Relativistic hydrodynamics for spin-polarized media

Radoslaw Ryblewski

THE HENRYK NIEWODNICZAŃSKI
INSTITUTE OF NUCLEAR PHYSICS
POLISH ACADEMY OF SCIENCES

primary reference:
W. Florkowski, R. R., A. Kumar, ***Prog. Part. Nucl. Phys.*** **108** (2019) 103709 [1811.04409]

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LHC Physics: Standard Model and Beyond
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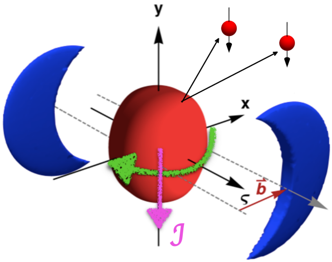
Jan. 7-10, 2020

Angular momentum in non-central heavy-ion collisions

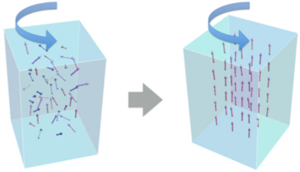
Non-central heavy-ion collisions (HIC) at low/intermediate energies create fireballs with large orbital angular momenta $L_{\rm init} \sim 10^5\hbar$
F. Becattini, F. Piccinini, J. Rizzo, PRC 77 (2008) 024906



Some part of the angular momentum of the QCD matter can be transferred from the orbital to the spin part
Barnett, Rev. Mod. Phys. 7, 129 (1935)

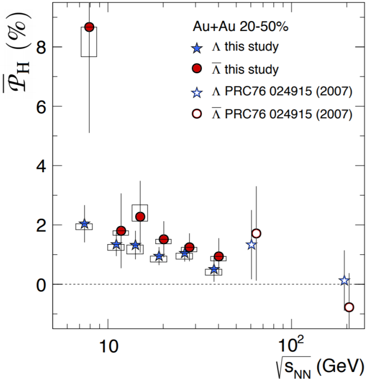
$$J_{\rm init} = L_{\rm init} \qquad = \qquad L_{\rm final} + S_{\rm final}$$

Should be reflected in the polarization of observed hadrons
Z.-T. Liang, X.-N. Wang, PRL 94 (2005) 102301; PLB 629 (2005) 20–26
J.-H. Gao, et al., PRC 77 (2008) 044902
S.-W. Chen, J. Deng, J.-H. Gao, Q. Wang, Front. Phys. China 4 (2009) 509–516
B. Betz, M. Gyulassy, G. Torrieri, PRC 76 (2007) 044901



First HIC experiments that measured spin polarization in Dubna, CERN and BNL reported negative results
M. K. Anikina, et al., Z. Phys. C25 (1984) 1–11
J. Bartke, et al., Z. Phys. C48 (1990) 191–200
B. I. Abelev, et al., PRC 76 (2007) 024915

First positive measurements of global spin polarization of Λ hyperons by STAR



thermal approach $\longrightarrow$ $P_\Lambda \approx \frac{1}{2}\frac{\omega}{T} + \frac{\mu_\Lambda B}{T}$ $\quad P_{\overline{\Lambda}} \approx \frac{1}{2}\frac{\omega}{T} - \frac{\mu_\Lambda B}{T}$

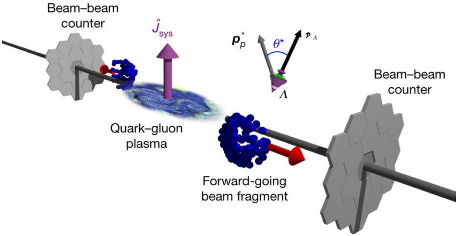
Becattini, F., Karpenko, I., Lisa, M., Upsal, I. , Voloshin, S. , PRC 95, 054902 (2017)

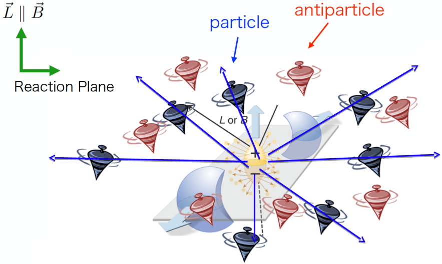
. . . the hottest, least viscous – and now, most vortical – fluid produced in the laboratory . . .

$$\omega = \left(P_\Lambda + P_{\overline{\Lambda}}\right) k_B T/\hbar \sim 0.6 - 2.7 \times 10^{22} \mathrm{s}^{-1}$$

L. Adamczyk et al. (STAR) (2017), Nature 548 (2017) 62-65

Measurement of global spin polarization of Λ hyperons



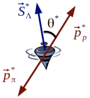


Parity-violating decay of hyperons

Daughter baryon is preferentially emitted in the direction of hyperon's spin (opposite for anti-particle)

$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha_{\mathrm{H}} \mathbf{P}_{\mathrm{H}} \cdot \mathbf{p}_{\mathbf{p}}^*)$$

P_H: Λ polarization
p_p^*: proton momentum in the Λ rest frame
α_H: Λ decay parameter
($\alpha_\Lambda = -\alpha_{\bar{\Lambda}} = 0.642 \pm 0.013$)



$$\Lambda \rightarrow p + \pi^-$$

(BR: 63.9%, cτ ~7.9 cm)

C. Patrignani et al. (PDG), Chin. Phys. C 40, 100001 (2016)

Credit: T.Niida, The 5th Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions, 2019

Theory succees - global polarization

Spin DoF are locally equilibrated
F.Becattini, F.Piccinini,Ann.Phys.323,2452(2008)
F.Becattini, F.Piccinini, J. Rizzo, PRC 77, 024906 (2008)

Polarization given by **thermal vorticity** (GEQ)
F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, Ann. Phys. 338, 32 (2013)

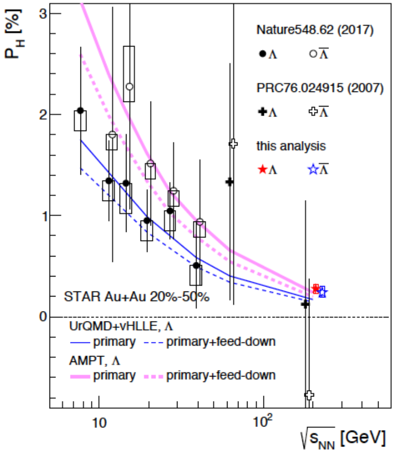
$$\omega^{\mu\nu} \quad \Leftrightarrow \quad \varpi^{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right)$$

$$\beta_\nu = \frac{u_\nu}{T}$$

Frozen at the hadronization stage using hydrodynamics/transport without spin

Possible 15%-20% dilution of primary Λ polarization due to **feed-down effect**
F. Becattini, I. Karpenko, M. Lisa, I. Upsal, S. Voloshin, PRC 95 (2017) no.5, 054902
X-L Xia, H. Li, X-G Huang, H. Z. Huang [1905.03120]
F. Becattini, G. Cao, E. Speranza [1905.03123]

J. Adam, et al., Phys. Rev. C 98, 014910 (2018)



UrQMD+vHLLE: I. Karpenko, F. Becattini, EPJC 77, 213 (2017)
AMPT: H. Li, L. Pang, Q. Wang, and X. Xia, PRC 96, 054908 (2017)
also agree with:
Y. Xie, D. Wang, and L. P. Csernai, PRC 95, 031901 (2017)
Y. Sun and C. M.Ko, PRC 96, 024906 (2017)

also more recently
D-X Wei, W-T. Deng., X-G.Huang, PRC 99 (2019) no.1, 014905

Phenomenological prescription used to describe the data

Statistical approach quite appealing!

Algorithm is:

1) Run any type of hydro, perfect or viscous, or transport, or whatsoever, without spin

2) Find $\beta_\mu(x) = u_\mu(x)/T(x)$ on the freeze-out hypersurface (defined often by the condition T=const)

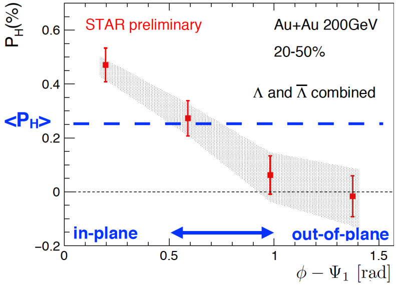
3) Calculate thermal vorticity $\varpi_{\alpha\beta}(x) \neq$ const

4) Identify thermal vorticity with the spin polarization tensor $\omega_{\mu\nu}$

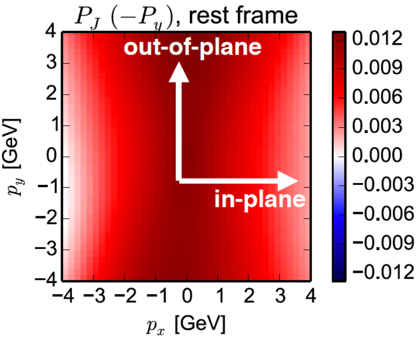
5) Make predictions about spin polarization

As we have seen such a method describes very well the global polarization of Λ
but let us look differential ...

Theory failures - azimuthal angle dependence of P_y



Credit: T.Niida, The 5th Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions, 2019

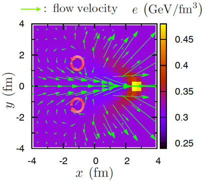


I. Karpenko, F. Becattini, EPJC 77, 213 (2017)

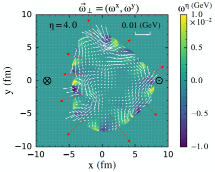
"thermal vorticity–based" approach fails to describe the azimuthal dependence of P_y component

Theory failures - azimuthal angle dependence of P_z

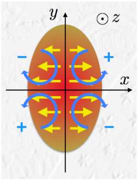
Non-trivial flow structure in the transverse plane (jet, ebe fluctuations etc)
generates longitudinal polarization



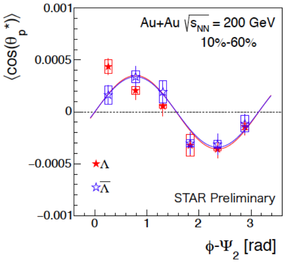
Y. Tachibana and T. Hirano, NPA904-905 (2013) 1023

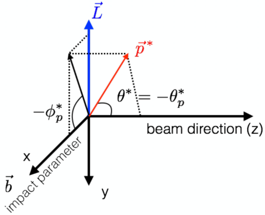


L.-G. Pang, H. Peterson, Q. Wang, and X.-N. Wang PRL117, 192301 (2016)



F. Becattini and I. Karpenko, PRL120.012302 (2018)
S. Voloshin, EPJ Web Conf.171, 07002 (2018)





$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha_{\mathrm{H}} \mathbf{P_H} \cdot \mathbf{p}_p^*)$$

$$\langle \cos\theta_p^* \rangle = \int \frac{dN}{d\Omega^*} \cos\theta_p^* d\Omega^*$$

$$= \alpha_{\mathrm{H}} P_z \langle (\cos\theta_p^*)^2 \rangle$$

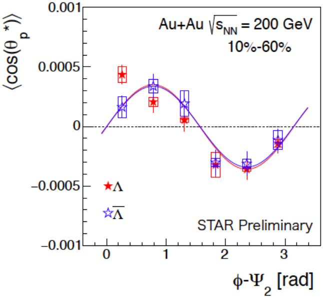
$$\therefore \ P_z = \frac{\langle \cos\theta_p^* \rangle}{\alpha_{\mathrm{H}} \langle (\cos\theta_p^*)^2 \rangle}$$

$$= \frac{3\langle \cos\theta_p^* \rangle}{\alpha_{\mathrm{H}}} \quad \text{(if perfect detector)}$$

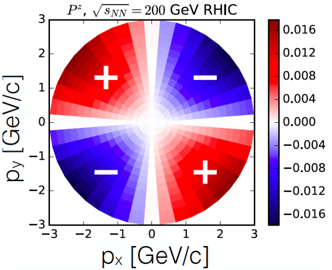
α_{H}: hyperon decay parameter
θ_p^*: θ of daughter proton in Λ rest frame

Credit: T.Niida, The 5th Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions, 2019

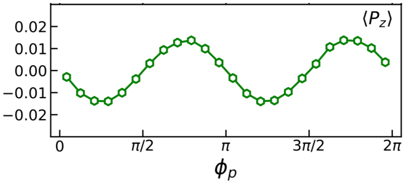
Theory failures - azimuthal angle dependence of P_z



T. Niida, NPA 982 (2019) 511514



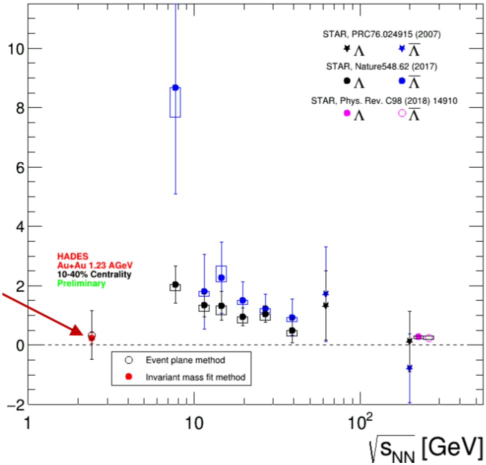
UrQMD+vHLLE: F. Becattini, I. Karpenko, PRL 120 (2018) no.1, 012302,



AMPT: X. Xia, H. Li, Z. Tang, Q. Wang, PRC98.024905 (2018)

"thermal vorticity–based" approach fails to describe the quadrupole structure of P_z component

Surprise from HADES - vanishing global polarization in Au+Au collisions at $\sqrt{s_{NN}}$= 2.4 GeV



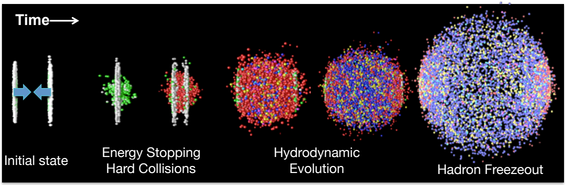
Credit: F. Kornas, International Workshop XLVII on Gross Properties of Nuclei and Nuclear Excitations, 2019

is there a threshold effect at very low energies ?

Fluid dynamics with spin?

space-time dynamics of spin polarization?

relativistic hydrodynamics forms the basis of HIC models



T. K. Nayak, Lepton-Photon 2011 Conference

perfect fluid dynamics = local equilibrium + conservation laws

for particles with spin the conservation of angular momentum is not trivial
new hydrodynamic variables should be introduced

Conservation laws of canonical currents

Constructions of the hydrodynamic frameworks rely on the local conservation laws

Noether's theorem:
for each continuous symmetry of the action there is a corresponding conserved (canonical) current

Conservation of charge (baryon number, electric charge, ...)

$$\partial_\mu \widehat{N}^\mu(x) = 0 \qquad \text{(1 equation/charge)}$$

Conservation of energy and momentum

$$\partial_\mu \widehat{T}^{\mu\alpha}_C(x) = 0 \qquad \text{(4 equations)}$$

Conservation of total angular momentum

$$\partial_\mu \widehat{J}^{\mu,\alpha\beta}_C(x) = 0 \qquad \widehat{J}^{\mu,\alpha\beta}_C(x) = -\widehat{J}^{\mu,\beta\alpha}_C(x) \qquad \text{(6 equations)}$$

Orbital and spin parts of angular momentum

Total angular momentum $\widehat{J}_C^{\mu,\alpha\beta}(x)$ may be decomposed into **orbital** and **spin** parts

$$\widehat{J}_C^{\mu,\alpha\beta}(x) = \underbrace{x^\alpha \widehat{T}_C^{\mu\beta}(x) - x^\beta \widehat{T}_C^{\mu\alpha}(x)}_{\widehat{L}_C^{\mu,\alpha\beta}(x)} + \widehat{S}_C^{\mu,\alpha\beta}(x)$$

Conservation of total angular momentum gives

$$\partial_\mu \widehat{J}_C^{\mu,\alpha\beta}(x) = 0 \qquad \Rightarrow \qquad \partial_\mu \widehat{S}_C^{\mu,\alpha\beta}(x) = \widehat{T}_C^{\beta\alpha}(x) - \widehat{T}_C^{\alpha\beta}(x)$$

$\widehat{T}_C^{\mu\alpha}(x) \neq \widehat{T}_C^{\alpha\mu}(x)$ **– canonical energy-momentum tensor is not symmetric**

$\Downarrow$

canonical spin tensor is not conserved separately

Pseudo-gauge transformations

Inclusion of the conservation of angular momentum is connected with the problem of the localization of energy and spin densities

Pseudo-gauge transformation

W. Hehl, Rept. Math. Phys. 9 (1976) 55–82; F. Becattini, L. Tinti, Phys. Rev. D 84 (2011) 025013; Phys. Rev. D 87(2) (2013) 025029

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\widehat{\Phi}^{\lambda,\mu\nu} - \widehat{\Phi}^{\mu,\lambda\nu} - \widehat{\Phi}^{\nu,\lambda\mu}\right)$$

$$\widehat{S}'^{\lambda,\mu\nu} = \widehat{S}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu}$$

- ⤳ preserve $\widehat{P}^{\mu} = \int d^3\Sigma_\lambda \widehat{T}^{\lambda\mu}(x)$ $\quad \widehat{J}^{\mu\nu} = \int d^3\Sigma_\lambda \widehat{J}^{\lambda,\mu\nu}(x)$
- ⤳ conservation laws unchanged

Energy-momentum and spin tensors are not uniquely defined

Belinfante-Rosenfeld construction (choosing superpotential $\widehat{\Phi} = \widehat{S}_C^{\lambda,\mu\nu}$)

Belinfante, F. J. (1939): Physica 6. 887-898, (1940); Rosenfeld, L. (1940): Mem. Acad. Roy. Belgique, cl. SC., tome 18, fasc. 6

$$\widehat{T}_B^{\mu\nu} = \widehat{T}_C^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\widehat{S}_C^{\lambda,\mu\nu} + \widehat{S}_C^{\mu,\nu\lambda} - \widehat{S}_C^{\nu,\lambda\mu}\right) \qquad \widehat{S}_B^{\lambda,\mu\nu} = 0$$

- ⤳ gives exactly symmetric Hilbert $T^{\mu\nu}$ acting as the source of gravity in GR
- ⤳ long-standing problem of physical significance of the spin tensor
- ⤳ spin tensor is used by the community that studies the spin of proton
  X.S. Chen, X.F. Lu, W.M. Sun, F. Wang, T. Goldman, Phys. Rev. Lett. 100 (2008) 232002;
  E. Leader, C. Lorce, Phys. Rep. 541 (2014) 163.

Local equilibrium density operator - Belinfante

In quantum statistical mechanics LE can be defined as a maximum of the von Neumann entropy

D.N. Zubarev, Sov. Phys. Dokl. 10 (1966) 850; D.N. Zubarev, A.V. Prozorkevich, S.A. Smolyanskii, Theor. Math. Phys. 40 (1979) 821.
Ch.G. Van Weert, Ann. Phys. 140 (1982) 133;
F. Becattini, Phys. Rev. Lett. 108 (2012) 244502.;F. Becattini, L. Bucciantini, E. Grossi, L. Tinti, Eur. Phys. J. C 75(5) (2015) 191;

$$S = -\mathrm{tr}(\widehat{\rho}_B \log \widehat{\rho}_B)$$

Constraints of given mean densities of conserved currents over some spacelike hypersurface Σ

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_B \,\widehat{j}^\mu(x)\right] = n_\mu j^\mu(x)$$

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_B \,\widehat{T}_B^{\mu\nu}(x)\right] = n_\mu T_B^{\mu\nu}(x)$$

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_B \,\widehat{J}_B^{\mu,\lambda\nu}(x)\right] = n_\mu \mathrm{tr}\left[\widehat{\rho}_B \left(x^\lambda \widehat{T}_B^{\mu\nu}(x) - x^\nu \widehat{T}_B^{\mu\lambda}(x)\right)\right] = n_\mu J_B^{\mu,\lambda\nu}(x)$$

$$\widehat{S}_B^{\lambda,\mu\nu} = 0$$

$$\Downarrow$$

condition on total angular momentum is redundant

n^μ - vector orthogonal to Σ

Resulting LE density operator in this case is

$$\widehat{\rho}_B = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}_B^{\mu\nu}(x)\beta_\nu(x) - \widehat{j}^\mu(x)\xi(x)\right)\right] \qquad \beta_\nu = \frac{u_\nu}{T} \qquad \xi = \frac{\mu}{T}$$

Local equilibrium density operator - canonical

Again, use maximization of entropy

$$S = -\mathrm{tr}(\widehat{\rho}_C \log \widehat{\rho}_C)$$

Constraints on charge, energy, momentum and angular momentum

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_C \, \widehat{j}^\mu(x)\right] = n_\mu j^\mu(x)$$

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_C \, \widehat{T}^{\mu\nu}_C(x)\right] = n_\mu T^{\mu\nu}_C(x)$$

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_C \, \widehat{J}^{\mu,\lambda\nu}_C(x)\right] = n_\mu \mathrm{tr}\left[\widehat{\rho}_C \left(x^\lambda \widehat{T}^{\mu\nu}_C(x) - x^\nu \widehat{T}^{\mu\lambda}_C(x) + \widehat{S}^{\mu,\lambda\nu}_C\right)\right] = n_\mu J^{\mu,\lambda\nu}_C(x)$$

$$\widehat{S}^{\lambda,\mu\nu}_C \neq 0$$

$$\Downarrow$$

non-trivial constraint on total angular momentum

$$\Downarrow$$

need to introduce additional Lagrange multipliers (spin polarization tensor $\omega_{\lambda\nu}$)

Resulting LE density operator is

$$\widehat{\rho}_C = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu}_C(x)\beta_\nu(x) - \widehat{j}^\mu(x)\xi(x) - \frac{1}{2}\widehat{S}^{\mu,\lambda\nu}_C(x)\omega_{\lambda\nu}(x)\right)\right]$$

$$\widehat{\rho}_C = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu}_B(x)\beta_\nu(x) - \widehat{j}^\mu(x)\xi(x) - \frac{1}{2}\widehat{S}^{\mu,\lambda\nu}_C(\omega_{\lambda\nu} - \varpi_{\lambda\nu})\right)\right]$$

$$\omega_{\lambda\nu} = \varpi_{\lambda\nu} = -\tfrac{1}{2}\left(\partial_\lambda\beta_\nu - \partial_\nu\beta_\lambda\right) \quad \Rightarrow \quad \widehat{\rho}_C = \widehat{\rho}_B$$

Description based on the Belinfante tensors is reduced compared to the description employing the canonical tensors

Global equilibrium conditions - canonical

In global equilibrium (GE) the density operator becomes stationary - divergence of the integrand must vanish

$$\widehat{\rho}_{\rm LEQ} = \frac{1}{Z} \exp\left[- \int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu}_C \left(\beta_\nu - \omega_{\nu\lambda} x^\lambda \right) - \frac{1}{2} \omega_{\lambda\nu} \widehat{J}^{\mu,\lambda\nu}_C - \xi \widehat{j}^\mu \right) \right]$$

Using conservation laws it leads to the conditions

$$\partial_\mu \omega_{\nu\lambda} = 0, \qquad \partial_\mu \beta_\nu = \omega_{\nu\mu}, \qquad \partial_\mu \xi = 0$$

and

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0 \quad \text{(Killing equation)} \quad \beta_\nu = b_\nu + \omega_{\nu\lambda} x^\lambda$$

which means that

$$\begin{aligned} \xi &= \text{const.} \\ \omega_{\lambda\nu} &= \varpi_{\lambda\nu} = -\frac{1}{2}(\partial_\lambda \beta_\nu - \partial_\nu \beta_\lambda) = \text{const} \end{aligned}$$

The spin polarization tensor is equal to thermal vorticity (this is NOT the case for a symmetric energy-momentum tensor)

General concept of perfect-fluid hydrodynamics with spin

If GE conditions are not satisfied, the integral over Σ depends on its choice. The integrals over hypersurfaces Σ_1 and Σ_2 differ by the volume integral which describes dissipative phenomena. If we neglect dissipation we may treat the LE operator $\widehat{\rho}_{\rm LEQ}$ as constant.

D.N. Zubarev, Sov. Phys. Dokl. 10 (1966) 850; D.N. Zubarev, A.V. Prozorkevich, S.A. Smolyanskii, Theor. Math. Phys. 40 (1979) 821.

In this case we define the expectation values of the conserved currents through the expressions

$$T^{\mu\nu} = {\rm tr}\left(\widehat{\rho}_{\rm LEQ}\,\widehat{T}^{\mu\nu}\right), \quad S^{\mu,\lambda\nu} = {\rm tr}\left(\widehat{\rho}_{\rm LEQ}\,\widehat{S}^{\mu,\lambda\nu}\right), \quad j^{\mu} = {\rm tr}\left(\widehat{\rho}_{\rm LEQ}\,\widehat{j}^{\mu}\right)$$

These tensors are all functions of β_μ, $\omega_{\mu\nu}$, and ξ which enter constitutive equations

$$T^{\mu\nu} = T^{\mu\nu}[\beta,\omega,\xi], \quad S^{\mu,\lambda\nu} = S^{\mu,\lambda\nu}[\beta,\omega,\xi], \quad j^{\mu} = j^{\mu}[\beta,\omega,\xi]$$

and satisfy the conservation laws

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}, \quad \partial_\mu j^{\mu} = 0$$

first works along these lines:
W. Florkowski, B. Friman, A. Jaiswal, E. Speranza, Phys. Rev. C 97(4) (2018) 041901
W. Florkowski, B. Friman, A. Jaiswal, R. R., E. Speranza, Phys. Rev. D 97 (2018) 116017

dissipative processes must drive the system to GE, hence ω should converge to ϖ eventually:
D. Montenegro, L. Tinti, G. Torrieri, Phys. Rev. D 96 (2017) 076016

Equilibrium distribution functions for particles with spin-1/2

Alternatively, hydrodynamics may be derived from the underlying distribution function

Phase-space distribution functions for massive spin-1/2 particles + (antiparticles -)

F. Becattini, V. Chandra, L. Del Zanna, E. Grossi , Annals Phys. 338 (2013) 32

$$f^{+}_{rs}(x,p) = \bar{u}_r(p) X^{+} u_s(p), \quad f^{-}_{rs}(x,p) = -\bar{v}_s(p) X^{-} v_r(p)$$

where (we restrict ourselves to classical statistics)

$$X^{\pm} = \exp\left[\pm\xi(x) - \beta_\mu(x) p^\mu \pm \frac{1}{2}\omega_{\mu\nu}(x)\widehat{\Sigma}^{\mu\nu}\right]$$

$\beta^\mu = u^\mu / T$

$\xi = \mu/T$

m - mass of particles
T - temperature
μ - chemical potential
u^μ - four velocity ($u^2 = 1$)
$\omega_{\mu\nu}$ - spin-polarization tensor
$\widehat{\Sigma}^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$ - spin operator

Keep in mind that it is an ansatz; has not been derived from any underlying microscopic theory

Equilibrium Wigner functions

Phase-space distribution functions á la Becattini et al. can be used to determine explicit expressions for the corresponding equilibrium Wigner functions

W. Florkowski, A. Kumar, R. R., Phys. Rev. C98 (2018) 044906

Equilibrium Wigner functions - De Groot, van Leeuwen, van Weert (GLW) framework

De Groot, van Leeuwen, van Weert: *Relativistic Kinetic Theory. Principles and Applications*, 1980.

$$\mathcal{W}^{+}_{\rm eq}(x,k) = \frac{1}{2}\sum_{r,s=1}^{2}\int dP\, \delta^{(4)}(k-p) u^r(p)\bar{u}^s(p) f^{+}_{rs}(x,p)$$

$$\mathcal{W}^{-}_{\rm eq}(x,k) = -\frac{1}{2}\sum_{r,s=1}^{2}\int dP\, \delta^{(4)}(k+p) v^s(p)\bar{v}^r(p) f^{-}_{rs}(x,p)$$

$$\mathcal{W}_{\rm eq}(x,k) = \mathcal{W}^{+}_{\rm eq}(x,k) + \mathcal{W}^{-}_{\rm eq}(x,k).$$

$$\mathcal{W}^{\pm}_{\rm eq}(x,k) \quad = \quad \frac{e^{\pm\xi}}{4m}\int dP\, e^{-\beta\cdot p}\;\; \delta^{(4)}(k\mp p)\left[2m(m\pm \not{p})\cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta}\,\omega_{\mu\nu}\,(\not{p}\pm m)\Sigma^{\mu\nu}(\not{p}\pm m)\right].$$

The presence of the Dirac delta functions indicates that, to large extent, $\mathcal{W}^{\pm}_{\rm eq}(x,k)$ describe classical motion - they cannot be regarded as complete, quantum equilibrium distributions.

Nevertheless, the functions $\mathcal{W}^{\pm}_{\rm eq}(x,k)$ incorporate spin DOF's and may serve to construct the formalism of hydrodynamics with spin (in the LO in $\hbar$).

Clifford-algebra expansion

Clifford-algebra expansion
expansion in terms of 16 independent generators of the Clifford algebra (with real coefficients)
C. Itzykson, J. B. Zuber, Quantum Field Theory, McGraw-Hill, New York, 1980.

$$\mathcal{W}^{\pm}_{\rm eq}(x,k) = \frac{1}{4}\left[\mathcal{F}^{\pm}_{\rm eq}(x,k) + i\gamma_5\mathcal{P}^{\pm}_{\rm eq}(x,k) + \gamma^{\mu}\mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k) + \gamma_5\gamma^{\mu}\mathcal{A}^{\pm}_{{\rm eq},\mu}(x,k) + \Sigma^{\mu\nu}\mathcal{S}^{\pm}_{{\rm eq},\mu\nu}(x,k)\right].$$

The coefficient functions $\mathcal{X} \in \{\mathcal{F}, \mathcal{P}, \mathcal{V}_{\mu}, \mathcal{A}_{\mu}, \mathcal{S}_{\nu\mu}\}$ in the expansion can be obtained by calculating the trace of $\mathcal{W}^{\pm}_{\rm eq}(x,k)$ multiplied first by the matrices: $\{\mathbf{1}, -i\gamma_5, \gamma_{\mu}, \gamma_{\mu}\gamma_5, 2\,\Sigma_{\mu\nu}\}$.
D. Vasak, M. Gyulassy, H. T. Elze, Annals Phys. 173 (1987) 462–492
...

Global equilibrium Wigner function (I)

The Wigner function satisfies the kinetic equation

$$\left(\gamma_\mu K^\mu - m\right)\mathcal{W}(x,k) = C[\mathcal{W}(x,k)] \qquad K^\mu = k^\mu + \frac{i\hbar}{2}\,\partial^\mu$$

Let us assume in LE/GE the Wigner function $\mathcal{W}(x,k)$ exactly satisfies the equation

$$\left(\gamma_\mu K^\mu - m\right)\mathcal{W}(x,k) = 0$$

Semi-classical expansion
expansion in powers of $\hbar$ of the coefficient functions of the Clifford algebra expansion

$$\mathcal{X} = \mathcal{X}^{(0)} + \hbar\mathcal{X}^{(1)} + \hbar^2\mathcal{X}^{(2)} + \cdots . \qquad \mathcal{X} \in \{\mathcal{F}, \mathcal{P}, \mathcal{V}_\mu, \mathcal{A}_\mu, \mathcal{S}_{\nu\mu}\}$$

Global equilibrium Wigner function (II)

LO in $\hbar$:

- $\mathcal{F}_{(0)}$ and $\mathcal{A}^{\mu}_{(0)}$ may be treated as only independent coefficients provided

$$k_\nu \, \mathcal{A}^{\nu}_{(0)}(x,k) = 0$$

- $\mathcal{P}_{(0)}, \mathcal{V}^{\mu}_{(0)}, \mathcal{S}^{\mu\nu}_{(0)}$ may be obtained from $\mathcal{F}_{(0)}$ and $\mathcal{A}^{\mu}_{(0)}$
- the zeroth order is not sufficient to determine the evolution of the functions $\mathcal{F}_{(0)}$ and $\mathcal{A}^{\mu}_{(0)}$
- algebraic relations imposed by the kinetic equation at LO are satisfied by $\mathcal{W}^{\pm}_{\rm eq}(x,k)$, thus LO terms in $\hbar$ of the “true” equilibrium Wigner functions $\mathcal{W}^{\pm}(x,k)$ which satisfy the quantum kinetic equation can be identified with $\mathcal{W}^{\pm}_{\rm eq}(x,k)$

NLO in $\hbar$:

- kinetic equations to be satisfied by the $\mathcal{F}_{(0)}$ and $\mathcal{A}^{\mu}_{(0)}$ are

$$k^\mu \partial_\mu \mathcal{F}_{(0)}(x,k) = 0 \qquad\qquad k^\mu \partial_\mu \, \mathcal{A}^{\nu}_{(0)}(x,k) = 0$$

- $\mathcal{W}^{\pm}_{\rm eq}(x,k)$ itself specify only the LO terms in $\hbar$
- through the QKE equilibrium LO coefficients generate non-trivial corrections to NLO coefficients

$$\begin{aligned}
\mathcal{P}^{(1)} &= -\frac{1}{2m}\partial^\mu \mathcal{A}_{{\rm eq},\mu} \\
\mathcal{V}^{(1)}_{\mu} &= \frac{1}{m}\left(k_\mu \mathcal{F}^{(1)} - \frac{1}{2}\partial^\nu \mathcal{S}_{{\rm eq},\nu\mu}\right) \\
\mathcal{S}^{(1)}_{\mu\nu} &= \frac{1}{2m}\left(\partial_\mu \mathcal{V}_{{\rm eq},\nu} - \partial_\nu \mathcal{V}_{{\rm eq},\mu}\right) - \frac{1}{m}\epsilon_{\mu\nu\alpha\beta} k^\alpha \mathcal{A}^{\beta}_{(1)}
\end{aligned}$$

From global to local equilibrium

In **GE** the following equations are exactly fulfilled

$$k^\mu \partial_\mu \mathcal{F}_{\rm eq}(x,k) = 0, \quad k^\mu \partial_\mu \mathcal{A}^\nu_{\rm eq}(x,k) = 0, \quad k_\nu \mathcal{A}^\nu_{\rm eq}(x,k) = 0.$$

β_μ field has to be the Killing vector while the parameters ξ and $\omega_{\mu\nu}$ are constant. The kinetic equations do not constrain the spin polarization tensor $\omega_{\mu\nu}$ to be equal to the thermal vorticity $\varpi_{\mu\nu}$.

In **LE** one requires that only moments of the kinetic equations are satisfied by allowing for space-time dependence of β_μ, ξ, and $\omega_{\mu\nu}$

$$\partial_\alpha N^\alpha_{\rm eq}(x) = 0, \quad \partial_\alpha T^{\alpha\beta}_{\rm eq}(x) = 0, \quad \partial_\lambda S^{\lambda,\mu\nu}_{\rm eq}(x) = 0$$

$\widehat{T}^{\beta\alpha}_{\rm eq}(x) = \widehat{T}^{\alpha\beta}_{\rm eq}(x) \quad \Rightarrow$ **spin tensor is conserved**

Adopting the kinetic-theory framework derived by de Groot, van Leeuwen, and van Weert, one can show that $T^{\mu\nu}_{\rm GLW}(x) = T^{\mu\nu}_{\rm eq}(x)$ and $S^{\lambda,\mu\nu}_{\rm GLW} = S^{\lambda,\mu\nu}_{\rm eq}$

NLO corrections in $\hbar$

Since the spin tensor enters with an extra power of $\hbar$, it is important to examine the Wigner function up to the next-to-leading order (NLO)

LO generates corrections in the NLO (assume $\mathcal{F}_{(1)} = \mathcal{A}^{\mu}_{(1)} = 0$)

$$\mathcal{P}^{(1)} = -\frac{1}{2m}\,\partial^{\mu}\mathcal{A}_{\mathrm{eq},\mu}$$

$$\mathcal{V}^{(1)}_{\mu} = -\frac{1}{2m}\partial^{\nu}\mathcal{S}_{\mathrm{eq},\nu\mu}$$

$$\mathcal{S}^{(1)}_{\mu\nu} = \frac{1}{2m}\left(\partial_{\mu}\mathcal{V}_{\mathrm{eq},\nu} - \partial_{\nu}\mathcal{V}_{\mathrm{eq},\mu}\right)$$

IMPORTANT IF the canonical formalism is used

$$T^{\mu\nu}_{\mathrm{GLW}}(x) = \frac{1}{m}\mathrm{tr}_4\int d^4k\,k^{\mu}\,k^{\nu}\mathcal{W}(x,k) = \frac{1}{m}\int d^4k\,k^{\mu}\,k^{\nu}\mathcal{F}(x,k)$$

$$T^{\mu\nu}_{\mathrm{C}}(x) = \int d^4k\,k^{\nu}\mathcal{V}^{\mu}(x,k)$$

quantum corrections induce asymmetry $T^{\mu\nu}_{\mathrm{C}}(x) \neq T^{\nu\mu}_{\mathrm{C}}(x)$

From canonical to GLW case

Including the components of $\mathcal{V}^\mu(x,k)$ up to the first order in the equilibrium case we obtain

$$T^{\mu\nu}_{\rm C}(x) = T^{\mu\nu}_{\rm GLW}(x) + \delta T^{\mu\nu}_{\rm C}(x)$$

where

$$\delta T^{\mu\nu}_{\rm C}(x) = -\frac{\hbar}{2m}\int d^4k k^\nu \partial_\lambda \mathcal{S}^{\lambda\mu}_{\rm eq}(x,k) = -\partial_\lambda S^{\nu,\lambda\mu}_{\rm GLW}(x)$$

$$\partial_\alpha T^{\alpha\beta}_{\rm C}(x) = 0$$

The canonical energy-momentum tensor is conserved

$$\begin{aligned} S^{\lambda,\mu\nu}_{\rm GLW} &= \frac{\hbar}{4}\int {\rm d}^4k\, {\rm tr}_4\left[\left(\left\{\sigma^{\mu\nu},\gamma^\lambda\right\} + \frac{2i}{m}\left(\gamma^{[\mu}k^{\nu]}\gamma^\lambda - \gamma^\lambda\gamma^{[\mu}k^{\nu]}\right)\right)\mathcal{W}(x,k)\right] \\ S^{\lambda,\mu\nu}_{\rm C} &= \frac{\hbar}{4}\int d^4k\, {\rm tr}_4\left[\left\{\sigma^{\mu\nu},\gamma^\lambda\right\}\mathcal{W}(x,k)\right] = \frac{\hbar}{2}\epsilon^{\kappa\lambda\mu\nu}\int d^4k \mathcal{A}_\kappa(x,k) \\ S^{\lambda,\mu\nu}_{\rm C} &= S^{\lambda,\mu\nu}_{\rm GLW} + S^{\mu,\nu\lambda}_{\rm GLW} + S^{\nu,\lambda\mu}_{\rm GLW} \end{aligned}$$

The canonical spin tensor is not conserved!

$$\partial_\lambda S^{\lambda,\mu\nu}_{\rm C}(x) = T^{\nu\mu}_{\rm C} - T^{\mu\nu}_{\rm C} = \partial_\lambda S^{\nu,\lambda\mu}_{\rm GLW}(x) - \partial_\lambda S^{\mu,\lambda\nu}_{\rm GLW}(x)$$

Pseudo-gauge transformation: canonical $\leftrightarrow$ GLW

The canonical and GLW frameworks are connected by a pseudo-gauge transformation.
Similarly to Belinfante, it leads to a symmetric energy-momentum tensor,
however, does not eliminate the spin degrees of freedom

If we introduce the superpotential $\Phi_{\rm C}^{\lambda,\mu\nu}$ defined by the relation

$$\Phi_{\rm C}^{\lambda,\mu\nu} \equiv S_{\rm GLW}^{\mu,\lambda\nu} - S_{\rm GLW}^{\nu,\lambda\mu}$$

we can write

$$S_{\rm C}^{\lambda,\mu\nu} = S_{\rm GLW}^{\lambda,\mu\nu} - \Phi_{\rm C}^{\lambda,\mu\nu}$$

and

$$T_{\rm C}^{\mu\nu} = T_{\rm GLW}^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi_{\rm C}^{\lambda,\mu\nu} + \Phi_{\rm C}^{\mu,\nu\lambda} + \Phi_{\rm C}^{\nu,\mu\lambda}\right)$$

Conclusions and Summary

- At the moment the differential polarization data is a chalange for theory
- Fails of the current theories suggest that polarization evolves independently of the thermal vorticity and maybe finally converges to it
- GLW forms of the energy-momentum and spin tensors are natural candidates for formulating hydrodynamic equations (symmetric energy-momentum tensor; the spin tensor strictly conserved; follow from kinetic theory)
- Separate conservation of the spin tensor may be traced back to the locality of the collision term. Departure from the locality is necessary to include dissipative effects.
- Numerical implementation of the hydrodynamic formalism with spin and its applications are on the way !

Thank you for your attention!

Backup slides

Spin polarization – standard QM treatment

Expansion in terms of Pauli matrices

$$f^{\pm}(x,p) = e^{\pm\xi - p\cdot\beta}\left[\cosh(\zeta) - \frac{\sinh(\zeta)}{2\zeta}\,\boldsymbol{P}\cdot\boldsymbol{\sigma}\right]$$

average polarization vector

$$\boldsymbol{\mathcal{P}} = \frac{1}{2}\,\frac{\mathrm{tr}_2\left[(f^{+}+f^{-})\boldsymbol{s}\right]}{\mathrm{tr}_2\left[f^{+}+f^{-}\right]} = -\frac{1}{2}\tanh(\zeta)\,\frac{\boldsymbol{P}}{2\zeta}$$

$$\boldsymbol{\mathcal{P}} = -\frac{1}{2}\tanh\left[\frac{1}{2}\sqrt{\boldsymbol{b}_*\cdot\boldsymbol{b}_* - \boldsymbol{e}_*\cdot\boldsymbol{e}_*}\right]\frac{\boldsymbol{b}_*}{\sqrt{\boldsymbol{b}_*\cdot\boldsymbol{b}_* - \boldsymbol{e}_*\cdot\boldsymbol{e}_*}}$$

where the spin polarization is expressed by the matrix

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}$$

$*$ denotes the PARTICLE REST FRAME

Pauli-Lubański four-vector (phase-space density $\Pi_\mu(x,p)$)

$J^{\lambda,\nu\alpha}(x,p)$ is the phase-space density of the angular momentum of particles

$$E_p \frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\Delta\Sigma_\lambda(x)\,E_p \frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p}\,\frac{p^\beta}{m}$$

$$E_p \frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p} = \frac{\kappa}{2}\,p^\lambda\left(x^\nu p^\alpha - x^\alpha p^\nu\right)\mathrm{tr}_4(X^+ + X^-) + \frac{\kappa}{2}\,p^\lambda \mathrm{tr}_4\left[(X^+ - X^-)\,\Sigma^{\nu\alpha}\right]$$

particle density in the volume $\Delta\Sigma$

$$E_p \frac{d\Delta\mathcal{N}}{d^3p} = \frac{\kappa}{2}\,\Delta\Sigma\cdot p\,\mathrm{tr}_4\,\left(X^+ + X^-\right)$$

$$\pi_\mu(x,p) = \frac{\Delta\Pi_\mu(x,p)}{\Delta\mathcal{N}(x,p)}$$

By applying the Lorentz transformation we find that the PL four-vector calculated in the PRF agrees with the spin polarization (!)

$$\pi^0_* = 0, \quad \boldsymbol{\pi}_* = \boldsymbol{\mathcal{P}} = -\frac{1}{2}\tanh(\zeta)\bar{\boldsymbol{P}}$$

Clifford-algebra expansion

The Wigner functions $\mathcal{W}^{\pm}_{\rm eq}(x,k)$, can be always expanded in terms of the 16 independent generators of the Clifford algebra

D. Vasak, M. Gyulassy, and H. T. Elze, Annals Phys. 173 (1987) 462–492
C. Itzykson and J. B. Zuber, Quantum Field Theory. McGraw-Hill, New York, 1980

$$\mathcal{W}^{\pm}_{\rm eq}(x,k) = \frac{1}{4}\left[\mathcal{F}^{\pm}_{\rm eq}(x,k) + i\gamma_5 \mathcal{P}^{\pm}_{\rm eq}(x,k) + \gamma^\mu \mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k) + \gamma_5\gamma^\mu \mathcal{A}^{\pm}_{{\rm eq},\mu}(x,k) + \Sigma^{\mu\nu}\mathcal{S}^{\pm}_{{\rm eq},\mu\nu}(x,k)\right].$$

The coefficient functions in the equilibrium Wigner function expansion can be obtained by the respective traces:

$$\begin{aligned}
\mathcal{F}^{\pm}_{\rm eq}(x,k) &= {\rm tr}\left[\mathcal{W}^{\pm}_{\rm eq}(x,k)\right],\\
\mathcal{P}^{\pm}_{\rm eq}(x,k) &= -i\,{\rm tr}\left[\gamma^5\mathcal{W}^{\pm}_{\rm eq}(x,k)\right],\\
\mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k) &= {\rm tr}\left[\gamma_\mu\mathcal{W}^{\pm}_{\rm eq}(x,k)\right]\\
\mathcal{A}^{\pm}_{{\rm eq},\mu}(x,k) &= {\rm tr}\left[\gamma_\mu\gamma^5\mathcal{W}^{\pm}_{\rm eq}(x,k)\right],\\
\mathcal{S}^{\pm}_{{\rm eq},\mu\nu}(x,k) &= 2\,{\rm tr}\left[\Sigma^{\mu\nu}\mathcal{W}^{\pm}_{\rm eq}(x,k)\right].
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
\mathcal{F}^{\pm}_{\rm eq}(x,k) &= 2m\cosh(\zeta)\int dP\; e^{-\beta\cdot p\pm\xi}\;\delta^{(4)}(k\mp p),\\
\mathcal{P}^{\pm}_{\rm eq}(x,k) &= 0,\\
\mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k) &= \pm 2\cosh(\zeta)\int dP\, e^{-\beta\cdot p\pm\xi}\;\delta^{(4)}(k\mp p)\,p_\mu,\\
\mathcal{A}^{\pm}_{{\rm eq},\mu}(x,k) &= -\frac{\sinh(\zeta)}{\zeta}\int dP\, e^{-\beta\cdot p\pm\xi}\;\delta^{(4)}(k\mp p)\,\tilde{\omega}_{\mu\nu}\,p^\nu,\\
\mathcal{S}^{\pm}_{{\rm eq},\mu\nu}(x,k) &= \pm\frac{\sinh(\zeta)}{m\zeta}\int dP\, e^{-\beta\cdot p\pm\xi}\;\delta^{(4)}(k\mp p)\left[\left(p_\mu\omega_{\nu\alpha} - p_\nu\omega_{\mu\alpha}\right)p^\alpha +\right.
\end{aligned}$$

Relations between equilibrium coefficient functions

One can verify that the equilibrium coefficient functions satisfy the following set of constraints:

$$k^\mu \, \mathcal{V}^{\pm}_{\mathrm{eq},\mu}(x,k) = m \, \mathcal{F}^{\pm}_{\mathrm{eq}}(x,k)$$

$$k_\mu \, \mathcal{F}^{\pm}_{\mathrm{eq}}(x,k) = m \, \mathcal{V}^{\pm}_{\mathrm{eq},\mu}(x,k)$$

$$\mathcal{P}^{\pm}_{\mathrm{eq}}(x,k) = 0$$

$$k^\mu \, \mathcal{A}^{\pm}_{\mathrm{eq},\,\mu}(x,k) = 0$$

$$k^\mu \, \mathcal{S}^{\pm}_{\mathrm{eq},\,\mu\nu}(x,k) = 0$$

$$k^\beta \, \tilde{\mathcal{S}}^{\pm}_{\mathrm{eq},\mu\beta}(x,k) + m \, \mathcal{A}^{\pm}_{\mathrm{eq},\,\mu}(x,k) = 0$$

$$\epsilon_{\mu\nu\alpha\beta} \, k^\alpha \, \mathcal{A}^{\pm\,\beta}_{\mathrm{eq}}(x,k) + m \, \mathcal{S}^{\pm}_{\mathrm{eq},\,\mu\nu}(x,k) = 0$$

These hold for any form of the fields: $\beta_\mu(x)$, $\xi(x)$, and $\omega_{\mu\nu}(x)$.

Semi-classical expansion

Lets now use a general form of the Wigner function

$$\mathcal{W}(x,k) \quad = \quad \frac{1}{4}\Big[\mathcal{F}(x,k) + i\gamma_5\mathcal{P}(x,k) + \gamma^\mu\mathcal{V}_\mu(x,k) + \gamma_5\gamma^\mu\mathcal{A}_\mu(x,k) + \Sigma^{\mu\nu}\mathcal{S}_{\mu\nu}(x,k)\Big]$$

In the case where the effects of both the mean fields and collisions can be neglected, the Wigner function satisfies the equation of the form

$$\Big(\gamma_\mu K^\mu - m\Big)\mathcal{W}(x,k) = 0 \qquad\qquad K^\mu = k^\mu + \frac{i\hbar}{2}\,\partial^\mu$$

Real and imaginary parts give

$$\begin{aligned}
k^\mu\mathcal{V}_\mu - m\mathcal{F} &= 0\\
\frac{\hbar}{2}\partial^\mu\mathcal{A}_\mu + m\mathcal{P} &= 0\\
k_\mu\mathcal{F} - \frac{\hbar}{2}\partial^\nu\mathcal{S}_{\nu\mu} - m\mathcal{V}_\mu &= 0\\
-\frac{\hbar}{2}\partial_\mu\mathcal{P} + k^\beta\tilde{\mathcal{S}}_{\mu\beta} + m\mathcal{A}_\mu &= 0\\
\frac{\hbar}{2}\Big(\partial_\mu\mathcal{V}_\nu - \partial_\nu\mathcal{V}_\mu\Big) - \epsilon_{\mu\nu\alpha\beta}k^\alpha\mathcal{A}^\beta - m\mathcal{S}_{\mu\nu} &= 0
\end{aligned}$$

$$\begin{aligned}
\hbar\partial^\mu\mathcal{V}_\mu &= 0\\
k^\mu\mathcal{A}_\mu &= 0\\
\frac{\hbar}{2}\partial_\mu\mathcal{F} + k^\nu\mathcal{S}_{\nu\mu} &= 0\\
k_\mu\mathcal{P} + \frac{\hbar}{2}\partial^\beta\tilde{\mathcal{S}}_{\mu\beta} &= 0\\
\Big(k_\mu\mathcal{V}_\nu - k_\nu\mathcal{V}_\mu\Big) + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\alpha\mathcal{A}^\beta &= 0
\end{aligned}$$

This suggest to search for solutions in the form

$$\mathcal{X} = \mathcal{X}^{(0)} + \hbar\mathcal{X}^{(1)} + \hbar^2\mathcal{X}^{(2)} + \cdots . \qquad\qquad \mathcal{X} \in \{\mathcal{F},\mathcal{P},\mathcal{V}_\mu,\mathcal{A}_\mu,\mathcal{S}_{\nu\mu}\}$$

Semi-classical expansion (LO)

In the leading order in $\hbar$ one has

$$k^\mu \mathcal{V}^{(0)}_\mu - m\mathcal{F}^{(0)} = 0$$

$$\mathcal{P}^{(0)} = 0$$

$$k_\mu \mathcal{F}^{(0)} - m\mathcal{V}^{(0)}_\mu = 0$$

$$k^\beta \tilde{\mathcal{S}}^{(0)}_{\mu\beta} + m\mathcal{A}^{(0)}_\mu = 0$$

$$\epsilon_{\mu\nu\alpha\beta} k^\alpha \mathcal{A}^\beta_{(0)} + m\mathcal{S}^{(0)}_{\mu\nu} = 0$$

$$k^\mu \mathcal{A}^{(0)}_\mu = 0$$

$$k^\nu \mathcal{S}^{(0)}_{\nu\mu} = 0$$

$$k_\mu \mathcal{V}^{(0)}_\nu - k_\nu \mathcal{V}^{(0)}_\mu = 0$$

Semi-classical expansion (NLO)

In the next-to-leading order in $\hbar$ one has

$$
\begin{aligned}
k^\mu \mathcal{V}^{(1)}_\mu - m\mathcal{F}^{(1)} &= 0\\
\tfrac{1}{2}\partial^\mu \mathcal{A}^{(0)}_\mu + m\mathcal{P}^{(1)} &= 0\\
k_\mu \mathcal{F}^{(1)} - \tfrac{1}{2}\partial^\nu \mathcal{S}^{(0)}_{\nu\mu} - m\mathcal{V}^{(1)}_\mu &= 0\\
-\tfrac{1}{2}\partial_\mu \mathcal{P}_{(0)} + k^\beta \tilde{\mathcal{S}}^{(1)}_{\mu\beta} + m\mathcal{A}^{(1)}_\mu &= 0\\
\tfrac{1}{2}\left(\partial_\mu \mathcal{V}^{(0)}_\nu - \partial_\nu \mathcal{V}^{(0)}_\mu\right) - \epsilon_{\mu\nu\alpha\beta} k^\alpha \mathcal{A}^\beta_{(1)} - m\mathcal{S}^{(1)}_{\mu\nu} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\partial^\mu \mathcal{V}^{(0)}_\mu &= 0\\
k^\mu \mathcal{A}^{(1)}_\mu &= 0\\
\frac{1}{2}\partial_\mu \mathcal{F}^{(0)} + k^\nu \mathcal{S}^{(1)}_{\nu\mu} &= 0\\
k_\mu \mathcal{P}^{(1)} + \frac{1}{2}\partial^\beta \tilde{\mathcal{S}}^{(0)}_{\mu\beta} &= 0\\
k_\mu \mathcal{V}^{(1)}_\nu - k_\nu \mathcal{V}^{(1)}_\mu + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\alpha \mathcal{A}^\beta_{(0)} &= 0
\end{aligned}
$$

Hydrodynamics with spin based on entropy-current analysis

Phenomenological derivation of hydrodynamics based on the conservation laws

K. Hattori, M. Hongo, X-G Huang, M. Matsuo, H. Taya,
PLB 795 (2019) 100-106

$$\partial_\mu T^{\mu\nu} = 0 \qquad T^{\mu\nu} \equiv T^{\mu\nu}_{(s)} + T^{\mu\nu}_{(a)}$$
$$\partial_\mu J^{\mu\alpha\beta} = 0 \qquad J^{\mu\alpha\beta} = \left(x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}\right) + \Sigma^{\mu\alpha\beta}$$

Dynamical variables near local thermal equilibrium are assumed to satisfy the first law of thermodynamics generalized with finite spin density S. (spin potential ω is conjugate to the spin density S)

$$Ts = e + p - \omega_{\mu\nu} S^{\mu\nu}, \quad Tds = de - \omega_{\mu\nu} dS^{\mu\nu}$$

One may organize the constitutive relations on the basis of a derivative expansion $T^{\mu\nu}_{(1)} \sim \mathcal{O}(\partial^1)$

$$T^{\mu\nu} = e u^\mu u^\nu + p\Delta^{\mu\nu} + T^{\mu\nu}_{(1)}$$
$$\Sigma^{\mu\alpha\beta} = u^\mu S^{\alpha\beta} + \Sigma^{\mu\alpha\beta}_{(1)}$$

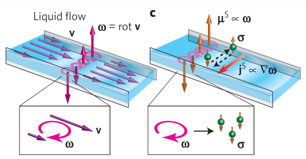
Lowest-order hydrodynamic equations of motion do not conserve the lowest-order entropy current (in contrast to the case of a fluid without spin)

$$\partial_\mu s^\mu_{(0)} = 2\beta\omega_{\alpha\beta} T^{\alpha\beta}_{(1a)}$$

The entropy production implies that spin density is inherently a dissipative quantity
$\Downarrow$
no counterpart of ideal spin-less hydrodynamics ?

new transport coefficients appear that control the relaxation time of spin density (rotational viscosity, boost heat conductivity)
$\Downarrow$
relativistic generalization of a non-relativistic micropolar hydrodynamics

R. Takahashi, et al. Nat. Phys. 12(1) (2016) 52
M. Matsuo, Y. Ohnuma, S. Maekawa, PRB 96 (2017) 020401

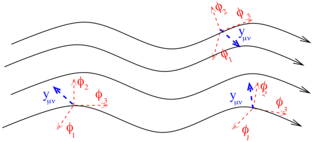


1st order dissipative corrections
$\Downarrow$
causality, stability?

Lagrangian formulation of relativistic fluid mechanics for spin-polarized systems

D. Montenegro, G. Torrieri, Phys.Rev. D94 (2016) no.6, 065042
D. Montenegro, L. Tinti, G. Torrieri, Phys. Rev. D 96(5) (2017) 056012; Phys. Rev. D 96(7) (2017) 076016
D. Montenegro, G. Torrieri, [1807.02796]

One constructs lagrangian which contains the information of the EoS including an entropy term derived from the fluid coordinate d.o.f. $b = \left(\det_{IJ}\left[\partial_\mu \phi_I \partial^\mu \phi_J\right]\right)^{1/2}$ **as well as polarization tensor** $y^{\mu\nu}$



D. Montenegro, R. Ryblewski, G. Torrieri, [1903.08729]

For small polarizations, the EoS reduces to

$$\mathcal{L} = F(b, y) = F\left(b\left(1 - c y_{\mu\nu} y^{\mu\nu}\right)\right)$$

$c > 0$ - ferromagnet, $c < 0$ - antiferromagnet

The ideal limit of hydrodynamics with polarization is generally non-causal (due to Ostrogradski theorem)

$$y_{\mu\nu} = \chi\left(b, \Omega_{\mu\nu}\Omega^{\mu\nu}\right)\Omega_{\mu\nu}, \qquad \Omega_{\mu\nu} = \nabla_{[\mu} u_{\nu]}$$

Causality can be cured by using Israel-Stewart type lagrangian, written in doubled coordinates and employing non-equilibrium polarization d.o.f. Y giving Maxwell-Cataneo type relation

$$\tau_Y \partial_\tau \delta Y_{\mu\nu} + \delta Y_{\mu\nu} = y_{\mu\nu}$$

causality can be fixed by a relaxation type term
$\Downarrow$
any material with a non-zero spin must have a minimum amount of dissipation