Karlsruhe Institute of Technology

TRR 257 - Particle Physics Phenomenology after the Higgs Discovery

Nested soft-collinear subtractions in NNLO QCD computations

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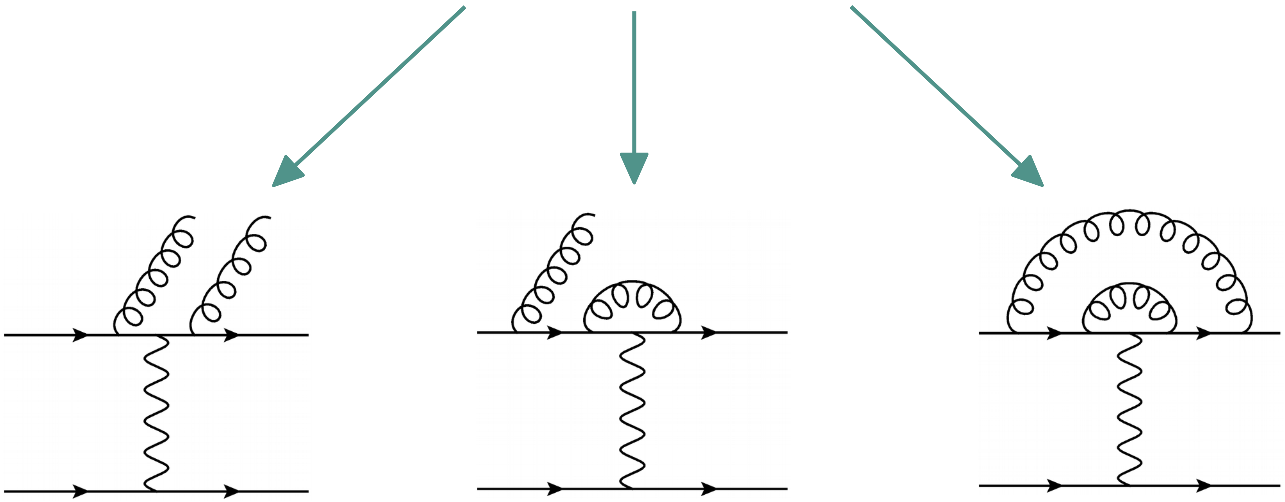
Differential cross section @NNLO

- To test the SM, predictions for fully-differential cross sections through higher orders in QCD are needed.
- Contributions to next-to-next-to-leading order partonic cross section

$$\mathrm{d}\sigma_{\mathrm{NNLO}} = \mathrm{d}\sigma_{\mathrm{rr}} + \mathrm{d}\sigma_{\mathrm{rv}} + \mathrm{d}\sigma_{\mathrm{vv}} + \mathrm{d}\sigma_{\mathrm{pdf}}$$



Differential cross section @NNLO

- UV and IR poles in different contributions

Contain ***explicit*** poles in 1/ε.

$$d\sigma_{\text{NNLO}} = \underbrace{d\sigma_{\text{rr}} + d\sigma_{\text{rv}}}_{} + d\sigma_{\text{vv}} + d\sigma_{\text{pdf}}$$

(the brace above spans $d\sigma_{\text{rv}} + d\sigma_{\text{vv}} + d\sigma_{\text{pdf}}$)

Contain singularities that become poles in 1/ε ***only*** upon phase space integration.

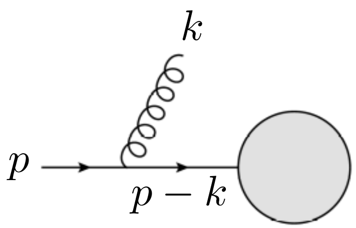
- In dimensional regularitzation $d = 4 - 2\epsilon$ the explicit poles of real-virtual (1-loop), double virtual (2-loop) and collinear renormalization contributions are known ***independent of the hard matrix element*** [Catani '98; Becher, Neubert '09; …]

$$\mathcal{M}_{\text{1-loop}}(\{p\}) = \left[\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\sum_i\left(\frac{1}{\epsilon^2} + \frac{g_i}{\mathbf{T}_i^2}\frac{1}{\epsilon}\right)\sum_{j\neq i}\frac{\mathbf{T}_i\cdot\mathbf{T}_j}{2}\left(\frac{\mu^2}{-s_{ij}}\right)^{\epsilon}\right]\mathcal{M}_{\text{tree}}(\{p\}) + \mathcal{M}^{\text{fin}}_{\text{1-loop}}(\{p\})$$

- We would like to find a process-independent description of poles that originate from real emission contributions ***without*** integrating over resolved phase-space and to demonstrate the cancellation of 1/ε infrared/collinear poles between real, virtual and collinear renormalization contributions in a ***general case***.

Singularities of real emission contributions

- Singularities of QCD amplitudes come in two varieties: soft ($E \to 0$) and collinear ($\vec{p}_i \parallel \vec{p}_j$).



$$\sim \frac{1}{(p-k)^2} \sim \frac{1}{E_p \times \underbrace{E_k}_{\text{Soft singularity}} \times \underbrace{(1 - \vec{n}_p \cdot \vec{n}_k)}_{\text{Collinear singularity}}} \longrightarrow \infty \quad \begin{cases} \text{for} & E_k \to 0 \\ \text{for} & \vec{n}_p \parallel \vec{n}_k \end{cases}$$

- The corresponding limits of amplitudes are generic and ***independent of a hard process***. E.g. in case of $q(p_1) + \bar{q}(p_2) \to Z + g(k)$ amplitude [Altarelli, Parisi, '77]

$$\rho_{ij} = 1 - \vec{n}_i \cdot \vec{n}_j$$

$$|M(\{p_1, p_2\}, k)|^2 \underset{E_k \to 0}{\approx} 2 g_{s,b}^2 C_F \times \boxed{\frac{1}{E_k^2} \times \frac{\rho_{12}}{\rho_{1k}\rho_{2k}}} \times |M(\{p_1, p_2\})|^2$$

Eikonal function

$$P_{qq}(z) = C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z) \right]$$

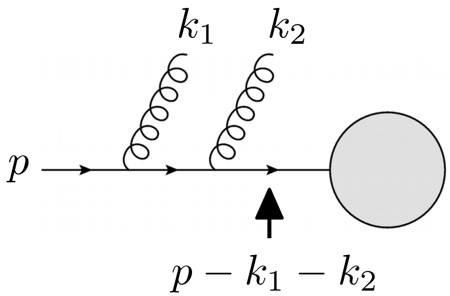
$$|M(\{p_1, p_2\}, k)|^2 \underset{k \parallel p_1}{\approx} -g_{s,b}^2 \times \frac{1}{E_1 E_k} \boxed{P_{qq}\left(\frac{E_1}{E_1 - E_6} \right)} \times \frac{1}{\rho_{1k}} \times \left| M\left(\left\{ \frac{E_1 - E_k}{E_1} \cdot p_1, p_2 \right\} \right) \right|^2$$

Splitting function

- Double-soft and double-collinear limits of the amplitude at NNLO are structurally similar to the NLO case and also known ***independent of a hard process.*** [Catani, Grazzini '99; …]

Absence of entangled soft and collinear limits

- Individual diagrams and propagators suggest that there are singular limits *beyond soft and collinear*.



$$\sim \frac{1}{(p-k_1-k_2)^2} \sim \frac{1}{2p\cdot k_1 + 2p\cdot k_2 - 2k_1\cdot k_2} \underset{\substack{k_1 \parallel p \\ k_2 \to 0}}{\longrightarrow} \infty$$

- For a ***given amplitude*** it can be checked explicitly that these singularities do not occur.
- Recently it was argued that this observation is general thanks to a phenomenon known as ***colour coherence*** (a soft gluon does not resolve details of a collinear splitting) [Caola, Melnikov, Röntsch, '17]
- As a result known soft and collinear limits must be sufficient to ***describe and regulate*** all singularities in scattering amplitudes.

An ideal subtraction scheme should be …

[Gehrmann-de Ridder, Gehrmann, Glover ‘05; Czakon ‘10, ‘11; Cacciari et al ‘15; Somogyi, Trócsányi, Del Duca ‘05; Caola, Melnikov, Röntsch ‘17; Herzog ‘18; Magnea et al ‘18; ...]

physically transparent	”physical” singularities and clear mechanism of cancellation;
local	subtracted matrix elements at each point in the phase-space are finite;
analytic	analytic formulas for integrated subtraction terms in each point of the resolved phase space;
modular	subtractions for complex processes are built from modules established in analysis of simpler processes;
efficient	efficient numerical evaluation.

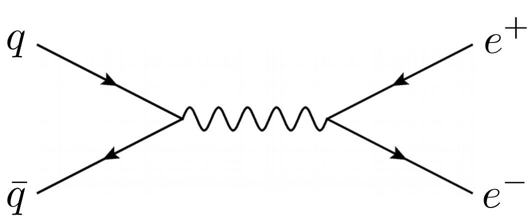
- **None of the existing subtraction schemes satisfies all these criteria.** Up to now this was not a showstopper for phenomenology. Nevertheless, for more complex processes, better subtraction schemes may become a necessity.

Nested soft-collinear subtraction scheme

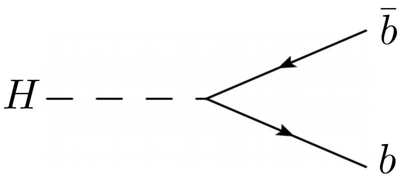
[Caola, Melnikov, Röntsch, ‘17]

- The **nested soft-collinear subtraction scheme** that possess many of the desired features.
- ***Non-trivial observation***: the non-trivial part of both soft and collinear limits depends at most on two external momenta. In a complex process these can be ***both initial states***, ***both finale states*** or ***one initial and one finale state***.
- Before dealing with complex processes it is therefore useful to study the subtraction scheme for simpler processes with only two external colour charged particles and check the formulas against analytic results. To cover all possible kinematic configurations we consider

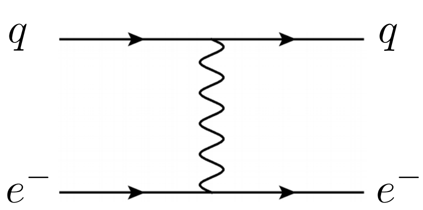
Drell-Yan process
both momenta are **initial states**
[Caola, Melnikov, Röntsch ‘19]



$H \to b\bar{b}$ **decay**
both momenta are **finale states**
[Caola, Melnikov, Röntsch ‘19]



Deep inelastic scattering
one momenta is an **initial** and one a **finale state**
[Asteriadis, Caola, Melnikov, Röntsch ‘19]

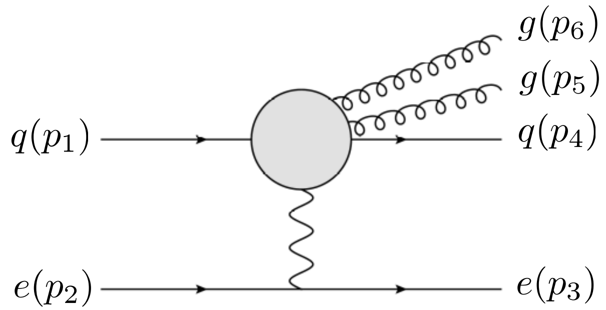


Considered in the following.

- Results can be used as ***building blocks*** for more complex processes.

Deep inelastic scattering

- As example we consider the real emission channel $q + e \to q + e + gg$



- We define

$$2s \cdot \mathrm{d}\sigma_{\mathrm{rr}} = \int \mathrm{dLips}(p_5, p_6)\; F_{\mathrm{LM}}(1,4,5,6) \equiv \langle F_{\mathrm{LM}}(1,4,5,6) \rangle$$

with

$$F_{\mathrm{LM}}(1,4,5,6) = \mathcal{N} \int \mathrm{dLips}(p_3, p_4)\; (2\pi)^d \delta^{(d)}(p_1 + p_2 - p_3 - p_4 - p_5 - p_6)$$
$$\times \, |M^{\mathrm{tree}}(\{p\}), p_5, p_6|^2 \times \boxed{\mathcal{O}(p_3, p_4, p_5, p_6)}$$

Arbitrary infrared safe observable

- The integral diverges and needs to be regulated. Due to the absence of entangled soft and collinear singularities all singularities can be subtracted **iteratively**.

Soft singularities

- **Double-soft singularity:** Introduce operator $\mathbb{S}$ that extracts the leading double soft singularity ($E_5 \sim E_6 \to 0$). We insert the unity $I = (I - \mathbb{S}) + \mathbb{S}$

$$2s \cdot \mathrm{d}\sigma_{\mathrm{rr}} = \underbrace{\int \mathrm{dLips}(p_5, p_6)\ (I - \mathbb{S}) F_{\mathrm{LM}}(1,4,5,6)}_{\text{double-soft singularity regulated}} + \underbrace{\int \mathrm{dLips}(p_5, p_6)\ \mathbb{S} F_{\mathrm{LM}}(1,4,5,6)}_{\text{extracted } 1/\varepsilon \text{ pole}}$$

- In the subtraction term the soft gluons decouple from the **matrix element** and the **observable**. Hence we can integrate it ***analytically*** over the phase space of gluons 5 & 6. [Caola, Delto, Frellesvig, Melnikov, Röntsch '18]

$$\int \mathrm{dLips}(p_5, p_6)\ \mathbb{S} F_{\mathrm{LM}}(1,4,5,6) = \boxed{\int \mathrm{dLips}(p_5, p_6)\ \mathrm{Eikonal}(1,2,5,6)} \times F_{\mathrm{LM}}(1,4)$$

factorized integration over 5 & 6

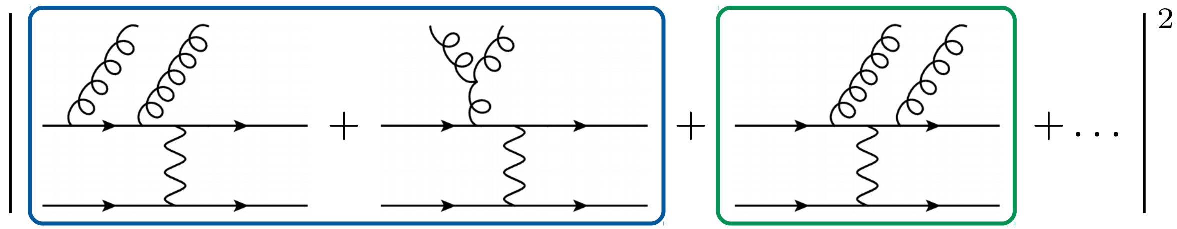
- The double-soft singularity is regulated ***locally*** at any point of the resolved phase-space. It still contains unregulated single-soft and collinear singularities.

- It is now straightforward to **regulate the remaining single-soft singularity iteratively**

$$\int \mathrm{dLips}(p_5, p_6)\ (I - \mathbb{S}) F_{\mathrm{LM}}(1,4,5,6) = \int \mathrm{dLips}(p_5, p_6)\ (I - S_6)(I - \mathbb{S}) F_{\mathrm{LM}}(1,4,5,6)$$
$$+ \int \mathrm{dLips}(p_5, p_6)\ S_6 (I - \mathbb{S}) F_{\mathrm{LM}}(1,4,5,6)$$

Collinear singularities

$$\left|M^{\text{tree}}(\{p\}, p_5, p_6)\right|^2 =$$



- In the collinear limits, many different singular configurations exist. However either *three defines partons* become collinear or *two pairs of partons* become collinear at once. They are separated using **partition functions**.

- In triple-collinear partitions the different single collinear singularities are separated in the phase-space. We isolated them by **splitting the angular phase-space into sectors**.

- In each partition and sector the structure of collinear singularities is now fully defined. Using the phase space parametrization from [Czakon ‘10, Phys.Lett. B693 (2010) 259-268] all singularities are made explicit.

- It is then straightforward to regulate remaining **collinear singularities interatively.**

Fully regulated double real contribution

- We obtain a fully-regulated formula for the double-real contribution

$$
\begin{aligned}
&\sum_{\substack{i,j=1,4\\ i\neq j}} \Big\langle [1-\mathbb{S}][1-S_6][1-C_{6j}][1-C_{5i}][\mathrm{d}g_5][\mathrm{d}g_6] w^{i5,j6} F_{LM}(1,4,5,6) \Big\rangle \\
&+ \sum_{i=1,4} \Big\langle [1-\mathbb{S}][1-S_6] \Big[\theta^{(a)}[1-\mathbb{C}_i][1-C_{6i}] + \theta^{(b)}[1-\mathbb{C}_i][1-C_{56}] \\
&\qquad + \theta^{(c)}[1-\mathbb{C}_i][1-C_{5i}] + \theta^{(d)}[1-\mathbb{C}_i][1-C_{56}] \Big] \\
&\qquad \times [\mathrm{d}g_5][\mathrm{d}g_6] w^{i5,i6} F_{LM}(1,4,5,6) \Big\rangle
\end{aligned}
$$

(phase-space sectors)

(partition functions)

- It is finite and can be used to compute arbitrary infra-red safe observables in d = 4 dimensions numerically.

- We have not discussed the integration of the subtraction terms. However it can be done, quite straightforward.

Numerical results

- Our results can be extensively tested against known analytic results. [Kazakov et al. ‘90; Zijlstra, van Neerven ‘92; Moch, Vermaseren ‘00; …]
- In the case of photon-induced deep-inelastic scattering with only up-quarks and gluons in the initial state and $\sqrt{s} = 100\,\mathrm{GeV},\ 10\,\mathrm{GeV} < Q < 100\,\mathrm{GeV},\ \mu_R = \mu_F = 100\,\mathrm{GeV}$ we obtain permille agreement for the ***NNLO contribution***

$$\sigma = \sigma_{\mathrm{LO}} + \sigma_{\mathrm{NLO}} + \boxed{\sigma_{\mathrm{NNLO}}}$$

channel	numerical result	analytic result
$\sigma_{\mathrm{q,ns}}^{\mathrm{NNLO}}$	$[33.1(2) - 2.18(1)\cdot n_f]\,\mathrm{pb}$	$[33.1 - 2.17\cdot n_f]\,\mathrm{pb}$
$\sigma_{\mathrm{q,s}}^{\mathrm{NNLO}}$	$9.19(2)\,\mathrm{pb}$	$9.18\,\mathrm{pb}$
$\sigma_{\mathrm{g}}^{\mathrm{NNLO}}$	$-142.4(4)\,\mathrm{pb}$	$-142.7\,\mathrm{pb}$

[Asteriadis, Caola, Melnikov, Röntsch ‘19]

- In general, we find that we can get per mill precision on the NNLO total cross section, corresponding to a few percent precision on the NNLO coefficient, running for a few hours on an 8-core machine.

Conclusion

Construction of the subtraction scheme was based on

- realization that known soft and collinear limits are sufficient. Property of gauge-invariant amplitudes not diagrams;
- iterative extraction of soft and collinear singularities;
- partitioning of angular phase-space into sectors to obtain unique collinear limits;
- the possibility to parametrize phase space in a way that makes analytic integration of subtraction terms possible.

Important to continue research into optimal subtractions since

- HL-LHC requires high precision theoretical predictions for collider processes at the LHC;
- despite progress with developing IR subtraction schemes, the "optimal" subtraction scheme is yet to be found.

What we already have

- the proposed nested soft-collinear scheme that already has many features of the "optimal" scheme;
- analytic formulas for colour singlet production, decay and DIS that can be used as building blocks to design subtractions for more complex LHC processes.

Backup slides

How to extract singularities without integration?

- We have seen that real emission contributions develop soft and collinear singularities. These singularities turn into 1/ε poles upon phase space integration

$$\int \frac{\mathrm{d}^{d-1}k}{2E} |M(\{p\},k)|^2 \sim \int \frac{\mathrm{d}E}{E^{1+\epsilon}} \; \frac{\mathrm{d}\theta}{\theta^{1+2\epsilon}} \times |M(\{p\})|^2 \sim \frac{1}{\epsilon^2}$$

- We would like to extract singularities ***without integration over resolved phase space***. Currently two approaches used: slicing and ***subtraction***.

- To illustrate the basic idea of subtraction, consider

$$I = \int_0^1 \mathrm{d}x \; \frac{1}{x^{1+\epsilon}} \mathrm{F}(x)$$

where $\mathrm{F}(x)$ is free of singularities. We then write

$$I = \int_0^1 \mathrm{d}x \; \frac{1}{x^{1+\epsilon}} \left[\mathrm{F}(x) - \mathrm{F}(0)\right] + \int_0^1 \mathrm{d}x \; \frac{1}{x^{1+\epsilon}} \mathrm{F}(0) = \underbrace{\int_0^1 \mathrm{d}x \; \frac{1}{x^{1+\epsilon}} \left[\mathrm{F}(x) - \mathrm{F}(0)\right]}_{\text{regulated, finite in the } \varepsilon \to 0 \text{ limit}} \underbrace{- \frac{1}{\epsilon}\mathrm{F}(0)}_{\text{extracted } 1/\varepsilon \text{ pole}}$$

Collinear singularities

In the collinear limits, many different singular configurations exist. However either **three defines partons** become collinear or **two pairs of partons** become collinear at once. The different configurations are separated in a first step:

1) Introduce partitioning

We use the unity

$$1 = \boxed{w^{51,61} + w^{54,64}} + \boxed{w^{51,64} + w^{54,61}}$$

with the limits

$$\lim_{5||l} w^{5i,6j} \sim \delta_{li}\,, \quad \lim_{6||l} w^{5i,6j} \sim \delta_{lj} \quad \text{and} \quad \lim_{5||i}\lim_{6||j} = 1$$

For instance $w^{51,64}|M|^2$ is then only singular when (5||1) and/or (6||4) but finite when (5||4), (6||1), (5||6), (5||6||1) and (5||6||4).

However, for instance $w^{51,61}|M|^2$ is still singular when (5||1), (6||1), (5||6) and (5||6||1) but finite for (5||4), (6||4) and (5||6||4).

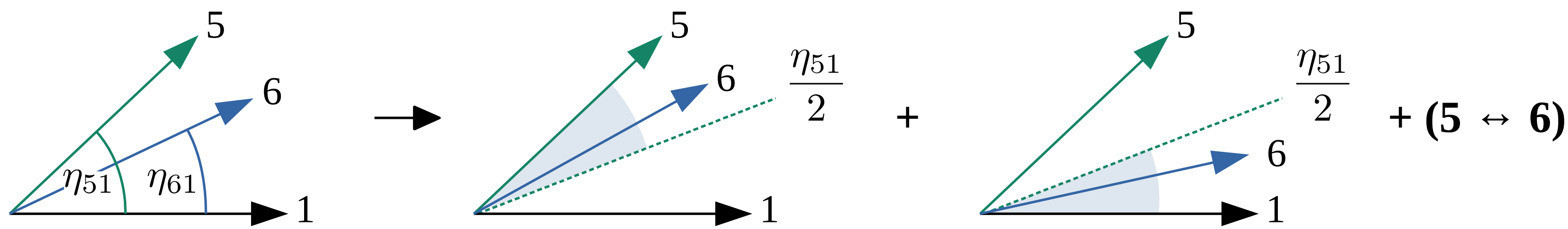
These singularities are separated in a second step.

Collinear singularities

Partitions $w^{51,61}$ and $w^{54,64}$ still contain double and single collinear configurations.

2) Split angular phase space into different sectors

As example consider partition $w^{51,61}$ (with $\eta_{ij} \equiv (1-\cos\theta_{ij})/2$). The angular phase-space is split into regions with defined collinear singularities:



Double-collinear singularity ***(6||5)*** and triple-collinear singularity ***(6||5||1)***.

Double-collinear singularity ***(6||1)*** and triple-collinear singularity ***(6||5||1)***.

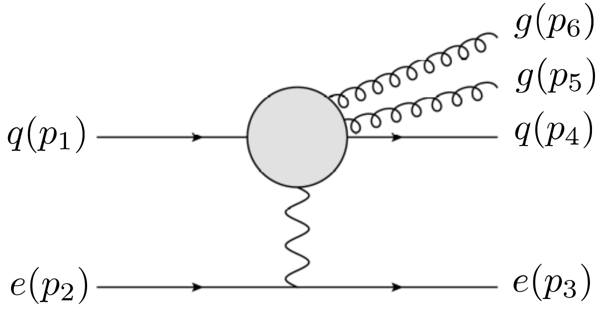
In practice this is done introducing the unity

$$1 = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{61} < \eta_{51}\right) + \theta\left(\eta_{51} < \frac{\eta_{61}}{2}\right) + \theta\left(\frac{\eta_{61}}{2} < \eta_{51} < \eta_{61}\right)$$
$$\equiv \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}$$

In each partition and sector the structure of collinear singularities is now fully defined. Using Czakon's phase space parametrization all singularities are made explicit.

Deep inelastic scattering: complete definition

- Consider a real emission process $q + e \to q + e + gg$ that possesses the most complicated singular structure.



- We define

Energy ordering

$$2s \cdot \mathrm{d}\sigma_{\mathrm{rr}} = \int [\mathrm{d}g_5][\mathrm{d}g_6] \boxed{\theta(E_5 - E_6)} F_{\mathrm{LM}}(1,4,5,6) \equiv \langle F_{\mathrm{LM}}(1,4,5,6) \rangle$$

with

$$F_{\mathrm{LM}}(1,4,5,6) = \mathcal{N} \int \mathrm{dLips}\ (2\pi)^d \delta^{(d)}(p_1 + p_2 - p_3 - p_4 - p_5 - p_6)$$
$$\times |M^{\mathrm{tree}}(\{p\}), p_5, p_6|^2 \times \mathcal{O}(p_3, p_4, p_5, p_6)$$

Needs to be sufficiently large but otherwise arbitrary!

$$[\mathrm{d}g_i] = \frac{\mathrm{d}^{d-1} p_i}{(2\pi)^{d-1} 2E_i} \boxed{\theta(E_{\mathrm{max}} - E_i)}$$

- The integral diverges and needs to be regulated. Due to the absence of entangled soft and collinear singularities all singularities can be subtracted **iteratively**.

Integrated subtraction terms: example 1

Double-soft regulated single-soft subtraction term:

Double-soft regulated single soft

$$\langle(1-\mathbb{S})F_{\mathrm{LM}}(1,4,5,6)\rangle = \langle(1-S_6)(1-\mathbb{S})F_{\mathrm{LM}}(1,4,5,6)\rangle + \langle S_6(1-\mathbb{S})F_{\mathrm{LM}}(1,4,5,6)\rangle$$

NLO kinematics, all singularities regulated, needs to be calculated numerically

$$\Big\langle [1-\mathbb{S}]\, S_6 F_{LM}(1,4,5,6)\Big\rangle = \Big\langle \hat{O}_{NLO} J_{145} F_{LM}(1,4,5)\Big\rangle$$

$$-\frac{[\alpha_s]^2 C_F}{\epsilon^3}\Big\langle \left[2C_F\left(\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right)\eta_{14}^{-\epsilon}K_{14} + C_A\left(\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\right)\right]$$

$$\times\left[\left(\frac{1}{2\epsilon}\left((2E_{max})^{-4\epsilon}-(2E_4)^{-4\epsilon}\right) - Z^{2,4}(2E_4)^{-4\epsilon}\right)F_{LM}(1,4)\right.$$

$$\left.+\frac{1}{2\epsilon}(2E_{max})^{-4\epsilon}F_{LM}(1,4) + (2E_1)^{-4\epsilon}\int \mathrm{d}z(1-z)^{-4\epsilon}\bar{P}_{qq}(z)\frac{F_{LM}(z\cdot 1,4)}{z}\right]\Big\rangle.$$

Born kinematics, contain poles explicitly

with

$$K_{ij} = \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\eta_{ij}^{1+\epsilon}\,{}_2F_1(1,1,1-\epsilon,1-\eta_{ij}) = 1 + \left[\mathrm{Li}_2(1-\eta_{ij}) - \frac{\pi^2}{6}\right]\epsilon^2 + \mathcal{O}(\epsilon^3)$$

$$Z^{n,m} = \frac{3}{2} + \frac{1}{12}\left[6+21m+15n-4n\pi^2\right]\epsilon + \mathcal{O}(\epsilon^2)$$

$$J_{145} = \frac{[\alpha_s]}{\epsilon^2}\left[(2C_F-C_A)\eta_{14}^{-\epsilon}K_{14} + C_A\left[\eta_{15}^{-\epsilon}K_{15}+\eta_{45}^{-\epsilon}K_{45}\right]\right](2E_5)^{-2\epsilon}$$

Integrated subtraction terms: example 2

Soft regulated single collinear finale state subtraction

$$\left\langle [1-\mathbb{S}][1-S_6]\left[C_{54}w^{54,61}+C_{64}w^{51,64}+\left(\theta^{(a)}C_{64}+\theta^{(c)}C_{54}\right)w^{54,64}\right][\mathrm{d}g_5][\mathrm{d}g_6]F_{LM}(1,4,5,6)\right\rangle$$

$$= \frac{[\alpha_s]C_F}{\epsilon}\left\langle \hat{O}_{NLO}\left[\left(\frac{1}{\epsilon}+Z^{2,2}\right)(2E_4)^{-2\epsilon}-\frac{1}{\epsilon}(2E_5)^{-2\epsilon}\right]\left[w_{\mathrm{DC}}^{51}+w_{\mathrm{TC}}^{54}\left(\frac{\rho_{54}}{4}\right)^{-\epsilon}\right]F_{LM}(1,4,5)\right\rangle$$

$$+\frac{[\alpha_s]^2C_F^2}{\epsilon^3}\left\langle\left[\left(\frac{1}{\epsilon}+Z^{2,2}\right)(2E_4)^{-2\epsilon}(2E_{max})^{-2\epsilon}-\frac{1}{2\epsilon}(2E_{max})^{-4\epsilon}\right]\right.$$

$$\left.\times\left[\langle\Delta_{51}\rangle_{S_5}-\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}-\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]F_{LM}(1,4)\right\rangle$$

$$+\frac{[\alpha_s]^2C_F^2}{\epsilon^2}\left[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right]\left[\frac{1}{\epsilon}+Z^{2,2}\right]\left[\frac{1}{\epsilon}+Z^{4,2}\right]\left\langle(2E_4)^{-4\epsilon}F_{LM}(1,4)\right\rangle$$

$$-\frac{[\alpha_s]^2C_F^2}{\epsilon^3}\left[\frac{1}{2\epsilon}+Z^{2,4}\right]\left\langle\left[\langle\Delta_{51}\rangle_{S_5}+\left(\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\right)\right](2E_4)^{-4\epsilon}F_{LM}(1,4)\right\rangle$$

$$-\frac{[\alpha_s]^2C_F^2}{\epsilon^2}\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\int\mathrm{d}z\left\langle\left[\left(\frac{1}{\epsilon}+Z^{2,2}\right)(2E_4)^{-2\epsilon}-\frac{1}{\epsilon}(2E_1)^{-2\epsilon}(1-z)^{-2\epsilon}\right]\right.$$

$$\left.\times(2E_1)^{-2\epsilon}(1-z)^{-2\epsilon}\bar{P}_{qq}(z)\frac{F_{LM}(z\cdot 1,4)}{z}\right\rangle.$$

NLO kinematics, all singularities regulated, needs to be calculated numerically

Born kinematics, contain poles

with

$$\langle\Delta_{51}\rangle_{S_5}=\frac{3}{2}+\epsilon\left(\frac{\ln 2}{2}-2\ln\eta_{14}\right)+\epsilon^2\left(-\frac{\pi^2}{3}-\ln 2+\frac{\ln^2 2}{4}-\frac{\ln\left(\frac{1+\sqrt{1-\eta_{14}}}{1-\sqrt{1-\eta_{14}}}\right)}{2\sqrt{1-\eta_{14}}}+\frac{\ln\eta_{14}}{2}-\ln 2\ln\eta_{14}+\frac{3\ln^2\eta_{14}}{2}+\frac{5}{2}\mathrm{Li}_2(1-\eta_{14})\right)+\mathcal{O}(\epsilon^3)\,.$$

Different subtraction schemes and slicing methods

qt	slicing	[Catani, Grazzini]
Jettiness	slicing	[Boughezal et al., Gaunt et al.]
Antenna	subtraction	[Gehrmann-de Ridder, Gehrmann, Glover et al.]
Projection-to-Born	subtraction	[Cacciari et al.]
Colorful NNLO	subtraction	[Del Duca, Troscanyi et al.]
Stripper	subtraction	[Czakon]
Nested soft-collinear	subtraction	[Caola, Melnikov, Röntsch]
Local Analytic Sector	subtraction	[Magnea, Maina et al.]
Geometric	subtraction	[Herzog]

	Analytic	FS Colour	IS Colour	Local
Antenna	✓	✓	✓	✗
qT	✓	✗	✓	✗ (slicing)
Colourful	✓	✓	✗	✓
Stripper	✗	✓	✓	✓
N-jettiness	✓	✓	✓	✗ (slicing)

Adapted from [Nigel Glover, Amplitudes '15]