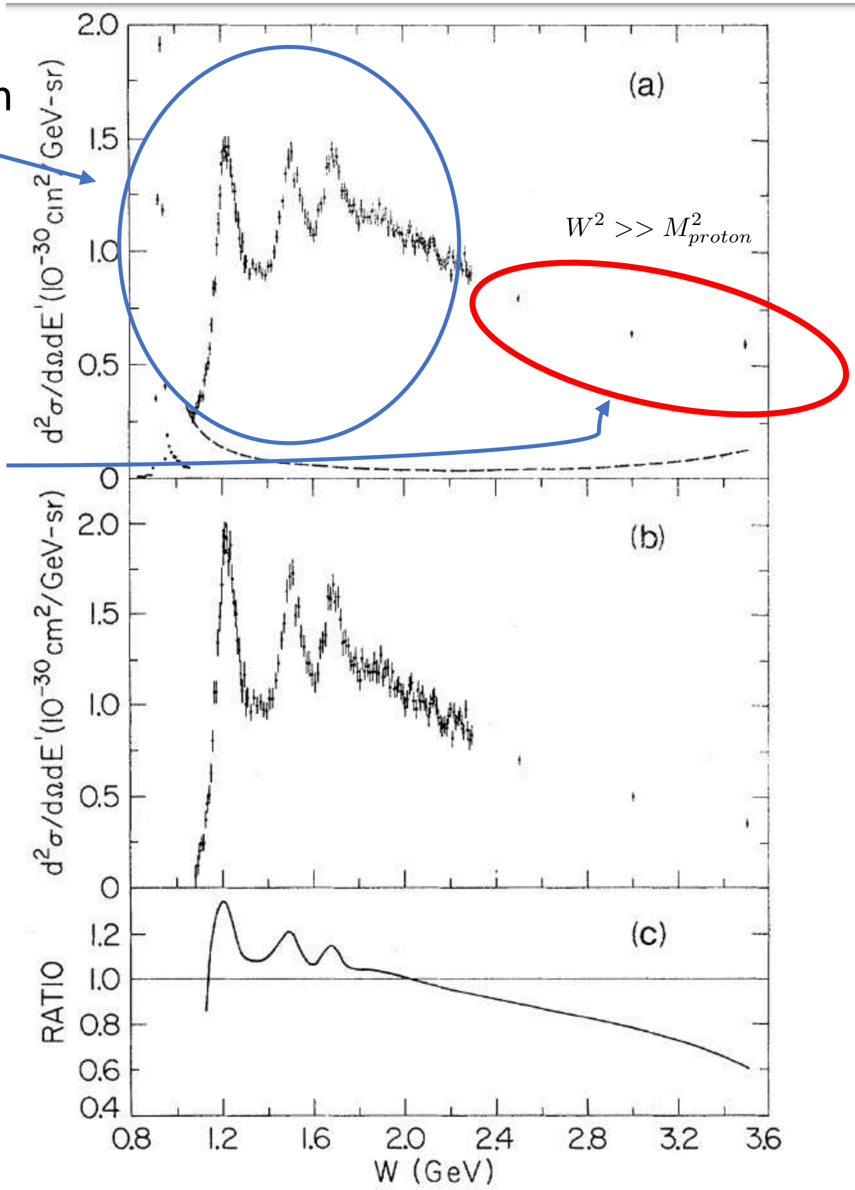
The form factors cannot be calculated from first principles.
They are functions extracted from measured differential cross sections.

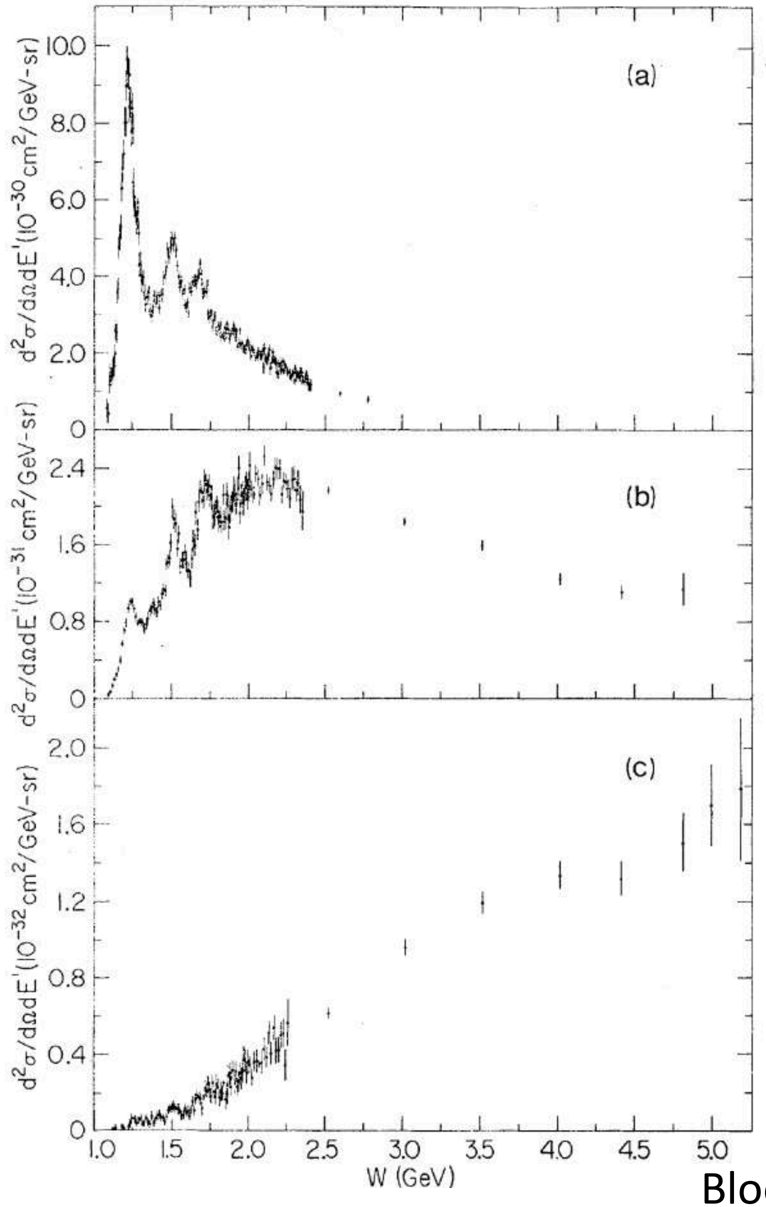
At larger Q, the functional form of G_E and G_M reflects the fact that the proton is an extended object with an inner structure.

DIS data from SLAC which performed an experiment with high-energy electron beams (7-18 GeV).
Scattering of electrons from a hydrogen target at 6^0 and 10^0 .

Resonance region

Deep
Inelastic region





$E = 7\ GeV$
$\theta = 6°$

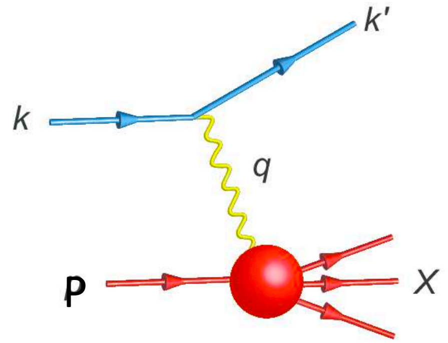
$E = 16\ GeV,$
$\theta = 6°$

$E = 17.7\ GeV,$
$\theta = 10°$

$$W^2 = (P+q)^2 = P^2 + 2P \cdot q + q^2 = M^2 + 2M\nu - Q^2$$

Bloom et al, PRL 1969

When the proton breaks apart (in DIS), the parametrization



$$\bar{u}(p')\left[F_1(Q^2)\gamma_\mu + F_2(Q^2) i\sigma_{\mu\nu}q^\nu\right]u(p)$$

is no longer good. Need to parametrize photon–proton–X interactions, where X is anything the proton can break up into.

Thus, it makes sense to parametrize the cross section (instead of the vertex) in terms of the momentum transfer q and the proton momentum P.

$$\left(\frac{d\sigma}{d\Omega\, dE'}\right)_{\text{lab}} = \frac{\alpha_e^2}{4\pi m_p q^4}\frac{E'}{E} L^{\mu\nu} W_{\mu\nu}$$

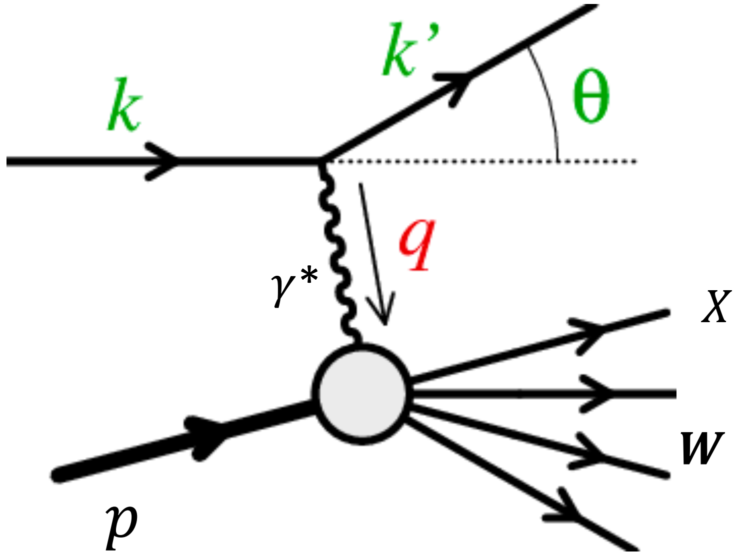
$$L_{\mu\nu} = \frac{1}{2}\text{Tr}\left[\not{k}'\gamma^\mu \not{k}\gamma^\nu\right] = 2(k'^\mu k^\nu + k'^\nu k^\mu - k\cdot k' g^{\mu\nu})$$ Leptonic tensor

$$e^2\epsilon_\mu \epsilon_\nu^\star W^{\mu\nu} = \frac{1}{2}\sum_{X,\text{spins}} \int d\Pi_X (2\pi)^4 \delta^4(q + P - p_X) \left|\mathcal{M}\left(\gamma^\star p^+ \to X\right)\right|^2$$

$$W^{\mu\nu} = W_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + W_2\left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)$$ Hadronic tensor

Deep Inelastic Scattering

$$e + p \to e + X$$



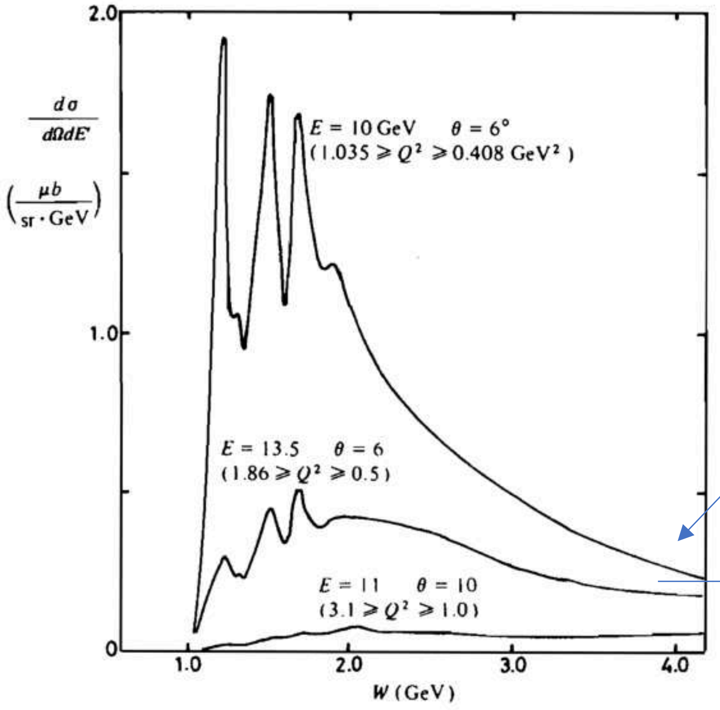
In deep inelastic collisions, E' and θ are independent (double diff. Xsec)

$$\frac{d\sigma}{d\Omega dE'} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \left[2W_1(\nu, Q^2)\tan^2\frac{\theta}{2} + W_2(\nu, Q^2)\right] / 2M$$

$$W^2 >> M^2_{proton}$$

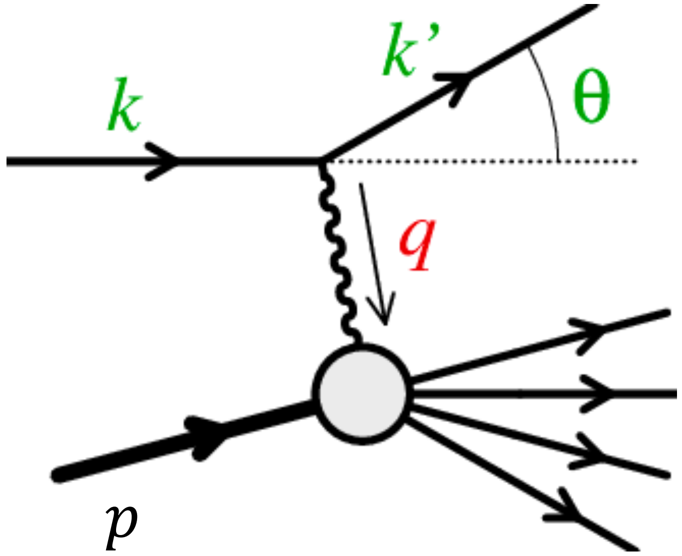
This expression contains the Mott cross-section as a factor and is analogous to the Rosenbluth formula.
It isolates the unknown shape of the nucleon target in two structure functions W_1 and W_2,
which are relativistic invariant functions of two independent variables ν and q^2. The structure functions correspond to the two possible polarization states of the virtual photon: longitudinal and transverse.
Longitudinal polarization exists only because photon is virtual.

- E', θ experimentally easy to understand
- x, Q^2 Lorentz invariant



Continuum (at large W) contribution to the Xsec sizeable even at large Q: indication of point like objects inside the proton

quasi Q-independent



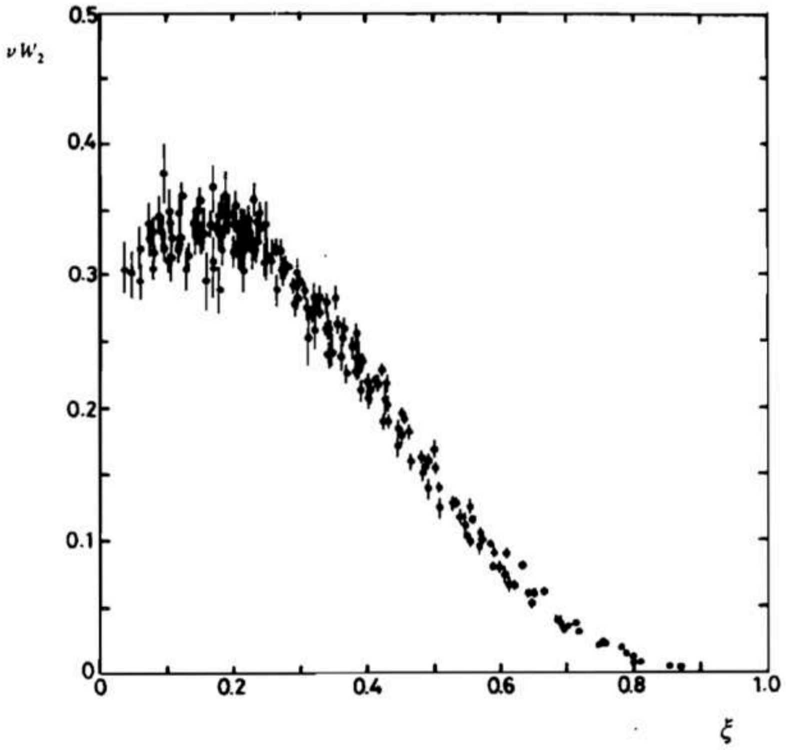
Bjorken scaling: the structure functions depend on E' and θ in one specific functional form

$$x = \frac{Q^2}{2p.q} \stackrel{\text{lab}}{=} \frac{2EE' \sin^2 \frac{\theta}{2}}{M(E - E')}$$

$$0 < x < 1$$

$$\lim_{Q^2 \to \infty,\ \nu/Q^2 \text{ fixed}} \nu W_2(Q^2, \nu) = M F_2(x)$$

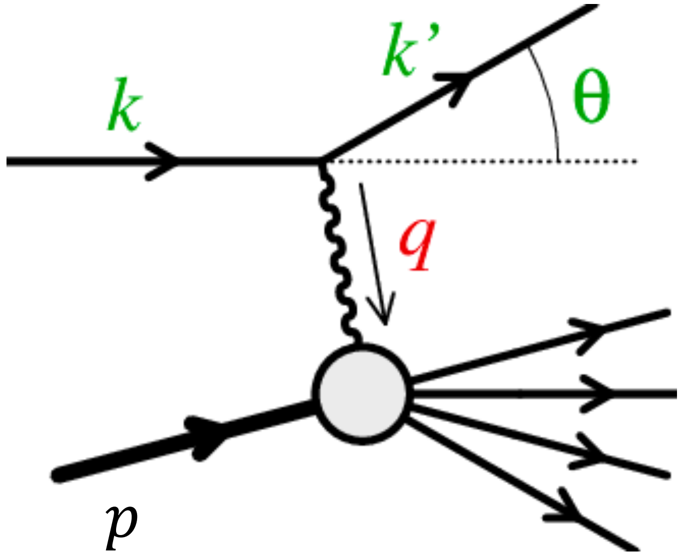
$$\lim_{Q^2 \to \infty,\ \nu/Q^2 \text{ fixed}} W_1(Q^2, \nu) = F_1(x)$$



Bjorken scaling

$$\nu W_2(\nu, Q^2) \simeq 2MF_2(\xi)$$

$$x = \xi$$

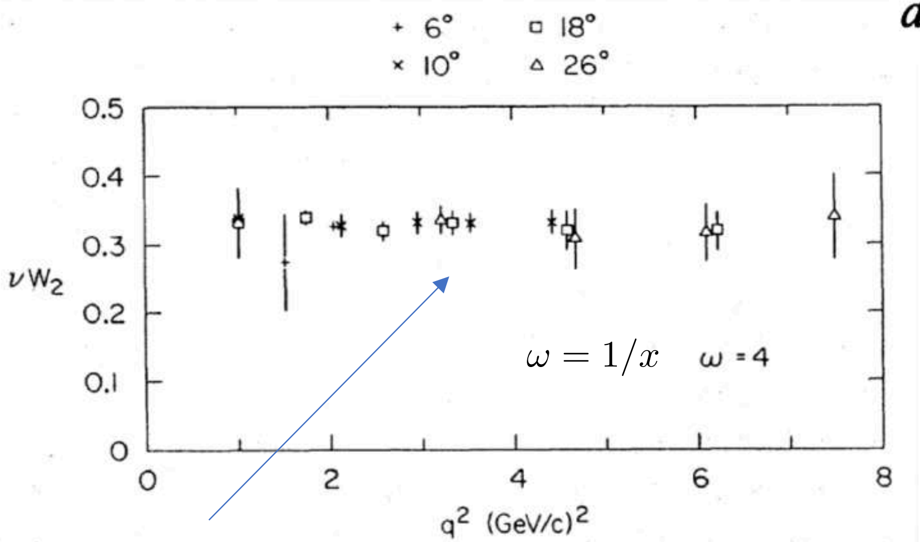


If the proton were point-like:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{M} \left(1 + \frac{Q^2}{2M^2}\tan^2\frac{\theta}{2}\right)\frac{E'}{E}$$

that is the Rosenbluth formula with G_E and G_M = 1. It can be rewritten as

$$\frac{d\sigma}{d\Omega dE'} = \left(\frac{d\sigma}{d\Omega}\right)_{M} \left(1 + \frac{Q^2}{2M^2}\tan^2\frac{\theta}{2}\right)\delta\left(\nu - \frac{Q^2}{2M}\right)$$



Almost no variation with Q!

What is the form of W1 and W2 in this case?

Bjorken scaling

$$W_1(\nu, Q^2) = \frac{Q^2}{2M}\delta\left(\nu - \frac{Q^2}{2M}\right)$$

$$W_2(\nu, Q^2) = 2M\,\delta\left(\nu - \frac{Q^2}{2M}\right)$$

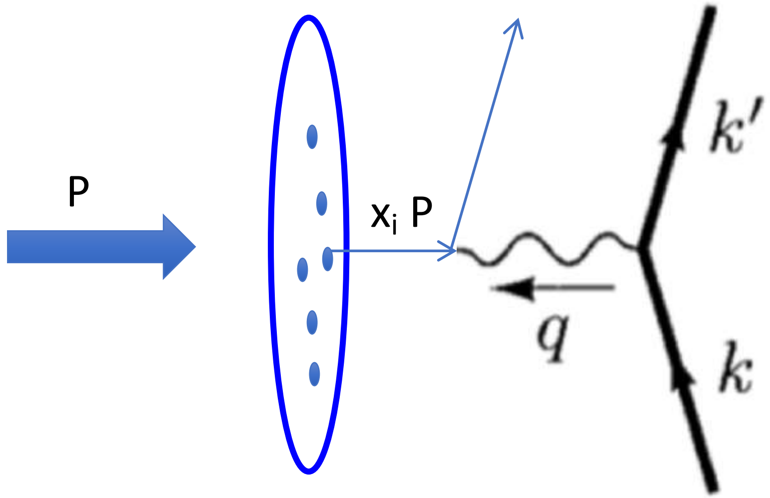
$$W_1(\nu, Q^2) = \xi\delta(1 - \xi)$$

$$\nu W_2(\nu, Q^2) = 2M\delta(1 - \xi)$$

But the proton is not point-like and to account for the Bjorken scaling we assume the proton is made of point-like constituents

$$p_i = x_i P$$

inelastic scattering as an incoherent sum of elastic scatterings on proton constituents which are massless almost free (asymptotic freedom)



Infinite momentum frame: proton is Lorentz contracted like a pancake (partons = chocolate chips).

$$0 \approx {p'}_i^2 = (p_i + q)^2 = (x_i P + q)^2 = x_i^2 M_i^2 + 2x_i P \cdot q + q^2$$

$$x_i = \frac{Q^2}{2P \cdot q}$$

electron-parton elastic scattering

$$\frac{d\sigma}{d\hat{t}}(e^- q \to e^- q) = \frac{2\pi\alpha^2 Q_f^2}{\hat{s}^2}\left[\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}\right]$$

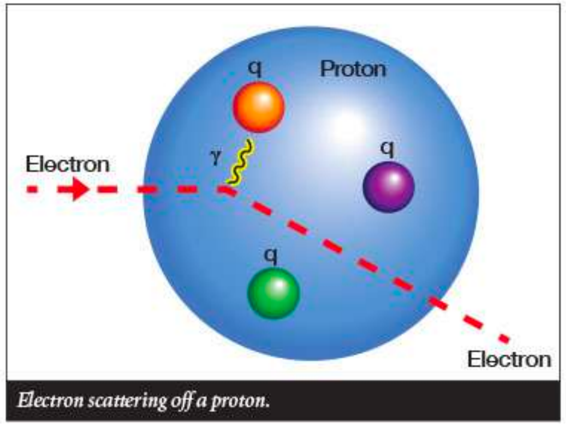
$$\hat{s} + \hat{t} + \hat{u} = 0.$$

$$\hat{t} = -Q^2$$

Mandelstam variables

$$\hat{s} = 2p \cdot k = 2\xi P \cdot k = \xi s.$$

Unpolarized Parton Distribution functions of the proton



Parton Distribution Function of the Nucleon

$$\sigma(e^- P^+ \to e^- X) = \sum_i \int_0^1 d\xi\, f_i(\xi) \hat{\sigma}(e^- p_i \to e^- X)$$

$$\xi = x, \qquad \text{where} \quad x \equiv \frac{Q^2}{2P \cdot q}$$

PDF: probability that a parton i is emitted by the proton, which carries a longitudinal fraction ξ of proton's momentum.

Xsec in the parton model

$$\left(\frac{d\hat{\sigma}(e^- q \to e^- q)}{d\Omega\, dE'}\right)_{\text{lab}} = \frac{\alpha_e^2 Q_i^2}{4E^2 \sin^4 \frac{\theta}{2}} \left[\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_q^2}\sin^2\frac{\theta}{2}\right] \delta\left(E - E' - \frac{Q^2}{2m_q}\right)$$

Qi = charge of the quark

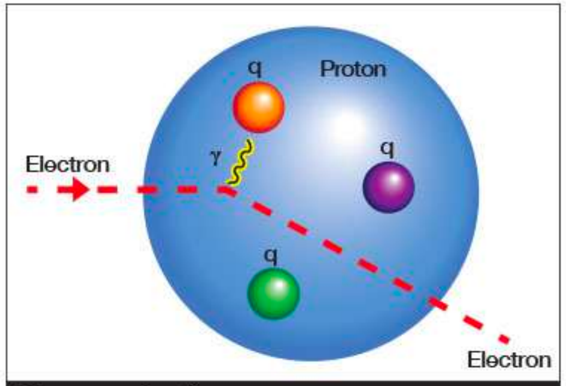
$$\delta\left(E - E' - \frac{Q^2}{2m_q}\right) = \delta\left(\frac{Q^2}{2m_p x} - \frac{Q^2}{2m_p \xi}\right) = \frac{2m_p}{Q^2} x^2 \delta(\xi - x)$$

$$\left(\frac{d\sigma(e^- P \to e^- X)}{d\Omega\, dE'}\right)_{\text{lab}} = \sum_i f_i(x) \frac{\alpha_e^2 Q_i^2}{4E^2 \sin^4 \frac{\theta}{2}} \left[\frac{2m_p}{Q^2} x^2 \cos^2\frac{\theta}{2} + \frac{1}{m_p}\sin^2\frac{\theta}{2}\right]$$

$$W_1(x, Q) = 2\pi \sum_i Q_i^2 f_i(x),$$

$$W_2(x, Q) = 8\pi \frac{x^2}{Q^2} \sum_i Q_i^2 f_i(x)$$

Unpolarized Parton Distribution functions of the proton



Electron scattering off a proton.

Parton Distribution Function of the Nucleon

$$\sigma(e^- P^+ \to e^- X) = \sum_i \int_0^1 d\xi\, f_i(\xi)\hat{\sigma}(e^- p_i \to e^- X)$$

$$\xi = x, \qquad \text{where} \quad x \equiv \frac{Q^2}{2P\cdot q}$$

PDF: probability that a parton i is emitted by the proton, which carries a longitudinal fraction ξ of proton's momentum.

Xsec in the parton model

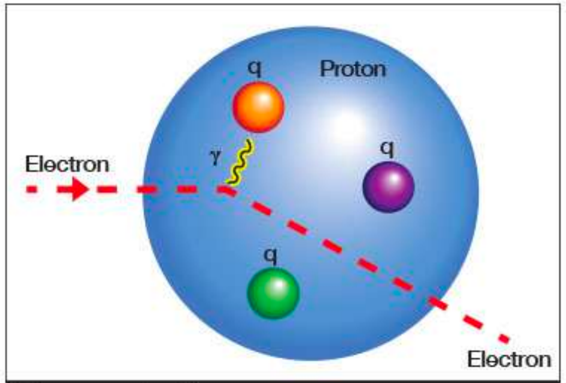
$$y = \frac{P\cdot q}{P\cdot k} = \frac{\nu}{E} \qquad dE' d\Omega = \frac{2m_p E}{E'}\pi y\, dx\, dy$$

Lorentz invariant formulation

$$\frac{d\sigma(e^- P \to e^- X)}{dx\, dy} = \frac{2\pi\alpha^2}{Q^4} s\left(1 + (1-y)^2\right)\sum_i Q_i^2 x f_i(x)$$

Physical justification of PDFs

- the momentum sloshes around among proton constituents at time scales $\approx \Lambda_{QCD}^{-1} \approx M_p^{-1}$
- these time scales are much slower than the time scales ~1/Q that the photon probes. The separation of scales $Q \gg \Lambda_{QCD}$ allows us to treat the parton wavefunctions within the proton as being decoherent, giving the **probabilistic interpretation**.
- to actually prove that this decoherence occurs, amounts to a proof of *factorization*.
- PDFs are non perturbative objects.



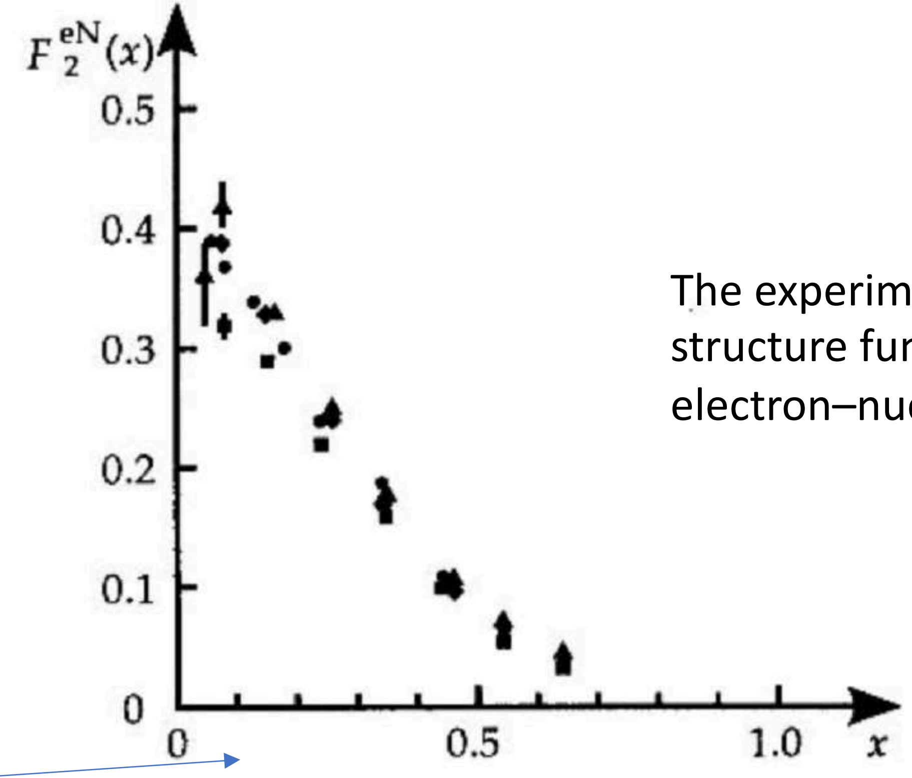
Electron scattering off a proton.

Comparing powers of y, we obtain

$$F_1^{\mathrm{eN}}(x) = \frac{1}{2}\sum_i f_i(x) q_i^2 \ , \qquad F_2^{\mathrm{eN}}(x) = \sum_i f_i(x) q_i^2 x$$

$$F_2^{\mathrm{eN}}(x) = 2x F_1^{\mathrm{eN}}(x)$$ Callan-Gross relation

$$F_2(x) = \tfrac{1}{2}(F_2^{\mathrm{ep}} + F_2^{\mathrm{en}})$$



The experimentally observed *F*2 structure function of deep inelastic electron–nucleon scattering .

Look at the x range

Write DIS X-section to zeroth order in α_s ('quark parton model'):

$$\frac{d^2\sigma^{em}}{dxdQ^2} \simeq \frac{4\pi\alpha^2}{xQ^4}\left(\frac{1+(1-y)^2}{2}F_2^{em} + \mathcal{O}(\alpha_s)\right)$$

$\propto F_2^{em}$ [structure function]

$$F_2 = x(e_u^2 u(x) + e_d^2 d(x)) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x)\right)$$

[$u(x)$, $d(x)$: parton distribution functions (PDF)]

<u>NB:</u>

- use perturbative language for interactions of up and down quarks
- but distributions themselves have a *non-perturbative* origin.

How do we access u and d PDFs?

Assumption ($SU(2)$ isospin): neutron is just proton with $u \Leftrightarrow d$:

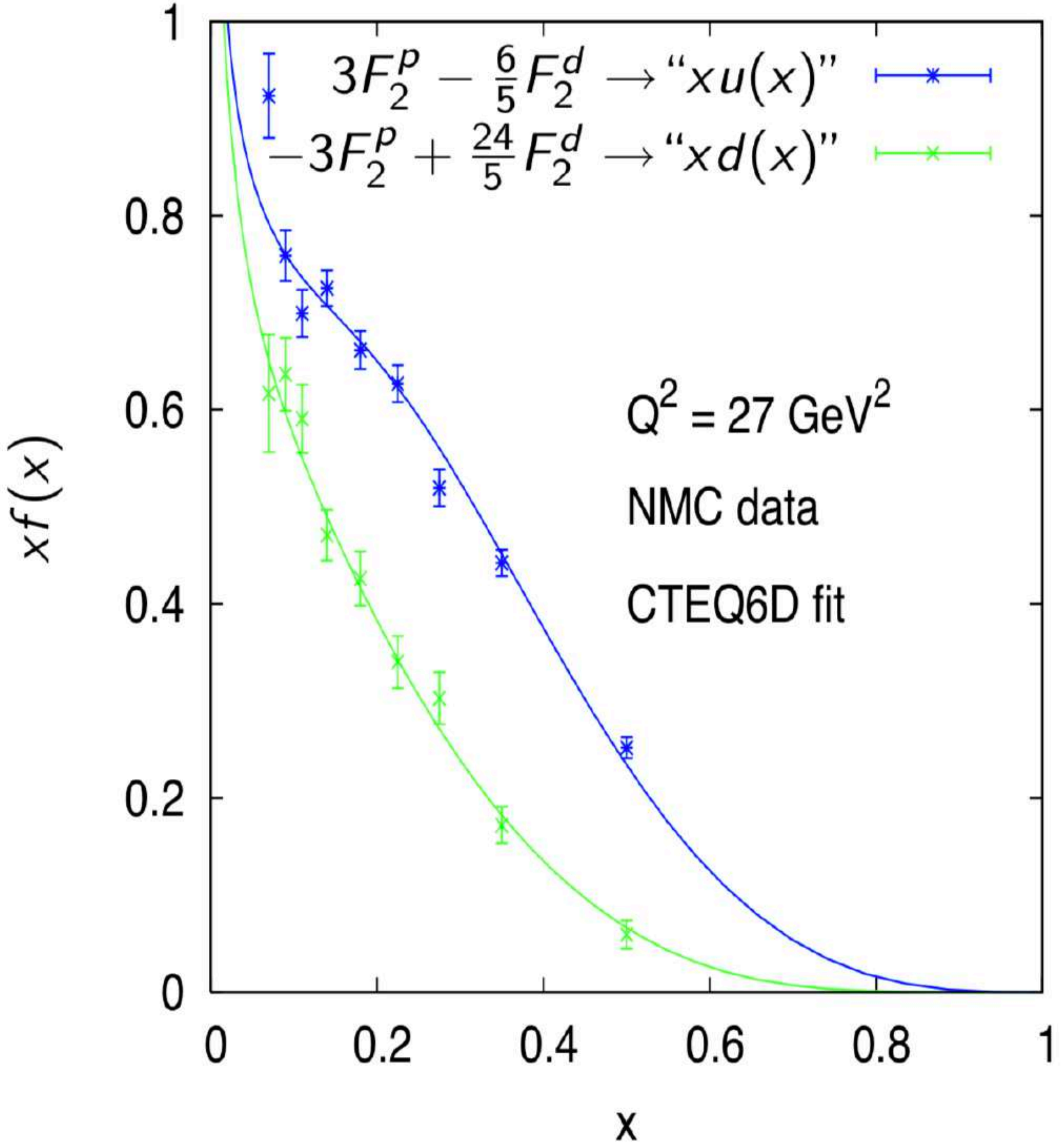
proton = uud; neutron = ddu

Isospin: $u_n(x) = d_p(x)\,, \qquad d_n(x) = u_p(x)$

$$F_2^p = \frac{4}{9}u_p(x) + \frac{1}{9}d_p(x)$$
$$F_2^n = \frac{4}{9}u_n(x) + \frac{1}{9}d_n(x) = \frac{4}{9}d_p(x) + \frac{1}{9}u_p(x)$$

Linear combinations of F_2^p and F_2^n give separately $u_p(x)$ and $d_p(x)$.

Experimentally, get F_2^n from deuterons: $F_2^d = \frac{1}{2}(F_2^p + F_2^n)$



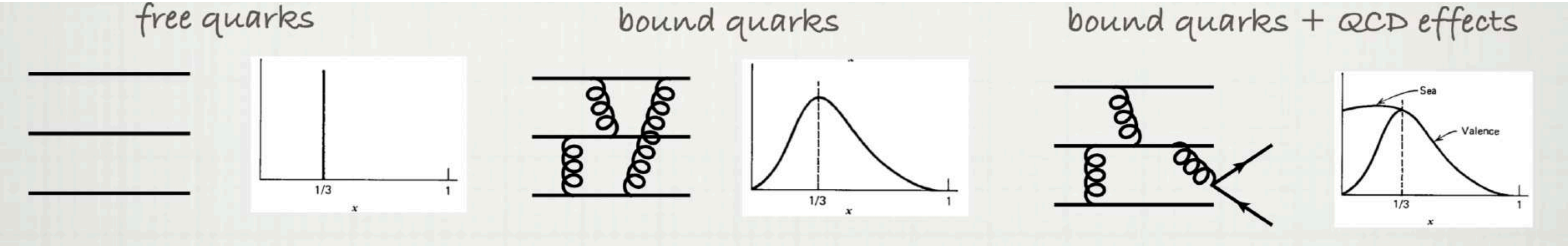
G. Salam

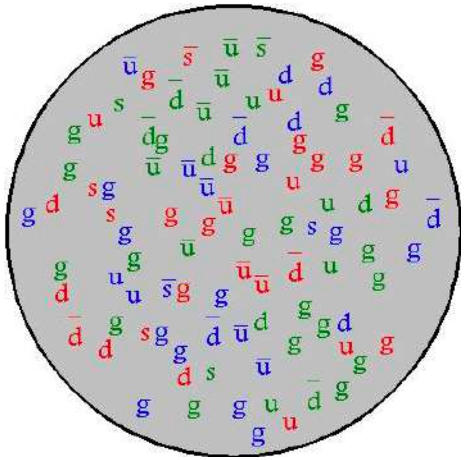
Combine F_2^p & F_2^d data, deduce $u(x)$, $d(x)$:

- Definitely more up than down (✓)

Look again at the x-range

But the proton wave function fluctuates!





$$F_2^{ep}(x) = \frac{4}{9}\left(u^p(x) + \bar{u}^p(x)\right)x + \frac{1}{9}\left(d^p(x) + \bar{d}^p(x)\right)x$$

$$F_2^{en}(x) = \frac{4}{9}\left(u^n(x) + \bar{u}^n(x)\right)x + \frac{1}{9}\left(d^n(x) + \bar{d}^n(x)\right)x$$

Antiquark distributions

$$F_2(x) = \frac{1}{2}\left(F_2^{ep}(x) + F_2^{en}(x)\right)$$
$$= \frac{1}{2}\left[\frac{5}{9}\left(u(x) + \bar{u}(x)\right) + \frac{5}{9}\left(d(x) + \bar{d}(x)\right)\right]x$$
$$= \frac{5}{18}\left[\left(u(x) + \bar{u}(x)\right) + \left(d(x) + \bar{d}(x)\right)\right]x\ .$$

u, d = valence quarks

s, c, b, ub, db, sb, … = sea quarks

For a proton (uud) we have the valence quark sum rules

$$\int_0^1 dx\,[u(x) - \bar{u}(x)] = 2$$

$$\int_0^1 dx\,\left[d(x) - \bar{d}(x)\right] = 1$$

In principle we should expect $$\sum_i \int dx\, x q_i(x) = 1$$ Momentum sum rules

$$\int_0^1 \mathrm{d}x\, x \left[(u(x) + \bar{u}(x) + d(x) + \bar{d}(x) + s(x) + \bar{s}(x) + \dots \right] = 1$$

→ ?

q_i	momentum
d_V	0.111
u_V	0.267
d_S	0.066
u_S	0.053
s_S	0.033
c_S	0.016
total	0.546

Something is missing here!

For a proton (uud) we have the valence quark sum rules

$$\int_0^1 dx\,[u(x) - \bar{u}(x)] = 2 \qquad \int_0^1 dx\,\left[d(x) - \bar{d}(x)\right] = 1$$

In principle we should expect $\sum_i \int dx\, x q_i(x) = 1$ Momentum sum rules

$$\int_0^1 \mathrm{d}x\; x\left[(u(x) + \bar{u}(x) + d(x) + \bar{d}(x) + s(x) + \bar{s}(x) + \ldots\right] = 1$$

THE GLUON

q_i	momentum
d_V	0.111
u_V	0.267
d_S	0.066
u_S	0.053
s_S	0.033
c_S	0.016
total	0.546

$$\int_0^1 dx\, x\left[f_u(x) + f_d(x) + f_{\bar{u}}(x) + f_{\bar{d}}(x) + f_g(x)\right] = 1$$

Computer simulations of the energy density of the gluon field fluctuation

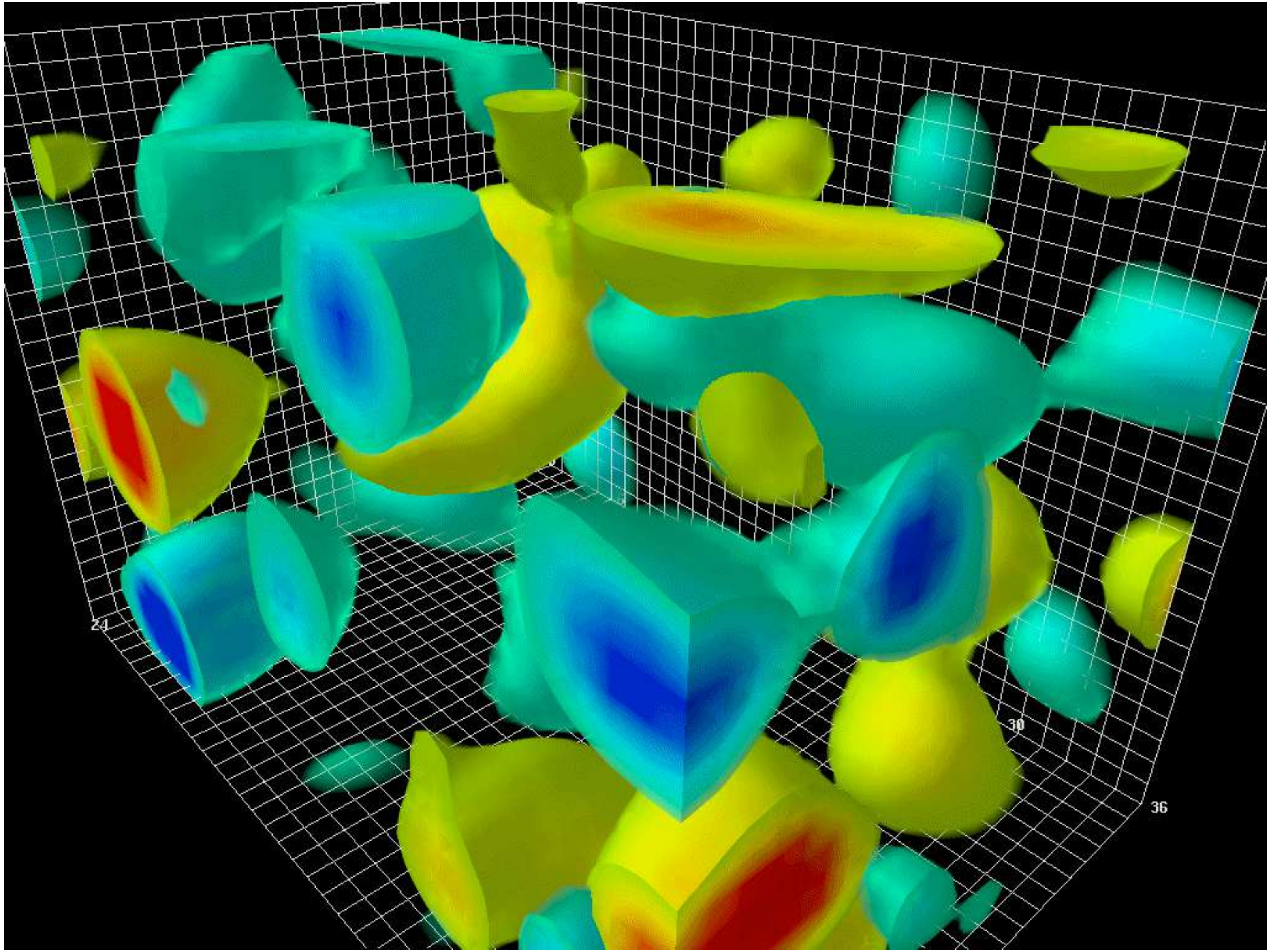
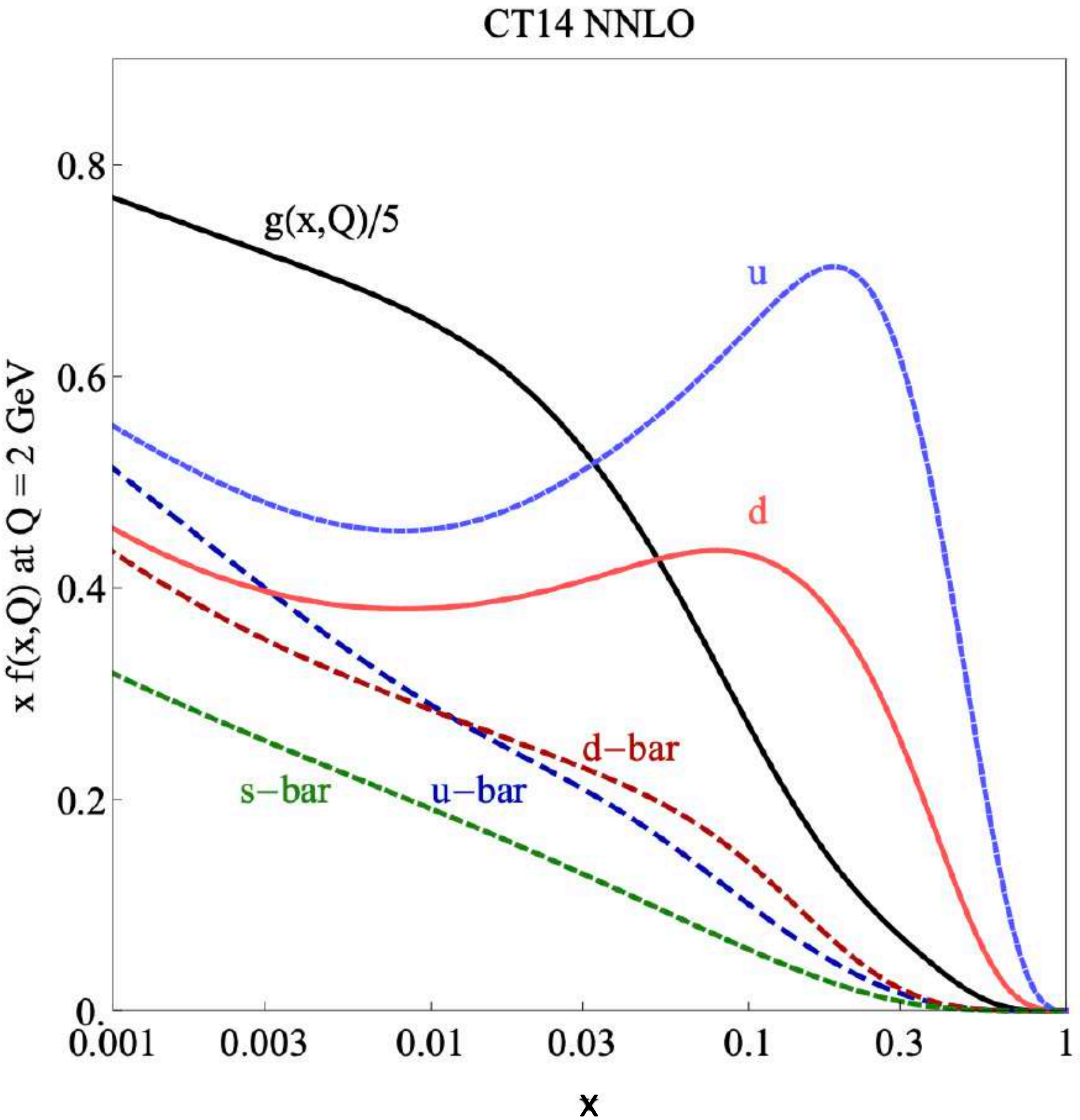


Image Credits: Derek Leinweber

$$\int_0^1 dx\, x\left[f_u(x) + f_d(x) + f_{\bar{u}}(x) + f_{\bar{d}}(x) + f_g(x)\right] = 1$$

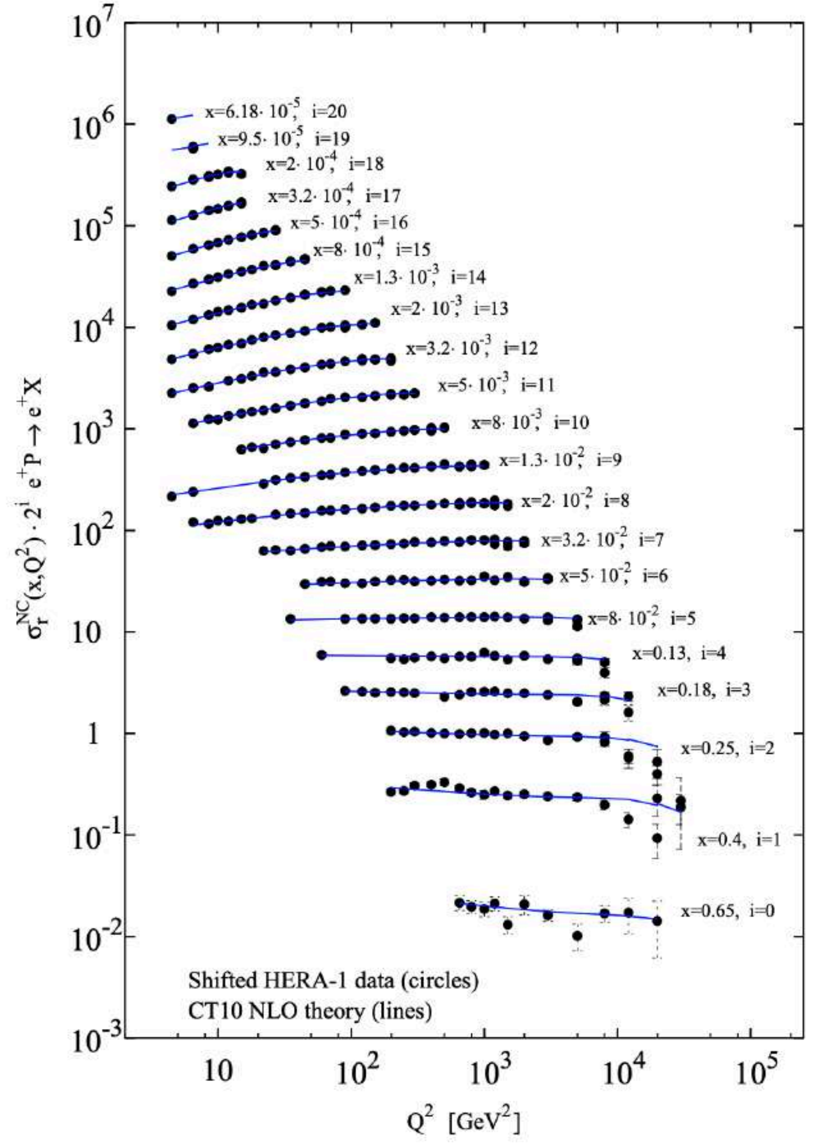


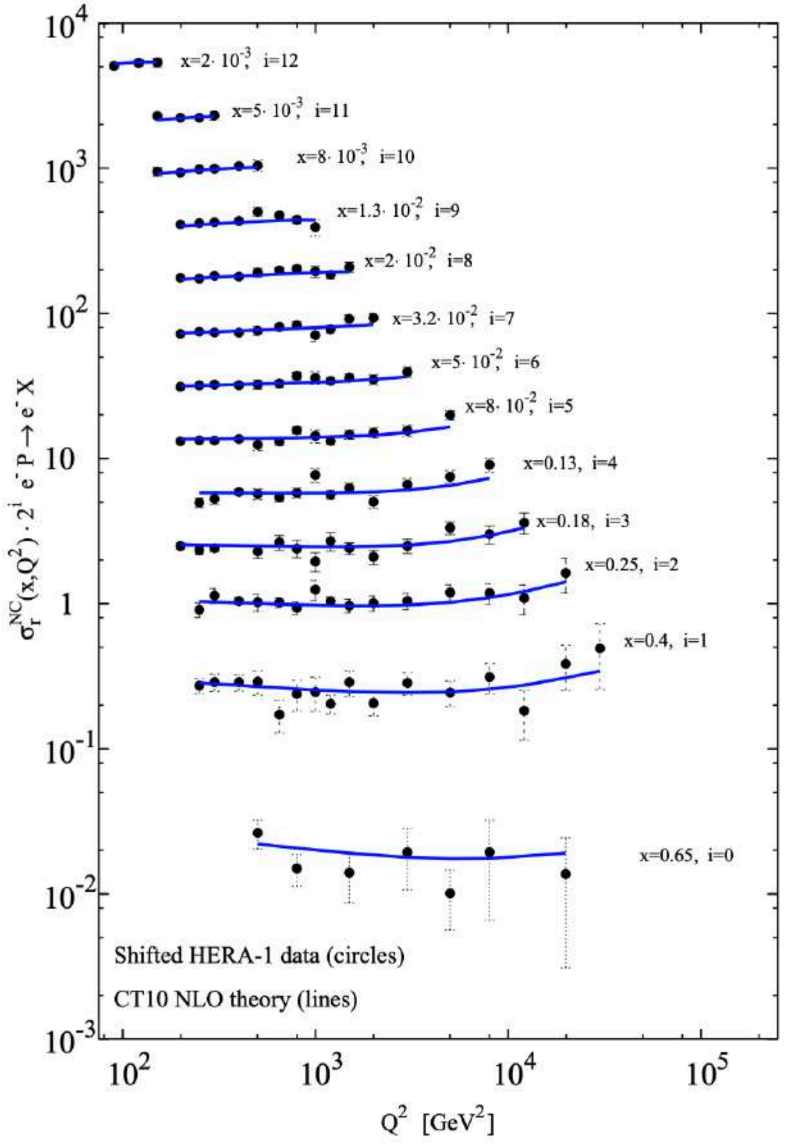
Wait: PDFs now depend on x and Q ?

To discuss gluons we must go beyond the leading order picture, and bring in more subtle features of QCD dynamics.

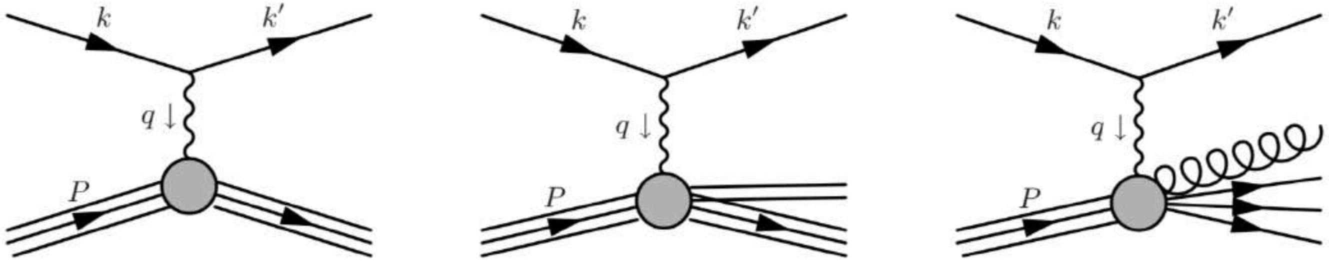
In fact, Bjorken scaling is actually violated at low x

HERA DIS data





PDF evolution: DGLAP equations



Let's go back to the DIS Xsec

$$W^{\mu\nu} = W_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + W_2\left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)$$

Hadronic tensor

partonic version of x

hard scattering

Let's rewrite it as

$$z \equiv \frac{Q^2}{2p_i\cdot q} \qquad p_i^\mu = \xi P^\mu$$

$$x = z\xi$$

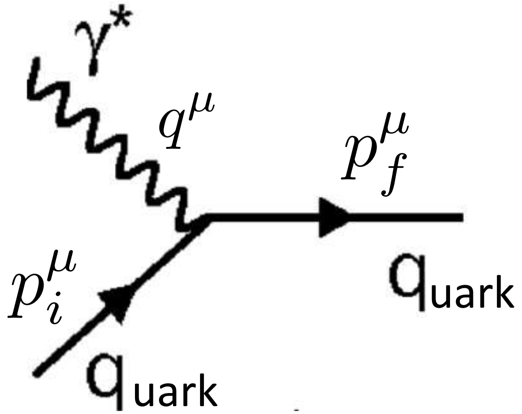
$$W^{\mu\nu}(x,Q) = \sum_i \int_0^1 dz \int_0^1 d\xi f_i(\xi)\hat{W}^{\mu\nu}(z,Q)\,\delta(x - z\xi)$$

$$= \sum_i \int_x^1 \frac{d\xi}{\xi} f_i(\xi)\hat{W}^{\mu\nu}\left(\frac{x}{\xi},Q\right).$$

Let's consider the partonic version of the hadronic tensor, given by $|\mathcal{M}(\gamma^\star q \to X)|^2$ integrated over final states

$$\hat{W}^{\mu\nu}(z,Q) = \frac{Q_i^2}{2}\int \frac{d^3\vec{p}_f}{(2\pi)^3}\frac{1}{2E_f}\mathrm{Tr}\left[\gamma^\mu \not{p}_i \gamma^\nu \not{p}_f\right](2\pi)^4\,\delta^4(p_i+q-p_f)$$

$$= 2\pi Q_i^2\left[\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)+\frac{4z}{Q^2}\left(p_i^\mu-\frac{p_i\cdot q}{q^2}q^\mu\right)\left(p_i^\nu-\frac{p_i\cdot q}{q^2}q^\nu\right)\right]\delta(1-z).$$



$$p_f^\mu = p_i^\mu + q^\mu$$

$$\mathcal{O}(\alpha_s^0) \quad \gamma^\star q \to q$$

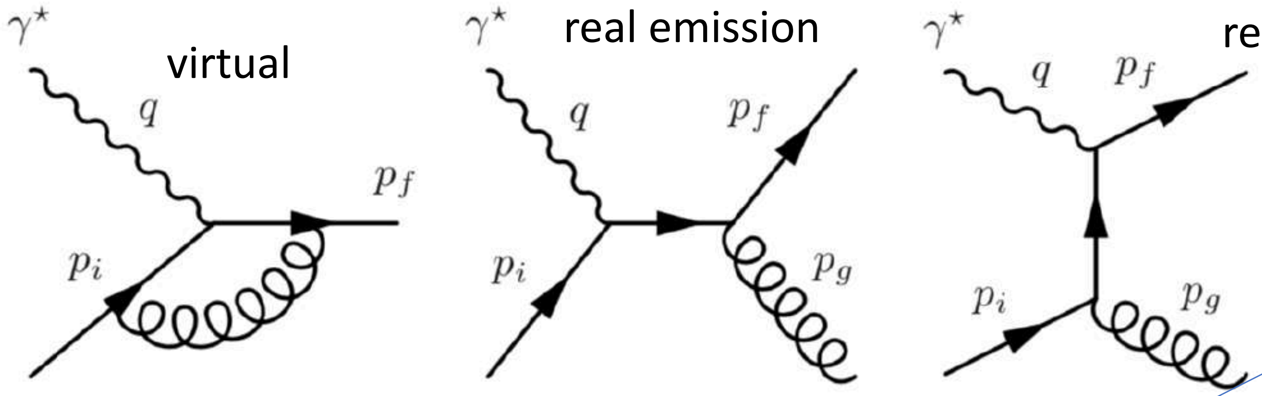
At LO, after the Lorentz contractions,

$$\hat{W}_1 = 2\pi Q_i^2\delta(1-z) = \frac{Q^2}{4z}\hat{W}_2$$

Callan –Gross relation

$$W_0 \equiv -g^{\mu\nu}W_{\mu\nu} \qquad W_0(x,Q) = 4\pi\sum_i Q_i^2 f_i(x)$$

At NLO we have other Feynman diagrams contributing



IR Soft and collinear divergence.
UV divergences cancelled by renormalization

$$\hat{W}_0^V = 4\pi Q_i^2 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{Q^2}\right)^{\frac{\varepsilon}{2}} \frac{\Gamma\left(1-\frac{\varepsilon}{2}\right)}{\Gamma\left(1-\varepsilon\right)} \left(-\frac{8}{\varepsilon^2} - \frac{6}{\varepsilon} - 8 - \frac{\pi^2}{3}\right) \delta(1-z)$$

$\overline{MS}$: we need a prescription to deal with singularities.

In D dim, D = 4-ε

$$\hat{W}_0^R = 4\pi Q_i^2 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{Q^2}\right)^{\frac{\varepsilon}{2}} \frac{\Gamma(1-\frac{\varepsilon}{2})}{\Gamma(1-\varepsilon)} \times \left\{3 + 2z - \frac{1+z^2}{1-z}\ln z \right.$$
$$\left. + \left(\frac{8}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{7}{2}\right)\delta(1-z) - \left(2\frac{1+z^2}{\varepsilon} + \frac{3}{2}\right)\left[\frac{1}{1-z}\right]_+ + (1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+ \right\}$$

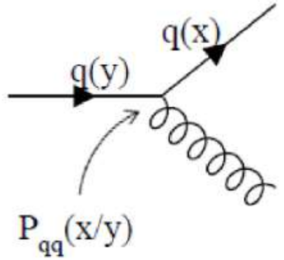
Soft and Collinear divergence

Putting all the contributions together the double $1/\varepsilon^2$ poles cancel out

$$\hat{W}_0 = \hat{W}_0^{\mathrm{LO}} + \hat{W}_0^{V} + \hat{W}_0^{R} = 4\pi Q_i^2 \Bigg\{ \Bigg[\delta(1-z) - \frac{1}{\varepsilon}\frac{\alpha_s}{\pi} P_{qq}(z) \left(\frac{4\pi\mu^2}{Q^2}\right)^{\frac{\varepsilon}{2}} \frac{\Gamma\left(1-\frac{\varepsilon}{2}\right)}{\Gamma(1-\varepsilon)} \Bigg]$$
$$+ \frac{\alpha_s}{2\pi} C_F \Bigg[(1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+ - \frac{3}{2}\left[\frac{1}{1-z}\right]_+ - \frac{1+z^2}{1-z}\ln z + 3 + 2z - \left(\frac{9}{2} + \frac{1}{3}\pi^2\right)\delta(1-z) \Bigg] \Bigg\}$$

$$P_{qq}(z) = C_F \left[(1+z^2)\left[\frac{1}{1-z}\right]_+ + \frac{3}{2}\delta(1-z) \right]$$

Plus distribution



DGLAP splitting function at LO.
(Dokshitzer, Gribov, Lipatov, Altarelli, Parisi)

$$\int_0^1 dz \frac{f(z)}{[1-z]_+} \equiv \int_0^1 dz \frac{f(z)-f(1)}{1-z}$$

Inserting $\hat{W}_0$ back into $W^{\mu\nu}(x,Q)$ we obtain

$$W_0(x,Q) = 4\pi \sum_i Q_i^2 \int_x^1 \frac{d\xi}{\xi} f_i(\xi) \left[\delta\left(1 - \frac{x}{\xi}\right) - \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right)\left(\frac{2}{\varepsilon} + \ln\frac{\tilde{\mu}^2}{Q^2}\right) + \text{finite} \right]$$

At fixed x the $1/\varepsilon$ pole does not cancel. We need to consider differences of cross sections to get a finite answer

$$W_0(x,Q) - W_0(x,Q_0) = 4\pi \sum_i Q_i^2 \int_x^1 \frac{d\xi}{\xi} f_i(\xi) \left[\frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right) \ln\frac{Q^2}{Q_0^2} \right]$$

finite!!!

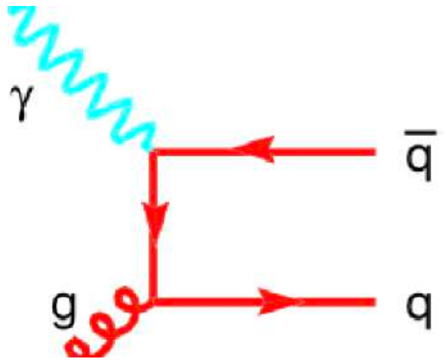
Q_0 is arbitrary. Renormalization Group Equation (RGE). Let's define, for every scale Q

$$W_0(x,Q) \equiv 4\pi \sum_i Q_i^2 f_i(x, \mu = Q) \quad \Rightarrow \quad f_i(x,\mu_1) = f_i(x,\mu) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} f_i(\xi,\mu_1)\, P_{qq}\left(\frac{x}{\xi}\right) \ln\frac{\mu_1^2}{\mu^2}$$

$$\mu \frac{d}{d\mu} f_i(x,\mu) = \frac{\alpha_s}{\pi} \int_x^1 \frac{d\xi}{\xi} f_i(\xi,\mu)\, P_{qq}\left(\frac{x}{\xi}\right)$$

DGLAP equation

At NLO in QCD, we also have the following diagram contributing to F2(x, Q)



$$\frac{F_2(x,Q^2)}{x} = \int dz\, d\xi\, f_g(\xi) \left\{ \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \left(-\frac{1}{\varepsilon}\right) \left(P_{qg}(z) + P_{\bar{q}g}(z)\right) + \frac{\alpha_s}{2\pi}\left[\left(z^2+(1-z)^2\right)\ln\frac{1-z}{z} + 6z(1-z)\right]\right\} \delta(x - z\xi)$$

Therefore DGLAP equations are mixed together and we have 2 Nf + 1 coupled integro-differential equations

$$\mu\frac{d}{d\mu}\begin{pmatrix} f_i(x,\mu) \\ f_g(x,\mu) \end{pmatrix} = \sum_j \frac{\alpha_s}{\pi}\int_x^1 \frac{d\xi}{\xi} \begin{pmatrix} P_{q_iq_j}(\frac{x}{\xi}) & P_{q_ig}(\frac{x}{\xi}) \\ P_{gq_j}(\frac{x}{\xi}) & P_{gg}(\frac{x}{\xi}) \end{pmatrix} \begin{pmatrix} f_j(\xi,\mu) \\ f_g(\xi,\mu) \end{pmatrix}$$

$$P_{qq}(z) = C_F\left[\frac{1+z^2}{[1-z]_+} + \frac{3}{2}\delta(1-z)\right], \quad P_{gg}(z) = 2C_A\left[\frac{z}{[1-z]_+} + \frac{1-z}{z} + z(1-z)\right] + \frac{\beta_0}{2}\delta(1-z)$$

LO DGLAP Splitting functions

$$P_{qg}(z) = T_F\left[z^2 + (1-z)^2\right],$$

$$P_{gq}(z) = C_F\left[\frac{1+(1-z)^2}{z}\right],$$

$$P_{ij}(x,\mu^2) = \frac{\alpha_s(\mu^2)}{4\pi}P_{ij}^{(0)}(x) + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2 P_{ij}^{(1)}(x) + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^3 P_{ij}^{(2)}(x) + \ldots$$

DGLAP: 2 Nf +1 coupled integro-differential equations

$$\mu\frac{d}{d\mu}\begin{pmatrix} f_i(x,\mu) \\ f_g(x,\mu)\end{pmatrix} = \sum_j \frac{\alpha_s}{\pi}\int_x^1 \frac{d\xi}{\xi}\begin{pmatrix} P_{q_i q_j}(\frac{x}{\xi}) & P_{q_i g}(\frac{x}{\xi}) \\ P_{g q_j}(\frac{x}{\xi}) & P_{gg}(\frac{x}{\xi})\end{pmatrix}\begin{pmatrix} f_j(\xi,\mu) \\ f_g(\xi,\mu)\end{pmatrix}$$

initial conditions determined through the fit to data.

We can use a different basis in the PDF space to decompose the equations in a useful way

$$g(x,Q^2)$$

$$q_S(x,Q^2) = \sum_{i=1}^{n_f}\left[q_i(x,Q^2) + \bar{q}(x,Q^2)\right]$$

Singlet

$$\frac{d}{d\log Q^2}\begin{pmatrix} q_S \\ g\end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg}\end{pmatrix} \otimes \begin{pmatrix} q_S \\ g\end{pmatrix}$$

$$q_V(x,Q^2) = \sum_{i=1}^{n_f}\left[q_i(x,Q^2) - \bar{q}(x,Q^2)\right]$$

$$q_{ij}^{\pm}(x,Q^2) = (q_i \pm \bar{q}_i) - (q_j \pm \bar{q}_j)$$

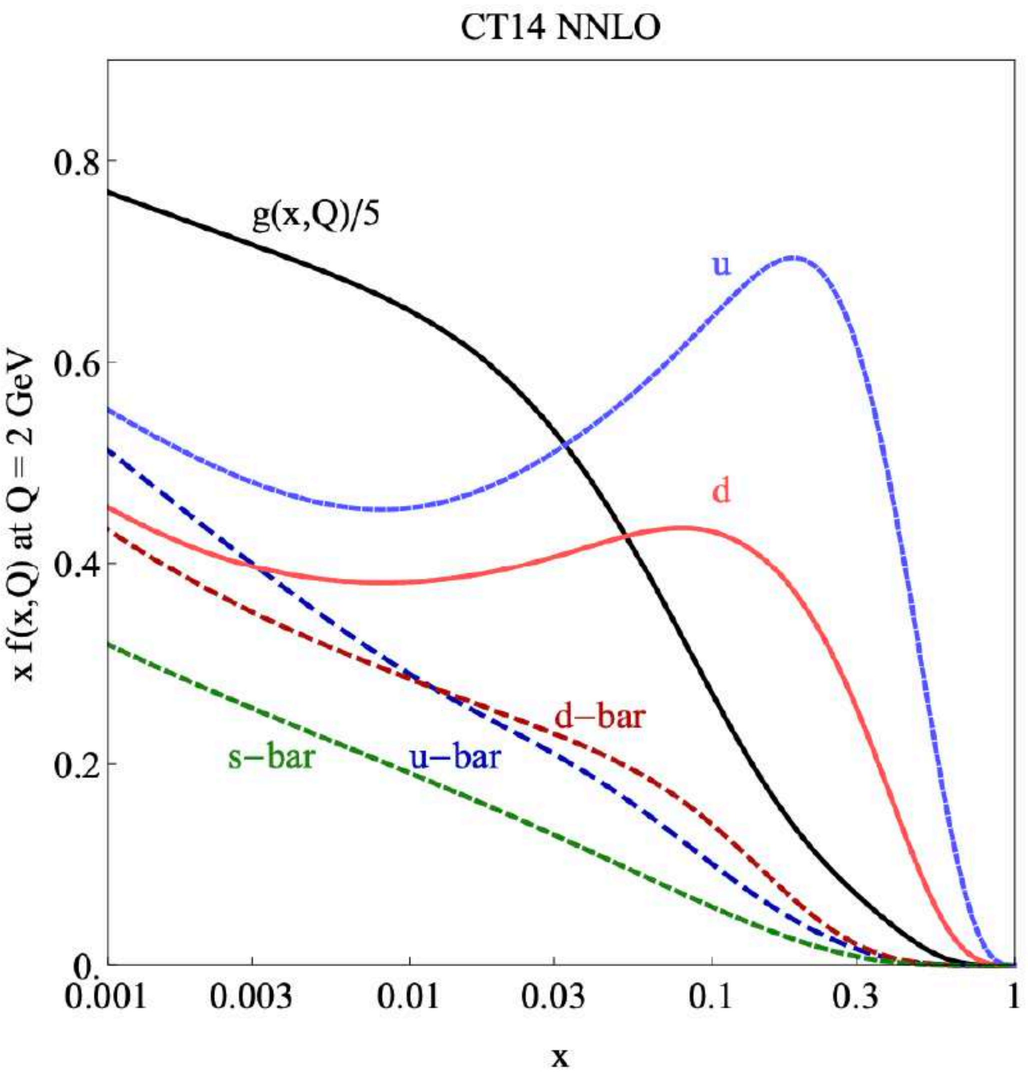
Non-Singlet

$$\frac{dq_{ij}^{\pm}}{d\log Q^2} = P_{\pm} \otimes q_{ij}^{\pm} \qquad \frac{dq_V}{d\log Q^2} = P_v \otimes q_V$$

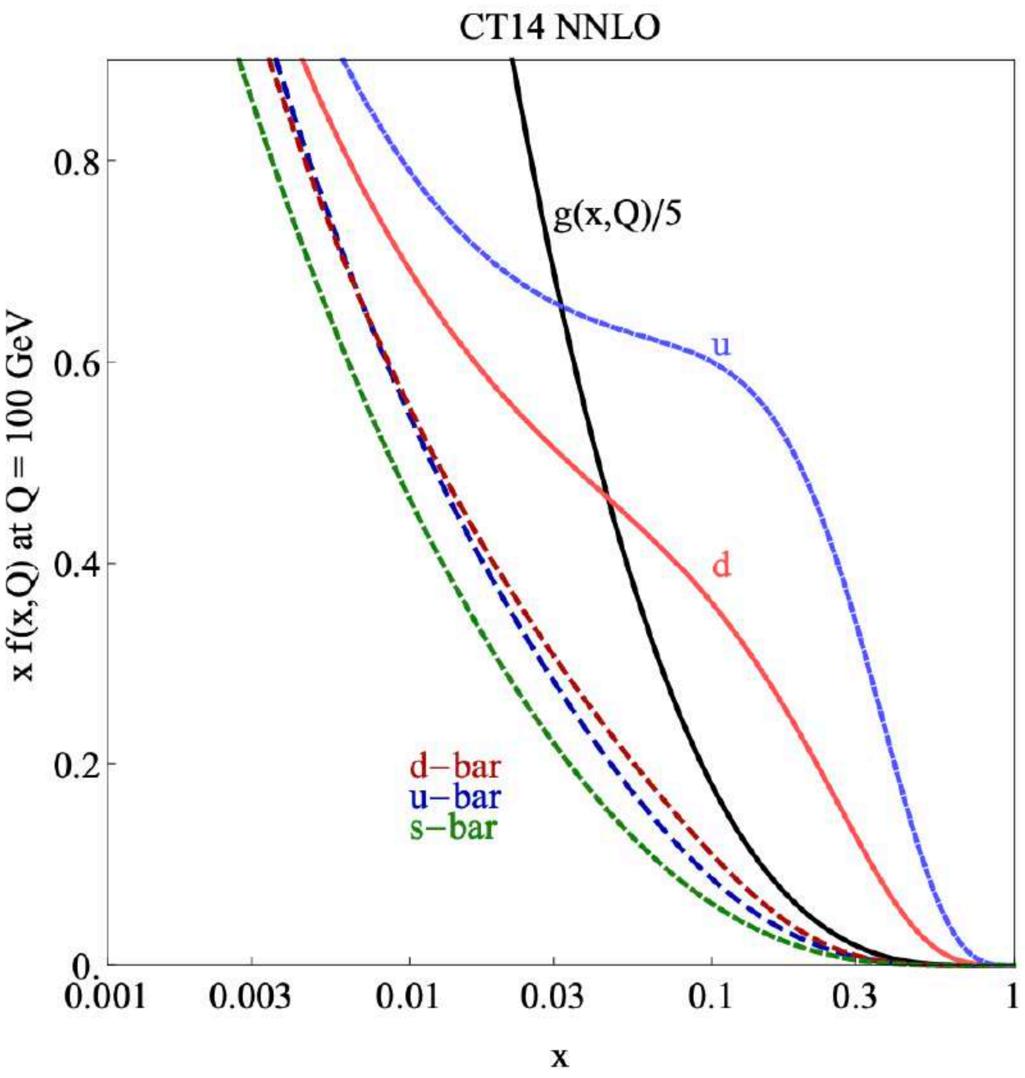
Different numerical codes to solve DGLAP evolution available on the market

- *Pegasus:* Mellin space, symbolic moment-space solutions on a one-fits-all Mellin inversion contour, A. Vogt; CPC 170 (2005);
- *Candia:* x-space, recursive solutions, C. Corianò, M.G., A. Cafarella, NPB748 (2006), CPC 179 (2008);
- *Hoppet:* x-space, Runge-Kutta, G. Salam, J. Rojo, CPC 180 (2009);
- *QCDNum:* x-space, eqns numerically solved on a discrete n×m grid in x and μ^2 , M. Botje, CPC 182 (2011);
- *Apfel*: x-space Runge-Kutta + higher order interpolations, V. Bertone, S. Carrazza, J. Rojo, CPC 185, (2014);
- *CHILIPDF*: x-space Chebyshev interpolation, M. Diehl, R. Nagar, F. Tackmann, 1906.10059 [hep-ph]

Q=2 GeV



Q=100 GeV



PDF's evolution from Q = 2 GeV up to Q = 100 GeV

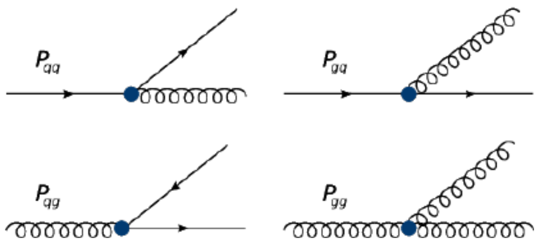
$$P_{ij}(x,\mu^2) = \frac{\alpha_s(\mu^2)}{4\pi} P_{ij}^{(0)}(x) + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2 P_{ij}^{(1)}(x) + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^3 P_{ij}^{(2)}(x) + \ldots$$

Higher orders are difficult to calculate

$$
\begin{aligned}
P_{ns}^{(0)}(x) &= C_F\big(2p_{qq}(x) + 3\delta(1-x)\big)\\
P_{ps}^{(0)}(x) &= 0\\
P_{qg}^{(0)}(x) &= 2n_f\, p_{qg}(x)\\
P_{gq}^{(0)}(x) &= 2C_F\, p_{gq}(x)\\
P_{gg}^{(0)}(x) &= C_A\Big(4p_{gg}(x) + \frac{11}{3}\delta(1-x)\Big) - \frac{2}{3}n_f\,\delta(1-x)
\end{aligned}
$$

LO 1973

Here is the result for the quantum corrections to the splitting functions in QCD



LO

$$
\begin{aligned}
P_{ns}^{(1)+}(x) &= 4C_AC_F\Big(p_{qq}(x)\Big[\frac{67}{18} - \zeta_2 + \frac{11}{6}H_0 + H_{0,0}\Big] + p_{qq}(-x)\Big[\zeta_2 + 2H_{-1,0} - H_{0,0}\Big]\\
&\quad + \frac{14}{3}(1-x) + \delta(1-x)\Big[\frac{17}{24} + \frac{11}{3}\zeta_2 - 3\zeta_3\Big]\Big) - 4C_Fn_f\Big(p_{qq}(x)\Big[\frac{5}{9} + \frac{1}{3}H_0\Big] + \frac{2}{3}(1-x)\\
&\quad + \delta(1-x)\Big[\frac{1}{12} + \frac{2}{3}\zeta_2\Big]\Big) + 4C_F^2\Big(2p_{qq}(x)\Big[H_{1,0} - \frac{3}{4}H_0 + H_2\Big] - 2p_{qq}(-x)\Big[\zeta_2 + 2H_{-1,0}\\
&\quad - H_{0,0}\Big] - (1-x)\Big[1 - \frac{3}{2}H_0\Big] - H_0 - (1+x)H_{0,0} + \delta(1-x)\Big[\frac{3}{8} - 3\zeta_2 + 6\zeta_3\Big]\Big)\\
P_{ns}^{(1)-}(x) &= P_{ns}^{(1)+}(x) + 16C_F\Big(C_F - \frac{C_A}{2}\Big)\Big(p_{qq}(-x)\Big[\zeta_2 + 2H_{-1,0} - H_{0,0}\Big] - 2(1-x)\\
&\quad - (1+x)H_0\Big)\\
P_{ps}^{(1)}(x) &= 4C_Fn_f\Big(\frac{20}{9}\frac{1}{x} - 2 + 6x - 4H_0 + x^2\Big[\frac{8}{3}H_0 - \frac{56}{9}\Big] + (1+x)\Big[5H_0 - 2H_{0,0}\Big]\Big)\\
P_{qg}^{(1)}(x) &= 4C_An_f\Big(\frac{20}{9}\frac{1}{x} - 2 + 25x - 2p_{qg}(-x)H_{-1,0} - 2p_{qg}(x)H_{1,1} + x^2\Big[\frac{44}{3}H_0 - \frac{218}{9}\Big]\\
&\quad + 4(1-x)\Big[H_{0,0} - 2H_0 + xH_1\Big] - 4\zeta_2x - 6H_{0,0} + 9H_0\Big) + 4C_Fn_f\Big(2p_{qg}(x)\Big[H_{1,0} + H_{1,1} + H_2\\
&\quad - \zeta_2\Big] + 4x^2\Big[H_0 + H_{0,0} + \frac{5}{2}\Big] + 2(1-x)\Big[H_0 + H_{0,0} - 2xH_1 + \frac{29}{4}\Big] - \frac{15}{2} - H_{0,0} - \frac{1}{2}H_0\Big)\\
P_{gq}^{(1)}(x) &= 4C_AC_F\Big(\frac{1}{x} + 2p_{gq}(x)\Big[H_{1,0} + H_{1,1} + H_2 - \frac{11}{6}H_1\Big] - x^2\Big[\frac{8}{3}H_0 - \frac{44}{9}\Big] + 4\zeta_2 - 2\\
&\quad - 7H_0 + 2H_{0,0} - 2H_1x + (1+x)\Big[2H_{0,0} - 5H_0 + \frac{37}{9}\Big] - 2p_{gq}(-x)H_{-1,0}\Big) - 4C_Fn_f\Big(\frac{2}{3}x\\
&\quad - p_{gq}(x)\Big[\frac{2}{3}H_1 - \frac{10}{9}\Big]\Big) + 4C_F^2\Big(p_{gq}(x)\Big[3H_1 - 2H_{1,1}\Big] + (1+x)\Big[H_{0,0} - \frac{7}{2} + \frac{7}{2}H_0\Big] - 3H_{0,0}\\
&\quad + 1 - \frac{3}{2}H_0 + 2H_1x\Big)\\
P_{gg}^{(1)}(x) &= 4C_An_f\Big(1 - x - \frac{10}{9}p_{gg}(x) - \frac{13}{9}\Big(\frac{1}{x} - x^2\Big) - \frac{2}{3}(1+x)H_0 - \frac{2}{3}\delta(1-x)\Big) + 4C_A^2\Big(27\\
&\quad + (1+x)\Big[\frac{11}{3}H_0 + 8H_{0,0} - \frac{27}{2}\Big] + 2p_{gg}(-x)\Big[H_{0,0} - 2H_{-1,0} - \zeta_2\Big] - \frac{67}{9}\Big(\frac{1}{x} - x^2\Big) - 12H_0\\
&\quad - \frac{44}{3}x^2H_0 + 2p_{gg}(x)\Big[\frac{67}{18} - \zeta_2 + H_{0,0} + 2H_{1,0} + 2H_2\Big] + \delta(1-x)\Big[\frac{8}{3} + 3\zeta_3\Big]\Big) + 4C_Fn_f\Big(2H_0\\
&\quad + \frac{2}{3}\frac{1}{x} + \frac{10}{3}x^2 - 12 + (1+x)\Big[4 - 5H_0 - 2H_{0,0}\Big] - \frac{1}{2}\delta(1-
\end{aligned}
$$

NLO: 1980

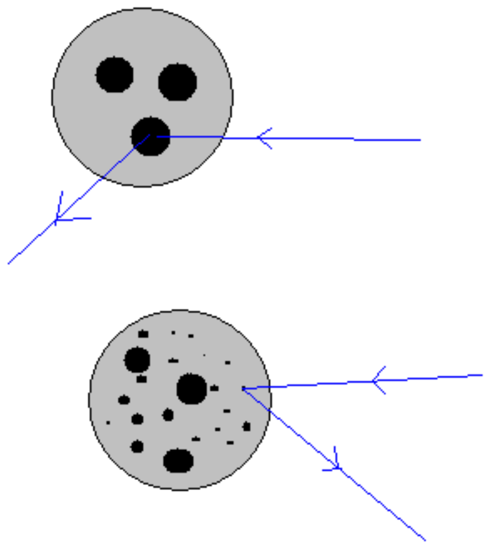
It took 24 years to calculate the third term in the P_{ij} perturbative expansion

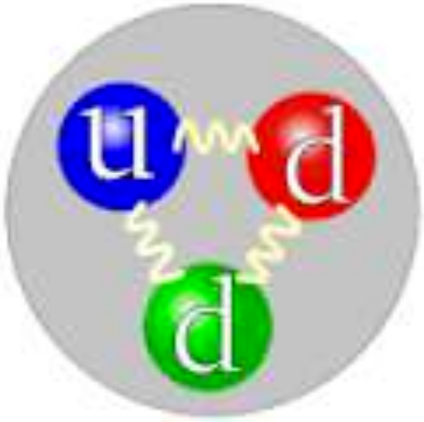
NNLO: next-to-next-to leading order.   NNNLO: work is in progress

Concluding remarks: Proton's inner structure

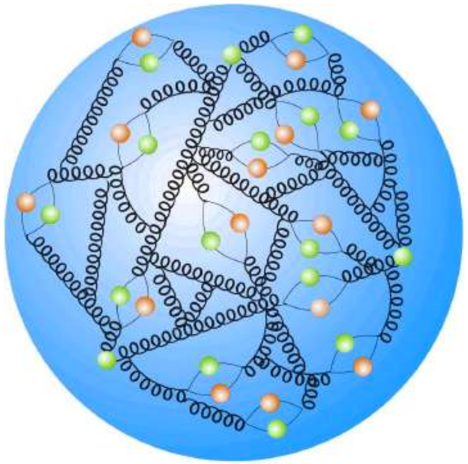
Parton Distribution Functions (PDFs) of the proton map out the longitudinal momentum distribution of proton's constituent quarks and gluons (aka partons).

When protons collide at very high energies, PDFs can be interpreted as the probability for finding a parton in the proton with a certain longitudinal momentum fraction **x** at resolution energy scale **Q**.





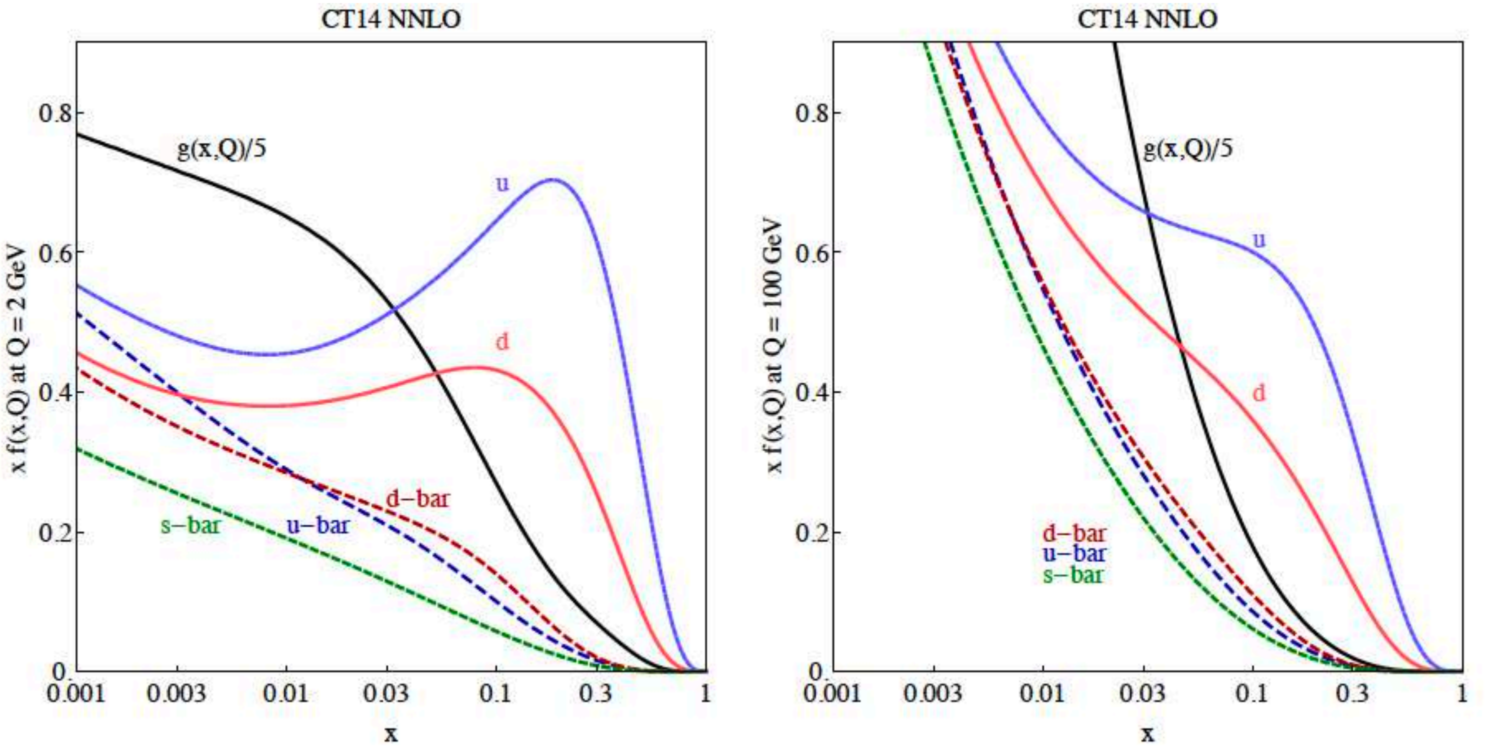
Resolving in energy



at energy scale Q_0

at energy scale $Q > Q_0$

High energy collisions at the LHC provide us with a tool to investigate the proton’s PDFs and the dynamics of quarks and gluons



PDFs cannot
be fully determined
analytically!

PDFs are determined by global analyses of world experimental data using a variety of statistical methods.

They represent one of the major sources of uncertainties for theory predictions and simulations at the LHC

THANK YOU!

BACK UP

Further reading and references

- Collins Soper Sterman, Factorization of Hard Processes in QCD, hep-ph/0409313
- G. Sterman, Quantum Field theory;
- J. Collins, Perturbative QCD;
- T. Muta, Foundations of Quantum Chromodynamics;
- M .Peskin, D. Schroeder, An introduction to QFT;
- Greiner, Schramm, Stein, QCD;
- Full details of the calculation in slide 30-32 can be found in: M. D. Schwartz, Quantum Field Theory and the Standard Model;
- T.-P. Cheng and L.F. Li, Gauge theory of the elementary particles;
- T. Plehn, TASI Lectures 2008
- G. Salam, 2009 European School of High-Energy Physics, Bautzen

The proton is an extended object

$$G_M(Q^2)/\mu = G_E(Q^2) = 1/(1 + Q^2/0.7)^2$$

Very well known experimentally

$$\langle r^2 \rangle = \int dx^3 r^2 j_0(x)$$

Spatial distribution

$$\langle r^2 \rangle = -6G'_E(0) = 0.67 \times 10^{-26}\,\mathrm{cm}^2$$

Asymptotically we have that: $G_M(Q^2), \quad G_E(Q^2) \sim (Q^2)^{1-n}$

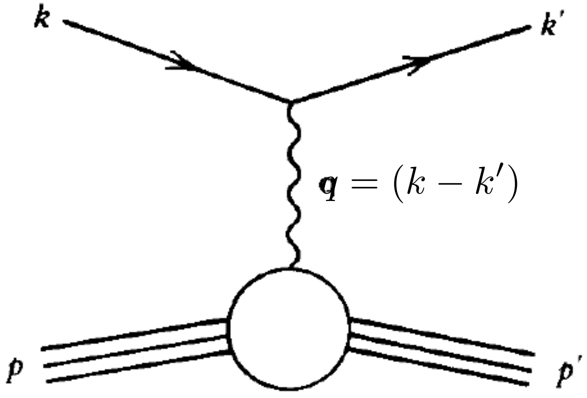
where in the case of the proton $n = 3$ is in agreement with the data

Kinematics 1 - Elastic electron proton scattering kinematic

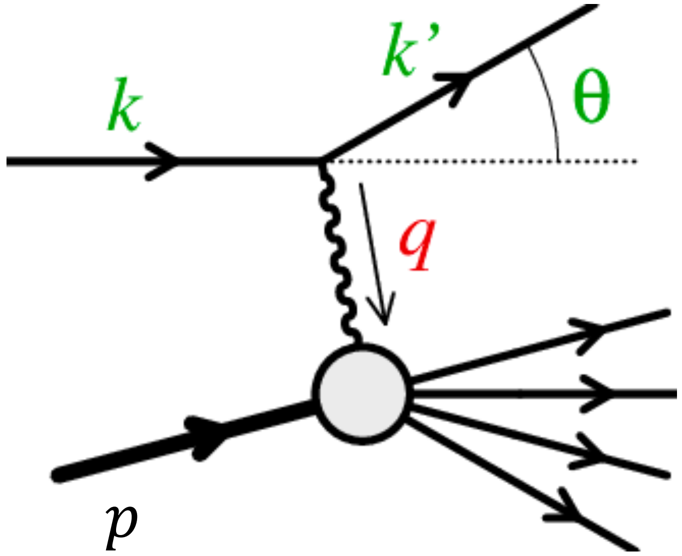
$$k_\mu = (E, \vec{k}_i), \quad k'_\mu = (E', \vec{k}'_i), \quad p_\mu = (M, \vec{0})$$

$$(q+p)^2 = q^2 + 2q \cdot p + M^2 = M^2$$ in the elastic scattering limit

$$2M(E - E') = -q^2$$



Kinematics 2 - Lorentz invariant quantities



$$q = k - k'$$

$$x = \frac{Q^2}{2M\nu} = \frac{Q^2}{2p \cdot q}$$

$$\nu = E - E' = \frac{p \cdot q}{M}$$

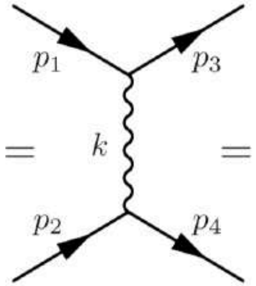
$$q^2 = (E - E')^2 - (\vec{k} - \vec{k}') \cdot (\vec{k} - \vec{k}') =$$

$$= m_e^2 + m_{e'}^2 - 2(EE' - |\vec{k}||\vec{k}'|\cos\theta) =$$

$$\approx \boxed{-4EE'\sin^2\tfrac{\theta}{2} \equiv -Q^2}$$

QED Amplitude eμ → eμ vs ep → ep

$i\mathcal{M} =$



$$= (-ie)\bar{u}(p_3)\gamma^\mu u(p_1)\frac{-i\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}{(p_1-p_3)^2}(-ie)\bar{u}(p_4)\gamma^\nu u(p_2)$$

$$\mathcal{M} = \frac{e^2}{t}\bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma_\mu u(p_2)$$

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4t^2}\text{Tr}[(\not{p}_1 + m_e)\gamma_\nu(\not{p}_3 + m_e)\gamma^\mu]\text{Tr}[(\not{p}_4 + m_\mu)\gamma_\mu(\not{p}_2 + m_\mu)\gamma^\nu]$$

Spin sum

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{2e^4}{t^2}\left[u^2 + s^2 + 4t(m_e^2 + m_p^2) - 2\left(m_e^2 + m_p^2\right)^2\right]$$

ep ->ep

mp >>me limit

$$p_1^\mu = (E, \vec{p}_i), \quad p_2^\mu = (m_p, 0), \quad p_3^\mu = (E, \vec{p}_f), \quad p_4^\mu = (m_p, 0)$$

$$\vec{p}_i \cdot \vec{p}_f = p^2 \cos\theta = v^2 E^2 \cos\theta$$

$$p = |\vec{p}_i| = |\vec{p}_f|$$

$$v = \frac{p}{E} = \sqrt{1 - \frac{m_e^2}{E^2}}$$

To the LO in me/mp

$$p_{13} = E^2(1 - v^2 \cos\theta),$$

$$p_{12} = p_{23} = p_{34} = p_{14} = E m_p$$

$$p_{24} = m_p^2,$$

$$t = (p_1 - p_3)^2 = -(\vec{p}_i - \vec{p}_f)^2 = -2p^2(1 - \cos\theta)$$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{t^2}\left[p_{14}p_{23} + p_{12}p_{34} - m_p^2 p_{13} - m_e^2 p_{24} + 2m_e^2 m_p^2\right]$$

$$= \frac{8e^4}{4v^4 E^4 (1-\cos\theta)^2}\left[E^2 m_p^2 + E^2 m_p^2 v^2 \cos\theta + m_e^2 m_p^2\right]$$

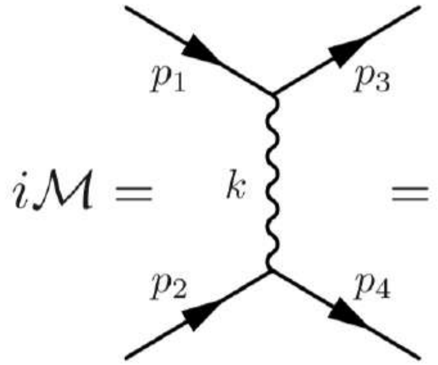
$$= \frac{2e^4 m_p^2}{v^4 E^2 (1-\cos\theta)^2}\left[2 - v^2(1 - \cos\theta)\right]$$

$$= \frac{e^4 m_p^2}{v^4 E^2 \sin^4\frac{\theta}{2}}\left[1 - v^2 \sin^2\frac{\theta}{2}\right].$$

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 v^2 p^2 \sin^4\frac{\theta}{2}}\left(1 - v^2 \sin^2\frac{\theta}{2}\right), \qquad E \ll m_p$$

$v \ll 1$ and $p \ll E \sim m_e$ → $$\frac{d\sigma}{d\Omega} = \frac{e^4 m_e^2}{64\pi^2 p^4 \sin^4\frac{\theta}{2}}$$

E >>me limit: we follow the same steps as before, but me=0 this time.



$$p_1^\mu = (E, \vec{p}_i), \quad p_2^\mu = (m_p, 0), \quad p_3^\mu = (E', \vec{p}_f), \quad p_4^\mu = p_1^\mu + p_2^\mu - p_3^\mu$$

$$|\vec{p}_i| = E \text{ and } |\vec{p}_f| = E'$$

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E} \left(\cos^2\frac{\theta}{2} + \frac{E - E'}{m_p} \sin^2\frac{\theta}{2} \right)$$

Mass dimension

$$g = g_0 \mu^{2-D/2} \quad ?$$

$$\mathscr{L} = a_1 F^{\mu\nu} F_{\mu\nu} + a_2 \bar{\psi} \gamma^\mu D_\mu \psi + a_3 \bar{\psi} \psi$$
$$+ a_4 \bar{\psi} \sigma^{\mu\nu} F_{\mu\nu} \psi + a_5 F^{\mu\nu} F_\nu{}^\lambda F_{\lambda\mu} + \quad \ldots$$

$S = \int d^D x \mathscr{L}$ Is dimensionless, thus $\dim[\mathscr{L}] = D$

By looking at the the kinetic terms for F and ψ

$$\dim[A_\mu] = \frac{D-2}{2}$$

$$\dim[\psi] = \frac{D-1}{2}$$

→

$$\dim[\bar{\psi}\psi] = D - 1 = 3 \quad ,$$
$$\dim[\bar{\psi}\gamma^\mu D_\mu \psi] = D = 4 \quad ,$$
$$\dim[F^{\mu\nu} F_{\mu\nu}] = D = 4 \quad ,$$
$$\dim[\bar{\psi}\sigma^{\mu\nu} F_{\mu\nu} \psi] = D + 1 = 5 \quad ,$$
$$\dim[F^{\mu\nu} F_\nu{}^\lambda F_{\lambda\mu}] = \frac{3}{2} D = 6 \quad , \ldots$$

$$\dim[g] + 2\dim[\psi] + \dim[A^a_\mu] = D$$

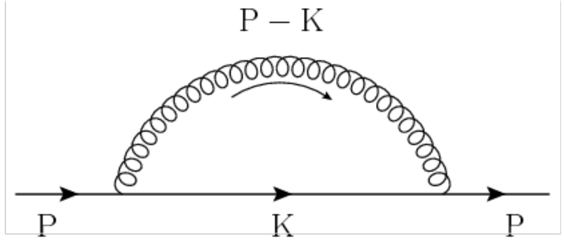
↓

$$\dim[g] = 2 - D/2$$

↓

$$g = g_0 \mu^{2-D/2}$$

UV divergence regularization: μR



$$\int \frac{d^4q}{16\pi^2} \cdots \longrightarrow \mu^{2\epsilon} \int \frac{d^{4-2\epsilon}q}{16\pi^2} \cdots = \frac{i\mu^{2\epsilon}}{(4\pi)^2} \left[\frac{C_{-1}}{\epsilon} + C_0 + C_1\,\epsilon + \mathcal{O}(\epsilon^2)\right]$$

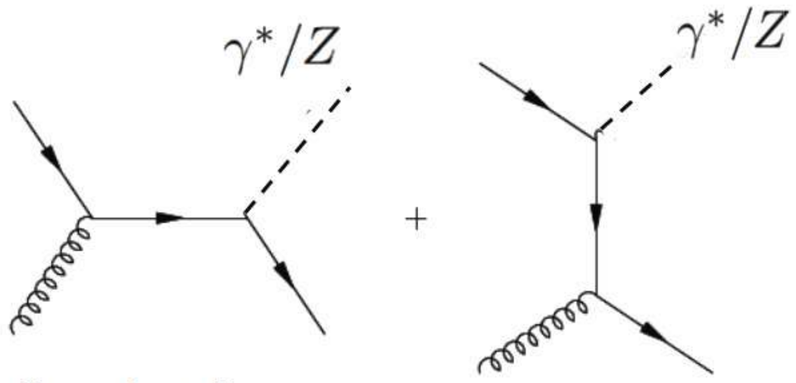
After renormalization, the poles cancel out but the scale is unmatched between the UV divergence and the counter term

$$\begin{aligned}\mu^{2\epsilon} \left[\frac{C_{-1}}{\epsilon} + C_0 + \mathcal{O}(\epsilon)\right] &= e^{2\epsilon\log\mu} \left[\frac{C_{-1}}{\epsilon} + C_0 + \mathcal{O}(\epsilon)\right] \\ &= \left[1 + 2\epsilon\log\mu + \mathcal{O}(\epsilon^2)\right] \left[\frac{C_{-1}}{\epsilon} + C_0 + \mathcal{O}(\epsilon)\right] \\ &= \frac{C_{-1}}{\epsilon} + C_0 + 2\log\mu\, C_{-1} + \mathcal{O}(\epsilon)\end{aligned}$$

The pole C_{-1}/ε gives a finite contribution to the cross section, involving the renormalization scale μR ≡ μ

IR divergence regularization: μF

Let’s consider qg-> Zq in Drell-Yan: at NLO we have no virtual correction -> no UV div



Compton diagram

Gluon emission angle $y = (1+\cos\theta)/2$

$\tau = m_Z^2/s$

$$\overline{|\mathcal{M}|^2} \sim 8\left[\frac{s^2 - 2sm_Z^2 + 2m_Z^4}{s(s-m_Z^2)}\;\frac{1}{y} - \frac{2m_Z^2}{s} \;+ \mathcal{O}(y)\right]$$

divergent for collinear gluon radiation: y -> 0 (emitted in the beam direction)

Following the procedure adopted to regularize UV divergences, we introduce a scale μ. DR now means we have to write the two-particle phase space in D = 4−2ε dimension

$$s\,\frac{d\sigma}{dy} = \frac{\pi(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{m_Z^2}\right)^{\epsilon}\frac{\tau^{\epsilon}(1-\tau)^{1-2\epsilon}}{y^{\epsilon}(1-y)^{\epsilon}}\overline{|\mathcal{M}|^2} \sim \left(\frac{\mu^2}{m_Z^2}\right)^{\epsilon}\frac{\overline{|\mathcal{M}|^2}}{y^{\epsilon}(1-y)^{\epsilon}} \quad ; \qquad \frac{1}{\sigma_{\text{tot}}}\,d\sigma \sim \frac{\alpha_s}{2\pi}\,\frac{dy}{y}\,dx\,P_j(x)$$

Here μ arises from the IR regularization of the phase space integral and is referred to as factorization scale μF .

PLUS (+) DISTRIBUTION

$$\hat{W}_0^R = 4\pi Q_i^2 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{Q^2}\right)^{\frac{\varepsilon}{2}} \frac{\Gamma(1-\frac{\varepsilon}{2})}{\Gamma(1-\varepsilon)}$$
$$\times \left\{3z + z^{\frac{\varepsilon}{2}}(1-z)^{-\frac{\varepsilon}{2}}\left(-\frac{2}{\varepsilon}\frac{1+z^2}{1-z} + 3 - z - \frac{3}{2}\frac{1}{1-z} - \frac{7}{4}\frac{\varepsilon}{1-z}\right)\right\}$$

The expansion around $\varepsilon = 0$ gives a distribution

$$\frac{1}{(1-z)^{1+\varepsilon}} = -\frac{1}{\varepsilon}\delta(1-z) + \frac{1}{[1-z]_+} - \varepsilon\left[\frac{\ln(1-z)}{1-z}\right]_+ + \sum_{n=2}^{\infty}\frac{(-\varepsilon)^n}{n!}\left[\frac{\ln^n(1-z)}{1-z}\right]_+$$

Properties

$$\int_0^1 dz \frac{f(z)}{[1-z]_+} \equiv \int_0^1 dz \frac{f(z)-f(1)}{1-z}$$

$$\int_0^1 dz\, f(z)\left[\frac{\ln^n(1-z)}{1-z}\right]_+ \equiv \int_0^1 dz(f(z)-f(1))\,\frac{\ln^n(1-z)}{1-z}$$