

# Introducing the Helicity-Flow Method

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*Work in Progress*

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# Outline

- 1 Calculating an Amplitude
- 2 Building a Helicity-Flow Picture
- 3 (Massless) QED Helicity Flow + Examples
- 4 (Massless) QCD Helicity Flow + Example
- 5 Summary & Outlook

# How to Calculate a (Massless) Scattering Amplitude

- QCD often factorise colour, use helicity basis for kinematics

$$\mathcal{M}_h(1^{h_1}, \dots, n^{h_n}) = \sum_i C_i A_i(p_1^{h_1}, \dots, p_n^{h_n})$$

- $C_i \equiv$  colour factor
  - QED:  $C_i = 1$
- $A_i \equiv$  kinematic amplitude
  - Cross incoming particles to outgoing
  - Each particle  $j$  is given a specific helicity  $h_j$
  - Since massless, helicity  $\sim$  chirality

# Calculating $A_i$ : the Spinor-Helicity Idea

- Standard Feynman diagrams have matrix structure
  - Takes much effort to simplify traces of  $\gamma^\mu$
- Spinor-helicity diagram is a complex number
  - Easy to square
- Lorentz algebra  $so(3,1) \cong su(2) \otimes su(2)$
- Dirac spinors reducible into two irreps of diff chirality
  - Weyl rep of Dirac algebra naturally separates the two irreps

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# Calculating $A_i$ : the Spinor-Helicity Pieces

- Lorentz algebra  $so(3, 1) \cong su(2) \otimes su(2)$ 
  - $\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^{\mu, \dot{\alpha}\beta} \\ \sqrt{2}\bar{\tau}^{\mu}_{\dot{\alpha}\beta} & 0 \end{pmatrix}$ ,  $\sqrt{2}\tau^{\mu, \dot{\alpha}\beta} = \sigma^{\mu, \dot{\alpha}\beta}$
  - $v(p) = \begin{pmatrix} \tilde{\lambda}_p^{\dot{\alpha}} \\ \lambda_{p, \alpha} \end{pmatrix}$ ,  $\bar{u}(p) = (\tilde{\lambda}_{p, \dot{\alpha}} \quad \lambda_p^\alpha)$
  - $\varepsilon_+^\mu(p, r) = \frac{\tilde{\lambda}_{p, \dot{\alpha}} \tau^{\mu, \dot{\alpha}\beta} \lambda_{r, \beta}}{\langle rp \rangle}$ ,  $\varepsilon_-^\mu(p, r) = \frac{\lambda_p^\alpha \bar{\tau}^{\mu}_{\dot{\alpha}\beta} \tilde{\lambda}_r^{\dot{\beta}}}{[pr]}$
- Final result in terms of inner products:
  - $\lambda_i^\alpha \lambda_{j\alpha} \equiv \langle ij \rangle$ ,  $\tilde{\lambda}_{i, \dot{\beta}} \tilde{\lambda}_j^{\dot{\beta}} \equiv [ij]$ ,  $\langle ij \rangle, [ij] \sim \sqrt{s_{ij}}$
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• e.g.

$$= \frac{2ie^2}{s_{e^+e^-}} [e^- \mu^+] \langle \mu^- e^+ \rangle$$

# Define Problem

- Can we still improve on this?
    - Deriving spinor inner products  $\langle ij \rangle, [kl]$  requires at least 2 steps
      - Re-write every object as spinors
      - Use Fierz identity  $\bar{\tau}_{\alpha\dot{\beta}}^\mu \tau_{\mu}^{\dot{\alpha}\beta} = \delta_{\alpha}^{\dot{\beta}} \delta^{\dot{\alpha}}_{\beta}$
    - Not intuitive which inner products we obtain
  - In SU(N) use graphical reps for calculations, e.g. Fierz id.
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$$\underbrace{\begin{array}{c} i \quad \bar{l} \\ \diagdown \quad / \\ \text{---} \text{---} \text{---} \text{---} \\ / \quad \diagdown \\ \bar{j} \quad k \end{array}}_{t_{i\bar{j}}^a t_{k\bar{l}}^a} = \underbrace{\begin{array}{c} i \quad \bar{l} \\ \diagdown \quad / \\ \text{---} \text{---} \\ / \quad \diagdown \\ \bar{j} \quad k \end{array}}_{\delta_{i\bar{j}} \delta_{k\bar{l}}} = \frac{1}{N} \underbrace{\begin{array}{c} i \quad \bar{l} \\ \diagdown \quad / \\ \diagup \quad \diagdown \\ / \quad \diagdown \\ \bar{j} \quad k \end{array}}_{\delta_{i\bar{j}} \delta_{k\bar{l}}}$$

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# Creating a Helicity Flow for QED: Part 1

- Key difference:

- Colour  $\equiv$  single  $SU(N)$ : generators  $t^a \rightarrow \delta$ 's
- Spinor-hel  $\equiv su(2) \otimes su(2)$ :  $\tau^\mu, \bar{\tau}^\mu, \lambda, \tilde{\lambda}, \epsilon_{\pm}^\mu \rightarrow \langle ij \rangle, [kl]$

- First step: Spinors and their inner products

- $\lambda_i^\alpha \lambda_{j,\alpha} = \langle ij \rangle = -\langle ji \rangle =$

- $\tilde{\lambda}_{i,\dot{\beta}} \tilde{\lambda}_j^{\dot{\beta}} = [ij] = -[ji] =$

- $\lambda_{j,\alpha} =$  ,  $\lambda_i^\alpha =$  ,  $\delta_\alpha^\beta =$

- $\tilde{\lambda}_{i,\dot{\alpha}} =$  ,  $\tilde{\lambda}_j^{\dot{\alpha}} =$  ,  $\delta_{\dot{\alpha}}^{\dot{\beta}} =$

- Second step: Fermion propagators

- $\not{p} = \sqrt{2} p^\mu \tau_\mu^{\dot{\alpha}\beta} \stackrel{p^2=0}{=} \tilde{\lambda}_p^{\dot{\alpha}} \lambda_p^\beta =$

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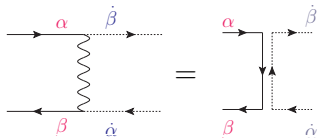
# Creating a Helicity Flow for QED: Part 2

- Third step: Vertices and vector propagators
    - vertices  $\frac{\gamma^\mu}{\sqrt{2}} \rightarrow \tau^\mu, \bar{\tau}^\mu$  contracted with vector propagator  $g_{\mu\nu}$
    - Fierz identity with indices:  $\bar{\tau}_{\alpha\dot{\beta}}^\mu \tau_{\mu}^{\dot{\alpha}\beta} = \delta_{\alpha}^{\beta} \delta_{\dot{\beta}}^{\dot{\alpha}}$
    - Fierz identity with flow:
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- $\Rightarrow \tau^{\mu, \dot{\alpha}\beta} = \quad , \quad \bar{\tau}_{\alpha\dot{\beta}}^\mu =$
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  - Fierz identity already utilised in flow rule

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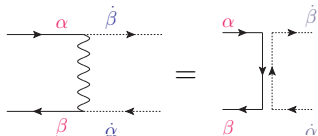
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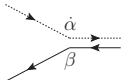
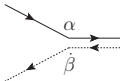
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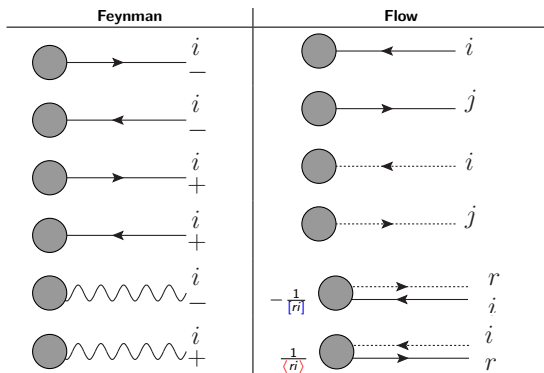


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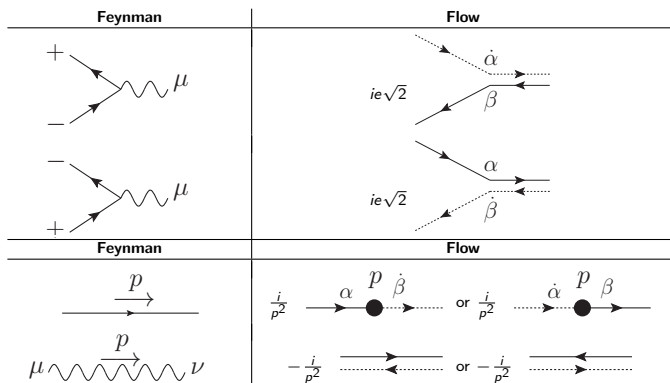
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# The QED Flow Rules: External Particles



Everything already Fierzed, in terms of spinors

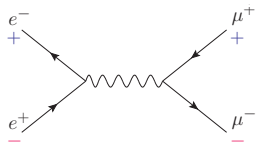
# The QED Flow Rules: Vertices and Propagators



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# Simplest QED Example

- Regular spinor-helicity  $\equiv$  easy

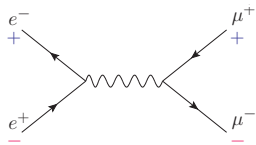


$$\begin{aligned}
 &= \frac{2ie^2}{s_{e^+e^-}} (\tilde{\lambda}_{e^-, \dot{\alpha}} \tau_{\mu}^{\dot{\alpha}\beta} \lambda_{e^+, \beta}) (\lambda_{\mu^-, \alpha} \bar{\tau}_{\alpha\dot{\beta}}^{\mu} \tilde{\lambda}_{\mu^+, \dot{\beta}}) \\
 &= \frac{2ie^2}{s_{e^+e^-}} \tilde{\lambda}_{e^-, \dot{\alpha}} \tilde{\lambda}_{\mu^+, \dot{\alpha}} \lambda_{\mu^-, \beta} \lambda_{e^+, \beta} \equiv \frac{2ie^2}{s_{e^+e^-}} [e^- \mu^+] \langle \mu^- e^+ \rangle
 \end{aligned}$$

- Helicity flow  $\equiv$  super easy and intuitive

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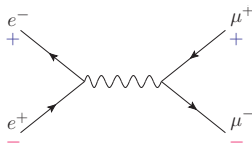
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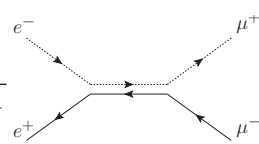


$$= \frac{2ie^2}{s_{e^+e^-}} (\tilde{\lambda}_{e^-, \dot{\alpha}} \tau_{\mu}^{\dot{\alpha}\beta} \lambda_{e^+, \beta}) (\lambda_{\mu^-, \alpha} \bar{\tau}_{\alpha\dot{\beta}}^{\mu} \tilde{\lambda}_{\mu^+, \dot{\beta}})$$

$$= \frac{2ie^2}{s_{e^+e^-}} \tilde{\lambda}_{e^-, \dot{\alpha}} \tilde{\lambda}_{\mu^+, \dot{\alpha}} \lambda_{\mu^-, \beta} \lambda_{e^+, \beta} \equiv \frac{2ie^2}{s_{e^+e^-}} [e^- \mu^+] \langle \mu^- e^+ \rangle$$

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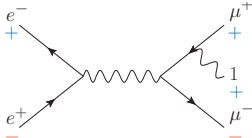


$$= \frac{2ie^2}{s_{e^+e^-}}$$




## Next Simplest QED Example

- Regular spinor-helicity  $\equiv$  easy



$$\begin{aligned}
 &= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+\mu^-}} \left( \tilde{\lambda}_{e^-, \dot{\alpha}} \tau_{\mu}^{\dot{\alpha}\beta} \lambda_{e^+, \beta} \right) \left( \lambda_{\mu^-, \bar{\tau}}^{\alpha} \bar{\tau}_{\alpha\dot{\beta}}^{\mu} (\not{p}_1 + \not{p}_{\mu^+})^{\dot{\beta}\delta} \not{\epsilon}_{\delta\dot{\gamma}}(1, r) \tilde{\lambda}_{\mu^+}^{\dot{\gamma}} \right) \\
 &= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+\mu^-} \langle r1 \rangle} \left( \tilde{\lambda}_{e^-, \dot{\alpha}} \tau_{\mu}^{\dot{\alpha}\beta} \lambda_{e^+, \beta} \right) \tilde{\lambda}_{1, \dot{\delta}} \tilde{\lambda}_{\mu^+}^{\dot{\delta}} \\
 &\quad \times \left( \lambda_{\mu^-, \bar{\tau}}^{\alpha} \tilde{\lambda}_{\alpha\dot{\beta}}^{\dot{\beta}} \tilde{\lambda}_1^{\delta} \lambda_{r, \delta} + \lambda_{\mu^-, \bar{\tau}}^{\alpha} \tilde{\lambda}_{\alpha\dot{\beta}}^{\dot{\beta}} \tilde{\lambda}_{\mu^+}^{\delta} \lambda_{r, \delta} \right) \\
 &\sim \lambda_{\mu^-, \beta}^{\beta} \lambda_{e^+, \beta} \left( \tilde{\lambda}_{e^-, \dot{\alpha}} \tilde{\lambda}_1^{\dot{\alpha}} \lambda_1^{\delta} \lambda_{r, \delta} + \tilde{\lambda}_{e^-, \dot{\alpha}} \tilde{\lambda}_{\mu^+}^{\dot{\alpha}} \lambda_{\mu^+}^{\delta} \lambda_{r, \delta} \right) \tilde{\lambda}_{1, \dot{\delta}} \tilde{\lambda}_{\mu^+}^{\dot{\delta}}
 \end{aligned}$$

## Correct Answer

$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+\mu^-} \langle r1 \rangle} \left( [e^- 1] \langle 1r \rangle + [e^- \mu^+] \langle \mu^+ r \rangle \right) [1\mu^+] \langle \mu^- e^+ \rangle$$

## Next Simplest QED Example

- Helicity flow  $\equiv$  super easy and intuitive

The diagrammatic equation shows the helicity flow of a photon. On the left, an incoming electron  $e^-$  (helicity  $+$ ) and an incoming positron  $e^+$  (helicity  $-$ ) meet at a vertex. A wavy line representing a photon with helicity  $1$  connects this vertex to another vertex where an outgoing muon  $\mu^+$  (helicity  $+$ ) and an outgoing antimuon  $\mu^-$  (helicity  $-$ ) emerge. This is equal to the same process where the photon is represented by a vertex with a dot, and the outgoing muon and antimuon lines are shifted relative to each other. The dot is labeled with  $p_1 + p_{\mu^+}$ . The photon helicity is  $1$  and the muon helicity is  $r$ .

$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+\mu^-}} \langle r1 \rangle$$

- Immediately read off inner products

## Correct Answer

$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+\mu^-}} \left( [e^-1]\langle 1r \rangle + [e^- \mu^+]\langle \mu^+ r \rangle \right) [1\mu^+]\langle \mu^- e^+ \rangle$$

# Next Simplest QED Example

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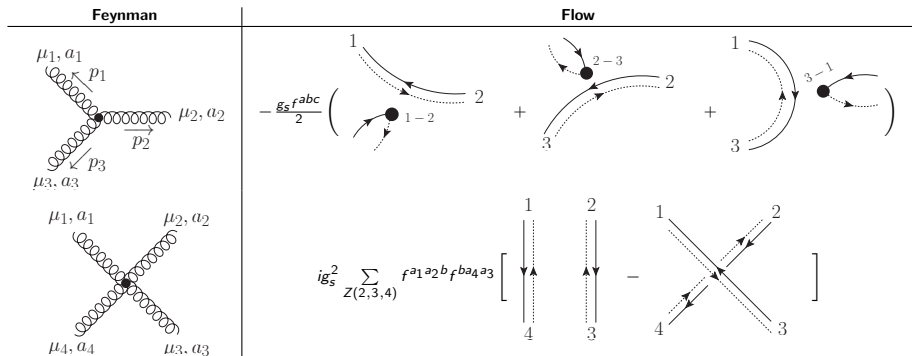
$$\begin{array}{c} e^- \\ + \\ \nearrow \\ e^+ \\ - \end{array} \begin{array}{c} \mu^+ \\ + \\ \nearrow \\ \mu^- \\ - \end{array} \begin{array}{c} 1 \\ \text{---} \\ 1 \end{array} = \frac{-i2\sqrt{2}e^3}{s_{e^+e^-} s_{\mu^+1} \langle r1 \rangle} \begin{array}{c} e^- \\ \text{---} \\ \nearrow \\ e^+ \end{array} \begin{array}{c} \mu^+ \\ \text{---} \\ \nearrow \\ \mu^- \end{array} \begin{array}{c} 1 \\ \text{---} \\ r \end{array}$$

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## Correct Answer

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# The Massless QCD Flow Rules



# QCD Example: $q_1 \bar{q}_1 \rightarrow q_2 \bar{q}_2 g$

- Triple-gluon vertex provides new structures

$$\begin{aligned}
 & \text{Tree-level diagram with triple-gluon vertex} \\
 &= \frac{ig_s^3}{2s_{q_1 \bar{q}_1} s_{q_2 \bar{q}_2} \langle r1 \rangle} \left[ \begin{aligned}
 & \text{Diagram 1: } q_1^- \text{ and } \bar{q}_2^+ \text{ exchange a gluon in the } s\text{-channel, } q_1^+ \text{ and } q_2^+ \text{ exchange a gluon in the } t\text{-channel.} \\
 & \text{Diagram 2: } q_1^- \text{ and } q_2^+ \text{ exchange a gluon in the } t\text{-channel, } q_1^+ \text{ and } \bar{q}_2^+ \text{ exchange a gluon in the } s\text{-channel.} \\
 & \text{Diagram 3: } q_1^- \text{ and } q_2^+ \text{ exchange a gluon in the } u\text{-channel, } q_1^+ \text{ and } \bar{q}_2^+ \text{ exchange a gluon in the } s\text{-channel.}
 \end{aligned} \right]
 \end{aligned}$$

# Summary

- Helicity flow allows for single-line calculation of Feynman diagram
- Also gives transparent/intuitive picture of inner products
- In contrast, spinor-hel method:
  - Requires multiple steps
  - Final result intransparent/unintuitive
- Massless QED and QCD tree-level done
- Useful for any generator using diagrams to avoid dealing with Lorentz algebra

# Outlook

- Initial paper coming soon
- Complete the SM at tree level
- Loop calculations
- Applications within generator(s)
- Amplitude-level calculations

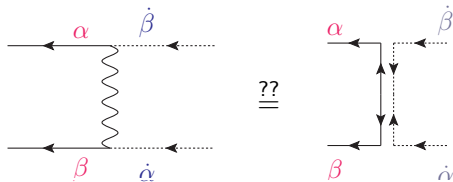
# Calculating $A_i$ : the Spinor-Helicity Method

- Lorentz algebra  $so(3, 1) \cong su(2) \otimes su(2)$
- Weyl representation of Dirac algebra naturally separates the two reps
  - $\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^{\mu, \dot{\alpha}\beta} \\ \sqrt{2}\bar{\tau}^{\mu}_{\alpha\dot{\beta}} & 0 \end{pmatrix}$ ,  $\sqrt{2}\tau^\mu = (1, \vec{\sigma})$ ,  $\sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma})$
  - $\text{Tr}(\tau^\mu \bar{\tau}^\mu) = g^{\mu\nu}$ ,  $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $P_\pm = \frac{1}{2}(1 \pm \gamma^5)$
  - $u(p) = \begin{pmatrix} u_-(p) \\ u_+(p) \end{pmatrix} = \begin{pmatrix} v_+(p) \\ v_-(p) \end{pmatrix} = \begin{pmatrix} \tilde{\lambda}_p^{\dot{\alpha}} \\ \lambda_{p, \alpha} \end{pmatrix}$ ,  $\bar{u}(p) = (\tilde{\lambda}_{p, \dot{\alpha}} \quad \lambda_p^\alpha)$
- Final result in terms of inner products:
  - $\lambda_i^\alpha \lambda_{j\alpha} \equiv \langle ij \rangle$ ,  $\tilde{\lambda}_{i, \dot{\beta}} \tilde{\lambda}_j^{\dot{\beta}} \equiv [ij]$ ,  $\langle ij \rangle, [ij] \sim \sqrt{s_{ij}}$
- $\varepsilon_+^\mu(p, r) = \frac{\tilde{\lambda}_{p, \dot{\alpha}} \tau^{\mu, \dot{\alpha}\beta} \lambda_{r, \beta}}{\langle rp \rangle}$ ,  $\varepsilon_-^\mu(p, r) = \frac{\lambda_p^\alpha \bar{\tau}^{\mu}_{\alpha\dot{\beta}} \tilde{\lambda}_r^{\dot{\beta}}}{[pr]}$
- No complicated traces of  $\gamma$  matrices, rather simple identities like:
  - $(\tilde{\lambda}_{i, \dot{\alpha}} \tau_\mu^{\dot{\alpha}\beta} \lambda_{j, \beta})(\lambda_k^\gamma \bar{\tau}^{\mu}_{\gamma\dot{\delta}} \tilde{\lambda}_l^{\dot{\delta}}) = \lambda_i^\beta \lambda_{k\beta} \tilde{\lambda}_{l, \dot{\alpha}} \tilde{\lambda}_j^{\dot{\alpha}} = \langle ik \rangle [lj]$



# Fun with Arrows and the Fierz Identity

- Sometimes have to contract  $\tau^\mu \tau_\mu$  or  $\bar{\tau}^\mu \bar{\tau}_\mu$
- This would lead to arrows pointing towards each other, e.g.



- To fix, use charge conservation of a current:

$$\lambda_i^\alpha \bar{\tau}_{\alpha\dot{\beta}}^\mu \tilde{\lambda}_j^{\dot{\beta}} = \tilde{\lambda}_{j,\dot{\alpha}} \tau^{\mu,\dot{\alpha}\beta} \lambda_{i,\beta}$$

- Or in pictures:

