Five-loop massive tadpoles

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based on recent work with
Thomas Luthe, Andreas Maier, Peter Marquard

and earlier work with
J. Möller, C. Studerus

Cosmology+Particles, UBB Chillán, June 2019

Motivation

- LHC: era of precision QFT
  - ▷ large SM backgrounds
  - ▷ dominant effects: mainly QCD
  - ▷ high precision required for BSM searches

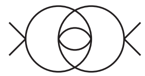
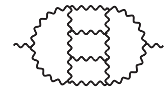
- theory working horse: renormalizable QFTs
  - ▷ perturbative expansions $\Rightarrow$ Feynman diagrams
  - ▷ higher-loop effects important (e.g. $g_\mu - 2$; m_q; H production; ...)

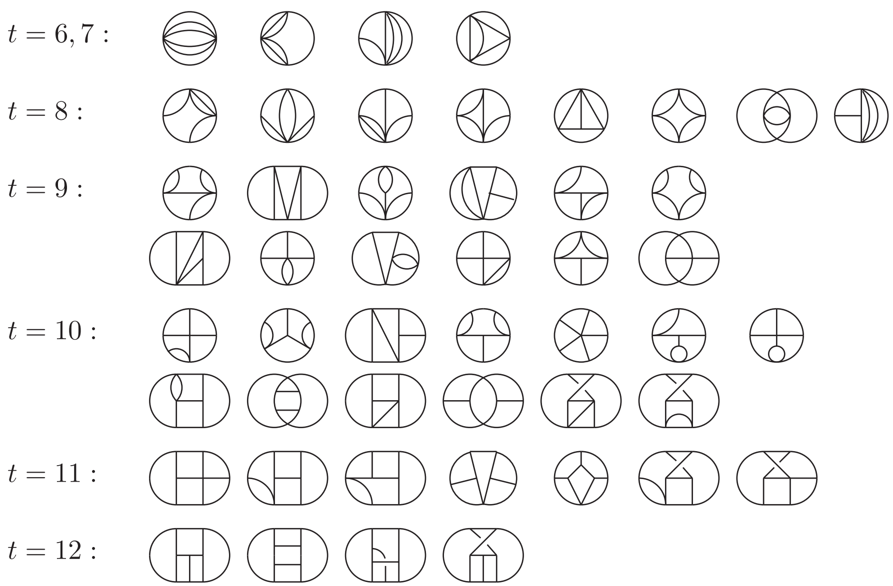
- lots of machinery developed recently
  - ▷ high automatization of complicated perturbative calculations
  - ▷ algebraic handling of Feynman diagrams
  - ▷ reduction of Feynman integrals to masters
  - ▷ numerical and/or algebraic determination of masters

Strategy

- for higher-loop precision, need to regularize and renormalize theory
  - ▷ work in dimensional reg. $d^4x \to d^dx$ and in $\overline{\text{MS}}$ scheme
  - ▷ evaluate all independent RCs: absorb *divergences*
  - ▷ e.g. QCD fields: $\psi_b = \sqrt{Z_2}\psi_r$, $A_b = \sqrt{Z_3}A_r$, $c_b = \sqrt{Z_3^c}c_r$
  - ▷ e.g. QCD couplings: $m_b = Z_m m_r$, $g_b = \mu^\epsilon Z_g g_r$
  - ▷ equivalently, def anomalous dimensions: $\gamma_i = -\partial_{\ln\mu^2} \ln Z_i$
- keep gauge group general
- generate all Fy diagrams; perform algebra: group-, Lorentz-, ... [Nogueira QGRAF; Vermaseren FORM]
- project all Fy integrals to unique set of scalar massive vacuum ints
  - ▷ exact decomposition of propagators ($m \in \{0, m, M\}$) [Chetyrkin/Misiak/Münz 1998]
  - ▷ $\frac{1}{(k-p)^2+m^2} = \frac{1}{k^2+M^2} + \frac{2kp-p^2+M^2-m^2}{(k^2+M^2)((k-p)^2+m^2)}$
  - ▷ recursively lower degree of UV div
  - ▷ other IR regularization schemes: e.g. (local/global) R* [Chetyrkin 1984]
- map all integrals to minimal set: IBP [Chetyrkin/Tkachov 1981; Laporta 2000]
- evaluate this minimal set (analytically/numerically to high accuracy) [with Luthe since 2011]

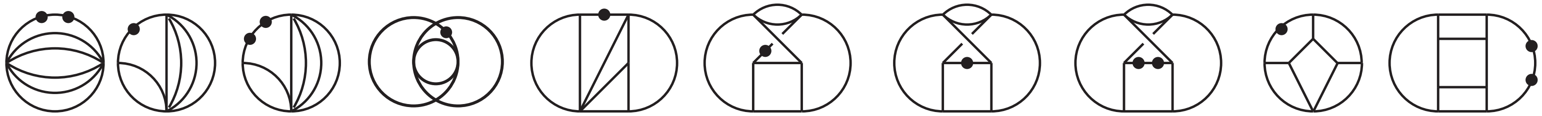
Reduction

- motivation: 5-loop Φ^4:  1 integral; YM:  1T integrals $(2^{14}6^{10})$
- complexity reduction via IBP (integration by parts, in d dim) [Chetyrkin/Tkachov 1981]
  - ▷ systematically use $0 = \int d^dk\, \partial_{k_\mu} f_\mu(k)$ [Laporta, Baikov, Gröbner]
  - ▷ key idea: lexicographic ordering among all loop integrals [Laporta 2000]
- arrive at rep in terms of irreducible ($\equiv$ master) integrals: $\sum_i \frac{\text{poly}_i(d,\xi)}{\text{poly}_i(d)}\, \text{Master}_i(d)$
  - ▷ e.g. 1234-loop: 1+1+3+(10+3)= 18 (fully massive) master ints [Laporta 2002]
- now consider fully massive 5-loop tadpoles
  - ▷ Euclidean space-time
  - ▷ same mass in all propagators $\Rightarrow 1/(q_i^2+1)$
- arrive at 48 unique 5-loop sectors (+19 factorized ones not shown)



5-loop Masters

- a (small) IBP reduction reveals that some sectors contain multiple master integrals
  - ▷ need in addition 62 (+3 factorized ones) masters with 'dots'. some examples:



  - ▷ recall that at 1/2/3/4-loop there were 0/0/0/3 masters with 'dots'

- how to evaluate these 48+62 (+19+3) zero-scale master integrals as fct(d)?
  various methods, e.g.
  - ▷ explicit integration in x-space
  - ▷ differential eqs (in mass ratio); solve iteratively with HPLs
  - ▷ solve dimensional recurrences, regarding $d \in \mathbb{C}$ [Lee 2009]
  - ▷ explicit solution of low-order difference equations: ${}_PF_{P-1}$ etc.
  - ▷ numerical solution of difference equations via factorial series [Laporta 2000]

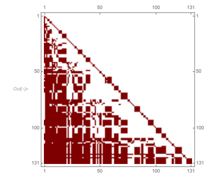
- Mathematical structure
  - ▷ interested in the coefficients of an ϵ expansion
  - ▷ e.g. harmonic sums $S(N)$ [Vermaseren 1998; Blümlein/Kurth 1998]
  - ▷ e.g. harmonic polylogarithms HPL(x) [Remiddi/Vermaseren 2000]
  - ▷ e.g. elliptic multiple zetas eMZV [Levin/Brown 2007]
  - ▷ if solution numerical: use some PSLQ

Generalities

- parametric representation
  - ▷ write LC of propags as $\sum x_i D_i = k^T M k + 2P \cdot k + Q$
  - ▷ get Symanzik polys as $\mathcal{U} = \det M$ and $\mathcal{F} = (\det M)(P^T M^{-1} P + Q)$
- simplification for fully massive integrals without external legs
  - ▷ $P = 0$ and $Q = \sum x_i \quad \Rightarrow \quad \mathcal{F} = \mathcal{U}Q = \mathcal{U}$
- L-loop massive tadpole in d dimensions, having N lines with powers $a_1, \ldots, a_N$

$$\frac{I_a}{J^L} = \frac{\Gamma(A - Ld/2)}{[\Gamma(1-d/2)]^L} \int_0^\infty d^N x \; \delta(1-X) \, \frac{p_a(x)}{[\mathcal{U}(x)]^{d/2}}$$

  - ▷ normalization by $J = \int d^d k/(k^2+1) = \pi^{d/2}\,\Gamma(1-d/2)$
  - ▷ numerators $p_a(x) = \prod_{i=1}^N x_i^{a_i - 1}/\Gamma(a_i)$
- dimensional shifts via $I_a^{(d)} = \mathcal{U}(x_i \to \partial_{m_i^2}) \, I_a^{(d+2)}$ [Tarasov 1996]

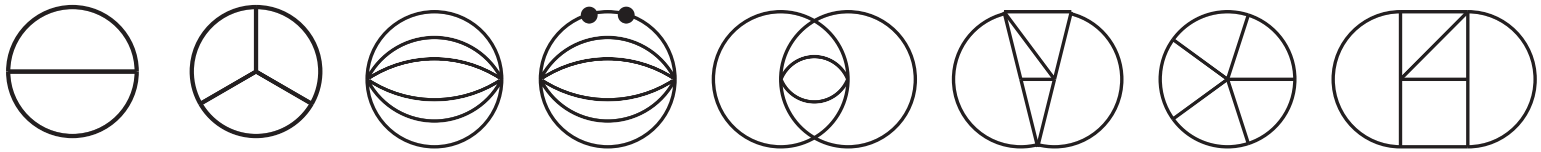


  - ▷ can be exploited to solve for (some) masters
  - ▷ at 1 loop: large-d asymptotics to fix fcts [Tarasov 2000]
  - ▷ $>$ 1 loop: solve up to periodic fct; compute special points in fixed dims [Lee 2009 ff]

Evaluation

- perform IBP reduction with symbolic power x on one line
- derive difference equation for generalized master $I(x) \equiv \int \frac{1}{D_1^x D_2 \ldots D_N}$
  - ▷ generic form: $\sum_{j=0}^{R} p_j(x) I(x+j) = F(x)$
- typically, want $I(1)$; solve the difference equation
  - ▷ explicitly (e.g. if 1st order; or if soln nested sum)
  - ▷ numerically (very general setup) [Laporta 2000]
- solve via factorial series $I(x) = I_0(x) + \sum_{j=1}^{R} I_j(x)$,
  - ▷ where $I_j(x) = \mu_j^x \sum_{s=0}^{\infty} a_j(s) \frac{\Gamma(x+1)}{\Gamma(x+1+s-K_j)}$
- need boundary condition for fixing, say, $a_j(0)$: use decoupling at large x
  - ▷ $I(x) = \int_{k_1} g(k_1)/(k_1^2+1)^x \;\Rightarrow\; I(x) \sim (1)^x x^{-d/2} g(0)$
- deep expansions (ϵ^{20}) at 5 loops (132 masters) at high precision (>250 digits) [with Luthe 2016]

Sample results (4d)



$$I_{28686.1.1} = +(-3)\,\epsilon^0 + (-\frac{3}{2})\,\epsilon^1 + (\frac{13}{24})\,\epsilon^2 + (-\frac{1267}{1440})\,\epsilon^3 + (-\frac{4193}{3456})\,\epsilon^4 + $$
$$+135.95072868792871461956492733702218574897992953584\,\epsilon^5 + \ldots$$

$$I_{28686.1.3} = +(0)\,\epsilon^0 + (\frac{3}{2})\,\epsilon^1 + (-\frac{1}{2})\,\epsilon^2 + (-\frac{443}{360})\,\epsilon^3 + (\frac{95}{216})\,\epsilon^4 + $$
$$-38.292059175062436961881799538284449799148385376441\,\epsilon^5 + \ldots$$

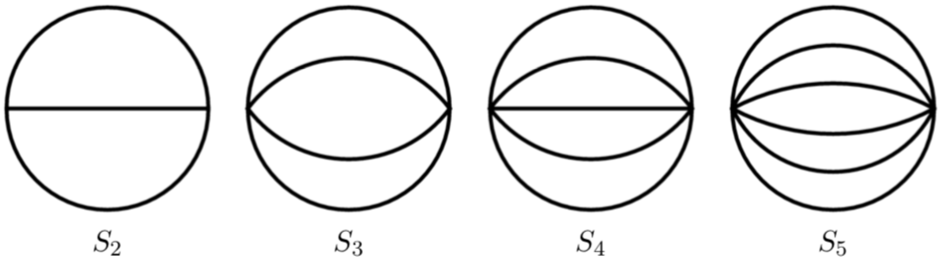
$$I_{30862.1.1} = +(-\frac{3}{5})\,\epsilon^0 + (-\frac{27}{10})\,\epsilon^1 + (-\frac{4\zeta_3}{5} - \frac{421}{60})\,\epsilon^2 + (-\frac{12\zeta_2^2}{25} + \frac{24\zeta_3}{5} + \frac{211}{24})\,\epsilon^3 + (\frac{72\zeta_2^2}{25} - 98\zeta_3 + \frac{32\zeta_5}{5} + \frac{12959}{48})\,\epsilon^4 +$$
$$+1143.1838307558764599466030303839590323268318605888\,\epsilon^5 + \ldots$$

$$I_{30231.1.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (\frac{3\zeta_3}{5})\,\epsilon^2 + (\frac{9\zeta_2^2}{25} + \frac{21\zeta_3}{5} + 3\zeta_5)\,\epsilon^3 + (-36H_2\zeta_3 + \frac{12\zeta_2^3}{7} + \frac{63\zeta_2^2}{25} - \frac{21\zeta_3^2}{5} + 27\zeta_3 - \frac{24\zeta_5}{5})\,\epsilon^4$$
$$-531.32391547725635267943444561495368318398901378435\,\epsilon^5 + \ldots$$

$$I_{32596.1.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (0)\,\epsilon^2 + (0)\,\epsilon^3 + (-14\zeta_7)\,\epsilon^4 + $$ [Wheel: Broadhurst 1985]
$$+235.07729596783467131454388080950411779239347239580\,\epsilon^5 + \ldots$$

$$I_{32279.3.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (0)\,\epsilon^2 + (0)\,\epsilon^3 + (-\frac{441\zeta_7}{40})\,\epsilon^4 + $$ [Zigzag: Broadhurst/Kreimer 1995; Brown/Schnetz 2012]
$$+181.78223928612340820790788236018642961198741994209\,\epsilon^5 + \ldots$$

Nice subsets: sunsets



- much-studied class; $S_{L>3}$ should be elliptic [Laporta 2002]
- at 2 loops: hypergeom soln (for all d) [Davydychev/Tausk 1993]
- at 2 loops: expansions contain HPL$(e^{2\pi i/6})$, HPL$(e^{2\pi i/3})$ [see Broadhurst 1998]

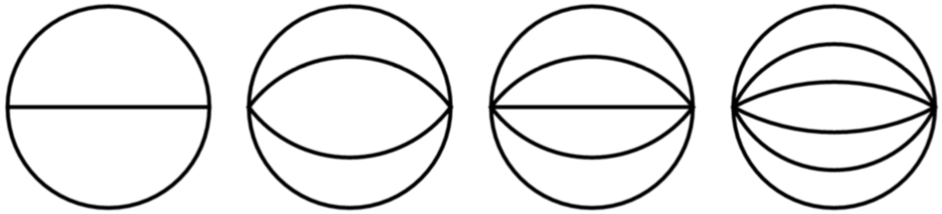
$$S_2/J^2 \stackrel{2-2\epsilon}{=} 6H_2\,\epsilon^2 - 12H_3\,\epsilon^3 + 24H_4\,\epsilon^4 - 48H_5\,\epsilon^5 + 96H_6\,\epsilon^6 + \ldots$$

$$S_2/J^2 \stackrel{1-2\epsilon}{=} \tfrac{3}{4}\Big[\tfrac{4}{9} - G_1\,\epsilon + G_2\,\epsilon^2 - G_3\,\epsilon^3 + G_4\,\epsilon^4 - G_5\,\epsilon^5 + \ldots\Big]$$

- ▷ $H_n = h_n + h_1\hat{C}_{n-1}\Big(1 - \frac{3^{\epsilon/2}\Gamma(1-\epsilon)}{\Gamma^2(1-\epsilon/2)}\Big)$ where $h_n = {}_{n+1}F_n\Big(\frac{1}{2},\ldots,\frac{1}{2};\frac{3}{2},\ldots,\frac{3}{2};\frac{3}{4}\Big)$
- ▷ e.g. $3H_1 = 3h_1 = 2\pi/\sqrt{3}$, $3H_2 = \sqrt{3}\,\mathrm{Cl}_2(2\pi/3)$ with $\mathrm{Cl}_n(x) = \mathrm{Im}\,\mathrm{Li}_n(e^{ix})$
- ▷ $G_n = g_n + \frac{16}{9}\hat{C}_n\Big(\frac{\pi 3^{\epsilon}\Gamma(1-2\epsilon)}{\Gamma^2(1/2-\epsilon)} - 1\Big)$ where $g_n = {}_{n+2}F_{n+1}\Big(\frac{3}{2},1,\ldots,1;2,\ldots,2;\frac{3}{4}\Big)$
- ▷ e.g. $9G_1 = -32\ln\frac{3}{2}$, $9G_2 = 16(3\mathrm{Li}_2(\frac{1}{4}) - \zeta_2 + 3\ln^2 2 - \ln^2\frac{3}{2})$

- at 2 loops: dim-shift rel $\hat{S}_2^{(d+2)} = \frac{3d}{4(d-1)}[\hat{S}_2^{(d)} - 1]$

Nice subsets: sunsets



- at 3 loops: expansions contain HPL(1)

$$S_3/J^3 \overset{2-2\epsilon}{=} 7\zeta_3\epsilon^3 + [48A_4 - 51\zeta_4]\epsilon^4 + [288A_5 + 306\zeta_4 \ln 2 - 465\zeta_5/2]\epsilon^5$$
$$+[1728A_6 + 720s_6 - 918\zeta_4 \ln^2 2 - 1134\zeta_6 - 305\zeta_3^2]\epsilon^6 + \ldots$$

$$S_3/J^3 \overset{1-2\epsilon}{=} \tfrac{1}{4} + \tfrac{3}{2}[\ln 2]\epsilon + \tfrac{3}{2}[4\ln 2 - \ln^2 2 - \zeta_2]\epsilon^2 + \ldots$$

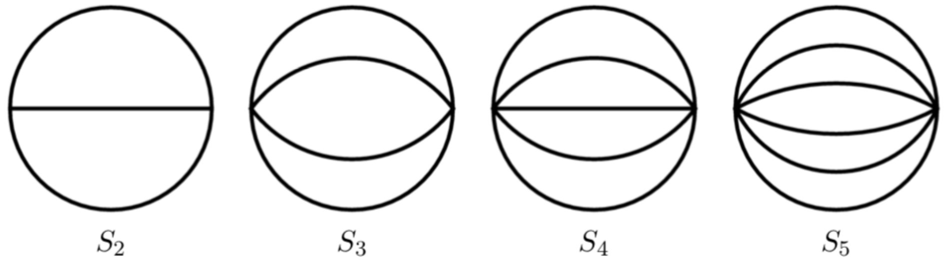
▷ where $A_n = \mathrm{Li}_n(\frac{1}{2}) + \mathrm{LC}(\zeta_2, \ln 2)$ and $s_6 = S_{-5,-1}(\infty) \approx 0.98744\ldots$

- at 3 loops: dim-shift rel $\hat{S}_3^{(d+2)} = \frac{2d^2}{3(d-1)(3d-4)(3d-2)}[8(d-2)\hat{S}_3^{(d)} - (11d-16)]$

- at 4 loops: numerical expansion

$$S_4/J^4 \overset{4-2\epsilon}{=} -\frac{5}{2} - \frac{5}{3}\epsilon - \frac{5}{144}\epsilon^2 - \frac{625}{864}\epsilon^3$$
$$-16.14646921346643369526363642598358965658209440 9714\,\epsilon^4$$
$$-302.68724953271679694373481692172522196605049592725\,\epsilon^5 + \ldots$$

Nice subsets: sunsets



- at 5 loops: numerical expansion

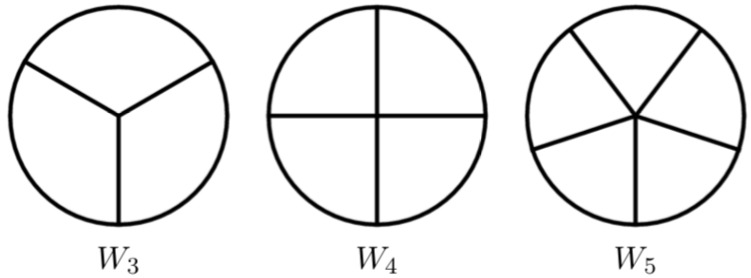
$$
\begin{aligned}
S_5/J^5 \quad \overset{4-2\epsilon}{=} \quad & -3 - \frac{3}{2}\,\epsilon + \frac{13}{24}\,\epsilon^2 - \frac{1267}{1440}\,\epsilon^3 - \frac{4193}{3456}\,\epsilon^4 \\
& +135.950728687928714619564927337022185748979929535 84\,\epsilon^5 \\
& +2603.8178589306305816799512636574088079352965131038\,\epsilon^6 + \ldots
\end{aligned}
$$

- interesting structure arises in d=2: leading term is a Bessel moment

$$\tilde{S}_L = \frac{S_L^{(2d)}/(\epsilon J)^L}{\Gamma(L+2)} = \frac{2^L}{\Gamma(L+2)} \int_0^\infty \mathrm{d}x\, x\, [K_0(x)]^{L+1}$$

  - ▷ known evaluations are $\tilde{S}_1 = 1/2\ ,\quad \tilde{S}_2 = H_2\ ,\quad \tilde{S}_3 = 7\zeta_3/24$
  - ▷ and $\tilde{S}_\infty = \exp(-2\gamma_E)$

Nice subsets: wheels



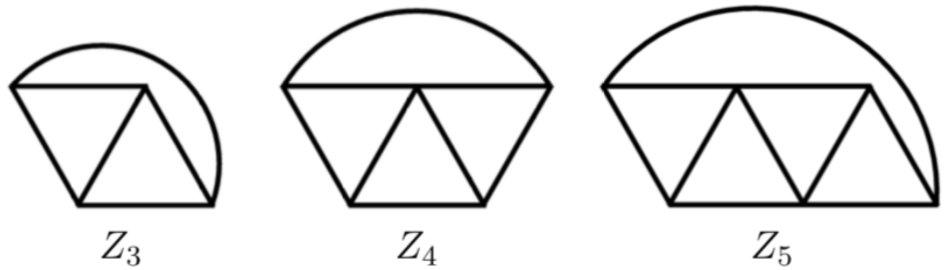
- leading terms known (at d=4) to all loop orders [Broadhurst 1985]

$$\frac{W_L}{J^L} = \frac{(-1)^L\,\Gamma(2L-1)}{\Gamma(L)\,\Gamma(L+1)}\,\zeta(2L-3)\,\epsilon^{L-1} + \mathcal{O}(\epsilon^L)$$

- can use this as check on numerics; for W_3 and W_4 OK to tens of thousands digits
- at 5 loops $W_5/J^5 = -14\zeta_7\epsilon^4 + \mathcal{O}(\epsilon^6)$ to 250 digits (plus many more orders numerically)

$$\begin{aligned} W_5/J^5 \overset{4-2\epsilon}{=} \quad & -14.11688988334691957575716569789715463439808 9847913\,\epsilon^4 \\ & +235.07729596783467131454388080950411779239347239580\,\epsilon^5 \\ & -2267.7386832930084122962994480580205855487545413738\,\epsilon^6 + \ldots \end{aligned}$$

Nice subsets: zigzags



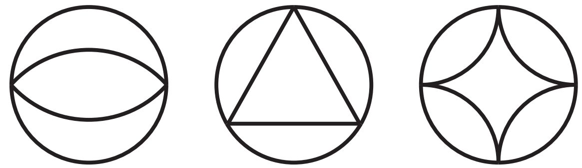
- note $Z_3 = W_3$ and $Z_4 = W_4$
- leading terms known (at d=4) to all loop orders [Kazakov 1984; Broadhurst 1995; Brown/Schnetz 2012]

$$\frac{Z_L}{J^L} = \frac{W_L}{J^L}\,\frac{4}{L}\left(1 + \frac{(-1)^L - 1}{2^{2L-3}}\right) + \mathcal{O}(\epsilon^L)$$

- can use this as check on numerics
- at 5 loops $Z_5/J^5 = -\frac{441}{40}\zeta_7\epsilon^4$ to 250 digits (plus many more orders numerically)

$$\begin{aligned} Z_5/J^5 \stackrel{4-2\epsilon}{=} \; & -11.117050783135699165908767987094009274588495755231\,\epsilon^4 \\ & +181.78223928612340820790788236018642961198741994209\,\epsilon^5 \\ & -1725.9996137403520805951673992442117288607704957542\,\epsilon^6 + \ldots \end{aligned}$$

Nice subsets: strings of bubbles



P_3 P_4 P_5

- $P_3(=S_3)$ and P_4 contain HPL(1) only, to all orders in ϵ (proof via differential eq in M/m)

$$P_4/J^4 \overset{2-2\epsilon}{=} 3\zeta_3\epsilon^4 + [48A_4 + 30\zeta_3 - 57\zeta_4]\epsilon^5 + [96A_4 + 288A_5 + 306\zeta_4 \ln 2 - 276\zeta_3 - 78\zeta_4 + 39\zeta_5/2]\epsilon^6 + \ldots$$

- PSLQ-fitted numerical result for P_5 contains HPL(1) only

$$P_5/J^5 \overset{4-2\epsilon}{=} -\frac{6}{5} - \frac{23}{5}\epsilon - \frac{91}{10}\epsilon^2 + \left[\frac{28\zeta_3}{5} + \frac{697}{20}\right]\epsilon^3 + \left[\frac{84\zeta_2^2}{25} - \frac{1066\zeta_3}{5} + \frac{22189}{40}\right]\epsilon^4$$
$$+\left[-1536A_4 + \frac{16482\zeta_2^2}{25} - \frac{12001\zeta_3}{5} - \frac{7716\zeta_5}{5} + \frac{354017}{80}\right]\epsilon^5 + \ldots$$

- dim-shift relations are closed within this set:

$$\hat{P}_3^{(d+2)} = \frac{2d^2}{3(d-1)(3d-4)(3d-2)}\left[8(d-2)\hat{P}_3^{(d)} - (11d-16)\right]$$

$$\hat{P}_4^{(d+2)} = \frac{d^3}{32(d-1)^3(2d-3)}\left[\frac{6(d-3)(3d-7)(3d-5)}{(d-2)(2d-5)}\hat{P}_4^{(d)} - \frac{295d^3-1614d^2+2855d-1620}{(2d-5)(3d-4)}\hat{P}_3^{(d)} + \frac{2(19d^2-53d+36)}{(3d-4)}\right]$$

$$\hat{P}_5^{(d+2)} = r_5(d)\hat{P}_5^{(d)} + r_4(d)\hat{P}_4^{(d)} + r_3(d)\hat{P}_3^{(d)} + r_1(d) \quad , \quad \text{etc.}$$

- general L-loop ϵ-expansions possible?

d=4: Quantum Chromodynamics

- drive renormalization program of QCD to five loops [with Luthe/Maier/Marquard]
- long-term effort of three independent groups [also: Chetyrkin/Baikov; Ueda/Vermaseren/Vogt]
- notation: strong coupling constant $a = \frac{C_A\, g^2(\mu)}{16\pi^2}$
  - ▷ color: $n_f \equiv \frac{T_F\, N_f}{C_A}$, $c_f \equiv \frac{C_F}{C_A}$, $d_{FF} \equiv \frac{[sTr(T^a T^b T^c T^d)]^2}{N_A\, T_F^2\, C_A^2}$, $\dots$
- result: 5-loop QCD β-function [with Luthe/Maier/Marquard]
  - ▷ $\beta = \partial_{\ln \mu^2}\, a = -a\left[\epsilon + \frac{11-4\,n_f}{3}\, a + b_1\, a^2 + b_2\, a^3 + b_3\, a^4 + b_4\, a^5 + \dots\right]$
  - ▷ $3^5\, b_4 = n_f^4\left[c_1\, c_f + c_2\right]$ [Gracey 1996]
    $+\, n_f^3\left[c_3\, c_f^2 + c_4\, c_f + c_5\, d_{FF} + c_6\right]$ [LMMS 2016]
    $+\, n_f^2\left[\dots\right] + n_f\left[\dots\right] + \left[c_{22}\, d_{AA} + c_{23}\right]$ [Herzog/Ruijl/Ueda/Vermaseren/Vogt 2017]
  - ▷ the n_f^4 term agrees exactly with known result
  - ▷ all c_i in terms of Zeta values,
    e.g. $c_6 = -3(6231 + 9736\zeta_3 - 3024\zeta_4 - 2880\zeta_5)$
  - ▷ last row confirmed [LMMS 2017]

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- drive renormalization program of QCD to five loops [with Luthe/Maier/Marquard]
- notation: strong coupling constant $a = \frac{C_A\, g^2(\mu)}{16\pi^2}$
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- result: 5-loop QCD quark mass anomalous dimension [LMMS 2016]
  - ▷ $\gamma_m = \partial_{\ln \mu^2} \ln m_q = -a c_f \Big[3 + \gamma_1\, a + \gamma_2\, a^2 + \gamma_3\, a^3 + \gamma_4\, a^4 + \ldots \Big]$
  - ▷ as usual, $\gamma_1 = (9c_f + 97)/6 - 10/3\, n_f$, ... ,
  - ▷ and now $\gamma_4 = n_f^4 \Big[c_1 \Big] + n_f^3 \Big[c_2\, c_f + c_3 \Big] + n_f^2 \Big[c_4\, c_f^2 + c_5\, c_f + c_6\, d_{FF} + c_7 \Big] + \ldots$
  - ▷ know all c_i analytically (Zetas only), for all n_f and color structures
  - ▷ the n_f^4 and n_f^3 terms agree exactly with known results [Gracey 1996]
- completion of 5-loop renormalization program
  - ▷ besides β and γ_m, have also $Z_{\psi\psi}$, Z_{cc} and Z_{ccg} (Fy gauge + ξ^1) [LMMS 2017]
  - ▷ all other RCs follow from these five, due to gauge invariance
  - ▷ full gauge dependence now also available [Chetyrkin/Falcioni/Herzog/Vermaseren 2017]

Conclusions

- recent advance in methods allows high-loop renormalization of quantum field theories
  - ▷ highly automated computer-algebra approaches
  - ▷ improved IBP algorithms (finite fields, . . .)
  - ▷ new insight into functional content of Feynman integrals

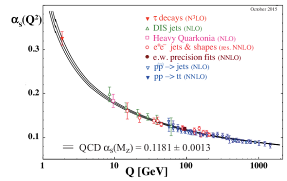
- difference equations (+ lots of CPU) are powerful enough to achieve 5 loops
  - ▷ non-trivial algorithmic fine-tuning
  - ▷ more? memory seems to become an issue

- applications to a host of QFTs in various fields
  - ▷ CM theory / effective models
  - ▷ mathematical physics/ SUSY
  - ▷ phenomenology / QCD+SM
  - ▷ . . .

- applications in various space-time dimensions
  - ▷ 2d, 3d, 4d, . . . , fractional, . . .

Apparent convergence

- one might be concerned about 'asymptotic series' behavior at 5 loops
  - ▷ e.g. QM: 1d anharmonic oscillator $\hat{H} \sim \hat{p}^2 + \hat{x}^2 + g\hat{x}^4$
  - ▷ ground state energy $E_0 = \sum e_n\, g^n$, with $|\frac{e_{n+1}}{e_n}| \overset{n\gg 1}{\sim} \frac{3n}{4}$, diverges [Bender/Wu 1973]
  - ▷ (however defines unique function $E_0(g)$ via Borel; compare $\frac{1}{1-x} = \sum x^n$)

- now QCD. $N_c = 3$, $\alpha_s = \frac{g^2(Q)}{4\pi} =$  ≈ 0.1 (at EW scale)

  - ▷ Beta function: [see also Baikov/Chetyrkin/Kühn 2016]

$$\beta_{N_f=3} = -0.17\,\alpha_s^2\,[1 + 0.57\,\alpha_s + 0.45\,\alpha_s^2 + 0.68\,\alpha_s^3 + 0.58\,\alpha_s^4 + \ldots]$$
$$\beta_{N_f=4} = -0.16\,\alpha_s^2\,[1 + 0.49\,\alpha_s + 0.31\,\alpha_s^2 + 0.49\,\alpha_s^3 + 0.28\,\alpha_s^4 + \ldots]$$
$$\beta_{N_f=5} = -0.15\,\alpha_s^2\,[1 + 0.40\,\alpha_s + 0.15\,\alpha_s^2 + 0.32\,\alpha_s^3 + 0.08\,\alpha_s^4 + \ldots]$$
$$\beta_{N_f=6} = -0.13\,\alpha_s^2\,[1 + 0.30\,\alpha_s - 0.03\,\alpha_s^2 + 0.18\,\alpha_s^3 + 0.002\,\alpha_s^4 + \ldots]$$

  - ▷ Quark mass anomalous dimension: [see also Baikov/Chetyrkin/Kühn 2014]

$$\gamma_{N_f=3} = -0.32\,\alpha_s\,[1 + 1.21\,\alpha_s + 1.26\,\alpha_s^2 + 1.43\,\alpha_s^3 + 2.04\,\alpha_s^4 + \ldots]$$
$$\gamma_{N_f=4} = -0.32\,\alpha_s\,[1 + 1.16\,\alpha_s + 1.01\,\alpha_s^2 + 0.88\,\alpha_s^3 + 1.15\,\alpha_s^4 + \ldots]$$
$$\gamma_{N_f=5} = -0.32\,\alpha_s\,[1 + 1.12\,\alpha_s + 0.75\,\alpha_s^2 + 0.36\,\alpha_s^3 + 0.43\,\alpha_s^4 + \ldots]$$
$$\gamma_{N_f=6} = -0.32\,\alpha_s\,[1 + 1.07\,\alpha_s + 0.49\,\alpha_s^2 - 0.15\,\alpha_s^3 - 0.10\,\alpha_s^4 + \ldots]$$

Fractional dimensions

- difference eqs carry full information on d
  - ▷ expand massive tadpoles e.g. around $d = \frac{10}{3} - 2\epsilon$?
- motivation: renormalization in critical dimension (where [g]=0)
  - ▷ e.g. O(N) scalar ϕ^n theory: $\partial_\mu\phi\,\partial^\mu\phi + g\phi^n$ [Gracey 2017]
  - ▷ critical dim $D_n = \frac{2n}{n-2} \Rightarrow D_{\{3,4,5,6,7,\ldots,\infty\}} = \{6, 4, \frac{10}{3}, 3, \frac{14}{5}, \ldots, 2\}$
  - ▷ near FP (nontriv zero of Beta fct), RG fcts carry info on phase transitions
- study RG fcts in non-integer dimensions
  - ▷ renormalization 'as usual', dim. reg. natural (angular ints?!)
  - ▷ fewer diagrams (e.g. ϕ^5: LO 2pt diag has 3 loops)
  - ▷ numbers: $\frac{p}{q} \rightarrow \Gamma(\frac{p}{q})$ at LO; $\zeta(s) \rightarrow$ Dirichlet β fct at NLO ($\in$HPL(i)) [Hager 2002]
- sample results in $d = \frac{10}{3} - 2\epsilon$ [Luthe]
  - ▷ as usual we normalize by $1/J^{loop}$; here, $J \sim \Gamma(1-\frac{d}{2}) = -\frac{3}{2}\Gamma(\frac{1}{3}) + \ldots$
  - ▷ 2-loop sunset $= -2.4882241714632542542\,\epsilon^0 - 8.6116306893649818141\,\epsilon^1 + \ldots$
  - ▷ 3-loop merc. $= -0.0305966721641989027\,\epsilon^0 + 0.0149523122719722312\,\epsilon^1 + \ldots$
  - ▷ 4-loop non-pl $= +0.0006880032418228675\,\epsilon^0 + 0.0023853718027957011\,\epsilon^1 + \ldots$
  - ▷ etc.