Probing Nuclear Superfluidity with Neutron Stars

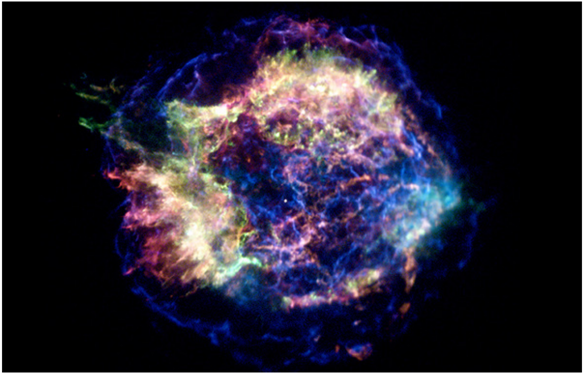
Nicolas Chamel

Institute of Astronomy and Astrophysics
Université Libre de Bruxelles, Belgium

Karpacz, 24 February 2020

Neutron stars: laboratories for dense matter

Formed in gravitational core-collapse supernova explosions, neutron stars are the most compact stars in the Universe. They are initially very hot ($\sim 10^{12}$ K) but cool down rapidly by releasing neutrinos.



Their dense matter is thus expected to undergo various phase transitions, as observed in terrestrial materials at low-temperatures.

Outline

1. Superfluidity and superconductivity in the laboratory
   - Basic phenomenology and historical context
   - Theoretical understanding of these phenomena

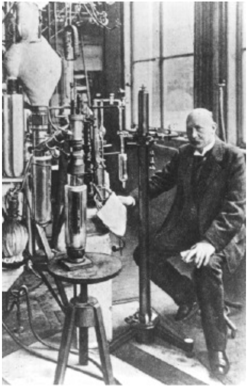
2. Superfluidity and superconductivity in neutron stars
   - Dynamics at the nuclear scale
   - Global hydrodynamic models
   - Astrophysical manifestations (pulsar frequency glitches)

Disclaimer: these lectures are not intended to be an extensive review of superfluidity and superconductivity, but aim at providing a basic understanding of these phenomena in neutron stars.

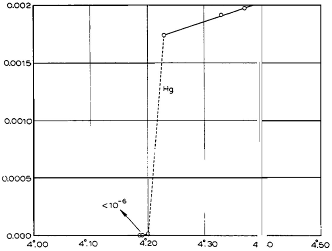
Part 1: Superfluidity and superconductivity in the laboratory

"Suprageleider"

Heike Kamerlingh Onnes and his collaborators were the first to liquefy helium in 1908.



On April 8th, 1911, H. K. Onnes and Gilles Holst discovered that the electric resistance of mercury dropped to almost zero at $T_\mathrm{c} \simeq 4.2$ K



Onnes was awarded the Nobel Prize in 1913.

The year later, tin and lead were found to be also superconducting.

Persistent electric currents

In 1914, Heike Kamerlingh Onnes designed an experiment to measure the decay time of a magnetically induced electric current in a superconducting lead ring.

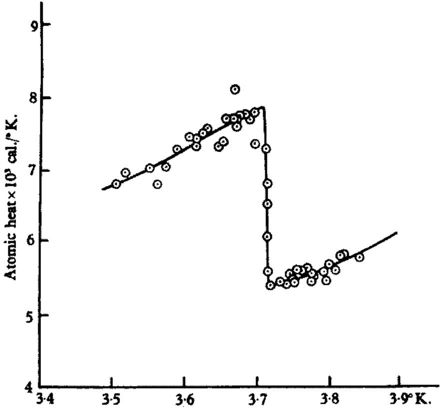
He noted "*During an hour, the current was observed not to decrease perceptibly*".

In superconducting rings, the decay time of induced electric currents is not less than 100 000 years !

J. File and R. G. Mills, PRL 10, 93 (1963)

Order of the phase transition

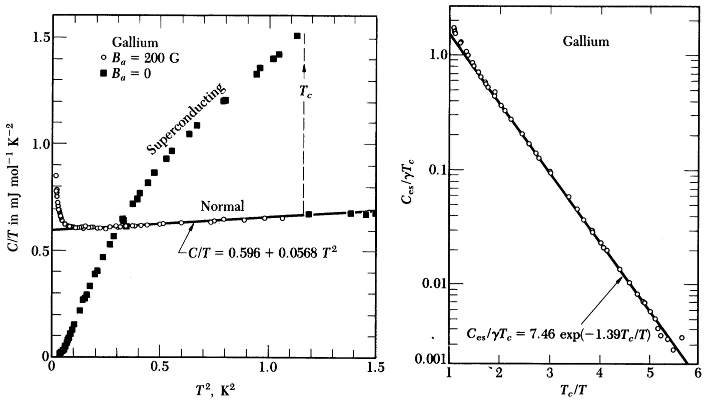
In 1932, Keesom and Kok found that the heat capacity of tin exhibits a discontinuity at T_c thus showing that the superconducting transition is of second order.



Keesom and Kok, Proc. Roy. Acad. Amsterdam 35, 743 (1932).

Energy gap

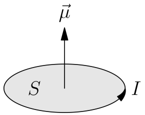
At temperatures $T < T_c$ the electron heat capacity is exponentially suppressed suggesting the existence of a **gap in the electron energy spectrum**.



Kittel, Introduction to Solid State Physics

Intermission: magnetostatics in a magnetic material

In a magnetic material, the set of microscopic magnetic dipole moments $\boldsymbol{\mu}$ give rise to a **magnetization current** $\boldsymbol{j}_m = c\nabla \times \boldsymbol{M}$ (cgs), where the **magnetization** $\boldsymbol{M}$ is the macroscopic density of magnetic moments.



Introducing the **auxiliary magnetic field** $\boldsymbol{H} \equiv \boldsymbol{B} - 4\pi\boldsymbol{M}$ (to avoid confusion $\boldsymbol{B}$ is usually referred to as the **magnetic induction**), Maxwell-Ampere's equation can be expressed as

$$\nabla \times \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j}_{\text{free}}$$

with $\boldsymbol{j}_{\text{free}} = \boldsymbol{j} - \boldsymbol{j}_m$ is the electric current of "free" charged particles associated with the applied field.

But $\boldsymbol{H}$ is not uniquely determined by $\boldsymbol{j}_{\text{free}}$ since

$$\nabla \cdot \boldsymbol{H} = -4\pi\nabla \cdot \boldsymbol{M}$$

However, $\boldsymbol{H}$ approximately coincides with the field applied along the symmetry axis of a thin long sample.

Intermission: magnetic susceptibility

In order to completely determine the magnetic field in the material, additional **constitutive equations** must be provided.

Let us consider that the following relation for **isotropic materials**

$$4\pi \boldsymbol{M} = \chi \boldsymbol{H}$$

where χ is the **magnetic susceptibility** of the material. In such case, we have

$$\boldsymbol{B} = \boldsymbol{H} + 4\pi \boldsymbol{M} = (1+\chi)\boldsymbol{H}$$

A material is

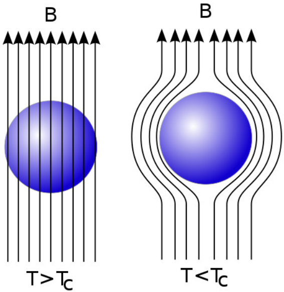
- **paramagnetic** if $\chi > 0$ under an applied field,
- **diamagnetic** if $\chi < 0$ under an applied field.

Typically $|\chi_{\text{diamagnetic}}| \ll \chi_{\text{paramagnetic}}$.

Some (e.g. ferromagnetic) materials may have a permanent magnetization even in the absence of an applied field.

Meissner-Ochsenfeld effect

When placed in a weak magnetic field, a superconductor acts as a perfect diamagnet: $\chi = -1$ therefore $B = (1+\chi)H = 0$.



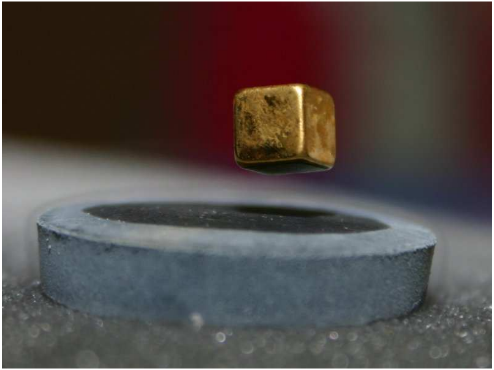
In 1933, Walther Meissner and Robert Ochsenfeld discovered that when a material initially placed in a magnetic field is cooled below the critical temperature, the **magnetic flux is expulsed**.

This phenomenon showed that a superconductor is not just a perfect conductor but correspond to a new thermodynamic state of matter.

Indeed, Ohm's law $\boldsymbol{j} = \sigma \boldsymbol{E}$ implies $\boldsymbol{E} = 0$ if $\sigma \to +\infty$. From Maxwell Faraday equation, $\dfrac{\partial \boldsymbol{B}}{\partial t} = -c\nabla \times \boldsymbol{E} = 0$: $\boldsymbol{B}$ should not change.

Magnetic levitation

As a spectacular consequence of the Meissner-Ochsenfeld effect, a magnet can be levitated over a superconducting material.



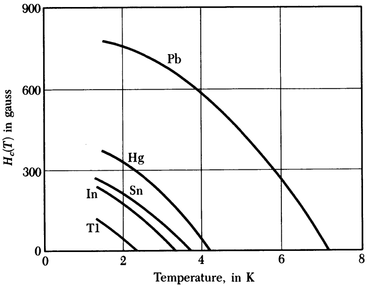
http://www.mn.uio.no/fysikk/english/research/groups/amks/superconductivity/levitation/

Magnetic levitation

Magnetic levitation

Critical magnetic field

Kamerlingh Onnes also discovered that superconductivity is destroyed if the magnetic field H exceeds some critical value H_c.



Experimentally, it is found that

$$H_c(T) = H_0\left[1 - \left(\frac{T}{T_c}\right)^2\right]$$

Kittel, Introduction to Solid State Physics

The critical magnetic field implies the existence of a **critical electric current**.

The existence of a critical magnetic field is a consequence of the Meissner-Ochsenfeld effect: expulsing the flux requires some energy however the energy gain of the phase transition is finite.

Thermodynamics of a superconductor

The thermodynamic state of a superconductor at a given temperature T and magnetic intensity H is determined by the **generalized free energy**

$$F(T,H) = U - TS - \frac{1}{4\pi}\boldsymbol{H}\cdot\boldsymbol{B}.$$

Using the laws of thermodynamics and assuming the material is incompressible, we find $dF = -SdT - \frac{1}{4\pi}\boldsymbol{B}\cdot d\boldsymbol{H}$.

- in the normal phase: $M \sim 0$ therefore $B \approx H$
  $F_N(T,H) - F_N(T,0) = -\frac{1}{4\pi}\int_0^H \boldsymbol{B}\cdot\boldsymbol{dH} = -\frac{H^2}{8\pi}$
- in the superconducting phase: $B = 0$ $(H < H_c)$
  $F_S(T,H) - F_S(T,0) = 0$

Moreover, we have $F_S(T,H_c(T)) = F_N(T,H_c(T))$.

$$F_S(T,H) - F_N(T,H) = \frac{1}{8\pi}\left(H^2 - H_c(T)^2\right) \leq 0 \text{ since } H \leq H_c(T)$$

Thermodynamics of a superconductor

Noting that $S = -\left.\frac{\partial F}{\partial T}\right|_H$, we obtain for the latent heat of the transition

$$\mathcal{L} = T(S_N - S_S) = -\frac{1}{4\pi} T \frac{dH_c}{dT}$$

This shows that the transition is first order for $H < H_c$ ($\mathcal{L} \neq 0$) and second order for $H = H_c$ ($\mathcal{L} = 0$).

The heat capacity is given by $C = T\left.\frac{\partial S}{\partial T}\right|_H$.

Assuming $C_N \approx \gamma T$ (normal metal at low temperatures), we find

$$C_S \approx \left(\gamma - \frac{H_0^2}{4\pi T_c^2}\right) T + \frac{3H_0^2}{4\pi T_c^4} T^3$$

Experimentally, C_S is exponentially suppressed therefore $\gamma = \frac{H_0^2}{4\pi T_c^2}$

$$\left.\frac{C_S}{C_N}\right|_{T=T_c} = 3$$

London theory


In 1934, Gorter and Casimir introduced a two-component model for superconductors:

- "superconducting" electrons with density n_s
- "normal" electrons with density n_n.

In 1935, Fritz and Heinz London proposed the constitutive equation: $\nabla \times \boldsymbol{j} = -\dfrac{n_s e^2}{m}\boldsymbol{B}$

Using the curl of Maxwell-Ampere's equation $\nabla \times \boldsymbol{B} = \dfrac{4\pi}{c}\boldsymbol{j}$ and noting that $\nabla \times \nabla \times \boldsymbol{B} = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2\boldsymbol{B} = -\nabla^2\boldsymbol{B}$ lead to

$$\lambda_L \nabla^2 \boldsymbol{B} = \boldsymbol{B} \text{ with } \lambda_L = \sqrt{\frac{mc^2}{4\pi n_s e^2}}.$$

Application: semi-infinite superconductor

Let us consider a semi-infinite superconducting material occupying the space with $x \geq 0$. A magnetic field is applied along the z axis. The solution of London's equation in the superconductor is given by

$$\lambda_L \frac{d^2 B}{dx^2} = B \Rightarrow B(x) = B(0)\exp(-x/\lambda_L).$$

The magnetic field penetrates inside the superconductor only within distances of the order of λ_L, called the **London penetration length**.

The electron current is mainly located in the surface since

$$j_y(x) = -\frac{c}{4\pi}\frac{dB}{dx} = \frac{cB(0)}{4\pi\lambda_L}\exp(-x/\lambda_L),$$ and $j_x = j_z = 0$.

Note that in thin films with thickness $d \ll \lambda_L$, the Meissner-Ochsenfeld effect is not complete therefore the thermodynamic approach breaks down. The critical field H_c parallel to the film is very high.

Pippard theory

London's equation can be alternatively written as $\boldsymbol{j}(\boldsymbol{r}) = -\frac{n_s e^2}{m}\boldsymbol{A}(\boldsymbol{r})$.

Brian Pippard introduced the **coherence length** ξ to allow for **non-local effects**:

$$\boldsymbol{j}(\boldsymbol{r}) = -\frac{3n_s e^2}{4\pi m \xi_0}\int d^3r' \, \frac{\boldsymbol{R}(\boldsymbol{R}\cdot\boldsymbol{A}(\boldsymbol{r'}))}{R^4}\exp\left(-\frac{R}{\xi}\right)$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r'}$.

London's theory corresponds to the limit $\xi \to 0$:

$$j_i(\boldsymbol{r}) \approx -\frac{3n_s e^2}{4\pi m \xi_0} A_j(\boldsymbol{r}) \int d^3r' \, \frac{R_i R_j}{R^4}\exp\left(-\frac{R}{\xi_0}\right)$$

Pippard theory

London's equation can be alternatively written as $\boldsymbol{j}(\boldsymbol{r}) = -\frac{n_s e^2}{m}\boldsymbol{A}(\boldsymbol{r})$.

Brian Pippard introduced the **coherence length** ξ to allow for **non-local effects**:

$$\boldsymbol{j}(\boldsymbol{r}) = -\frac{3n_s e^2}{4\pi m\xi_0}\int d^3r' \, \frac{\boldsymbol{R}(\boldsymbol{R}\cdot\boldsymbol{A}(\boldsymbol{r'}))}{R^4}\exp\left(-\frac{R}{\xi}\right)$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r'}$.

London's theory corresponds to the limit $\xi \to 0$:

$$j_i(\boldsymbol{r}) \approx -\frac{3n_s e^2}{4\pi m\xi_0}A_j(\boldsymbol{r})\int d^3R\,\frac{1}{3}\delta_{ij}\frac{\exp\left(-R/\xi_0\right)}{R^2}$$

Pippard theory

London's equation can be alternatively written as $\boldsymbol{j}(\boldsymbol{r}) = -\frac{n_s e^2}{m}\boldsymbol{A}(\boldsymbol{r})$.

Brian Pippard introduced the **coherence length** ξ to allow for **non-local effects**:

$$\boldsymbol{j}(\boldsymbol{r}) = -\frac{3n_s e^2}{4\pi m\xi_0}\int d^3r'\,\frac{\boldsymbol{R}(\boldsymbol{R}\cdot\boldsymbol{A}(\boldsymbol{r'}))}{R^4}\exp\left(-\frac{R}{\xi}\right)$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r'}$.

London's theory corresponds to the limit $\xi \to 0$:

$$\boldsymbol{j}(\boldsymbol{r}) \approx -\frac{nq^2}{m\xi_0}\boldsymbol{A}(\boldsymbol{r})\int_0^{+\infty} dR\,\exp\left(-\frac{R}{\xi_0}\right) \to -\frac{n_s e^2}{m}\boldsymbol{A}(\boldsymbol{r})$$

Pippard theory

London's equation can be alternatively written as $\boldsymbol{j}(\boldsymbol{r}) = -\frac{n_s e^2}{m}\boldsymbol{A}(\boldsymbol{r})$.

Brian Pippard introduced the **coherence length** ξ to allow for **non-local effects**:

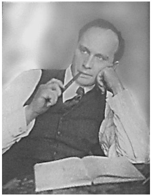
$$\boldsymbol{j}(\boldsymbol{r}) = -\frac{3 n_s e^2}{4\pi m \xi_0} \int d^3 r' \, \frac{\boldsymbol{R}(\boldsymbol{R} \cdot \boldsymbol{A}(\boldsymbol{r'}))}{R^4} \exp\left(-\frac{R}{\xi}\right)$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r'}$.

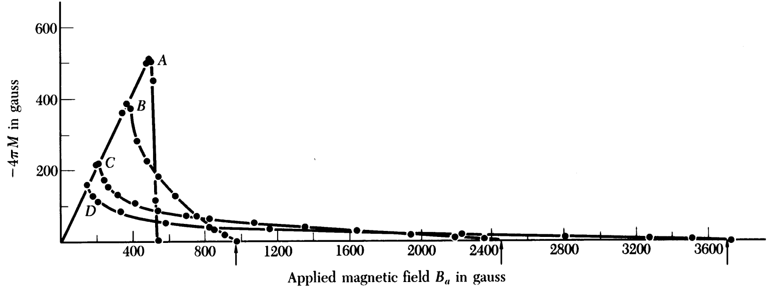
London's theory corresponds to the limit $\xi \to 0$:

London-Pippard's theory explains the electrodynamics of superconductors.

"Soft" vs "hard" superconductors

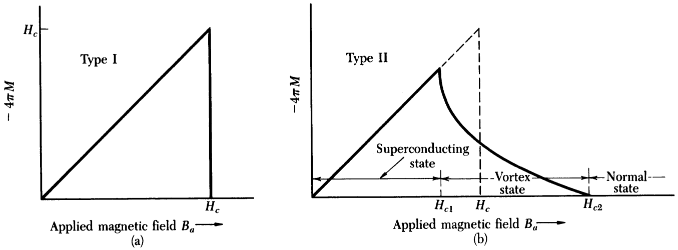


In 1935, Lev Vasilievich Shubnikov at the Kharkov Institute of Science and Technology in Ukraine discovered that some so called "hard" or type II superconductors (as opposed to "soft" or type I superconductors) exhibit **two critical fields**.



Superconducting magnetization curves of annealed polycrystalline lead and lead-indium alloys at 4.2 K. (A) lead; (B) lead-2.08 wt. % indium; (C) lead-8.23 wt. % indium; (D) lead-20.4 wt.% indium. From Kittel, Introduction to Solid State Physics.

"Soft" vs "hard" superconductors



Kittel, Introduction to Solid State Physics

The Meissner effect is incomplete between H_{c1} and H_{c2} ($B \neq 0$).

- H_{c2} is generally much higher than H_c in "soft" superconductors, T_c is also higher. "Hard" superconductors are thus used to generate strong magnetic fields.
- H_{c2} is limited by spin paramagnetism of conduction electrons, see *Clogston, PRL 9, 266 (1962).*
- Except for vanadium, technetium and niobium, "hard" superconductors consist of metallic compounds and alloys.

Hydrodynamic approach

A superconductor can be treated as a mixture of two fluids:

- a viscous fluid of "normal" electrons (and ions),
- an **irrotational ideal fluid of charged particles**: $\nabla \times \boldsymbol{\pi_s} = \mathbf{0}$ where $\boldsymbol{\pi_s} = m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A}$ is the generalized momentum per particle.

- The fluid remains irrotational at any time since

$$\frac{\partial}{\partial t}\nabla \times \boldsymbol{\pi_s} = \nabla \times (\boldsymbol{v} \times \nabla \times \boldsymbol{\pi_s}) = \mathbf{0}$$

- Using $\boldsymbol{j} = n_s q \boldsymbol{v}$ leads to London's equation $\nabla \times \boldsymbol{j} = -\frac{n_s q^2}{m}\boldsymbol{B}$
- The condition $\nabla \times \boldsymbol{\pi_s} = \mathbf{0}$ is more fundamental: it explains the induction of a uniform magnetic field $\boldsymbol{B} = -\frac{2m}{q}\boldsymbol{\omega}$ inside a superconductor rotating at the angular velocity $\boldsymbol{\omega}$.

Discovery of superfluidity

During the 1930s, it was found by several groups that below $T_\lambda = 2.17$ K, helium does not behave like an ordinary liquid.

"*by analogy with superconductors, the helium below the* λ*-point enters a special state which might be called* ***superfluid***."

Kapitza, Nature 141, 74 (1938).

Kapitza received the Nobel Prize in 1978.

"*the observed type of flow most certainly cannot be treated as laminar or even as ordinary turbulent flow.*"

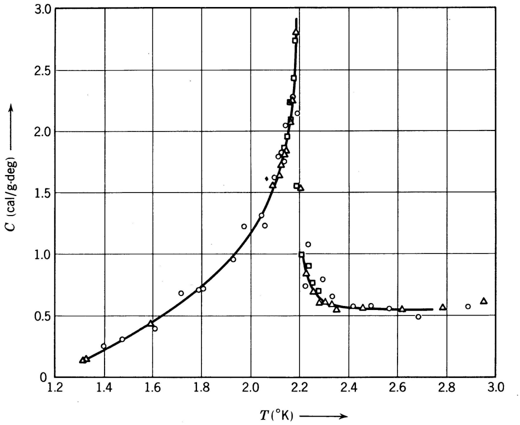
Allen and Misener, Nature 141, 75 (1938).

About the history of superfluidity:

Balibar in "History of Artificial Cold, Scientific, Technological and Cultural Issues", Boston Studies in the Philosophy and History of Science 299 (Springer, 2014), pp.93-114; Balibar, J. Low Temp. Phys. 146, 441 (2007).

Lambda point

The specific heat of helium exhibits a sudden change at $T_\lambda = 2.17$ K:



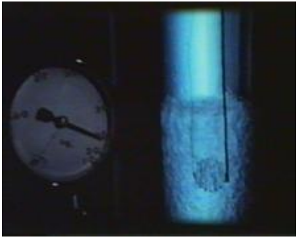
Keesom and Clusius, Proc. Roy. Acad. Amsterdam 35, 307 (1932).

Singularities in the specific heat are generally associated with order-disorder transitions (e.g. ferromagnetic transition).

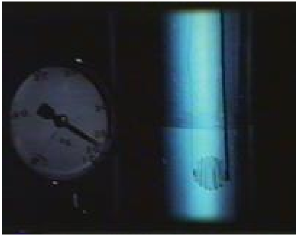
Heat transport in He II

Contrary to ordinary liquids,

- He II does not follow Fourier's law for the heat current $\mathcal{J} = -\lambda \nabla T$, except in extremely fine slits or capillaries. Actually, the ratio $\mathcal{J}/|\nabla T|$ diverges as $|\nabla T| \to 0$!
- He II does not boil:



$T > T_\lambda$



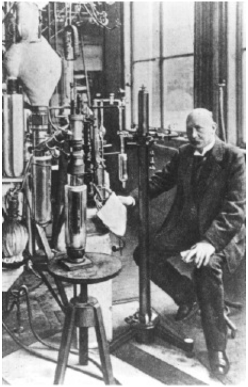
$T < T_\lambda$

http://www.lps.ens.fr/~balibar/Allen-boiling.mpg

"super heat conductivity", Keesom.

Heat in He II is not transported according to classical laws.

Onnes and Dana observations about liquid helium



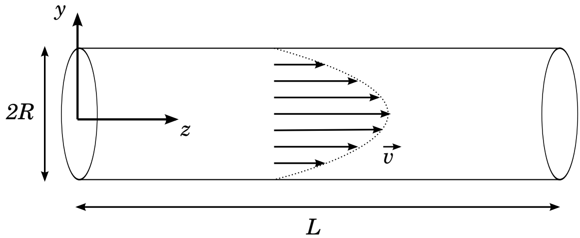
Incidentally, Kamerlingh Onnes and his collaborators also discovered superfluidity without realizing it the same day they discovered superconductivity in April 1911!

Onnes noted about liquid helium:
"*Just before the lowest temperature [about 1.8 K] was reached, the boiling suddenly stops...*"

Leo Dana, a visiting student at Onnes' lab measured the lambda transition in the specific heat in 1922 but no one paid attention!

About the history of superconductivity:
van Delft&,Kes, Phys. Today, 63, 9, 38 (2010)

Intermission: viscous flow



In ordinary liquids, the mass flow Q of liquid through a pipe of length $L \gg R$ is given by **Hagen-Poiseuille law**

$$Q = \rho \frac{\pi R^4 |\Delta P|}{8 \eta L}$$

where ρ is the mass density, $|\Delta P|$ is the pressure drop and η the shear viscosity.

"Superfluidity"

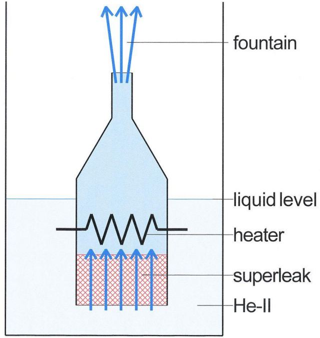
- **"superleak"**: He II can flow without resistance through very narrow slits and capillaries, almost independently of the pressure drop.
- **"superflow"**: persistent flow of He II (note the similarity with persistent currents in superconductors)

  Reppy and Depatie, PRL 12, 187 (1964)
- "superfluidity" disappears beyond some **critical velocity** (note the similarity with critical currents in superconductors)
- on the other hand, He II exhibits similar viscosity as He I in experiments with oscillating disks.

`http://www.lps.ens.fr/~balibar/Allen-superflow.mpg`

He II does not follow the classical laws of hydrodynamics.

Fountain effect



Allen and Jones, Nature 141, 243 (1938)

`http://www.lps.ens.fr/~balibar/Allen-fountain.mpg`

The superfluid flows from the cooler to the hotter region. From the second law of thermodynamics, we thus conclude that the superfluid carries no heat (no entropy).

Hess-Fairbank effect

Fritz London predicted in 1954 the analog of the Meissner effect for superfluid helium, which was experimentally observed by Hess and Fairbank in 1967.

Hess and Fairbank, PRL 19, 216 (1967)

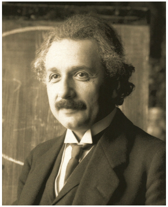
Initially at rest, He II remains at rest if the container is set into (slow) rotation as for a perfect fluid with no viscosity.

Now, liquid helium is initially set into rotation with angular frequency $\omega < \omega_c$.

- at $T > T_\lambda$ the liquid rotates classically with angular momentum $L_0 = I_0\,\omega$ where I_0 is the moment of inertia
- at $T < T_\lambda$ the superfluid rotates with a reduced angular momentum $L(T) = I(T)\omega$ with $I(T) < I_0$ and $I(0) = 0$.

This phenomenon shows that a superfluid is not just a perfect fluid but corresponds to a new thermodynamic state of matter.

Superfluidity and Bose-Einstein condensation



Satyendra Nath Bose and Albert Einstein predicted in 1925 that at low enough temperatures an ideal gas of bosons condense into a **macroscopic quantum state**. But this prediction was largely ignored or considered as incorrect.

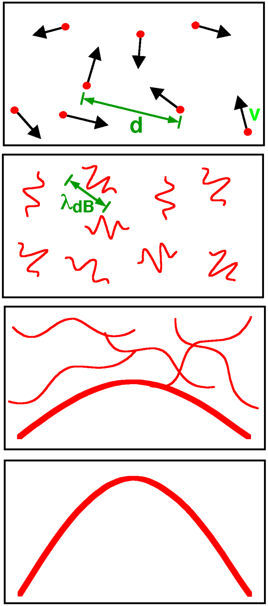
Einstein himself was skeptical: *"The theory is pretty, but is there anything true in it?"*

The connection with superfluidity was first advanced by Fritz London in 1938:

"a model which is so far from reality that it simplifies liquid helium to an ideal gas [...] [but] it seems difficult not to imagine a connection with the condensation phenomenon of Bose–Einstein statistics."

London, Nature 141, 643 (1938)

What is Bose-Einstein condensation (BEC)?



High Temperature T:

thermal velocity v
density d^{-3}

"Billiard balls"

Low Temperature T:

De Broglie wavelength
$\lambda_{dB} = h/mv \propto T^{-1/2}$

"Wave packets"

$T = T_{crit}$:
Bose-Einstein Condensation

$\lambda_{dB} \approx d$

"Matter wave overlap"

T=0:
Pure Bose condensate

"Giant matter wave"

Illustration of BEC from MIT group

Ideal Bose gas

Let us consider an ideal Bose gas of N noninteracting particles. At $T = 0$, all particles lie in the lowest single-particle energy state $\varepsilon = 0$. The occupancy of this state still remains macroscopic at temperature

$$T < T_c = \frac{2\pi\hbar^2}{m\zeta(3/2)^{2/3}} n^{2/3} \approx 3 \text{ K for helium}$$

At $T > 0$, the occupancy of the state $\varepsilon_0 = 0$ is given by the Bose-Einstein distribution $N_0 = \dfrac{1}{\exp[\beta(\varepsilon_0 - \mu)] - 1}$, where $\beta = 1/(k_B T)$ and the chemical potential $\mu \sim -\dfrac{k_B T}{N} \to 0$ as $T \to 0$.

The occupancy of excited states $\varepsilon > \varepsilon_0$ is given by

$\displaystyle\int_0^{+\infty} d\varepsilon \frac{\mathcal{N}(\varepsilon)}{\exp(\beta\varepsilon) - 1} \approx N \left(\frac{T}{T_c}\right)^{3/2}$ with $\mathcal{N}(\varepsilon)$ the density of states.

At $T = T_c$, $N_0 = 0$, while at $T = 0$, $N_0 = N$. In liquid helium, $N_0/N \approx 6 - 8\%$ at $T = 0$ due to interactions between atoms.

Two-fluid model

Following the suggestion of Fritz London that superfluidity is related to Bose-Einstein condensation, Laszlo Tisza postulated that **He II contains two disctinct components:**

- a superfluid that carries no entropy (condensate)
- a normal viscous fluid.

This model explained all the observed phenomena and predicted thermomechanical effects like **"temperature waves"**. *Tisza, Nature 141, 913 (1938).*

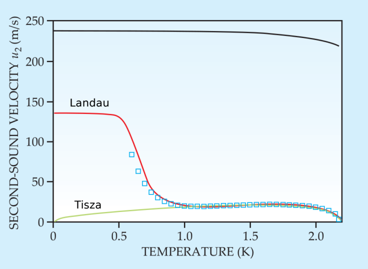
Although Landau did not believe that superfluidity is related to BEC (he never cited F. London!), he developed the two-fluid model based on **"quasiparticle" excitations in quantum fluids**.

Landau, Phys. Rev. 60, 356 (1941)

About the history: *Balibar, C. R. Physique 18, 586 (2017)*

Landau vs Tisza and London

Although the two-fluid models of Tisza and Landau were very similar, they led to **different predictions** for the speed u_2 of temperature waves (which Landau called “second sound”) at low temperatures.



Measurements by Vasilii Peshkov in 1960 showed that Landau was right.

Peshkov, Sov. Phys. JETP 11, 580 (1960).

Donnely, Physics Today 62, 34 (2009).

But London and Tisza original ideas that superfluidity is related to BEC later proved to be correct.

Tisza considered that the normal fluid was made of non-condensed atoms while for Landau it was made of “quasiparticles”. The density of non-condensed atoms is a property of the liquid at rest (ground state) while the density of “quasiparticles” is a property of the superflow (excited state).

Landau's theory of He II

Although helium atoms are strongly interacting, Landau assumed that at low temperatures He II can be described in terms of weakly-interacting **"quasiparticles"**, which do not correspond to material particles but to complex many-body motions (excitations).

Let us consider a *macroscopic* body of mass M_0 flowing through the superfluid. At low T, its velocity $\boldsymbol{V}$ can be changed if a quasiparticle of energy $E(p)$ and momentum $\boldsymbol{p}$ is created.

- energy conservation $\frac{1}{2}M_0 V^2 > \frac{1}{2}M_0 V'^2 + E(p)$
- momentum conservation $M_0\boldsymbol{V} = M_0\boldsymbol{V'} + \boldsymbol{p}$

$$\Rightarrow E(p) < \boldsymbol{V}\cdot\boldsymbol{p} - \frac{p^2}{2M_0} \approx \boldsymbol{V}\cdot\boldsymbol{p} \text{ since } M_0 \text{ is macroscopic.}$$

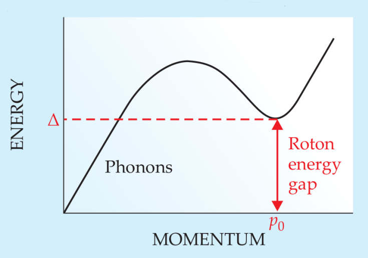
The flow is resistanceless if $V < V_c = \min\left\{\frac{E(p)}{p}\right\}$.

Landau's theory of He II

For a gas of noninteracting particles, the "quasiparticle" excitations have energies $E(p) = \dfrac{p^2}{2m}$ therefore $V_c = 0$: the ideal Bose gas is not superfluid.

For He II, Landau assumed two different kinds of "quasiparticles":

- **phonons** at low p
  $E(p) \approx c_s p$ (sound waves)
- **rotons** at high p
  $E(p) \approx \Delta + \dfrac{(p - p_0)^2}{2m_0}$



The critical velocity is given by $V_c = \dfrac{\Delta}{p_0} \approx 60\ \mathrm{m\ s^{-1}}$. This value was confirmed by ion propagation experiments.

Ellis & McClintock, Philos. Trans. R. Soc. London, Ser. A, 315, 259 (1985).

Phonons and rotons

In 1947, Bogoliubov calculated the energy of quasiparticles in a **weakly interacting dilute Bose gas** using many-body techniques:

$$E(p) = \sqrt{\left(\frac{p^2}{2m}\right)^2 + p^2 c_s^2} \approx c_s p \text{ at low } p$$

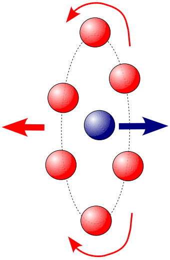
J. Phys.(USSR) 11, 23 (1947)

This shows that a BEC of *interacting* particles is superfluid.

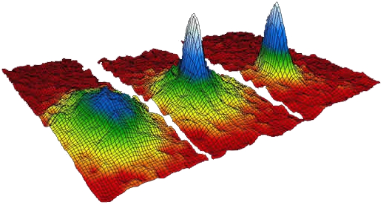
Landau thought that rotons are related to vortices. Feynman argued that rotons are **atomic size "smoke rings"**.

Rotons have also been interpreted as a characteristic feature of density fluctuations marking the **onset of crystallization** ("ghosts of Bragg spots", Nozières).

J. Low Temp. Phys. 137, 45 (2004).

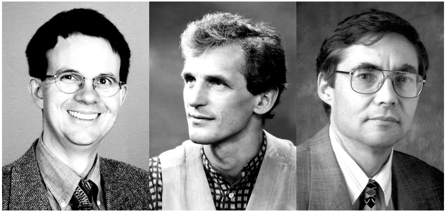


Bose-Einstein condensation in dilute atomic gases



On June 5, 1995, the first dilute BEC was produced by Eric Cornell and Carl Wieman at the University of Colorado at Boulder NIST-JILA, with $\sim$ 2000 rubidium ^{87}Ru atoms cooled to 170 nK.

Shortly thereafter, Wolfgang Ketterle's team at MIT obtained a BEC of $\sim 5 \times 10^5$ sodium ^{23}Na atoms cooled to 2 μK.



For their achievements, Cornell, Ketterle and Wieman were awarded the 2001 Nobel Prize in Physics.

BEC have been produced by other groups using various kinds of atoms and their superfluid properties have been demonstrated.

Flow quantization and vortices

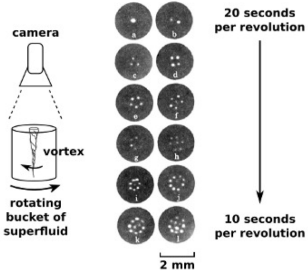
Onsager-Feynman quantization of the superflow:

$$\oint \boldsymbol{v_s} \cdot d\ell = \frac{Nh}{m} \text{ with } N = 0, 1, \text{ etc.}$$

where $\boldsymbol{v_s}$ is the "superfluid velocity" introduced by Landau.

A superfluid rotating at angular frequency ω in a bucket of radius R is threaded by $N = \dfrac{2m\pi R^2\omega}{h}$ **quantized vortex lines**, each carrying an angular momentum $\hbar$.

In between vortices, the flow is "irrotational" $\nabla \times \boldsymbol{v_s} = 0$.

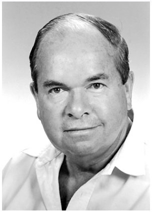


Yarmchuk et al., PRL43, 214 (1979)

$\boldsymbol{v_s} = \boldsymbol{\pi_s}/m$ is actually a momentum: the Onsager-Feynman condition is nothing but the **Bohr-Sommerfeld quantization** $\oint \boldsymbol{\pi_s} \cdot d\ell = Nh$.

Carter & Khalatnikov,Phys.Rev.D45,4536(1992)

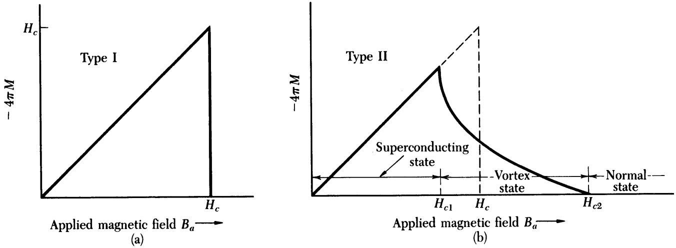
Abrikosov state



In 1957, Alekseï Alekseïevitch Abrikosov predicted that a "hard" superconductor is threaded by a **regular array of magnetic flux tubes** for $H_{c1} < H < H_{c2}$. He was awarded the Nobel Prize in Physics in 2003.

Abrikosov, Soviet Physics JETP 5, 1174 (1957)

Abrikosov state

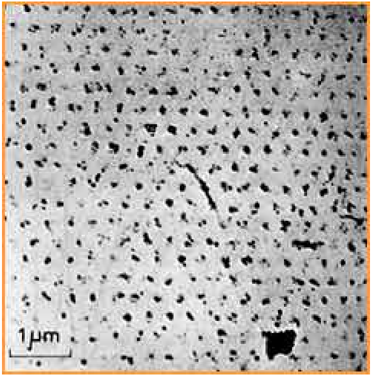


Kittel, Introduction to Solid State Physics

- Below H_{c1}, the magnetic flux is expelled inside the superconductor.
- At $H = H_{c1}$, the first magnetic flux tubes penetrate the superconductor.
- For $H_{c1} < H < H_{c2}$, flux tubes arrange themselves on a regular array with the lattice spacing determined by H (Shubnikov state) .
- At $H = H_{c2}$, the core of magnetic flux tubes overlap and superconductivity disappears.

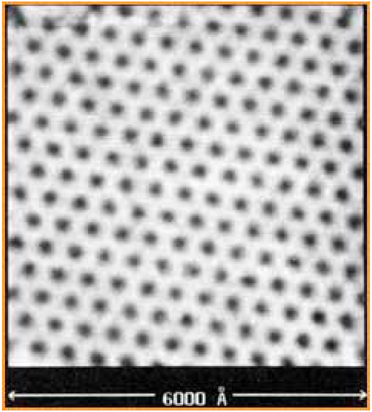
Abrikosov vortex state

Pb-4at%In



Essmann & Trauble, Phys. Lett. 24A, 526 (1967)

$NbSe_2$



Hess et al., PRL 62, 214 (1989)

Magnetic flux quantization

F. London predicted in 1948 that the magnetic flux inside a superconducting loop must be quantized.

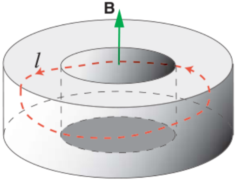


Figure 1.3: Magnetic flux through the hole in a superconductor is quantized.

This was experimentally confirmed in 1961 by Bascom Deaver (PhD) under the supervision of William Fairbank at Stanford University, and independently by Robert Doll and Martin Näbauer at the Low Temperature institute in Hersching (Bavaria).

Deaver & Fairbank, PRL 7, 43 (1961); Doll & Näbauer, PRL 7, 51 (1961)

See also *100 Years of Superconductivity, published by Horst Rogalla, Peter H. Kes, CRC Press, Taylor & Francis group, 2012, p.161*

Magnetic flux quantization

Let us consider the **Bohr-Sommerfeld quantization rule**:

$$\oint \boldsymbol{\pi_s} \cdot \boldsymbol{d\ell} = Nh \text{ with } N = 0, 1, \text{ etc.}$$

With $\boldsymbol{j} = nq\boldsymbol{v}$ and $\boldsymbol{\pi_s} = m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A}$,

$$\oint \left(m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A} \right) \cdot \boldsymbol{d\ell} = \frac{m}{nq} \int \nabla \times \boldsymbol{j} \cdot \boldsymbol{dS} + \frac{q}{c} \int \boldsymbol{B} \cdot \boldsymbol{dS},$$

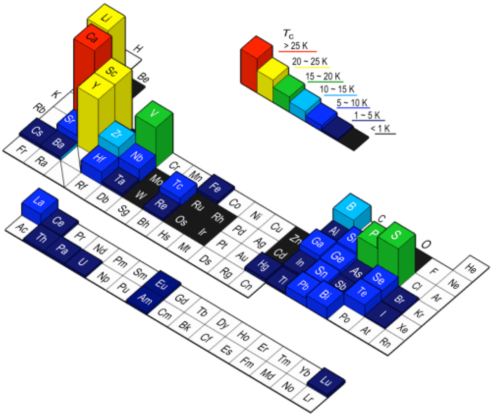
However, $\boldsymbol{B} = 0$ therefore $\nabla \times \boldsymbol{j} = 0$ (Meissner effect).

$$\Rightarrow \Phi = \int \boldsymbol{B} \cdot \boldsymbol{dS} = N\Phi_0 \text{ with } \Phi_0 = hc/q.$$

Remarks:

- Φ_0 is called a **"fluxoid" or "fluxon"**.
- $\Phi = \Phi_{\mathrm{ext}} + \Phi_{\mathrm{s}}$. Since Φ_{ext} is not quantized, Φ_{s} must adjust itself accordingly !
- Experimentally $\Phi_0 = hc/(2e)$ therefore the **superconducting particles carry a charge $q = 2e$**.
- The superconducting current will persist unless the flux changes.

Superconducting elements

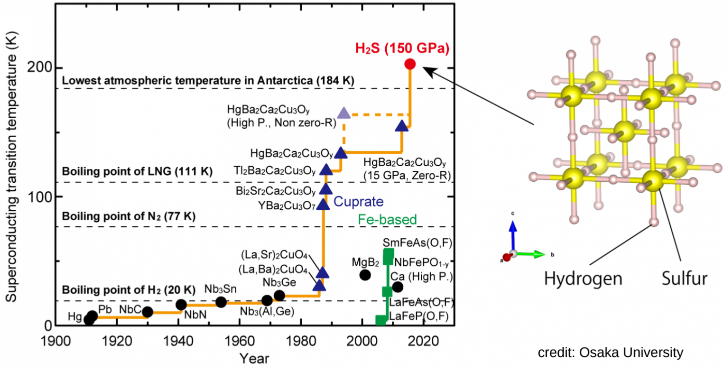


Credit: Osaka University

Copper, silver and gold, three of the best metallic conductors, are not superconducting ! The microscopic explanation had to wait for the Bardeen-Cooper-Schrieffer (BCS) theory in 1957.

Timeline of superconductor discoveries

High-T_c cuprate superconductors were discovered in 1986 by IBM researchers G. Bednorz and K.A. Müller (Nobel Prize in 1987).

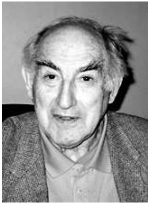


credit: Osaka University

Very recently LaH_{10} has been found to be superconducting (under high pressures) at almost room temperature ($T_c = 260$ K)!

Somayazulu et al., PRL 122, 027001 (2019)

Towards a microscopic theory of superconductivity



In 1950, Landau and Ginzburg developed a **phenomenological theory** of superconductivity.

Landau received the Nobel Prize in 1962. Ginzburg shared the 2003 Nobel Prize in Physics with Abrikosov.

A **microscopic theory** was proposed in 1957 by Bardeen, Cooper and Schrieffer.

In 1959, Gorkov showed that the Ginzburg-Landau equations can be derived from the BCS theory.

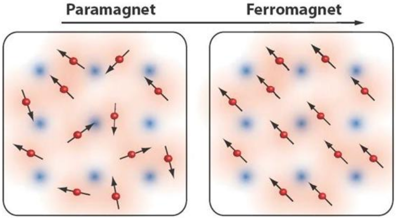
BCS shared the 1972 Nobel Prize in Physics.

Ginzburg-Landau theory of superconductivity

Second-order phase transitions are associated with **spontaneous symmetry breaking**, and can be characterized by an **order parameter** η, such that $\eta(T \geq T_c) = 0$ and $\eta(T < T_c) \neq 0$.

Examples:

- liquid-gas phase transition at the critical point $\eta = v_{\mathrm{liq}} - v_{\mathrm{gas}}$
- ferromagnetic-paramagnetic transition $\eta = M$



Ginzburg and Landau postulated:

- η has the nature of a **wave function** Ψ,
- $|\Psi|^2 = n_s$, where n_s is the density of superconducting particles,

Ginzburg-Landau theory of superconductivity

For T close to T_c, the free energy can be expanded in a Taylor series:

$$F_S = F_N + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{2m}\left|\frac{\hbar}{\mathrm{i}}\nabla\Psi - q\boldsymbol{A}\Psi\right|^2 + \frac{B^2}{8\pi} + \dots$$

Note that $\frac{1}{2m}\left|\frac{\hbar}{\mathrm{i}}\nabla\Psi - q\boldsymbol{A}\Psi\right|^2$ is the lowest order gradient term that is gauge invariant.

Minimizing with respect to Ψ and $\boldsymbol{A}$ yields

- $\frac{1}{2m}\left[\frac{\hbar}{\mathrm{i}}\nabla - q\boldsymbol{A}\right]^2\Psi + \alpha\Psi + \beta|\Psi|^2\Psi = 0$
- $\boldsymbol{j} = \frac{q}{2m}\frac{\hbar}{\mathrm{i}}\left[\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right] - \frac{q^2}{m}|\Psi|^2\boldsymbol{A}$

Fluctuations of the order parameter

Let $\boldsymbol{A} = 0$ and $\boldsymbol{j} = 0$.

- $\boldsymbol{j} = \frac{q}{2m}\frac{\hbar}{\mathrm{i}}\left[\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*\right] = 0$ therefore Ψ is real.
- $-\frac{\hbar^2}{2m}\nabla^2\Psi + \alpha\Psi + \beta\Psi^3 = 0$, or equivalently setting $\varphi \equiv \sqrt{\frac{\beta}{|\alpha|}}\Psi$

$$\xi^2\nabla^2\varphi + \varphi(1 - \varphi^2) = 0 \text{ with } \xi \equiv \sqrt{\frac{\hbar^2}{2m|\alpha|}}$$

For a semi-infinite superconductor in $x \geq 0$, $\varphi(x) = \tanh\left(\frac{x}{\sqrt{2}\xi}\right)$.

Therefore $\varphi(0) = 0$ at the boundary between the normal and superconducting phases, while deep inside the superconductor $\varphi(x \to +\infty) = 1$ so that $\Psi(x \to +\infty) = \sqrt{n_s} = \sqrt{|\alpha|/\beta}$.

The **coherence length** ξ is the characteristic distance over which $\Psi(\boldsymbol{r})$ fluctuates (different from Pippard's coherence length).

London penetration length

Let us now assume $\Psi(\boldsymbol{r}) = \sqrt{n_s} = \sqrt{|\alpha|/\beta}$ (no spatial fluctuations) in a weak magnetic field $B \ll H_c$.

The second Ginzburg-Landau's equation reduces to London's equation :

$$\boldsymbol{j} = \frac{q}{2m}\frac{\hbar}{\mathrm{i}}\left[\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*\right] - \frac{q^2}{m}|\Psi|^2\boldsymbol{A} = -\frac{n_s q^2}{m}\boldsymbol{A}$$

$$\Rightarrow \lambda_L \nabla^2 \boldsymbol{B} = \boldsymbol{B} \text{ with } \lambda_L = \sqrt{\frac{mc^2}{4\pi n_s q^2}} = \sqrt{\frac{mc^2\beta}{4\pi|\alpha|q^2}}$$

The **London penetration length** λ_L is the characteristic distance over which $\boldsymbol{B}$ penetrates the superconductor.

Two characteristic length scales

The Ginzburg-Landau theory predicts that both λ_L and ξ scale like $|\alpha|^{-1/2} \propto (T_c - T)^{-1/2}$ but their ratio is constant

$$\kappa \equiv \frac{\lambda_L}{\xi} = \frac{mc}{q\hbar}\sqrt{\frac{\beta}{2\pi}}$$

One can show that

- if $\kappa < 1/\sqrt{2}$, the superconductor is "soft",
- if $\kappa > 1/\sqrt{2}$, the superconductor is "hard".

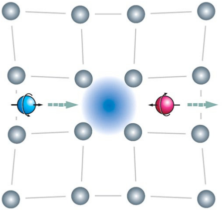
Roughly speaking, at H_{c1} the first fluxoid nucleates. It carries a quantum flux Φ_0: the magnetic field inside is $\sim H_{c1}$ and extends over a distance $\sim \lambda_L$. At H_{c2}, fluxoids are the most densely packed with a spacing $\sim \xi$ and the magnetic field penetrates almost uniformly the superconductor.

Therefore $H_{c1} \sim \dfrac{\Phi_0}{\pi\lambda_L^2}$ and $H_{c2} \sim \dfrac{\Phi_0}{\pi\xi^2}$. Note that $\dfrac{H_{c2}}{H_{c1}} \sim \dfrac{\lambda_L}{\xi}$.

If $\xi \gtrsim \lambda_L$, fluxoids cannot form.

BCS theory of superconductivity

The discovery of the **isotope effect**, $T_c \propto M^{-\alpha}$, suggested that crystal lattice dynamics play a role in superconductivity.



In a superconductor, the dynamical distorsions of the crystal lattice (phonons) can induce an **attractive effective interaction** between electrons of opposite spins.

Electrons form **pairs** which behave like **bosons** and can thus **condense** below a certain critical temperature. A superconductor can thus be viewed as a charged superfluid.

This suggested that fermionic atoms could also be superfluid. Osheroff found in 1971 that ^{3}He is superfluid below 2.5 mK.

Effective electron-electron interaction

Two electrons in vacuum repel each other due to the instantaneous Coulomb interaction $V(\boldsymbol{r_1}, t_1, \boldsymbol{r_2}, t_2) = \frac{e^2}{r}\delta(t)$ with $r = |\boldsymbol{r_1} - \boldsymbol{r_2}|$ and $t = t_1 - t_2$.

$$\widetilde{V}(\boldsymbol{q}, \omega) = \frac{1}{\Omega}\int dt \int d^3r\, V(\boldsymbol{r})e^{-\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{r}+\omega t)} = \frac{4\pi e^2}{\Omega q^2}.$$

Two conduction electrons in a solid interact with other electrons and with ions. Their "bare" interaction is thus "dressed" by the medium.

Typical scales in a solid:

- conduction electrons of density n (Fermi gas)
  Fermi energy $\varepsilon_F = \frac{1}{2}mv_F^2$ where $v_F = \frac{\hbar k_F}{m}$ is the Fermi velocity and $k_F = (3\pi^2 n)^{1/3}$
- low-energy longitudinal lattice vibrations (phonons)

  ion plasma frequency $\omega_p = \sqrt{\frac{4\pi Z^2 e^2 n_{\mathrm{I}}}{M}}$

Bardeen-Pines interaction

Approximating a solid by a "jelium", the **effective interaction** between electrons is approximately given by ($q \ll k_F$, $\omega \ll c_s k_F$, $c_s \ll v_F$)

$$V_{\mathrm{eff}}(\boldsymbol{q},\omega) = \underbrace{\frac{4\pi e^2}{q^2+q_{\mathrm{TF}}^2}}_{\text{screening}} + \underbrace{\frac{4\pi e^2}{q^2+q_{\mathrm{TF}}^2}\frac{\omega(\boldsymbol{q})^2}{\omega^2-\omega(\boldsymbol{q})^2}}_{\text{polarization}}$$

where $\omega(\boldsymbol{q}) \approx c_s q$, $c_s = \omega_p / q_{\mathrm{TF}}$ is the sound speed, and

$q_{\mathrm{TF}} = \sqrt{4\pi e^2 \dfrac{\partial n}{\partial \mu}}$ is the Thomas-Fermi wave vector.

- **Charge screening** makes the Coulomb interaction much less repulsive at large distances $V(r) = \dfrac{e^2}{r} \to V_{\mathrm{eff}}(r) = \dfrac{e^2}{r} e^{-q_{\mathrm{TF}} r}$
- **Polarization of the ion lattice** leads to retarded interaction: the distortion of the lattice induced by the first electron is felt at a later time by the second electron.

The effective electron-electron interaction induced by the polarization of the ion lattice is **attractive** for $\omega < \omega(\boldsymbol{q})$ and repulsive otherwise.

Cooper pairing and BCS gap equations

In conventional superconductors, electron with opposite spins form independent pairs with no momentum ($^1S_0$ pairing)

Each electron has a momentum $\hbar \boldsymbol{k}$ and an energy $\varepsilon(\boldsymbol{k}) = \dfrac{\hbar^2 k^2}{2m} - \varepsilon_F$.

The binding energy of a pair $(\boldsymbol{k}\uparrow, -\boldsymbol{k}\downarrow)$ is determined by the **highly non-linear** BCS gap equations

$$\Delta(\boldsymbol{k}) = -\sum_{\boldsymbol{k}'} \widetilde{V}_{\text{eff}}(\boldsymbol{k}' - \boldsymbol{k}) \frac{\Delta(\boldsymbol{k}')}{2E(\boldsymbol{k}')} \tanh\left[\frac{E(\boldsymbol{k}')}{2k_B T}\right],$$

$\widetilde{V}_{\text{eff}}(\boldsymbol{k}' - \boldsymbol{k})$ is the pairing interaction and $E(\boldsymbol{k}) \equiv \sqrt{\varepsilon(\boldsymbol{k})^2 + \Delta(\boldsymbol{k})^2}$

The solution of the BCS gap equations is highly **nonlinear**:

With Cooper interaction $\widetilde{V}_{\text{eff}}(|\varepsilon| < \hbar\omega_D) = -V_0/\Omega$, where $\omega_D = q_D c_s$ is the Debye frequency and $q_D = (6\pi^2 n_{\mathrm{I}})^{1/3}$, one finds

$$\Delta(T = 0) \approx 2\hbar\omega_D \exp\left(-\frac{1}{\mathcal{N}(0) V_0}\right)$$

where $\mathcal{N}(0)$ is the density of states on the Fermi surface.

Beyond BCS

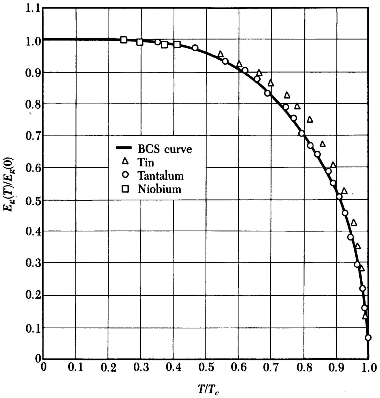
The BCS theory has been very successful in describing so called conventional superconductors with low T_c (weak coupling).

Universal relations

$$\frac{T_c}{\Delta(0)} = \frac{\exp(\gamma)}{\pi}$$

$$\frac{C_S}{C_N} \approx 1 + \frac{3}{2}\delta\exp(-2\gamma) \approx 2.5$$

(Euler-Mascheroni constant $\gamma \approx 0.577$).



Kittel, Introduction to Solid State Physics

The BCS theory is a mean-field approach. It was later reformulated and extended using quantum field theory.

Theory of inhomogeneous superconductors

The extension of the BCS theory to inhomogeneous superconductors leads to the **Bogoliubov-de Gennes equations**:

$$\begin{pmatrix} h_0(\boldsymbol{r}) + U(\boldsymbol{r}) & \Delta(\boldsymbol{r}) \\ \Delta(\boldsymbol{r})^* & -h_0(\boldsymbol{r})^* - U(\boldsymbol{r})^* \end{pmatrix} \begin{pmatrix} \varphi_{1k}(\boldsymbol{r}) \\ \varphi_{2k}(\boldsymbol{r}) \end{pmatrix} = E_k \begin{pmatrix} \varphi_{1k}(\boldsymbol{r}) \\ \varphi_{2k}(\boldsymbol{r}) \end{pmatrix}$$

$$h_0(\boldsymbol{r}) \equiv -\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla} - \frac{\mathrm{i}q}{\hbar c}\boldsymbol{A}\right)^2 - \varepsilon_F \text{ is the kinetic operator}$$

$U(\boldsymbol{r}) = U[n(\boldsymbol{r})]$ and $\Delta(\boldsymbol{r}) = \Delta[\tilde{n}(\boldsymbol{r})]$ are the **mean-field potentials**

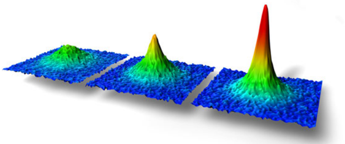
$$n(\boldsymbol{r}) = \sum_k \left\{ f_k |\varphi_{1k}(\boldsymbol{r})|^2 + (1 - f_k)|\varphi_{2k}(\boldsymbol{r})|^2 \right\}$$ is the **“normal” density**

$$\tilde{n}(\boldsymbol{r}) = \sum_k (2f_k - 1)\varphi_{2k}(\boldsymbol{r})\varphi_{1k}(\boldsymbol{r})^*$$ is the **“abnormal” density**

$$f_k = \frac{1}{1 + \exp(E_k/k_B T)}$$ is the Fermi occupation factor

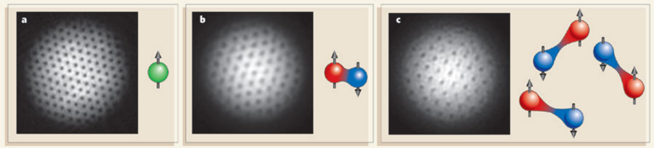
Fermionic condensates: from BEC to BCS

On December 16, 2003, the first dilute fermionic condensate was produced by Deborah Jin at JILA with 500 000 potassium ^{40}K atoms cooled to 50 nK.



By varying the pairing interaction with a magnetic field, it is possible to study the **crossover from a BEC to a BCS state**.

Leggett&Zhang in "The BCS-BEC Crossover and the Unitary Fermi Gas", Lecture Notes in Physics 836 (Springer, 2012), pp. 33-47



Quantized vortices in: (a) a BEC of bosonic sodium atoms, a fermionic condensate of ^{6}Li atoms in the BEC (b) and BCS (c) states.

Zwierlein et al, Nature 435, 1047 (2005)

Part 1: Summary

Superfluids and superconductors (charged superfluids) **cannot be explained by classical hydrodynamics/electromagnetism**.

Superconductivity and superfluidity are intimately related **macroscopic quantum phenomena**:

- absence of electric resistance/viscosity,
- persistent current/flow in rings,
- Meissner-Ochsenfeld effect ($\boldsymbol{B} = 0$)/Hess-Fairbank effect ($\boldsymbol{L} = 0$),
- critical current/velocity,
- quantized fluxoids/vortices $\oint \boldsymbol{\pi_s} \cdot \boldsymbol{d\ell} = Nh$

Superfluidity and superconductivity are associated with **Bose-Einstein condensation**. In the case of fermions, the condensation proceeds through the formation of pairs.