Jet Cross Sections, Shapes and Substructure

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PITT PAC

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1. Why and where are there jets?
2. The measures and structures of jets.

   We'll try and point out ways in which QCD jets are unique, yet part of a universal phenomenon in field theory.

What we're going to try and get across in Part 1: Why and where are there jets?

A. The intuition behind particle jets, and a sketch of their history in experiment.

B. Challenges at very high energy: why and how soft and collinear enhancements arise in long-time behavior

C. Why energy flow is a guide to calculable cross sections: infrared safety

In Part II, The measures and structures of jets, we'll discuss

A. How jets are found and their cross sections computed

B. Inside jets I: jet shapes, their resummations in and beyond perturbation theory

C. Inside jets II: fragmentation functions and evolution

1A. The intuition behind particle jets, and a sketch of their history in experiment.

Outline

- Quantitative comparisons of QCD to experiment began with fully inclusive processes.
- In a seeming paradox, inclusive cross sections can be related to elastic scattering of quarks (the parton model). Asymptotic freedom makes this plausible
- Electron positron annihilation to hadrons is dominated by two-jet events that clearly reflect quark pair creation. The observable called "thrust" helps identify jets and justify the use of the term jet.
- High energy accelerators, at energies far above (light) quark masses, all produce events consistent with this interpretation.

Prehistory of jets: the 1950's – 1960's

- The first observations of particle "jets" was in cosmic ray detection.

  Particle jets in cosmic rays ...

  "The average transverse momentum resulting from our measurements is p_T=0.5 BeV/c for pions ... Table 1 gives a summary of jet events observed to date ..." (B. Edwards et al, Phil. Mag. 3, 237 (1957))

- The era of high energy physics and the discovery of the Standard Model

  Once asymptotic freedom explained scaling (Feynman, Bjorken)

$$\sigma^{\rm incl}_{e\,{\rm proton}}\left(Q, x = \frac{Q^2}{2p\cdot q}\right) \to \sigma^{\rm excl}_{e\,{\rm parton}}(Q) \times F_{\rm proton}(x)\,,$$

- the question arose: what happens to partons in the final state?
  (Feynman, Bjorken & Paschos, Drell, Levy & Yan, 1969)
  Do "the hadrons 'remember' the directions along which the bare constituents were emitted? ... "the observation of such 'jets' in colliding beam processes would be most spectacular." (Bjorken & Brodsky, 1969) Or does confinement forbid a it?

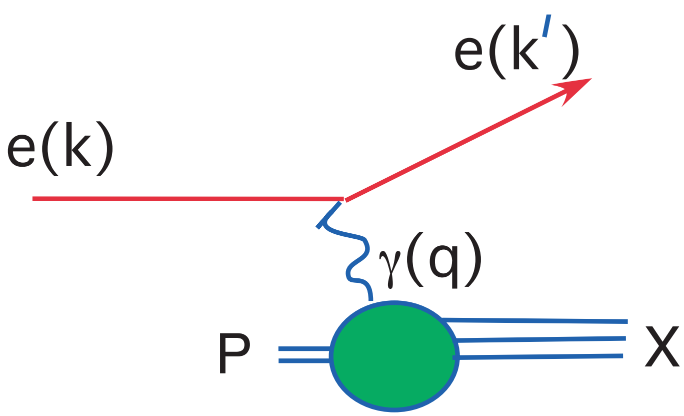
- The inclusive DIS cross section is described by exclusive partonic scattering. Could something similar happen in a less inclusive observable?

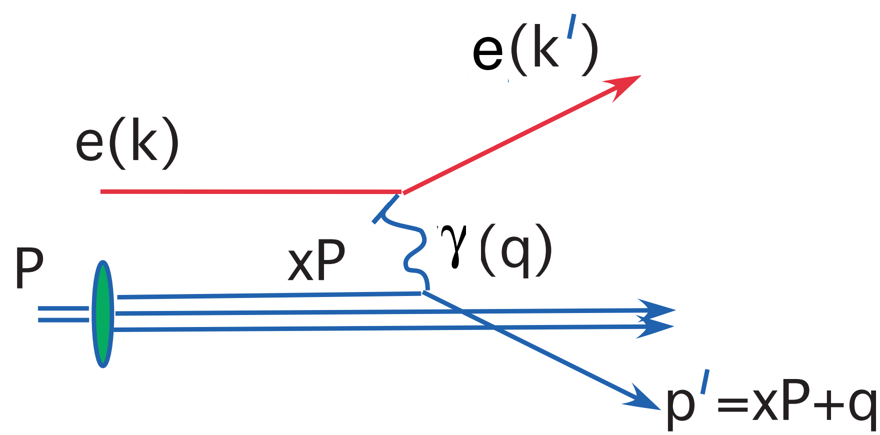
- To make this long story short: Quantum Chromodynamics (QCD) reconciled the irreconcilable. Here was the problem.

  1. Quarks and gluons explain spectroscopy, but aren't seen directly – confinement.

  2. In highly ("deep") inelastic, electron-proton scattering, the inclusive cross section was found to well-approximated by lowest-order elastic scattering of point-like (spin-1/2) particles (="partons" = quarks here) a result called "scaling":

$$\frac{d\sigma_{e+p}(Q, p\cdot q)}{dQ^2}\Big|_{\text{inclusive}} \quad \propto \quad F\left(x = \frac{Q^2}{2p\cdot q}\right) \frac{d\sigma^{\text{free}}_{e+\text{spin}\,\frac{1}{2}}}{dQ^2}\Big|_{\text{elastic}}$$





- If the "spin-$\frac{1}{2}$ is a quark, how can a confined quark scatter freely?

- This paradoxical combination of confined bound states at long distances and nearly free behavior at short distances was explained by asymptotic freedom: In QCD, the force between quarks behaves at short distances like

$$f(r) \sim \frac{\alpha_s(r)}{r^2}, \qquad \alpha_s(r^2) = \frac{4\pi}{\ln\left(\frac{1}{r^2\lambda^2}\right)}$$

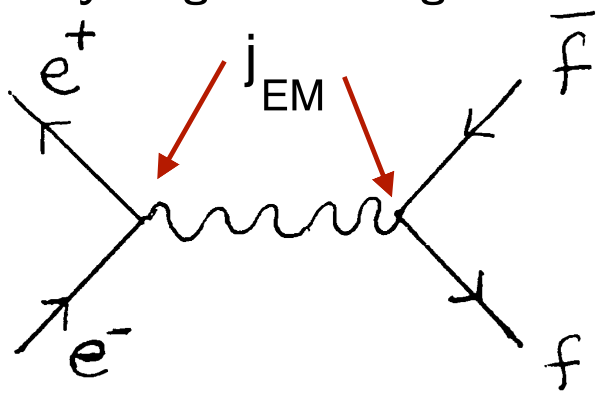
where $\Lambda \sim 0.2$ GeV. For distances much less than $1/(0.2GeV) \sim 10^{-8}$cm the force weakens. These are distances that began to be probed in deep inelastic scattering experiments at SLAC in the 1970s.

- The short explanation of DIS: Over the times $ct \leq \hbar/GeV$ it takes the electron to scatter from a quark-parton, the quark really does seem free. Later, the quark is eventually confined, but by then it's too late to change the probability for an event that has already happened.

- The function $F(x)$ is interpreted as the probability to find quark of momentum xP in a target of total momentum P – a parton distribution.

- **To explore further, SLAC used the quantum mechanical credo: anything that can happen, will happen.**

- **Quarks have electric charge, so if they are there to be produced, they will be. This can happen when colliding electron-positron pairs annihilate to a virtual photon, which (ungratefully) decays to just anything with charge.**



- **Of course because of confinement it's not that. But more generally, we believe that a virtual photon decays at a point through a local operator: $j_{\rm em}(x)$.**

- **This enables translating measurements into correlation functions . . . In fact, the cross section for electron-positron annihilation probes the vacuum with an electromagnetic current.**

- On the one hand, all final states are familiar hadrons, with nothing special about them to tell the tale of QCD, $|N\rangle = |\text{pions, protons}\ldots\rangle$,

$$\sigma_{e^+e^-\to\ \text{hadrons}}(Q)\ \propto\ \sum_N |\langle 0|j^\mu_{\text{em}}(0)|N\rangle|^2\,\delta^4(Q-p_N)$$

- On the other hand, $\sum_N |N\rangle\langle N| = 1$, and using translation invariance this gives

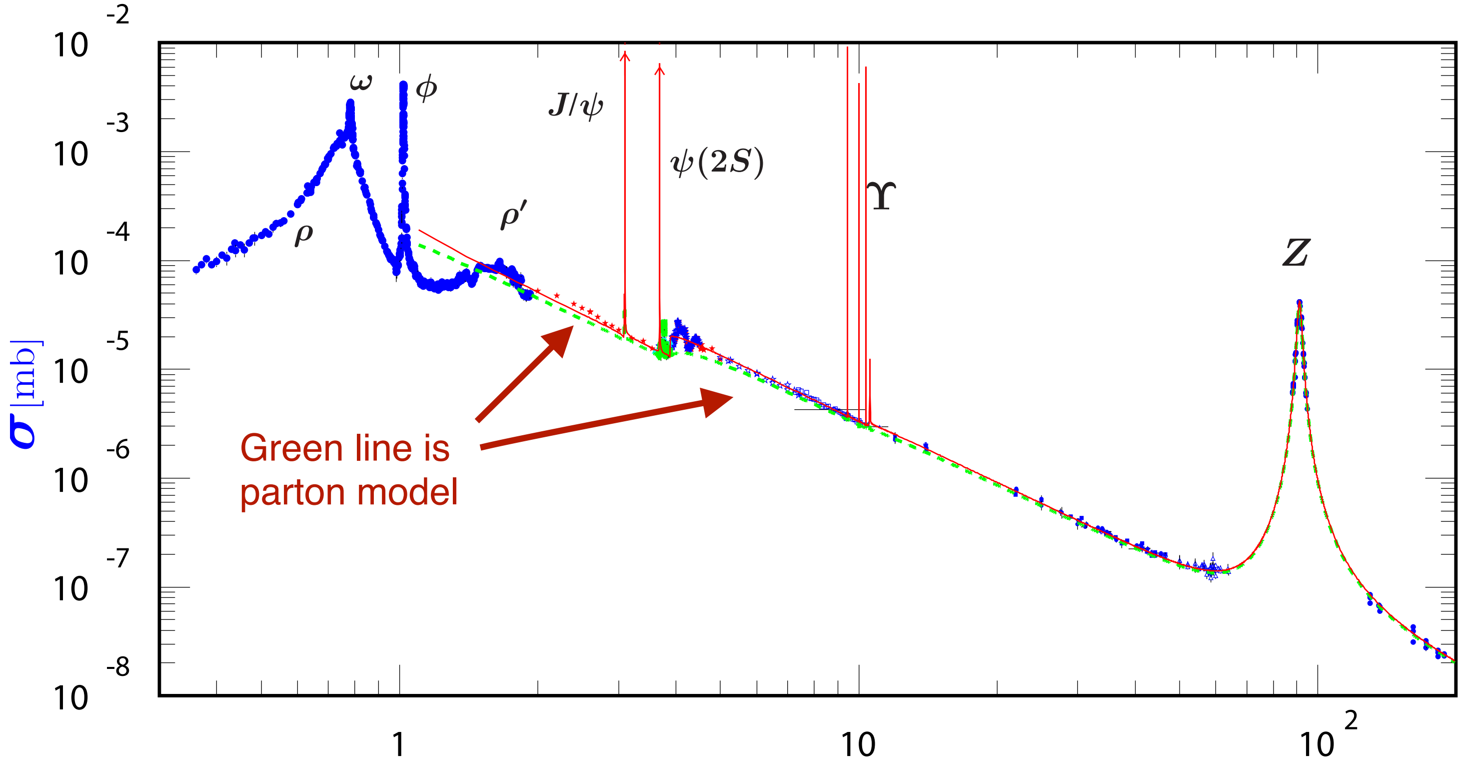
$$\sigma_{e^+e^-\to\ \text{hadrons}}(Q)\ \propto\ \int d^4x\, e^{-iQ\cdot x}\ \langle 0|j^\mu_{\text{em}}(0)\ j^\mu_{\text{em}}(x)|0\rangle$$

- We are probing the vacuum at short distances, imposed by the Fourier transform as $Q\to\infty$. The currents are only a distance $1/Q$ apart.

- Asymptotic freedom suggests a "free" result: QCD at lowest order ("quark-parton model") at cm. energy Q and angle θ

$$\sigma^{tot}_{e^+e^-\to\text{hadrons}}\ =\ \frac{4\pi\alpha^2_{\text{EM}}}{3Q^2}$$

- **This works for σ_{tot} to quite a good approximation (with calculable corrections)**



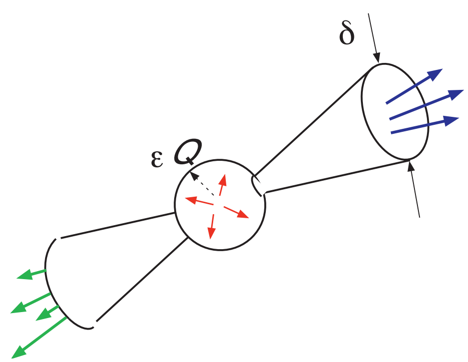
- **So the "free" theory again describes the inclusive sum over confined (nonperturbative) bound states – another "paradox".**

- **Is there an imprint on these states of their origin? Yes. What to look for? The spin of the quarks is imprinted in their angular distribution:**

$$\frac{d\sigma(Q)}{d\cos\theta} = \frac{\pi\alpha_{\rm EM}^2}{2Q^2}\left(1+\cos^2\theta\right)$$

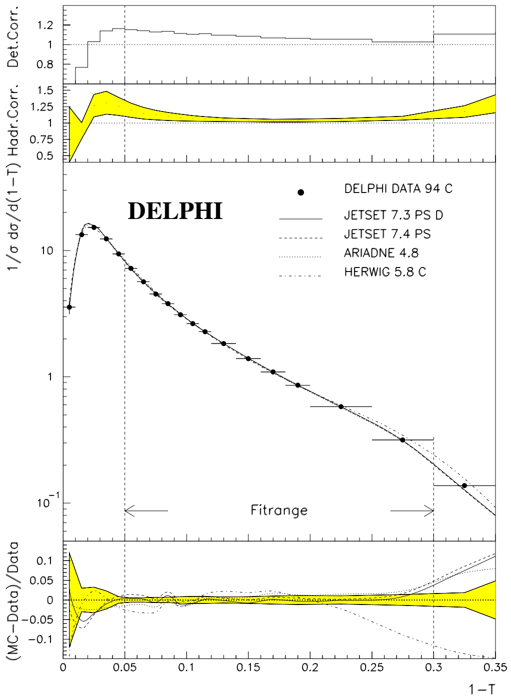
- **It's not quarks, but can look for a back to back flow of energy by finding an axis that maximizes the projection of particle momenta ("thrust") measuring a "jet-like" structure**

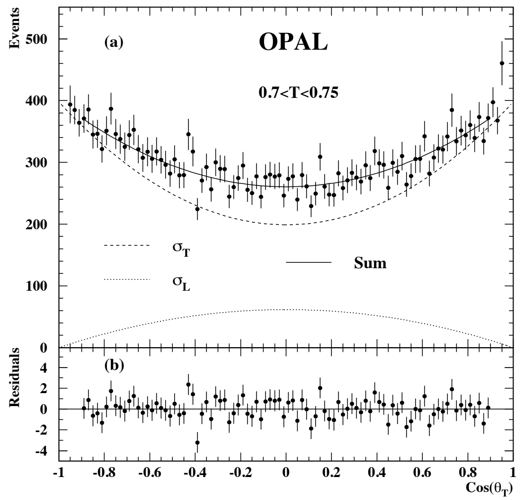
$$\frac{d\sigma_{\rm e^+e^-\to\ hadrons}(Q)}{dT} \propto \sum_N |\langle 0|j^\mu_{\rm em}(0)|N\rangle|^2\,\delta^4(Q-p_N)\ \delta\left(T - \frac{1}{Q}\max_{\hat{\rm n}}\sum_{i\in N}|\vec{p}_i\cdot\hat{\rm n}|\right)$$

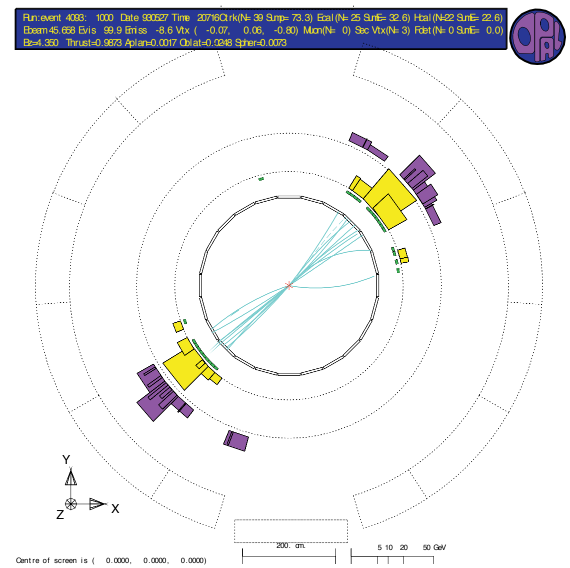


- **When the particles all line up $T\to 1$ (neglecting masses). So what happens?**

- **Here's what was found (from a little later, at LEP):**

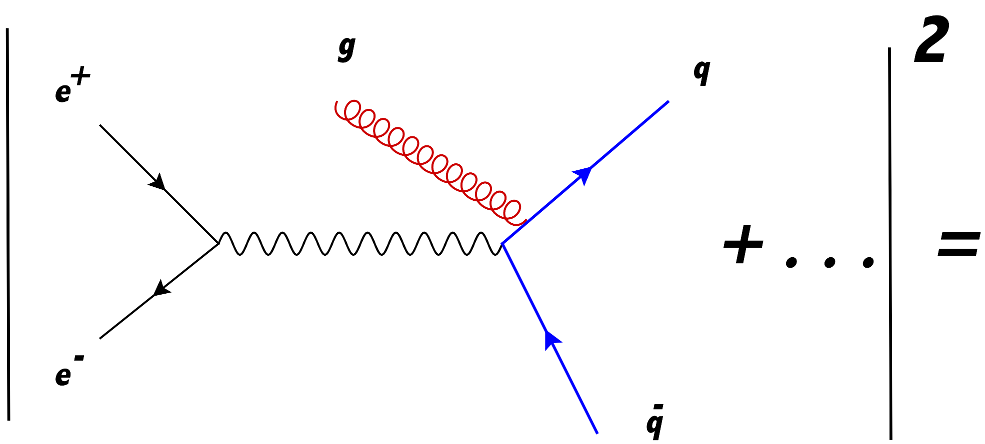


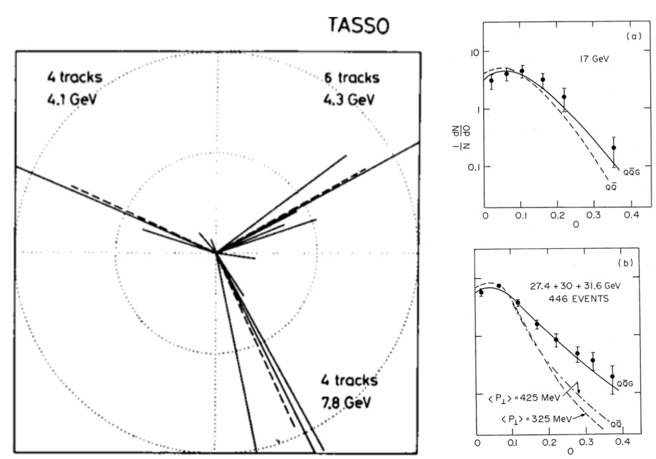




- **Thrust is peaked near unity and follows the $1 + \cos^2\theta$ distribution – reflecting the production of spin $\frac{1}{2}$ particles – back-to-back. All this despite confinement. Quarks have been replaced by "jets" of hadrons. What could be better? But what's going on? How can we understand persistence of short-distance structure into the final state, evolving over many many orders of magnitude in time?**

- Back to the Timeline . . . 1975 -1980: the first quark and gluon jets
- As we've seen: in electron-positron annihilation to hadrons, the angular distribution for energy flow follows the lowest-order ("Born") cross section for the creation of spin-1/2 pairs of quarks and antiquarks (As first seen by Hanson *et al*, at SLAC in 1975)
- Jets are "rare" because the high momentum transfer scattering of partons is rare (but calculable), but in e^+e^- annihilation to hadrons the "rarity" is in the likelihood of annihilation. Once that takes places, jets are nearly always produced.
- And then (Ellis, Gaillard, Ross (1976) Ellis, Karliner (1979)): hints of three gluons in Upsilon decay, and then unequivocal gluon jets at Petra (1979) (S.L. Wu (1984))



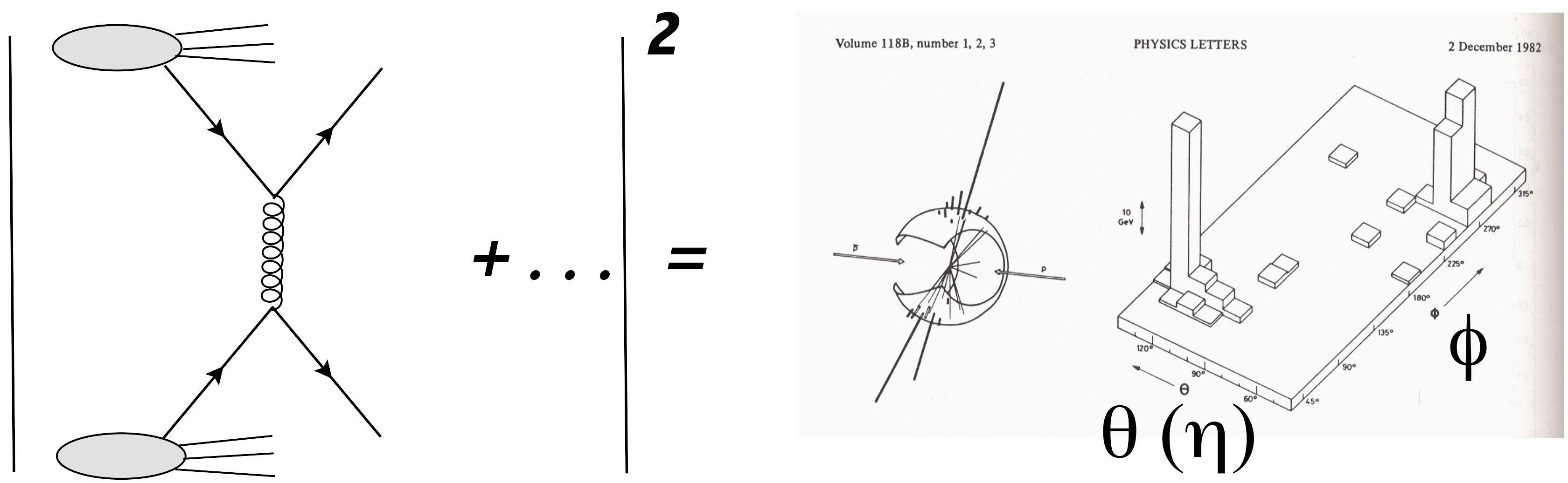


(a) (b)

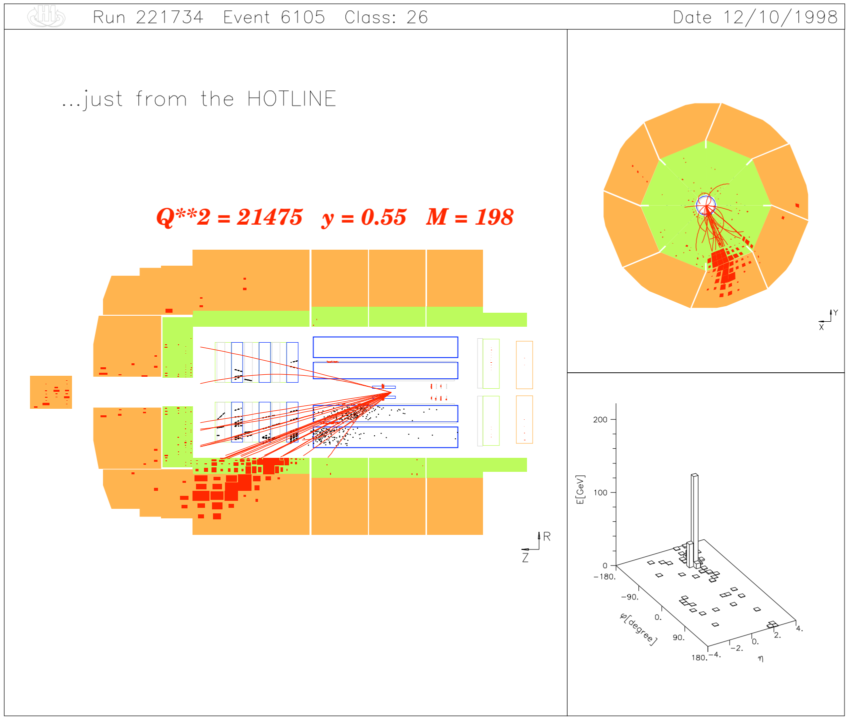
(On the right, O is oblateness, which measures the spread of energy in a plane.)

- confirmed color as a dynamical variable.

- **Jets at hadron colliders . . .**
- **80's: direct and indirect 'sightings' of scattered parton jets at Fermilab and the ISR at CERN, often in the context of single-particle spectra. Overall, however, an unsettled period until the SPS large angular coverage makes possible (UA2) 'lego plots' in terms of energy flow, and leads to the unequivocal observation of high-p_T jet pairs that represent scattered partons.**

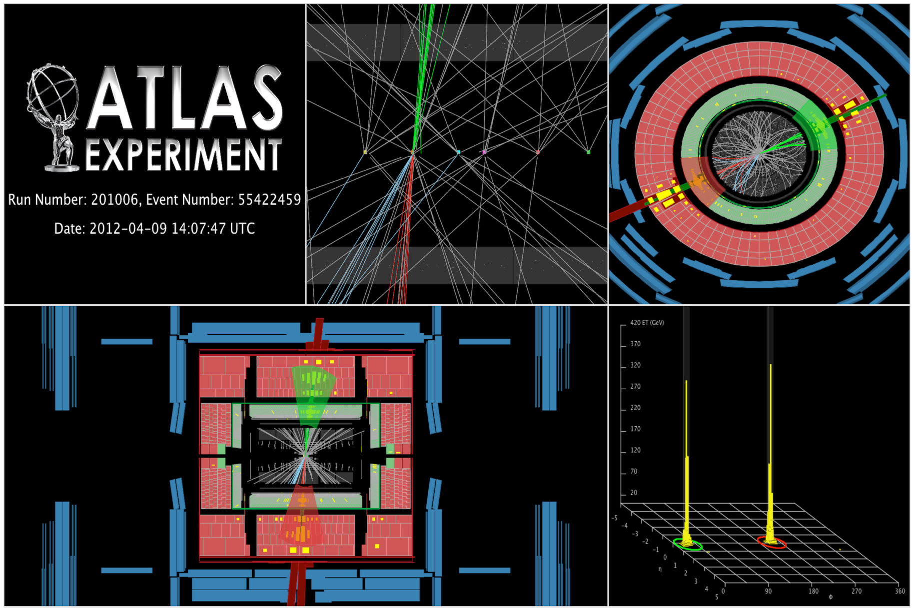


- **1990's – 2005: The great Standard Model machines: HERA, the Tevatron Run I, and LEP I and II provided jet cross sections over multiple orders of magnitude. The scattered quark appears.**



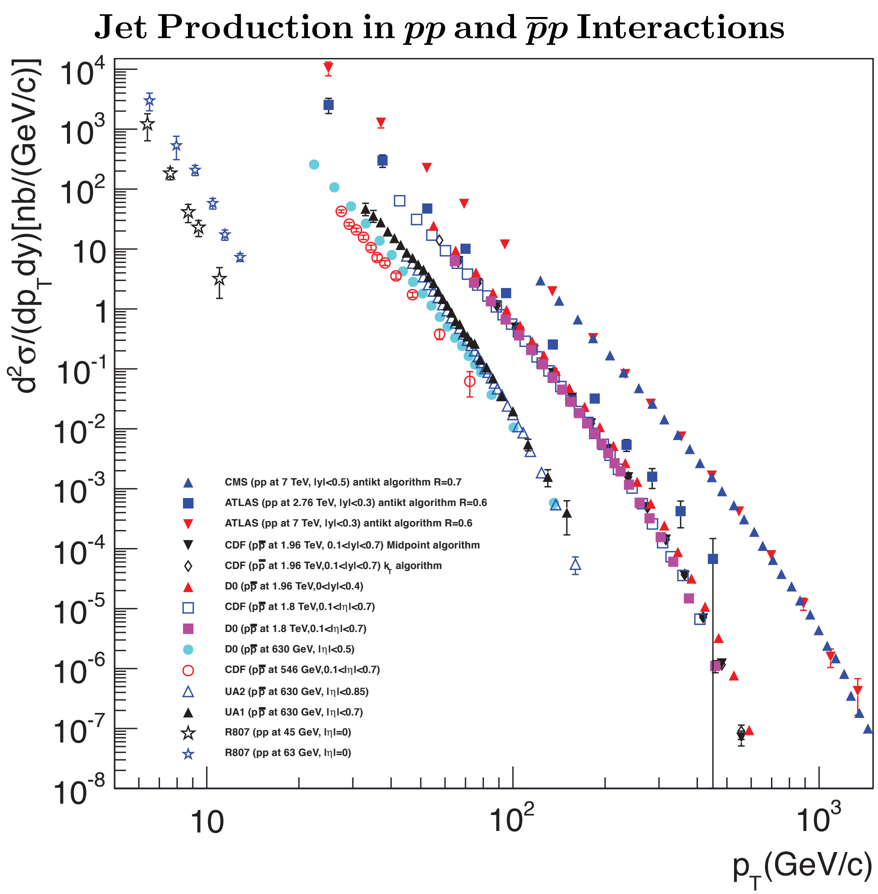
- **And now ... the era of jets at the anticipated limits of the SM, ushered in by Tevatron Run II, on to the LHC: 2 $\rightarrow$ 7 $\rightarrow$ 8 $\rightarrow$ 13 TeV .**

- **Events at the scale $\delta x \sim \frac{\hbar}{1\ \mathrm{TeV}} \sim 2 \times 10^{-19}$ meters ... observed about 10 meters away.**



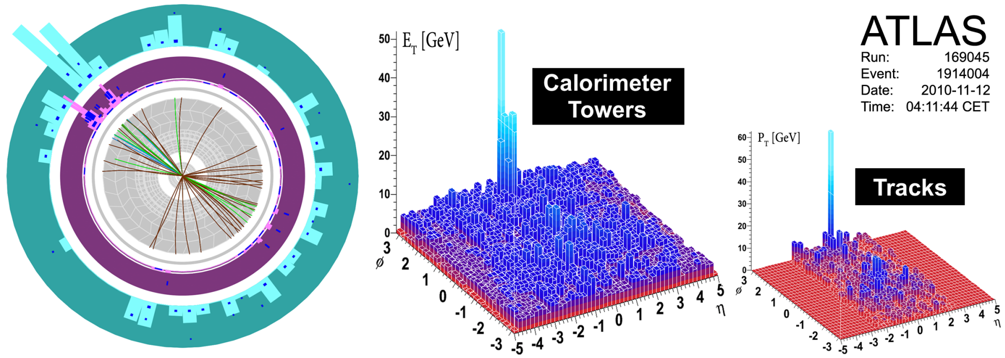
- **These jets can be remarkably narrow in an energy histogram, even if surrounded by a concentration of much softer particle tracks. This suggests a relation to QED bremsstrahlung.**

“REVIEW OF PARTICLE PROPERTIES” FIGURE: TEV JETS AND BEYOND



In brief, in their other life: shining from the inside, jets are probe of new phases of strongly-interacting matter in nuclear collisions at RHIC and the LHC,

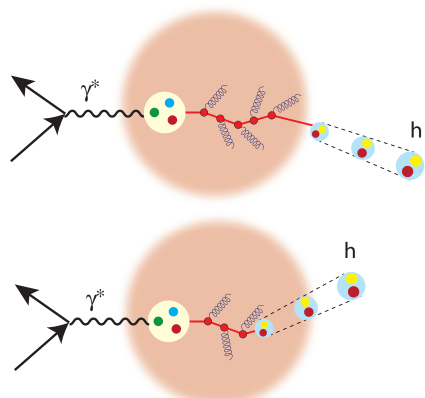
(Bjorken (1983) . . .)



(From 1011.6182)

And of "cold nuclei" in electron-ion collisions,

(A. Arccadi $et\ al.$**, Electron-ion Collider White Paper (1212.1701))**



1A. Summary

- **Quantitative comparisons of QCD to experiment began with fully inclusive processes.**
- **In a seeming paradox, inclusive cross sections can be related to elastic scattering of quarks (the parton model). Asymptotic freedom makes this plausible**
- **Electron positron annihilation to hadrons is dominated by two-jet events that clearly reflect quark pair creation. The observable "thrust" helps identify and justify the use of the term jet.**
- **High energy accelerators, at energies far above (light) quark masses, all produce events consistent with this interpretation.**

IB: Challenges at very high energy: why and how soft and collinear enhancements arise in long-time behavior

Outline

- In QCD, long-time dynamics is not accessible to perturbation theory.

- The example of QED suggests that partially inclusive cross sections can be calculable perturbatively by eliminating infrared divergences.

- At very high energies, divergences appear in scattering amplitudes when lines in virtual states become collinear as well as soft.

- Time-ordered perturbation theory provides a convenient picture of how an amplitude develops in time. It gives insight into both UV and IR behavior.

- At large times, the effects of interactions between high energy particles vanish, except for those between collinear-moving and/or soft particles.

How to use perturbation theory in QCD?

- How to go beyond totally inclusive cross sections in QCD? Can quarks and gluons be of help? At lowest order, $\mathrm{e^+e^-} \rightarrow \mathrm{q\bar{q}}$ is easy to calculate, but what can we do with $\mathrm{e^+e^-} \rightarrow \mathrm{q\bar{q}g}$? It is divergent when the energy of the gluon vanishes, and has logs of quark mass over total energy. (We'll see why.)

- And what to do about the running of the asymptotically free QCD coupling? If low-energy divergences imply sensitivity to long distances, doesn't the coupling blow up, making the entire process nonperturbative?

- Very analogous questions were phrased for strong interactions at high energy (think cosmic rays) in the 1930s, even before renormalization was invented. And back then the analysis of Bloch and Nordseick for QED was recognized as a possible way forward . . .

- **The glorious example of QED: At lowest order, electron-electron scattering is finite, but at next to leading order it is IR divergent for both virtual corrections and photon emission. But in a partially inclusive sum over soft photon emission only, the divergences cancel, and we derive a finite cross section.**

- **How? We introduce an "energy resolution", ϵE, below which we count all photons. Then divergences are replaced by factors $\alpha \ln(E_e/\Delta E)$, and this "inclusive" cross section is well-approximated by the lowest order (again). Schematically:**

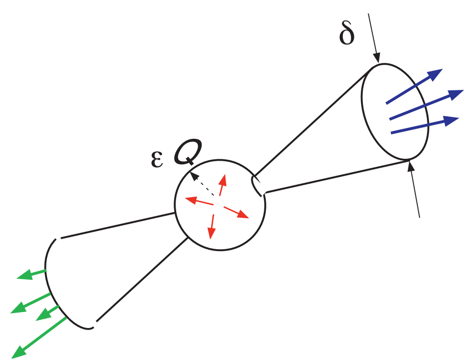
$$\frac{d\sigma_n^{(\mathrm{IR})}}{d\Omega} \sim \frac{d\sigma_0}{d\Omega} \times \frac{1}{n!} \left(\frac{\alpha_{\mathrm{EM}}}{\pi} \ln\left(\frac{E_e}{m_e}\right) \ln\left(\frac{\epsilon E_{\mathrm{tot}}}{m_\gamma}\right)\right)^n \exp\left[- \frac{\alpha_{\mathrm{EM}}}{\pi} \ln\left(\frac{E_e}{m_e}\right) \ln\left(\frac{E_e}{m_\gamma}\right)\right]$$

- **For $|\ln \epsilon| \ll 137$, this is very close to the Born ($n = 0$) cross section. All the higher orders cancel (corrected by well-behaved terms we've omitted here). The paradoxical lesson: "the more inclusive, the closer to the lowest order."**

- Once QCD was invented, QED served an inspiration for the treatment of strong interactions in the limit when energies and momentum transfers are much larger than masses.

- For QCD, at very high energy we had to introduce an energy resolution and another, "angular" resolution. We'll see why below, and how to generalize to a much larger set of observables.

- From now on, all our particles will be massless. Particles whose masses are of the order of the energy/momentum transfer scale can be treated at the same time, but require special attention. (Aside – this is treating QCD as though it were a conformal theory, with no intrinsic mass scale.) The picture:



- With ϵQ the energy resolution, an δ an angular resolution. Defines a "cone jet".

- **Looks promising, but how does it work? First, we have to isolate the problem, then show how the jet approach solves it.**

- **Let's remember what we'd like to calculate. It's a general "transition probability", or cross section, summed over final states "f", which we'll represent as**

$$
\begin{aligned}
P[S] &= \sum_f S[f] \, |\langle m_f | m_0 \rangle|^2 \\
&= \sum_f S[f] \sum_{n',n} \langle m_0 | m_f \rangle^{(n')} \langle m_f | m_0 \rangle^{(n)}
\end{aligned}
$$

The function $S[f]$ defines the cross section. It includes all the normalizations, and otherwise can be unity for some states, zero for others, or in between. Generally, we'll assume it's a smooth function.

- **To calculate $P[S]$, we'll start with the amplitude $\langle m_f | m_0 \rangle^{(n)}$ at fixed perturbative order (n) in QCD or some other theory. This is "just" a bunch of Feynman diagrams, but we'll consider a variation of this route.**

Perturbation theory "from the beginning"

- It really just follows from Schrödinger equation for mixing of free particle states $|m\rangle$,

$$i\hbar \frac{\partial}{\partial t}|\psi(t)>= \left(H^{(0)}+V\right)|\psi(t)>$$

Usually with free-state "IN" boundary condition :

$$|\psi(t=-\infty)>=|m_0>=|p_1^{\rm IN},p_2^{\rm IN}\rangle$$

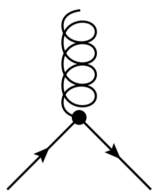
- Notation : $V_{ji}=\langle m_j|V|m_i\rangle$ (vertices)

- Theories differ in their list of particles and their (hermitian) Vs.

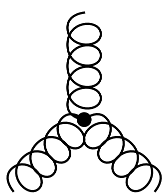
For QCD, the Lagrange density

$$
\begin{aligned}
\mathcal{L}_{\mathrm{QCD}} &= \overline{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i - \tfrac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a - g_{\mathrm{s}}\,\overline{\psi}_i\lambda_{ij}^a\psi_j\,\gamma^\mu A_\mu^a \\
F_a^{\mu\nu} &= \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - 2g_{\mathrm{s}}\,f_{abc}A_b^\mu A_c^\nu
\end{aligned}
$$

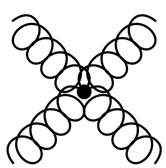
And vertices



$g_{\mathrm{s}}\,\bar{\psi}_i\lambda_{ij}^a\psi_j\gamma^\mu A_\mu^a$ quark-gluon vertex



$g_{\mathrm{s}}\,(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu)f_{abc}A_\mu^b A_\nu^c$ 3-gluon vertex



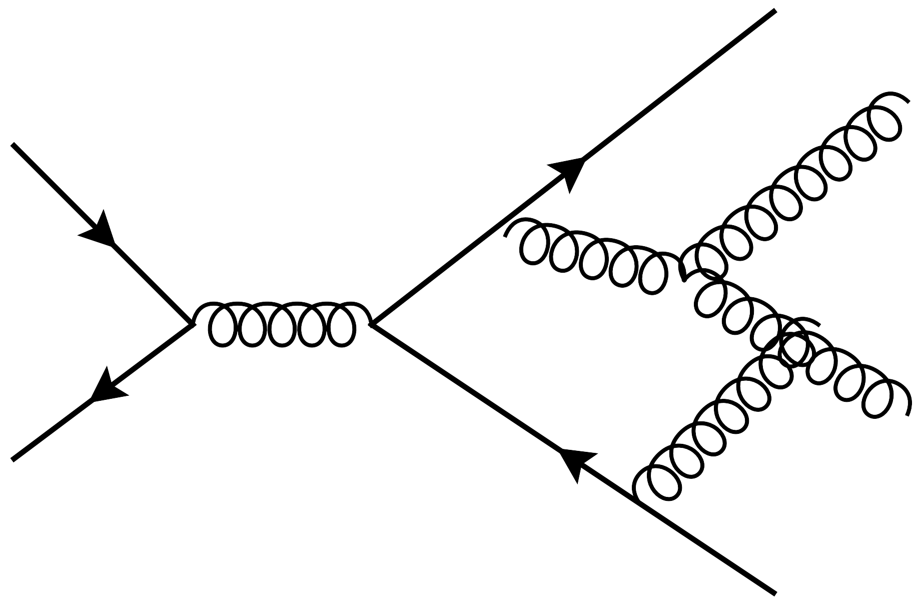
$g_{\mathrm{s}}^2\,f_{abc}A_b^\mu A_c^\nu\,f_{ade}A_\mu^d A_\nu^e$ 4-gluon vertex

- **Solutions to the Schrödinger equation are sums of ordered time integrals. "Old-fashioned perturbation theory."**

$$
\begin{aligned}
\langle m_F | m_0 \rangle^{(n)} = & \sum_{\tau\ \text{orders}} \int_{-\infty}^{\infty} d\tau_n \ldots \int_{-\infty}^{\tau_2} d\tau_1 \\
& \times \prod_{\text{loops}\ i} \int \frac{d^3\ell_i}{(2\pi)^3} \prod_{\text{lines}\ j} \frac{1}{2E_j} \times \prod_{\text{vertices}\ a} iV_{a\to a+1} \\
& \times \exp\left[i \sum_{\text{states}\ m} \left(\sum_{j\ \text{in}\ m} E(\vec{p}_j) \right) (\tau_m - \tau_{m-1}) \right]
\end{aligned}
$$

- **Perturbative QFT in a nutshell:** integrals are divergent in QFT from:
- $\tau_i \to \tau_j$ (UV) and $\tau_i \to \infty$ (IR).
- Renormalization takes care of coinciding times. We'll just assume this is done.

Each term in this expansion corresponds to a "time-ordered" diagram



Here the vertices are ordered at different times. Sums of orderings give (topologically equivalent) "Feynman diagrams", which exhibit the Lorentz invariance manifestly.

The integrals over loop momenta are exactly the sums over all virtual states.

- Once renormalized, infinities only come from large times in . . . (same formula)

$$\langle m_n | m_0 \rangle = \sum_{\tau \text{ orders}} \int_{-\infty}^{\infty} d\tau_n \ldots \int_{-\infty}^{\tau_2} d\tau_1$$
$$\times \prod_{\text{loops } i} \int \frac{d^3 \ell_i}{(2\pi)^3} \prod_{\text{lines } j} \frac{1}{2E_j} \times \prod_{\text{vertices } a} i V_{a \to a+1}$$
$$\times \exp \left[i \sum_{\text{states } m} \left(\sum_{j \text{ in } m} E(\vec{p}_j) \right) (\tau_m - \tau_{m-1}) \right]$$

- Divergences from $\tau_i \to \infty$ are "Infrared=IR". In some sense, their "solution" is jets.

- But – it's not as bad as it looks! Time integrals extend to infinity, but usually oscillations damp them and answers are finite. Long-time, "infrared" divergences (logs) come about when phases vanish and the time integrals diverge.

- **When does this happen? Here's the phase:**

$$\exp\left[\, i \sum_{\text{states}\, m} \left(\sum_{j \text{ in } m} E(\vec{p}_j) \right) (\tau_m - \tau_{m-1}) \right] =$$

$$\exp\left[\, i \sum_{\text{vertices}\, m} \left(\sum_{j \text{ in } m} E(\vec{p}_j) \;-\; \sum_{j \text{ in } m-1} E(\vec{p}_j) \right) \;\tau_m \right]$$

- **Divergences for $\tau_i \to \infty$ requires two things:**

i) (RHS) the phase must vanish $\leftrightarrow$ "degenerate states"

$$\sum_{j\, \in m} E(\vec{p}_j) = \sum_{j\, \in m+1} E(\vec{p}_j)\,, \quad \text{and}$$

ii) (LHS) the phase must be stationary in loop momenta (sums over states):

$$\frac{\partial}{\partial \ell_{i\mu}}\,[\,\text{phase}\,] = \sum_{\text{states}\, m}\; \sum_{j \text{ in } m} (\pm\beta_j^{\mu})(\tau_{m+1} - \tau_m) = 0$$

where the β_js are normal 4-velocities:

$$\beta_j = \pm\partial E_j / \partial \ell_i\,.$$

- **Condition of stationary phase:**

$$\sum_{\text{states } m} \sum_{j \text{ in } m} (\pm\beta_j^\mu)(\tau_{m+1} - \tau_m) = 0$$

- $\beta^\mu \Delta\tau = x^\mu$ **is a classical translation. For IR divergences, there must be free, classical propagation as** $t \to \infty$**. Easy to satisfy if all the** β_j**'s are equal.**

- **Whenever fast partons (quarks or gluons) emerge from the same point in space-time,**

  <u>they will rescatter for long times only with collinear partons</u>.

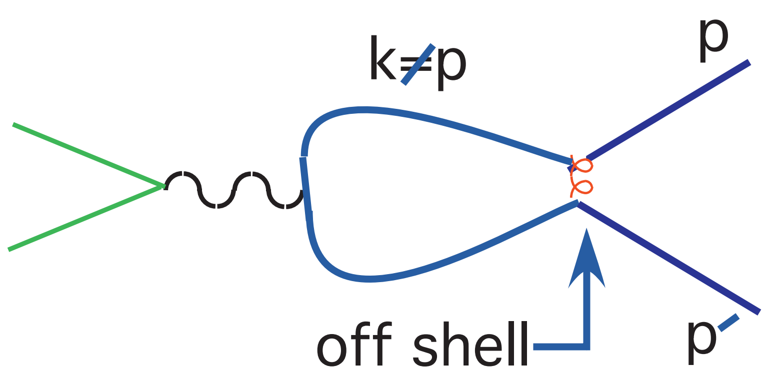
  Of course, radiating or absorbing zero momentum particles also don't affect the phase.

  Note, all the states we can reach by rescattering or zero momentum interactions describe the same energy flow.

  When we get to cross sections, this is where the conditions for infrared safety will come from.

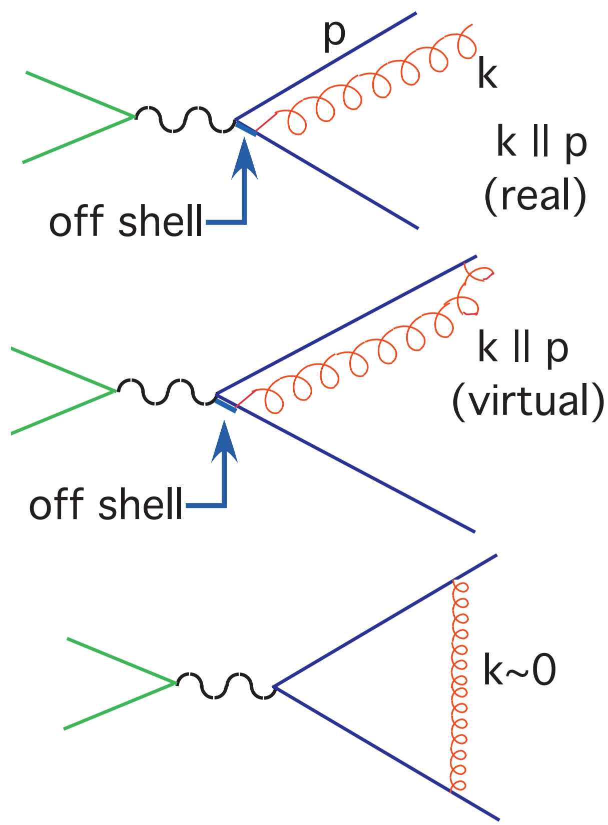
- **Let's illustrate the role of classical propagation.**

- **Example 1: degenerate states that cannot give long-time divergences:**



- **This makes identifying enhancements a lot simpler!**

- **RESULT: For particles emerging from a local scattering, (only) collinear or soft lines can give long-time behavior and enhancement.** **Example:**



- **This generalizes to any order, and any field theory, but gauge theories alone have soft ($k \to 0$) divergences.**

- **These are what we can't compute in pQCD (as physical processes). And we didn't want to, because they are never produced! Let's find out what we can compute.**

IB Summary

– In QCD, long-time dynamics is not accessible to perturbation theory.

– The example of QED suggests that partially inclusive cross sections can be calculable perturbatively by eliminating infrared divergences.

– At very high energies, divergences appear when lines become collinear as well as soft.

– Time-ordered perturbation theory provides a convenient picture of how an amplitude develops in time. It gives insight into both UV and IR behavior.

– At large times, the effects of interactions among high energy particles vanish, except for those between collinear-moving and/or soft particles.

IC. Why energy flow is a guide to calculable cross sections: infrared safety

Outline

- The integral of the largest time controls IR behavior.

- Particle emission or absorption requires a characteristic formation time, which diverges is the collinear limit.

- The momentum flow evolution of each jet is independent of the others.

- Time-ordered emissions provide ordered branching pictures.

- In cross sections, a free sum over states always cancels long-time behavior by use of the largest time equation.

- Infrared safe weight functions can provide perturbative cross sections, and properties of jets.

- **The role of the largest time:**

$$\langle m_F | m_0 \rangle^{(n)} = \sum_{\text{orders } \mathrm{m_1 \ldots m_n}} \prod_{\text{loops } i} \int \frac{d^3\ell_i}{(2\pi)^3} \prod_{\text{lines } j} \frac{1}{2E_j}$$

$$\times \prod_{\text{vertices } a=1}^{n} \int_{\tau_{a-1}}^{\tau_{a+1}} iV_{a\to a+1} \exp\left[i \left(\sum_{j \text{ in } a+1} E(\vec{p}_j) \right) (\tau_a - \tau_{a-1}) \right]$$

$$= \sum_{\text{orders } \mathrm{m_1 \ldots m_n}} \prod_{\text{loops } i} \int \frac{d^3\ell_i}{(2\pi)^3} \prod_{\text{lines } j} \frac{1}{2E_j}$$

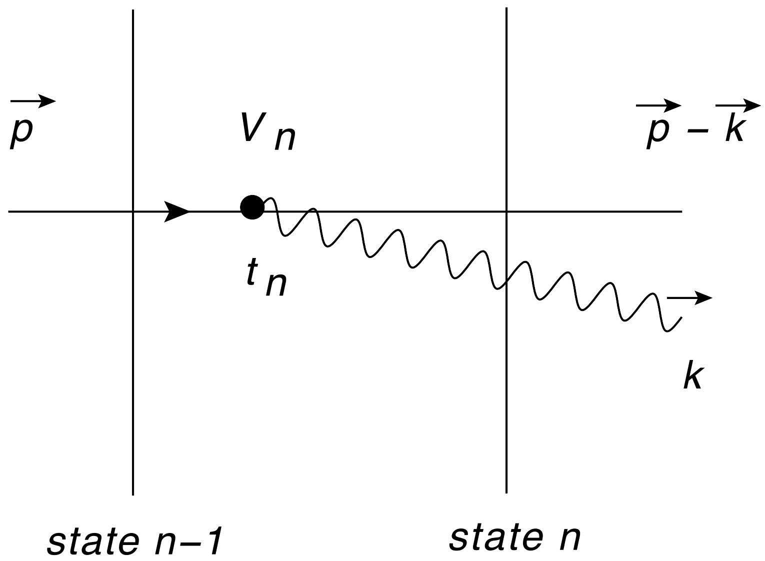
$$\times \prod_{\text{vertices } a=1}^{n} \int_{\tau_{a-1}}^{\tau_{a+1}} iV_{a\to a+1} \exp\left[i \left(\sum_{j \text{ in } a} E(\vec{p}_j) - \sum_{j \text{ in } a-1} E(\vec{p}_j) \right) \tau_a \right]$$

With $\tau_0 = -\infty$ **and** $\tau_{n+1} = \infty$

- **So large times are controlled by the** τ_n **integral: the "largest time".**

$$\int_{\tau_{n-1}}^{\infty} iV_{n-1\to F} \exp\left[i \left(\sum_{j \text{ in } F} E(\vec{p}_j) - \sum_{j \text{ in } n-1} E(\vec{p}_j) \right) \tau_n \right]$$

Say the final interaction is the splitting of one particle into two, all treated as massless:



Here state n = the final state f

All the other energies cancel, and the largest time integral is

$$\int_{\tau_{n-1}}^{\infty} d\tau_n i V_{n-1\to F} e^{i\left(\sum_{j\,\mathrm{in}\,n} E(\vec{p}_j)\;-\;\sum_{j\,\mathrm{in}\,n-1} E(\vec{p}_j)\right)\tau_n}$$

$$= \int_{\tau_{n-1}}^{\infty} d\tau_n i V_{n-1\to F} e^{i\Delta_n \tau_n}$$

Relabel: $p \to k_1,\ k \to k_2$:

$$\Delta_n \;=\; E(\vec{k}_1 - \vec{k}_2) + E(\vec{k}_2) - \;E(\vec{k}_1)$$

Can use the $i\epsilon$ prescription $\Delta \to \Delta + i\epsilon$ to make the integral converge. Or, we can observe that most of this integral cancels out "oscillation by oscillation" . Say $\tau_{n-1} \to 0$:

$$\begin{aligned}
\int_0^\infty d\tau_n e^{i\Delta_n \tau_n} &= \frac{1}{\Delta_n}\int_0^\infty dx\,[\cos x + i\sin x] \\
&= \frac{1}{\Delta_n}\int_0^\infty dx \frac{d}{dx}[\sin x - i\cos x] \\
&= -\frac{1}{\Delta_n}[\sin 0 - i\cos 0] \\
&= \frac{i}{\Delta_n}\int_0^{\pi/2} dx \sin x
\end{aligned}$$

– Only times smaller than $\pi/2\Delta_n$ really contribute to the amplitude.

– $1/\Delta_n$ is called the "formation time" of state n.

What is Δ_n and when does it vanish? When it does, we're going to have problems!

$$\Delta_n = E(\vec{k}_1 - \vec{k}_2) + E(\vec{k}_2) - E(\vec{k}_1)$$

– Kinematics

$$\vec{k}_1 = (P, \vec{0}_T)\,,\ \vec{k}_2 = (zP, \vec{k}_T)\,,\ k_T \leq zP \ll P$$

– Then

$$\Delta_n = \frac{k_T^2}{2zP} \Leftrightarrow \frac{1}{\Delta_n} = \frac{2zP}{k_T^2}$$

– Formation time grows with P/k_T (soft radiation) and with zP/k_T (collinear radiation).

– In terms of the angle: $k_T = zP \sin\theta$, for small θ,

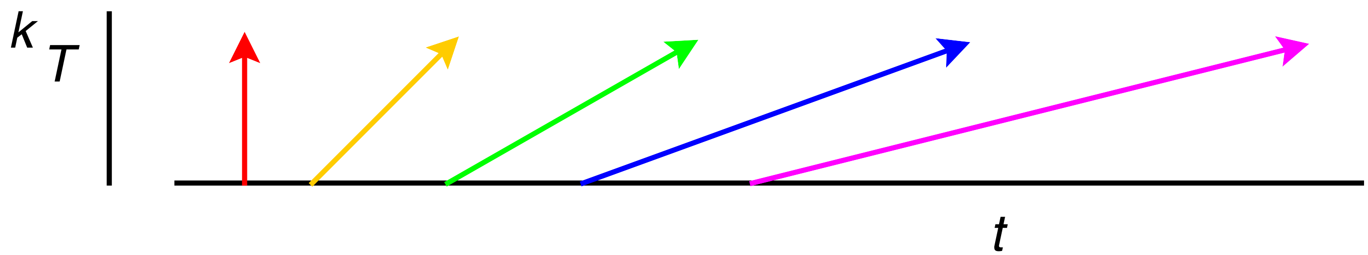
$$\frac{1}{\Delta_n} \sim \frac{1}{\theta^2 zP} \sim \frac{1}{\theta k_T}$$

– At fixed k_T, formation time increases as radiation becomes more forward.

– **This is a very general picture of the formation of final states.**

– **Because the final time limits all other integrals, particles produced at earlier times can only involve shorter formation times – wider angles and/or larger k_T.**

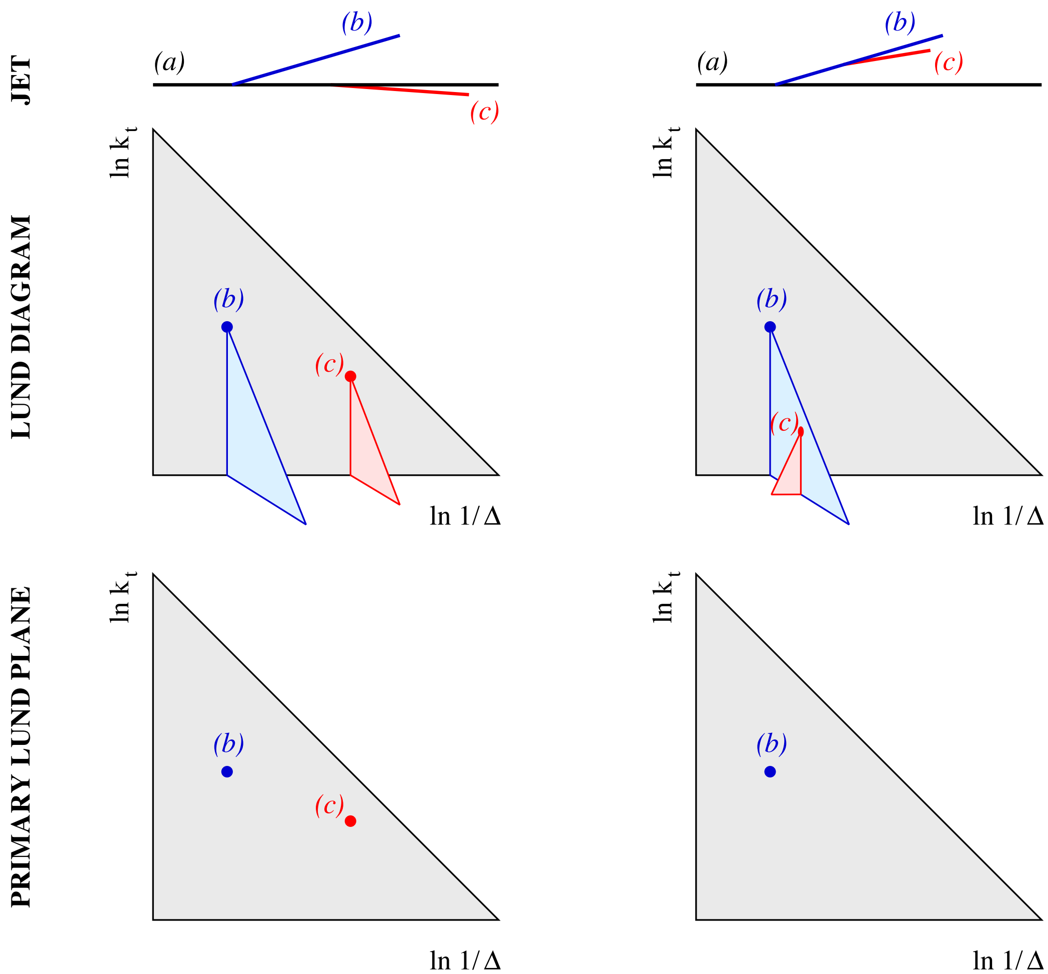
– **Gives a "branching" picture of radiation. At fixed k_T it starts with soft wide-angle, and moves on to smaller and smaller angles.**



– **When we probe smaller and smaller angle radiation, we look at states that took longer and longer to produce.**

– **As time increases, particle emission of each jet becomes more collimated. The jets evolve independently.**

– **Another popular way of representing radiation "branches": the "Lund Plane". Each branch is a point.**



from Dryer, Salam, Soyez 1807.04578

– **In the presence of massless particles, we encounter a divergent time integral whenever we find a $\Delta_n = 0$.**

– **The point $\Delta_n = 0$ is exactly a point of stationary phase in k_T.**

$$\begin{aligned} \int d^2k_T \int^{\infty} d\tau_n e^{i\Delta_n\tau_n} &= \int d^2k_T \int^{\infty} d\tau_n e^{i\tau_n k_T^2/2P} \\ &= \pi P \int^{\infty} \frac{d\tau_n}{\tau_n} \end{aligned}$$

– **$\Delta_n = 0$ when $z = 0$ and/or $k_T = 0$: Soft or collinear radiation.**

– **Now we can motivate the construction of IR finite cross sections.**

Finite-time cross sections and what they represent. Consider the probability for a sum over states f, each weighted by $S[f]$,

$$P[S] \ = \ \sum_f S[f] \sum_{n',n} \langle m_0 | m_f \rangle^{(n')} \langle m_f | m_0 \rangle^{(n)}$$

- Each matrix element and complex conjugate is a sum of ordered time integrals
- In any term of $P[S]$, there is a largest time.
- The largest time may be in the amplitude, or in the complex conjugate. We combine these two possibilities. Inside the sum over states, we find

$$\ldots \times \int_{\tau'_{n-2}}^{\tau'_n} e^{i\Delta_{n-1}\tau_{n-1}} (-iV'_{f-2\to f-1}) e^{-i\Delta_{n-1}\tau'_{n-1}} \quad \Leftarrow \text{times in } \langle m_0 | m_f \rangle$$

$$\times \int_{\tau_{f-1}}^{\infty} d\tau_n i V_{f-1\to f} e^{i\Delta_n \tau_n} \left(S[f] - S[f-1] \right)$$

$$\text{times in } \langle m_f | m_0 \rangle \Rightarrow \quad \times \int_{\tau_{n-2}}^{\tau_n} e^{i\Delta_{n-1}\tau_{n-1}} i V_{f-2\to f-1} e^{i\Delta_{n-1}\tau_{n-1}} \times \ldots$$

- When $S[f] = S[f-1]$ this vanishes! This is called the "largest time equation". It is an expression of unitarity – the sum of all probabilities has to be one.
- All that matters is the difference due to the last interaction: $V_{f-1\to f}$. When this produces a difference in $S[f]$, the result is nonzero.

- As in the introductory lectures, we define a set of smooth (symmetric) functions for which

$$S_{n+1}(p_1 \ldots (1-z)p_n, zp_n) \;=\; S_{n+1}(p_1 \ldots p_n)$$

Then, whenever $\Delta_n \to 0$, we only need

$$S_{n+1}[f] - S_n[f-1] \;\sim\; k_\perp^b s_f$$

for some constant s_f with $b > 0$. Then

$$\int d\tau_n \; e^{i\Delta_n \tau_n} \left(S_{n+1}[f] - S_n[f-1] \right) \;\to\; s_f \int d\tau_n \; k_\perp^b e^{i\Delta_n \tau_n}$$

- There is now suppression for large times:

$$s_f \int d^2k_T \, k_\perp^b \int^\infty d\tau_n e^{i\Delta_n \tau_n} \;=\; \pi s_f \Gamma(1+b) \int^\infty \frac{d\tau_n}{\tau_n^{1+b/2}}$$

- and the perturbative integral will be finite. The largest time integral converges, and so must the smaller ones.

- In summary, For any $S[f]$ that respects energy flow, we compute the cross section

$$\boldsymbol{P[S] \ = \ \sum_f S[f]\ |\langle m_f|m_0\rangle|^2}$$

- The same applies jet cross sections themselves if they are designed to respect the flow of energy. Here, S[f] is chosen to be unity for states that obey certain conditions in jet finding algorithms – which depend only on energy flows,

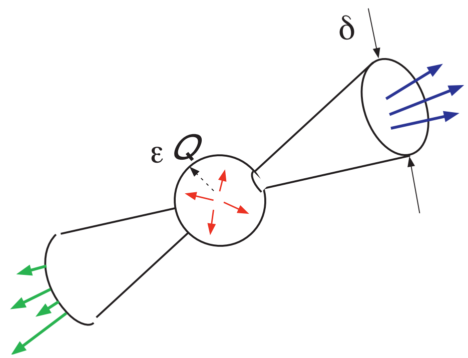
$$\boldsymbol{\sigma[S_{\rm n-jet}] \ = \ \sum_f \theta(S_{\rm n-jet}[f])\ |\langle m_f|m_0\rangle|^2}$$

- Once we have identified a set of jets, we can then explore their properties by using weight functions $w_{\rm n-jet}[f]$ that reveal their structure,

$$\boldsymbol{\langle w_{\rm n-jet}\rangle \ = \ \frac{\sum_f w_{\rm n-jet}[f]\,\theta(S_{\rm n-jet}[f])\ |\langle m_f|m_0\rangle|^2}{\sum_f \theta(S_{\rm n-jet}[f])\ |\langle m_f|m_0\rangle|^2}}$$

- These are what we can compute.

- An example is the cross section for a cone jet with a given energies,



- The smaller (larger) the "resolutions" ϵ and δ, the more (less) sensitivity to long times. We follow the story only to times like $1/Q\delta$.

- We'll turn to examples next.

IC. Summary

- The integral of the largest time controls IR behavior.
- Particle emission or absorption requires by a characteristic formation time, which diverges is the collinear limit.
- Jet evolution is independent.
- Time-ordered emissions provide angular-ordered branching pictures.
- In cross sections, a free sum over states always cancels long-time behavior by use of the largest time equation.
- Infrared safe weight functions can provide perturbative cross sections, and properties of jets.

In the second part, we'll discuss how these ideas have been and are being implemented and tested.