Scattering Amplitudes from First Principles

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Introduction

- Consider n gluon tree scattering amplitude,
  for instance for $n = 5$: $g + g \to g + g + g$

n	3	4	5	6	...	10
number Feynman diagrams	1	4	25	220	...	$> 10^6$

- How does the result look like?

$$A_n(1^- 2^- 3^+ \dots n^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

- Answer simple, calculation complicated.

- Feynman diagrams: each diagram of same order as the result.

- Gauge invariance violated; gauge redundancy.

- Virtual particles introduces - that is, violation of energy-momentum conservation ($p^2 \neq m^2$).

- Measurable quantities: (squared) amplitudes

- Closer look at $n = 3$ gluon amplitude. Apart from coupling constant, amplitude completely fixed by Lorentz invariance and locality!

$$A_3(1^- 2^- 3^+) = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$

- Particles with arbitrary helicity

$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = \begin{cases} [12]^{h_1+h_2-h_3} [23]^{h_2+h_3-h_1} [31]^{h_3+h_1-h_2}, & \text{for } h_1 + h_2 + h_3 > 0 \\ \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1}, & \text{for } h_1 + h_2 + h_3 < 0 \end{cases}$$

Weyl formalism

- 4-vectors in terms of 2×2 matrices

$$p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}}$$

- Since $\det(p_{a\dot{a}}) = m^2 = 0$, $\mathrm{rank}(p_{a\dot{a}})$ = 1,

$$p_{a\dot{a}} = -|p]_a \langle p|_{\dot{a}}$$

- Massive case, see Arkani-Hamed, Huang, Huang, arXiv:1709.04891
- Two 2-component vectors $|p]_a$, $|p\rangle_{\dot{a}}$
- For real p is $(p_{a\dot{a}})$ Hermitian,

$$\langle p|_{\dot{a}} = (|p]_a)^*$$

Little group scaling

- $p^2 = 0$, there is a frame

$$p = \left(p_0, 0, 0, p_0\right)^{\mathrm{T}}$$

- Little group is subgroup of Lorentz transformations Λ, keeping p invariant, $\Lambda p = p$:

$$SO(2) \sim U(1)$$

- Little group phase t (keeping def. invariant)

$$|p\rangle \to t|p\rangle, \qquad |p] \to t^{-1}|p]$$

- Transformation of Weyl spinors under little goup scaling fix the amplitudes A_3.

- Consider Feynman diagrams for instance $g + g \to g + g$

- External polarization "tensors" like ϵ^μ appear.

- Polarization "tensors" transform (transversal),

$$\Lambda p = p, \quad \Lambda \epsilon^\mu \to \epsilon^\mu + c(p) p^\mu$$

- Feynman diagrams are no Lorentz tensors!

- Amplitudes have correct transformation behavior.

- However A_3 vanishes for real momenta.

BCFW recursion

- Glueing together on-shell amplitudes – BCFW!

  Britto, Cachazo, Feng, Witten Rev. Lett. **94**, (2005)

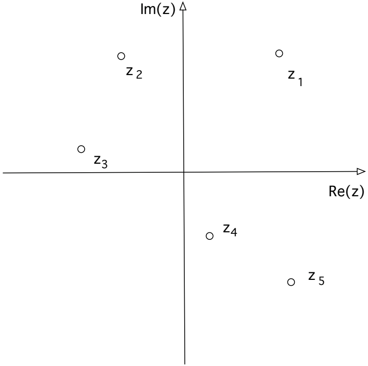
- We consider tree diagrams.
- Analytic continuation of external momenta, deformation.

  $$\hat{p}_i = p_i + z r_i, \quad \text{such that } \hat{p}_i^2 = 0, \quad \sum_i \hat{p}_i = 0$$

  with one common z

- $A_n \to \hat{A}_n(z) \qquad \text{with } A_n = \hat{A}_n(0)$
- $\hat{A}_n(z)$ very simple: no branch cuts; no logs, no square roots.
- Simple poles at $z_I \neq 0$ of on-shell internal lines.

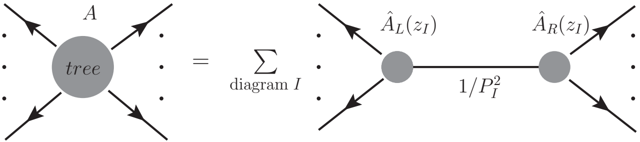
- Poles of $\hat{A}_n(z)$



- Consider $\hat{A}_n(z)/z$; simple pole at $z = 0$ with residue A_n.

- Consider contour circle at infinity.
- In case of $\hat{A}_n(z) \to 0$ for $z \to \infty$

$$A_n = -\sum_I \text{res}_{z=z_I} \frac{\hat{A}_n(z)}{z} = \sum_I \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)$$



- Recursion leads to complete factorization of amplitudes!
- Basic building block A_3.

- Construction of amplitudes by A_3.
- Boundary terms vanish in Yang Mills and gravity.

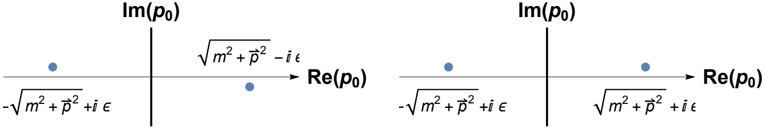
  Feng, Wang, Wang, Zhang, JHEP 1001 (2010)
- A_4 follows automatically. In particular 4-gluon vertex obsolete.
- Amplitudes fixed by locality, unitarity and Poincaré invariance.
- Parke-Taylor formula can be proven for any n.
- Drawback: limitation to tree amplitudes.

Feynman-tree theorem

- Consider Feynman and Advanced propagators:

$$G_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}, \qquad G_A(p) = \frac{i}{p^2 - m^2 - i\epsilon \, \mathrm{sign}(p_0)}$$

Poles



- With identity $\frac{1}{x \pm i\epsilon} = P.V. \left(\frac{1}{x} \right) \mp i\pi\delta(x)$

  we get

$$G_A(p) = G_F(p) - 2\pi\; \delta^{(+)}(p^2 - m^2)$$

  with $\delta^{(+)}(p^2 - m^2) = \theta(p_0)\delta(p^2 - m^2)$

- Consider a loop, replace $G_F(p)$ by $G_A(p)$.

- In loop momentum integration all poles of zero component lie above real axis.

- Closing integration contour below, Feynman-tree theorem:

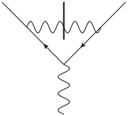
$$
\begin{aligned}
0 &= \int \frac{d^4q}{(2\pi)^4} N(q) \ \prod_i G_A^{(i)} \\
&= \int \frac{d^4q}{(2\pi)^4} N(q) \prod_i \left\{ G_F^{(i)} - 2\pi\delta^{(+)}((q - p_1 - \ldots - p_i)^2 - m^2) \right\}.
\end{aligned}
$$

- Expanding product - loop diagram expressed as cut diagrams.
- Recursive application – opening the loops

Example: $g-2$ electron

- $2^3 - 1$ cut diagrams.

1



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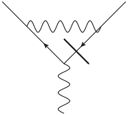

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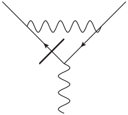

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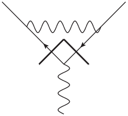

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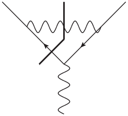

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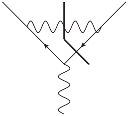

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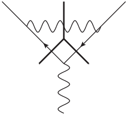
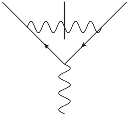


- Let us look at an explicit calculation, cut diagram 1

  1



- We have a cut of the photon line

- Singular diagram in the *forward limit*

- However FTT and BCFW are valid in D dimensions!

- Explicit Calculation:

$$I_1 = -\mu^{4-D} \int \frac{d^D q_1}{(2\pi)^D}\, 2\pi\delta^{(+)}(q_1^2)\, \frac{1}{(q_2^2 + i\epsilon)(q_3^2 + i\epsilon)}.$$

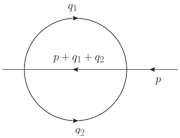
- Parametrization of on-shell loop momentum

$$q_1 = \frac{\sqrt{s_{12}}}{2}\xi \left(1, 2\sqrt{v(1-v)}\boldsymbol{e}_{\mathrm{T}}, 1-2v\right)^{\mathrm{T}}, \quad \xi \in [0,\infty[,\ v \in [0,1]$$

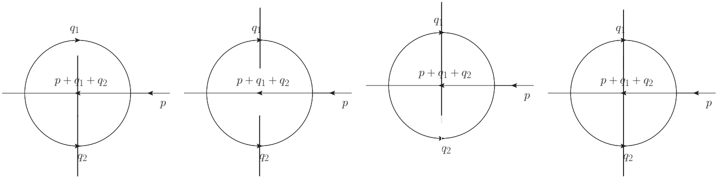
$$I_1 = \Omega_{D-2}\, \frac{\mu^{4-D}}{4(2\pi)^{D-1}} s_{12}^{D/2-3} \int d\xi\, \xi^{D-5} dv \frac{(v(1-v))^{(D-4)/2}}{v(1-v)} = 0.$$

- Integration over ξ is *scaleless* and vanishes.
- Only diagrams 2 and 3 contribute and we get the correct result.
  MM, Int.Jor.Mod.Phys.A '18

Sun rise two-loop diagram



- Recursive application of FTT, double and triple cuts



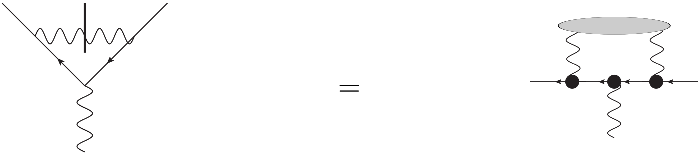
- Some tedious integrations, but the correct result.

$$I_S = -i\frac{\Gamma(2\epsilon-1)\Gamma(1-\epsilon)^3}{(4\pi)^D\Gamma(3-3\epsilon)}(-p^2-i\epsilon)^{1-2\epsilon} \quad (1)$$

Scattering Amplitude construction

- Reverse FTT followed by BCFW:
- Glue together A_3 on-shell subamplitudes
- Plug in propagator factors $1/P^2$ in between.
- Integrate over pairs of unobservable partices.

- Example eleectron–photon vertex:
  Cut diagram 1:



- No loops due to FTT.
- Factorization, that is, unitarity is explicit
- Every particle is on-shell.

Conclusions

- Goal: constuct amplitudes from first principles like, locality, unitarity, and Poincaré invariance.
- BCFW-recursion relations factorize tree amplitudes.
- Applicable to Yang Mills and gravity.
- Opening loops recursively with the FTT.
- No off-shell particles, no polarization ”tensors”.
- Construct scattering amplitudes by glueing together on-shell 3-point amplitudes which are fixed from first principles.

Our group in Chillán - Centro de Ciencias Exactas

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