Poisson type conformastat spherically symmetric spacetimes

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Abstract
Static spherically symmetric solutions of the Einstein's field equations in isotropic coordinates from Newtonian potential-density pairs are investigated. The approach is used in the construction of spherical matter distributions made of perfect fluid starting with the seed potential-density pairs corresponding to a harmonic oscillator (homogeneous sphere) and a massive spherical dark matter halo with a logarithm potential. Moreover, the geodesic motion of test particles in stable circular orbits around such structures is studied. The models considered satisfy all the energy conditions.

Poisson type spacetimes

The line element for a static spherically symmetric spacetime in isotropic coordinates is given by

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}(dr^2 + r^2 d\Omega^2), \tag{1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. The Einstein's field equations $G_{ab} = 8\pi G T_{ab}$ yield

$$T^t_t = \frac{e^{-2\lambda}}{4\pi G}\left[\nabla^2\lambda + \frac{1}{2}\nabla\lambda\cdot\nabla\lambda\right], \tag{2a}$$

$$T^r_r = \frac{e^{-2\lambda}}{8\pi G}\left[(\lambda')^2 + 2\lambda'\nu' + \frac{2}{r}(\lambda'+\nu')\right], \tag{2b}$$

$$T^\theta_\theta = T^\varphi_\varphi = \frac{e^{-2\lambda}}{8\pi G}\left[\lambda'' + \nu'' + (\nu')^2 + \frac{1}{r}(\lambda'+\nu')\right], \tag{2c}$$

where primes indicate differentiation with respect to r.
In terms of the orthonormal tetrad (comoving observer) $e_{(a)}{}^b = \{U^b, X^b, Y^b, Z^b\}$, where

$$U^a = \frac{1}{\sqrt{-g_{00}}}\delta^a_0, \qquad X^a = \frac{1}{\sqrt{g_{11}}}\delta^a_1, \tag{3a}$$

$$Y^a = \frac{1}{\sqrt{g_{22}}}\delta^a_2, \qquad Z^a = \frac{1}{\sqrt{g_{33}}}\delta^a_3. \tag{3b}$$

the energy density is $\rho = -T^0_0$ and the principal stresses (pressures or tensions) $p_i = T^i_i$.
By setting $e^\lambda = \left(1 - \frac{\phi(r)}{2}\right)^4$, we get for the energy density the following nonlinear Poisson type equation

$$\nabla^2\phi = 4\pi G\rho\left(1 - \frac{\phi}{2}\right)^5. \tag{4}$$

For a given physical energy density profile ρ, the metric function ϕ can be obtained by resolving this equation. A physically reasonable way to choose ρ is by requiring that in the Newtonian limit it reduces to its Newtonian value ρ_N. A simple particular form of ρ which satisfies such condition is

$$\rho = \frac{\rho_0}{\left(1 - \frac{\phi}{2}\right)^5}. \tag{5}$$

Replacing this expression in (4) one finds in this case that the pair (ϕ, ρ_0) is a solution of the Poisson's equation. Hence $(\phi, \rho_0) = (\Phi, \rho_N)$ for any physical system. Therefore,

$$\rho = \frac{\rho_N}{\left(1 - \frac{\Phi}{2}\right)^5}. \tag{6}$$

To obtain the other metric function ν an additional assumption must be imposed.

Motion of particles

The circular speed (rotation curves) of the particles around the structures is given by [1]

$$v_c^2 = \frac{r\nu_{,r}}{1 + r\lambda_{,r}}, \tag{7}$$

and the specific angular momentum is

$$L^2 = \frac{r^2 e^{2\lambda} v_c^2}{1 - v_c^2}. \tag{8}$$

This quantity can be used to analyze the stability of the particles against radial perturbations. The condition of stability,

$$\frac{d(L^2)}{dr} > 0, \tag{9}$$

is an extension of the Rayleigh criteria of stability of a fluid in rest in a gravitational field.

Perfect fluid spacetimes

For a perfect fluid source

$$T^{ab} = (\rho + p)U^a U^b + p g^{ab}, \tag{10}$$

and the the condition of pressure isotropy can be cast as

$$L\mathcal{G}_{,xx} = 2\mathcal{G}L_{,xx}, \quad L \equiv e^{-\lambda}, \quad \mathcal{G} \equiv Le^{\nu}, \quad x \equiv r^2. \tag{11}$$

In this case the metric function ν is obtained solving this equation.

Harmonic oscillator type spheres

In Newtonian gravity the gravitational potential of a sphere of radius a and constant mass density ρ_N is

$$\Phi = \begin{cases} -2\pi G\rho_N(a^2 - \frac{1}{3}r^2) & (r < a), \\ -\frac{4\pi G\rho_N a^3}{3r} & (r > a). \end{cases} \tag{12}$$

For $r < a$ the potential corresponds to a harmonic oscillator potential which has been used to model extended dark matter haloes with harmonic core. The circular speed is

$$v_{cN} = \sqrt{\frac{4\pi G\rho_N}{3}}r. \tag{13}$$

Solving the condition of pressure isotropy (11), the interior solution is [2]

$$e^\nu = \left[C_1\left(\frac{r^2}{a^2} - \frac{3(b+2)}{b}\right)^4 + C_2\left(\frac{r^2}{a^2} - \frac{3(b+2)}{b}\right)^{-3}\right] \times \left[1 + \frac{b}{2}\left(1 - \frac{1}{3}\frac{r^2}{a^2}\right)\right]^2, \tag{14a}$$

$$e^\lambda = \left[1 + \frac{b}{2}\left(1 - \frac{1}{3}\frac{r^2}{a^2}\right)\right]^2, \tag{14b}$$

where $b = 2\pi G\rho_N a^2/c^2$,

$$C_1 = -\frac{9b^4}{112(b+3)^6}, \tag{15a}$$

$$C_2 = \frac{432(b-4)}{7b^3}. \tag{15b}$$

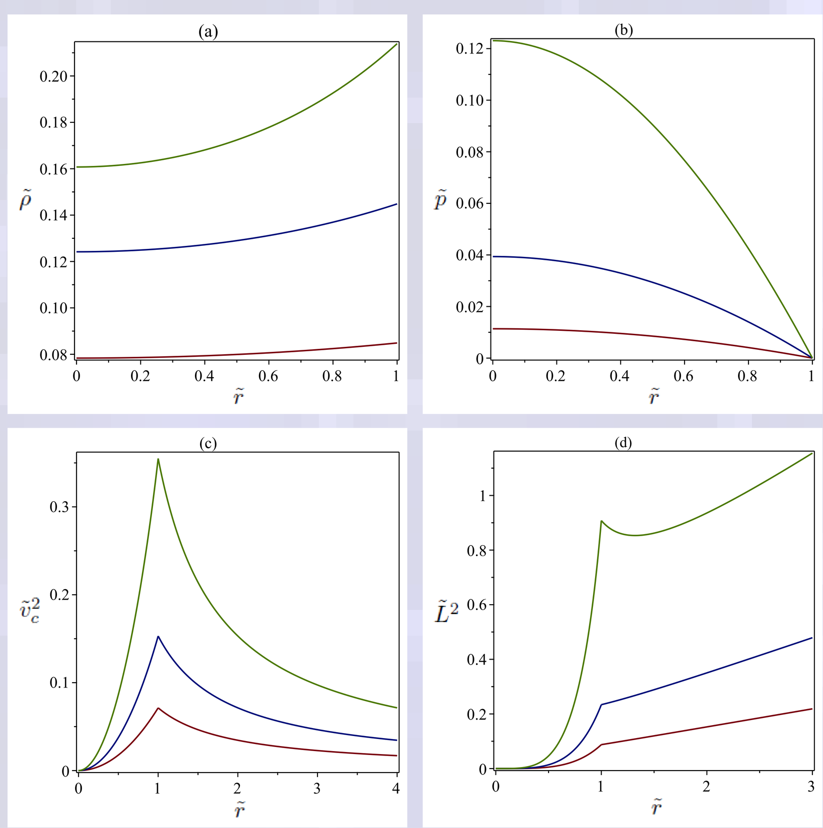

Figure 1: Graphs, as functions of $\tilde{r} = r/a$, of (a) $\tilde{\rho} = 2\pi Ga^2\rho/c^2$, (b) $\tilde{p} = (8\pi G/c^4)p$, (c) $\tilde{v}_c^2 = v_c^2/c^2$, and (d) $\tilde{L}^2 = L^2/(c^2a^2)$ with $b = 0.1$ (bottom curves), 0.2 , 0.4 (top curves).

Logarithmic potential type spheres

Dark matter haloes can be modeled in Newtonian theory with a logarithmic potential of the form

$$\Phi = \frac{1}{2}v_0^2 \ln\left(\frac{r^2 + a^2}{b}\right), \tag{16}$$

where a and b are constants, and v_0 is circular speed at large radii, also a constant. The mass density distribution is

$$\rho_N = \frac{v_0^2(r^2 + 3a^2)}{4\pi G(r^2 + a^2)^2}, \tag{17}$$

and the circular speed in radius r is

$$v_{cN} = \frac{v_0 r}{\sqrt{r^2 + a^2}}. \tag{18}$$

This potential yields an asymptotically flat rotation curve. Solving (11), a particular solution is [2]

$$e^{\frac{\nu}{2}} = C\left(r^2 + a^2\right)\left[1 - \frac{1}{4}\frac{v_0^2}{c^2}\ln\left(\frac{r^2 + a^2}{b}\right)\right]^6, \tag{19a}$$

$$e^{\frac{\lambda}{2}} = \left[1 - \frac{1}{4}\frac{v_0^2}{c^2}\ln\left(\frac{r^2 + a^2}{b}\right)\right]^4, \tag{19b}$$

where C is an integration constant.

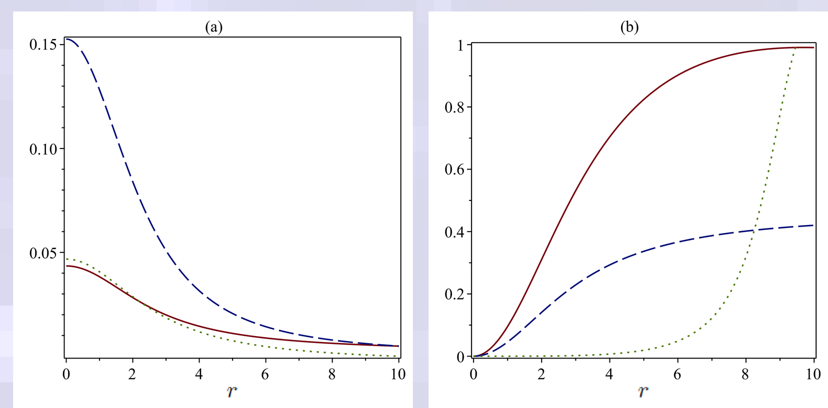

Figure 2: (a) $\tilde{\rho} = 4\pi G\rho$ (solid curve), $\tilde{\rho}_N = 4\pi G\rho_N$ (dashed curve), $\tilde{p} = 8\pi Gp$ (dotted curve), (b) v_c (solid curve), v_{cN}^2 (dashed curve) and $L^2 \times 10^{-4}$ (dotted curve), with parameters $\tilde{v}_0^2 = 0.458$, $a = 3$.

Conclusions

- Two perfect fluid sources for static spherically symmetric fields in isotropic coordinates based on the Newtonian potential-density pairs corresponding to a harmonic oscillator and a massive spherical dark matter halo with a logarithm potential were constructed.
- In all the cases we found stable circular orbits, but for harmonic oscillator type fields was observed that the increase in the gravitational field can make unstable the motion of the particles.
- For logarithmic potential type spheres we found that relativistic rotation curve is flattened after a certain value of the radial distance as observational data indicate.
- The models considered satisfy all the energy conditions.

References

[1] G. García-Reyes. Poisson type conformastat spherically symmetric anisotropic fluid spacetimes. *arXiv:1911.05512.*

[2] G. García-Reyes. Poisson type relativistic perfect fluid spheres. *arXiv:1812.04958.*