Analysis of a Real Time Approach to Quantum Tunneling

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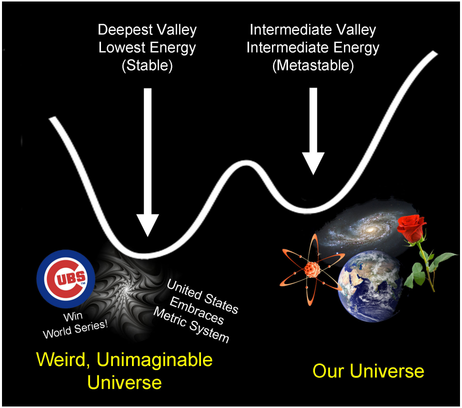
May 19, 2021

Mark P. Hertzberg, Fabrizio Rompineve, and Neil Shah. "Quantitative Analysis of the Stochastic Approach to Quantum Tunneling". In: *Phys. Rev. D* 102.7 (2020), p. 076003. DOI: 10.1103/PhysRevD.102.076003. arXiv: 2009.00017 [hep-th]

Overview

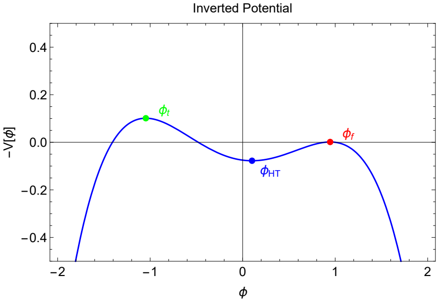
Tunneling

- Quantum tunneling central to physics
  - Higgs meta-stable, turnover at $E \sim \mathcal{O}(10^{11})$ GeV
  - String theory, exponentially many meta-stable vacua
  - Diodes, nuclear fusion
- In single particle QM tunneling from exact solution from Schrodinger Eq.
- In QFT, exact solution is difficult $\Rightarrow$ approximation techniques



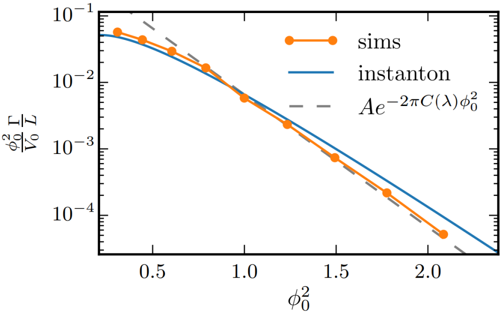
Instanton

- Standard approximation technique: Coleman Instanton [2]
- Involves classical solution in Euclidean spacetime ($t \to i\tau$)
- Instanton requires an $O(4)$ symmetry, broken in key regimes e.g. inflation
- If we can find real time tunneling method, can examine tunneling in time-dependent backgrounds!



Stochastic Method

- Past work by Linde [4] has shown parametric agreement between stochastic method and instanton
- Recent work by Braden, Johnson, Peiris, Pontzen, and Weinfurtner [1] claims excellent agreement between stochastic method and instanton



Stochastic Method

- Consider a potential $V(\phi)$ with at least two minima ϕ_f, ϕ_t.
- Initialize $\phi = \phi_f + \delta\phi$, $\pi = 0 + \delta\dot{\phi}$
- Draw $\delta\phi$ and $\delta\dot{\phi}$ from the free theory:

$$\Psi_{\text{free}}(\delta\phi) \propto \exp\left[-\frac{1}{2}\int \frac{d\boldsymbol{k}}{2\pi}\omega_{\boldsymbol{k}}|\delta\phi_{\boldsymbol{k}}|^2\right] \quad (1)$$

where $\omega_k^2 = \boldsymbol{k}^2 + V''(\phi_f) = \boldsymbol{k}^2 + m_f^2$

- The 2-point correlation functions for $\delta\phi_{\boldsymbol{k}}$ and $\delta\dot{\phi}_{\boldsymbol{k}}$ are:

$$\langle\delta\phi_{\boldsymbol{k}}^*\delta\phi_{\boldsymbol{k}'}\rangle = \frac{1}{2\omega_{\boldsymbol{k}}}(2\pi)\delta(\boldsymbol{k}-\boldsymbol{k}') \qquad \langle\delta\dot{\phi}_{\boldsymbol{k}}^*\delta\dot{\phi}_{\boldsymbol{k}'}\rangle = \frac{\omega_{\boldsymbol{k}}}{2}(2\pi)\delta(\boldsymbol{k}-\boldsymbol{k}') \quad (2)$$

- Use the Wigner distribution as a joint probability distribution to sample $\delta\phi_{\boldsymbol{k}}$ and $\delta\dot{\phi}_{\boldsymbol{k}}$ *simultaneously* to set initial conditions.
- Place ϕ in box of size L with periodic boundary conditions $\Rightarrow$ discrete k-modes $k_n = 2\pi n/L$ with cutoff n_{cut}.

Stochastic Method

- Initial conditions:

$$\delta\phi(x) = \frac{1}{\sqrt{L}} \sum_{n=1}^{n_{\text{cut}}} e^{ik_n x} \phi_{k_n} + c.c. \quad \Delta\phi_{k_n} = \sqrt{\langle|\phi_{k_n}|^2\rangle} = \epsilon_\phi \frac{1}{\sqrt{2\omega_{k_n}}} \tag{3}$$

$$\delta\dot{\phi}(x) = \frac{1}{\sqrt{L}} \sum_{n=1}^{n_{\text{cut}}} e^{ik_n x} \dot{\phi}_{k_n} + c.c. \quad \Delta\dot{\phi}_{k_n} = \sqrt{\langle|\dot{\phi}_{k_n}|^2\rangle} = \epsilon_\pi \sqrt{\frac{\omega_{k_n}}{2}} \tag{4}$$

where ϵ_ϕ, ϵ_π are "fudge factors" to control the amplitude of fluctuations

Braden Method

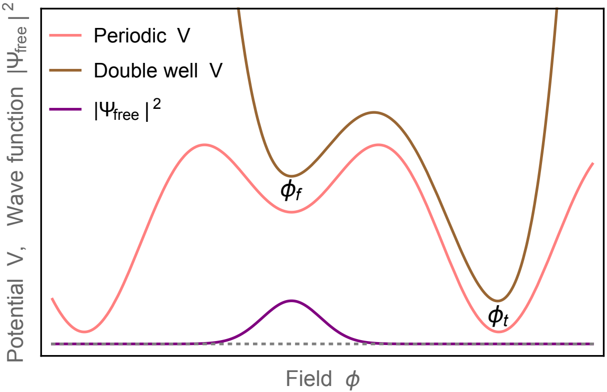
- Prepare an ensemble of classical fields $\{\phi_i\}$ with quantum initial conditions and evolve under the classical equations of motion:

$$\ddot{\phi}_i - \nabla^2\phi_i + V'(\phi_i) = 0 \tag{5}$$

- Determine tunneling rate by examining timescale over which classical fields "tunnel":
  - Let's define the following volume average for a field ϕ_i: $c_i(t)$
  - Choose some threshold $\mathcal{T}$, such that a field ϕ_i has "tunneled" at time t when $c_i(t) > \mathcal{T}$
  - Define the survival rate $F_{\text{survive}}(t) \equiv$ No. of fields that have not tunneled at time t.
  - Then using $F_{\text{survive}}(t) = e^{-\Gamma t}$, extract Γ as the tunneling rate.

Stochastic Method

- We will consider the follow potentials:
- For the periodic potential, $c_i(t) = \frac{1}{L}\int dx \cos(\phi_i(t,x)/\phi_0)$
- For the DW potential, $c_i(t) = \frac{1}{L}\int dx \phi_i(t,x)/\phi_0$



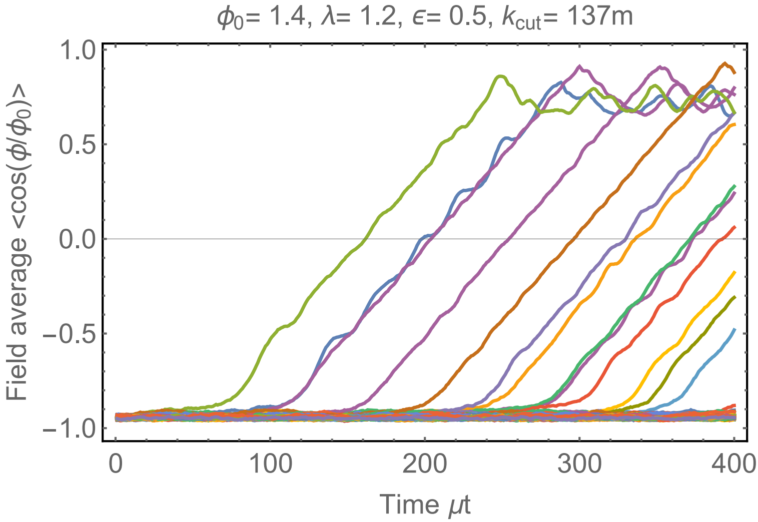
Cosine Potential

- The periodic potential has the form:

$$V(\phi) = V_0\left(\cos\left(\frac{\phi}{\phi_0}\right) + \frac{\lambda^2}{2}\sin^2\left(\frac{\phi}{\phi_0}\right)\right) \tag{6}$$

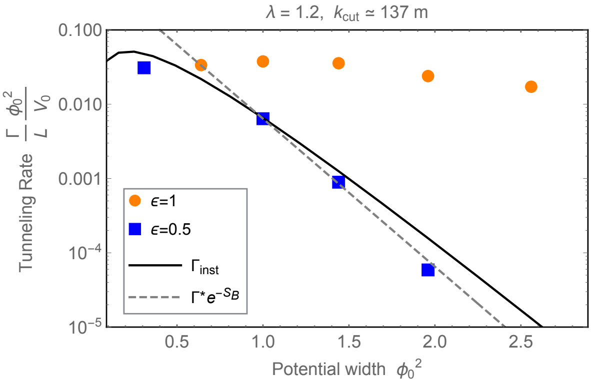
- This potential has infinite true and false vacua at $\phi_f = 2\pi m\phi_0$ and $\phi_\pi = m\pi\phi_0$ where $m \in \mathbb{N}$. Focus on two vacua $\phi_f = 0$ and $\phi_t = \pi\phi_0$
- $\cos(\phi/\phi_0)$ tracks tunneling: $\cos(\phi_f/\phi_0) = -1$, $\cos(\phi_t/\phi0) = 1$
- ϕ_0 controls potential width, λ controls potential height and mass, V_0 is normalized to $V_0 = 0.008\phi_0^2$
- Set $\epsilon_\phi = \epsilon_\pi = \epsilon$

Cosine Potential



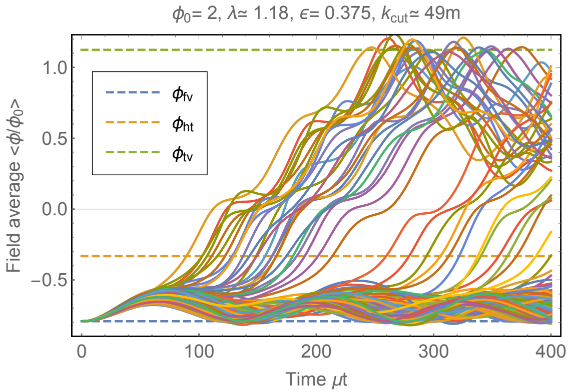
Cosine Potential

- Solid line is $\Gamma_{\text{inst}} = \Gamma_0 \left(\frac{S_B}{2\pi}\right)^2 e^{-S_B}$. Without renormalization, use $\Gamma_0 = Nm^2 L$
- Dashed line is $\Gamma^\star e^{-S_B}$ where $\Gamma^\star = \Gamma_{\text{stoch}}(\epsilon = 0.5, \phi_0 = 1.0)$



Double Well Potential

- Consider now: $V(\phi) = V_0\left(\left(1 - \frac{\phi^2}{\phi_0^2}\right)^2 + \lambda\left(1 - \frac{\phi}{\phi_0}\right)\right)$
- New tunneling threshold $\frac{1}{L}\int dx\phi(t,x)/\phi_0 \geq V(\phi_{\mathrm{HT}})$



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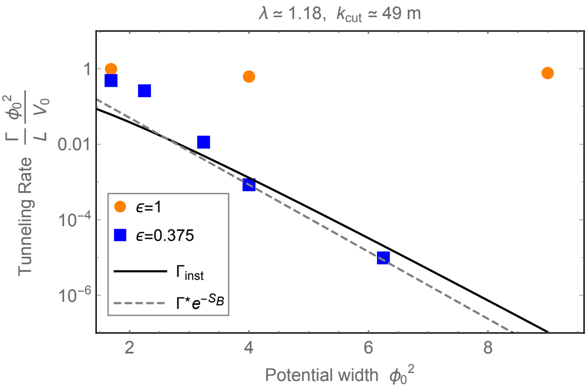


Renormalization

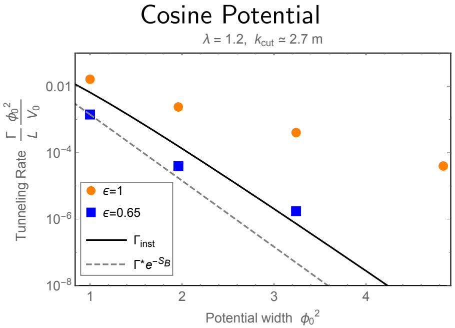
- The one-loop correction to the mass in 1+1 is:

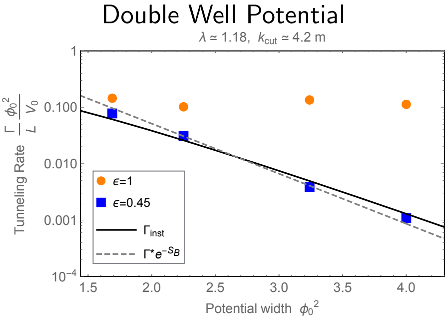
$$m_R^2 = m_B^2 + \frac{g}{8\pi} \log\left(\frac{k_{\text{cut}} + m_B^2}{m_B^2}\right) \tag{7}$$

where $g_{\cos} = V_0/\phi_0^4(1 - 4\lambda^2)$ and $g_{\text{DW}} = 24V_0/\phi_0^4$

- Requiring $|m_R^2 - m_B^2| < |m_B^2|$ gives us an upper bound on k_{cut}. Choosing new cutoffs, we get:

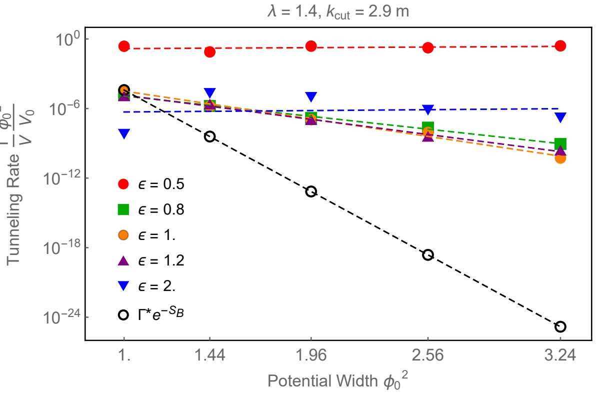
Renormalization





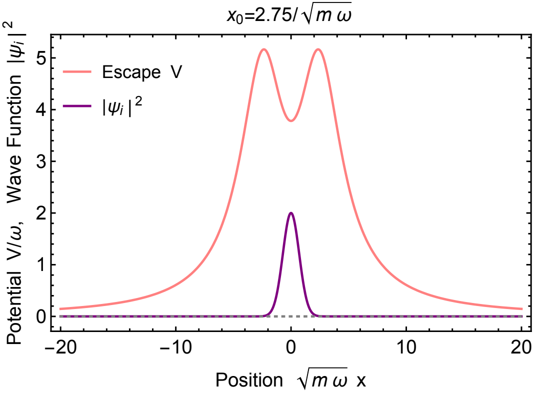
Other Physical States

- $\Delta\phi_k \Delta\dot{\phi}_k = \frac{\epsilon_\phi \epsilon_\pi}{2}$. So clearly $\epsilon_\phi = \epsilon_\pi = \epsilon < 1$ violates the uncertainty principle.
- Can modify fluctuation amplitudes while saturating uncertainty as follows: $\epsilon_\phi = 1/\epsilon_\pi = \epsilon$



Particle Escape

- The wavefunction starts in well, then spreads out. This is analogous to a particle escaping



Particle Escape

- Track variance in position over time:

$$\langle x^2 \rangle_Q(t) = \int_{-\infty}^{\infty} dx |\psi(x,t)|^2 x^2 \tag{8}$$

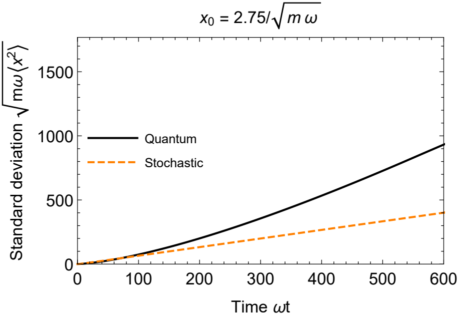
- Create an ensemble of 10^4 initial conditions for $\{x_i, p_i\}$ from gaussian distributions with variances:

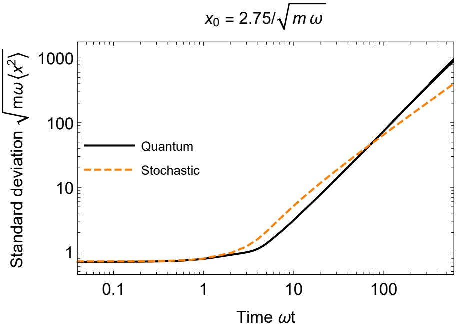
$$\sigma_{x,i}^2 = \frac{1}{2m\omega} \ , \ \sigma_{p,i}^2 = \frac{m\omega}{2} \tag{9}$$

- Then evolve each x_i classically over time and ensemble average to obtain $\langle x^2 \rangle_S(t)$

Particle Escape

- Choose large box to minimize edge effects: $x_0 = 2.75/\sqrt{m\omega}$, $L = 3536/\sqrt{m\omega}$





Conclusions

- The instanton is an imaginary-time approximation of tunneling rates that fails for certain time-dependent backgrounds
- Recent work introduced a real-time formalism that claimed excellent agreement to the instanton
- This isn't quite true, the stochastic method over-predicts tunneling rates unless fluctuations are artificially suppressed
- Various curing methods were applied, and the stochastic method continued to show only parametric agreement
- Future work: Develop a prescription for obtaining ideal "fudge factors"

References

Jonathan Braden et al. "New semiclassical picture of vacuum decay". In: *Physical review letters* 123.3 (2019), p. 031601.

Sidney Coleman. "Fate of the false vacuum: Semiclassical theory". In: *Physical Review D* 15.10 (1977), p. 2929.

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Andrei Linde. "Hard art of the universe creation". In: *arXiv preprint hep-th/9110037* (1991).

A "classical" perspective

- The thresholds are chosen as following:
- For the Braden periodic potential:
  - Recall the volume average for a field ϕ_i: $c_i(t) = \frac{1}{L} \int dx \cos(\phi_i(t,x)/\phi_0)$
  - Then, define the ensemble values: $\bar{c}_T/\Delta c_T \equiv$ Ensemble average/std. dev. of $\{c_i(0)\}$
  - Define the threshold $\mathcal{T}_{\text{Braden}} = \bar{c}_T + n_\sigma \Delta c_T$ where $5 \leq n_\sigma \leq 25$
- For the DW potential: $\mathcal{T}_{\text{DW}} = V(\phi_{\text{HT}})$

A "classical" perspective

- Let's define the Weyl transform of some operator $\hat{A}$:

$$\tilde{A}(q_{\pm\boldsymbol{k}}, \pi_{\pm\boldsymbol{k}}) = \int dx\, dy\, e^{-i\pi_{\boldsymbol{k}}x - i\pi_{-\boldsymbol{k}}y} \left\langle q_{\boldsymbol{k}} + \frac{x}{2}, q_{-\boldsymbol{k}} + \frac{y}{2} \middle| \hat{A} \middle| q_{\boldsymbol{k}} - \frac{x}{2}, q_{-\boldsymbol{k}} - \frac{y}{2} \right\rangle \tag{10}$$

- Define the *Wigner function* $W \equiv (2\pi)^{-2}\tilde{\rho}$. If $W \geq 0 \Rightarrow$ phase space distribution
- By correspondence, define corresponding function of $\hat{A}$ as:

$$\hat{q}_{\pm\boldsymbol{k}} \to q_{\pm\boldsymbol{k}}\ ,\ \hat{\pi}_{\pm\boldsymbol{k}} \to \pi_{\pm\boldsymbol{k}} \Rightarrow \hat{A}(\hat{q}_{\pm\boldsymbol{k}}, \hat{\pi}_{\pm\boldsymbol{k}}) = A_{\mathrm{C}}(q_{\pm\boldsymbol{k}}, \pi_{\pm\boldsymbol{k}}) \tag{11}$$

A "classical" perspective

- Define stochastic average of $\hat{A}$ as:

$$\langle A\rangle_{\text{stoch}} = \int A_{\text{C}}(\boldsymbol{s})W(\boldsymbol{s})d^4\boldsymbol{s} \tag{12}$$

where $\boldsymbol{s} \equiv (q_{\pm\boldsymbol{k}}, \pi_{\pm\boldsymbol{k}})$ is a 4-vector in phase space and $W(\boldsymbol{s})$ is the Wigner function.

- The Weyl transform has a key property we can use:

$$\langle\hat{A}\rangle = \text{Tr}(\hat{\rho}\hat{A}) = \frac{1}{(2\pi)^2}\int \tilde{A}(\boldsymbol{s})\tilde{\rho}(\boldsymbol{s})d^4\boldsymbol{s} = \int \tilde{A}(\boldsymbol{s})W(\boldsymbol{s})d^4\boldsymbol{s} \tag{13}$$

- Clearly $\langle A\rangle_{\text{stoch}} = \langle\hat{A}\rangle$ if $A_{\text{C}}(\boldsymbol{s}) = \tilde{A}(\boldsymbol{s})$

Single Particle QM

- In QFT, we draw ϕ from Wigner function as joint distribution, then evolve classically.
- What if we move to SPQM and draw directly from a Gaussian wavefunction?
- Consider the following wavefunction:

$$\psi(x) = \left(\frac{m\omega}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}m\omega\, x^2\right] \tag{14}$$

in the following potential:

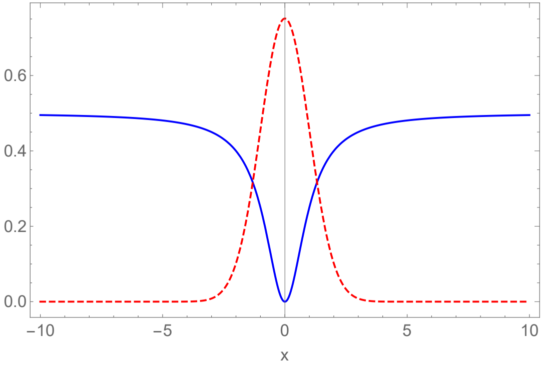
$$V(x) = \frac{1}{2}m\omega^2 x^2 \frac{1-\frac{1}{2}\left(\frac{x}{x_0}\right)^2}{1+\frac{1}{2}\left(\frac{x}{x_0}\right)^4} \tag{15}$$

Quantum Escape

- Define the potential:

$$V(x) = \frac{1}{2}\frac{x^2}{1 + x^2/\lambda^2} \tag{16}$$

- Initialize a Gaussian in the well, track its escape and compare to stochastic



Quantum Escape