INSTITUTO SUPERIOR TÉCNICO

PROJECTO MEFT

Scalar Fields, BH and Spherical Coordinates

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Research work performed for the Master in Engineering Physics

at

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June 13, 2019

1 Introduction

This paper serves as an introduction to the work to be done for obtaining the Masters degree in Engineering Physics. Here, an introduction to the relevant topics will be given, as well as a state-of-the-art review on the works regarding such topics. Finally, a plan of the work will be presented.

In this section several topics, of paramount relevance for the work being developed, will be introduced. First, an overview of the General Theory of Relativity is given, to which an introduction to BH follows. Thirdly a brief review on superradiance is given, and finally the field of NR is presented. At the end of this section, the goals of the work are enumerated.

Througout this work, we use greek letters for spacetime indices running between 0 and 3 and latin letters for spatial indices running between 1 and 3. Furthermore, we use natural units $c = G = 1$, unless stated otherwise.

1.1 General Relativity

In 1905, Albert Einstein wrote *Zur Elektrodynamik bewegter Körper* or, in english, *On the Electrodynamics of Moving Bodies*, an article in which he pointed out the inconsistencies between the results of Maxwell's theory of electrodynamics and the notions of relative and absolute motion, set forth by Galileo and used by Newton in his theory of mechanics [1]. On one hand, the theories seemed to make a clear distinction between the absolute motion of different bodies when treating physical processes, but on the other hand the observable results of such treatments were dependent only on the relative motion of the bodies involved. This led Einstein to formulate the "Principle of Relativity", which stated that no properties of physical phenomena correspond to the concept of absolute rest and, instead, all physical equations hold in all inertial frames of reference. Furthermore, Einstein postulated that light in a vacuum should travel at a constant speed c, regardless of the frame of reference. This was the birth of Special Relativity (SR).

Ten years later, in 1915, Einstein proposed a generalisation of both SR and Newton's theory of gravity, which was suitably named General Relativity (GR). The theory is condensed in Einstein's field equations which take the form

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{1}$$

These equations relate the curvature of spacetime with the matter and energy contents of the universe. Here, the metric tensor $g_{\mu\nu}$ is the unknown, and it describes the geometry of spacetime. $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar, respectively, which describe the curvature of the metric. The stress-energy tensor $T_{\mu\nu}$ describes the matter and energy contents.

Fast forward over 100 years and GR is still the most widely accepted theory of gravitation. It is paramount to a number of modern applications, such as the Global Positioning System (GPS), and has stood several tests since its creation. Amongst these tests, some of the most notable are the deflection of starlight by the sun during a total solar eclipse in May 1919 [2], which stands as the first test of the theory's predictions, and more recently the detection of gravitational waves by the Laser Interferometer Gravitational-Wave Observatory (LIGO) collaboration in 2015 [3] or the first image of a BH captured by the Event Horizon Telescope (EHT) collaboration in 2019 [4–9].

1.2 Black Holes

There are several very interesting implications one can derive from GR, such as the aforementioned bending of light around massive objects and gravitational waves, as well as various cosmological models which try to explain the evolution of our universe. In this work we shall focus our attention on Black Holes (BH). These are very well known solutions of Einstein's field equations (1), usually described as objects so massive that they exert such a strong gravitational pull that not even light can escape from them.

The first known solution for a metric compatible with GR, found by Karl Schwarzschild in 1916 [10], describes a spherically symmetric gravitational field. Its line element is given, in Schwarzschild coordinates $\{t, r, \theta, \phi\}$, by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right), \tag{2}$$

where M can be interpreted as the mass of the object responsible for the gravitational field. This metric clearly defines two regions: an interior region, where $r < r_S$, and an exterior region, where $r > r_S$, where $r_S = 2M$ is the Schwarzschild radius. It can be shown that the interior region is casually disconnected from the exterior (see, for example, reference [11]), and therefore no information about the interior can be extracted. Hence, the Schwarzschild metric (2) describes a static BH.

BH often arise as the end state of stellar evolution, due to gravitational collapse. The Schwarzschild solution (2) can be shown to be the end state of a spherically symmetric, electrically neutral star [11]. For a more general object, a set of no-hair and uniqueness theorems [12] state that all stationary, asymptotically flat BH solutions of GR can be fully described by three parameters, namely the mass M, electric charge Q and angular momentum S. The most general BH solution of the field equations (1) was found by Kerr in 1963 [13], and its line element is

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta \, d\phi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right) d\phi - a \, dt\right]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2, \tag{3}$$

where

$$\Delta \equiv r^2 - 2Mr + a^2 + Q^2, \tag{4a}$$

$$\rho^2 \equiv r^2 + a^2\cos^2\theta, \tag{4b}$$

and $a = S/M$ is the angular momentum per unit mass. This solution describes an spinning BH spacetime, which will be the focus of this work.

1.3 Superradiance

BH in GR are very special objects: for one, they are quite simple to describe, being only dependent on three parameters, but they also entail several interesting properties and associated phenomena, which have been in active research for the past decades. One of these interesting phenomena is BH superradiance.

Superradiance is, in general, the enhancement of radiation, and it is a phenomenon that extends itself to several areas of physics, as well as BH physics. In fact, the first studies of superradiance can be traced back to the early days of quantum mechanics, when, in 1929, Klein showed that the Dirac Equation allowed for the propagation of electrons through potential barriers without exponential damping, if the barrier was large enough [14]. Some years later, Hund, working with scalar fields and the Klein-Gordon equation, showed that this potential barrier could give rise to pair creation of charged particles. The term *superradiance* was coined by Dicke in 1954, describing phenomena of radiation amplification by coherence of emitters [15].

Despite the fact that the first studies of superradiance were done in quantum systems, it is also possible to find such effects in classical physics, as shown by Zel'dovich in 1972 [16], in his work regarding the amplification of cylindrical waves by rotating bodies. In BH physics, superradiance is allowed by dissipation at the event horizon. If the amplified radiation is confined to the vicinity of the BH, through mechanisms such as massive fields or magnetic fields, strong instabilities occur, giving rise to the so called "BH bombs". These may have applications in searches for physics beyond the Standard Model, such as the search for dark matter. They are also related to the formation of new "hairy" BH solutions and can provide insight into subjects of particle and condensed matter physics such as spontaneous symmetry breaking and superfluidity through the AdS/CFT correspondence. Therefore, BH superradiance is a very interesting topic to study, with a wide variety of applications.

1.4 Numerical Relativity

As with most theories in modern physics, GR is a highly complex subject. The field equations (1), despite the simple form in which they are written, are a set of ten coupled non-linear partial differential equations containing thousands of terms. Therefore, analytical solutions are only possible to find when assuming strong symmetries, such as the ones mentioned above. One must, then, resort to numerical methods when studying more general spacetimes. This need to solve Einstein's equations on computers gave birth to the field of Numerical Relativity (NR).

When solving differential equations in physics, one is usually interested in time evolution of the system in study, given a set of suitable initial and boundary conditions. In most fields of physics, one has an explicit time coordinate, so it is trivial to choose the direction of evolution. However, in GR, there is no clear distinction between time and space coordinates. Therefore, one of the main goals of NR has been to find a way to split the field equations (1) in a way that allows for their numerical evolution. There are several approaches to solve this problem, but the relevant one for this work is the 3+1 formalism (for an extensive review on the subject, see reference [17]), which splits spacetime into three-dimensional space and a time direction.

In recent years, NR has had its share of successes, from the first stable long-term numerical simulation of a BH binary system by Pretorius [18] to the prediction of the gravitational wave signal one should expect from a BH merger, which led to its detection by LIGO [3]. Besides predicting experimental results, NR can also help us understand some aspects of physics which are not easily accessible by experiment. That is the case when studying BH superradiance and BH bombs, which will be the focus of this work.

1.5 Goals

The main goal of this work is studying the phenomenon of superradiance of scalar and vector fields in NR, using a code in spherical coordinates. This will be done by accomplishing the following milestones:

- writing a 3+1 code in spherical coordinates for the evolution of scalar fields;
- using the code for numerical evolution in curved spacetimes;
- learning how to use the `Einstein Toolkit` and coupling the evolution code to this infrastructure;
- using the code to study systems of scalar fields coupled to BH spacetimes.

2 State of the Art

2.1 Rotational superradiance

In 1972, Zel'dovich published a paper where he investigated the amplification of electromagnetic radiation by a rotating body [16]. This work leveraged on previous works concerning the amplification of sound waves by reflection from a boundary that moves with supersonic velocity, as well as other early studies of superradiance. In all these studies, the limiting condition was that the linear velocity of the body that dissipates energy had to be larger than the fase velocity of the incident waves [19]. It is, therefore, easy to see that in a linear experiment, it is impossible to obtain superradiance of electromagnetic waves, as the body would have to superluminally.

However, one can consider a cylindrical or spherical geometry where a body, which rotates with angular velocity Ω, is hit by an electromagnetic pole wave, that is a wave of the form $\phi \propto e^{-i\omega t + im\varphi}$, where ω is the frequency, m is an azimuthal number and φ is the azimuthal coordinate. For these waves, the phase velocity is given by $v_{ph} = \omega/m$, and the condition for superradiance reads

$$\omega - m\Omega > 0. \tag{5}$$

Indeed, condition (5) is also a necessary one for BH rotational superradiance to occur.

2.2 Formulation of the Cauchy problem - equations of motion and initial data

In order to study numerically the superradiant instabilities of scalar fields coupled to BH, we must obtain the **equations of motion**, or evolution equations, and the **initial data** which is to be evolved. These are the basic ingredients of numerical simulation. For this, we follow the treatment of Okawa et al [20] and consider the Einstein-Hilbert action of GR coupled to a complex, massive scalar field Φ,

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{16\pi} + \mathcal{L}_\Phi\right), \text{ where} \tag{6a}$$

$$\mathcal{L}_\Phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi^*\partial_\nu\Phi - \frac{1}{2}\mu_S^2\Phi^*\Phi - V\left(\Phi\right), \tag{6b}$$

g is the metric determinant, $\mu_S = m_S/\hbar$ is the mass parameter, R is the Ricci scalar and $V\left(\Phi\right)$ is the scalar field potential. The equations of motion are obtained from the action through the Euler-Lagrange equations, and we obtain

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \tag{7a}$$

$$\left(\partial^\mu\partial_\mu - \mu_S^2\right)\Phi = V'\left(\Phi\right). \tag{7b}$$

These are, as expected, the Einstein field equations (1) and a Klein-Gordon equation with a potential on the right hand side. Here, the energy-momentum tensor will be given by

$$T_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}\left(\partial_\lambda\Phi^*\partial^\lambda\Phi + \mu_S^2\Phi^*\Phi\right) - g_{\mu\nu}V\left(\Phi\right) + \frac{1}{2}\left(\partial_\mu\Phi^*\partial_\nu\Phi + \partial_m u\Phi\partial_\nu\Phi^*\right). \tag{8}$$

GR treats all coordinates in the same way and there is no clear distinction between time and space. However, in order to evolve the spacetime using a computer, one must pick an evolutiuon coordinate and re-write the equations with this in mind. One of the ways to do this is to use the 3+1 formalism [17], which works by foliating the 4-dimensional spacetime into 3-dimensional spatial hypersurfaces, parametrized by a "time" coordinate t. These hypersurfaces are endowed with a spatial metric γ_{ij}, and the line element in this formalism reads

$$ds^2 = -\left(\alpha^2 - \beta_i\beta^i\right)dt^2 + 2\gamma_{ij}\beta^i dt dx^j + \gamma_{ij}dx^i dx^j, \tag{9}$$

where α and β^i are the lapse function and the shift vector, respectively, and they encode the coordinate degrees of freedom.

Performing the decomposition according to Ref. [20] will yield a system of constraint equations and evolution equations, which are only weakly hyperbolic, meaning they are likely to generate numerical instabilities when evolved. One of the ways to get around this is by using the BSSN formulation, named after Baumgarte and Shapiro [21] and Shibata and Nakamura [22], who developed this formalism independently. It turns out that the system of evolution equations that results from this is a well-posed one, meaning we can expect convergence in the numerical evolution. We will not go into detail on this formalism, as it would take much more than the length of this paper.

One last step is needed to ensure the Cauchy problem is well formulated: that of the initial conditions. Amongst the equations obtained from the 3+1 decomposition of Eqs. (7) are constraint equations, which are not to be evolved but still they must be satisfied at every point in the evolution. It can be shown analytically that if the constraints are satisfied at the initial time, then they will be satisfied at all later times, that is

$$\frac{d\mathcal{H}}{dt} = 0, \tag{10}$$

where $\mathcal{H}$ is such that the constriants take the form $\mathcal{H} = 0$. It is, therefore, very important to choose constraint satisfying initial data. This has been done by Okawa et al [20], where it was also shown that, for constraint violating initial data, the magnitude of constraint violation becomes larger at later times, whilst for constraint satisfying initial data, a free evolution - one in which the constraints are not actively enforced but rather monitored at all times - yields constraint satisfaction at all later times.

2.3 Review of past results

In recent years, the subject of superradiance of boson fields around BH has been a hot one, with many researchers devoting their time and effort to this topic. The results reported by these researchers have been very promising. One of the first studies of scalar field BH bombs was performed by Okawa et al [20], in which the aforementioned formalism for initial data and equations of motion has been developed, along with the first 3+1 evolution of these systems. However, the superradiant instability did not appear in the results, as the evolution time scales needed for such phenomena are much larger than what was possible. More recently, a more efficient method by Dolan [23] allowed for a 1+1 evolution of massless and massive scalars at much larger time scales. Their results appear to confirm the appearence of the superradiant instability in the massive scalar case, whilst being in accordance with previous results at a smaller time scale.

Other works have focused on the vector bosonic field rather than the scalar case. This is usually a better test case since the superradiant instability takes less time to develop, thus requiring a smaller amount of computational resources. The interaction between Proca fields and non-rotating BH was studied first by Zilhão et al [24], revealing that BH spacetimes can sustain vector boson condensates for extremely long timescales. Furthermore, the interaction between Proca fields and quickly rotating blach holes has been investigated by East [25], revealing an nonlinear growth and subsequent saturation of the field due to the extraction of rotational energy from the BH.

2.4 Spherical coordinates

In NR, it is customary to adopt Cartesian coordinate systems when solving Einstein's equations numerically. However, several problems of interest in this field, like binary BH mergers, have total or approximate spherical symmetry or axis symmetry, making a spherical coordinate system a much more natural choice. Cartesian coordinates some advantages, like the fact that they are regular, which means they do not possess coordinate singularities anywhere. However, Cartesian grids over-resolve in angular directions, meaning that the number of cells in the grid per unit angle increases massively as we move out from the centre. A way to solve this issue is by using mesh refinement infrastructures, which define several grids with varying cell sizes, offering more resolution in regions of greater interest, but also introducing some numerical error in the zones where the different grids are patched together.

Spherical coordinates offer an upper hand on some of their characteristics. For one, they can take advantage of the symmetries of the systems in study. Furthermore, the number of cells per unit angle does not grow as the distance from the centre increases, so that a mesh refinement routine is not necessary in principle, removing some noise from the simulations. However, a spherical coordinate system introduces coordinate singularities at the origin and along the $\theta = 0$ axis, which have to be dealt with. Furthermore, the Courant-Friedrichs-Lewy condition requires much smaller time steps since the cells at large radius cover large volumes.

To get the advantages of both coordinate systems whilst mitigating some of their shortcommings, one might choose to use a multipatch aproach. This consists on breaking up the computational domain in overlapping patches, choosing different coordinate systems in each of them, so that coordinate singularities do not appear but one is still able to take advantage of the natural symmetries of the problem.

In the near past, some have tried to use spherical coordinates to solve the Einstein field equations. We highlight the work of Mewes et al [26], which reports on the implementation of a thorn called `SphericalBSSN` in the `Einstein Toolkit`, an open-source infrastructure to tackle problems in NR. This thorn, as its name points out, is aimed at solving the BSSN equations in a spherical grid, implementing ways to treat all the shortcomings of this coordinate system. Furthermore, this work demonstrates that these coordinate systems reduce numerical errors by several orders of magnitude, making the development of codes like this paramount to precision simulation of physical systems in GR.

3 Work Plan and Calendar

The work plan for this project, along with the broad time windows for each of the steps, is as follows:

1. **August 2019**: Theoretical study of the formalism of NR, the 3+1 decomposition, evolution equations and constraints, initial data and extraction of results.
2. **August - September 2019**: Theoretical study of superradiance and the coupling of bosonic fields to BH from a perturbative standpoint.
3. **September 2019**: Familiarisation with numerical codes (`Einstein Toolkit`) and multipatch infrastructures.
4. **October 2019**: Simulation and analysis of results of rotating BH and matter fields using these codes, establishing a baseline for comparison.
5. **November 2019**: Theoretical study of the BSSN equations in spherical coordinates.
6. **November - December 2019**: Literature research on the availability of codes using the BSSN formalism in spherical coordinates.
7. **December 2019 - February 2020**: Coupling of the BSSN formalism to fundamental fields by writing a code for future simulation.
8. **February - March 2020**: Application of the code developed earlier to problems of physical interest in the study of superradiance, and analysis of results.
9. **March - May 2020**: Writing of the Masters Dissertation.

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