

# Probing Nuclear Superfluidity with Neutron Stars

Nicolas Chamél

Institute of Astronomy and Astrophysics  
Université Libre de Bruxelles, Belgium



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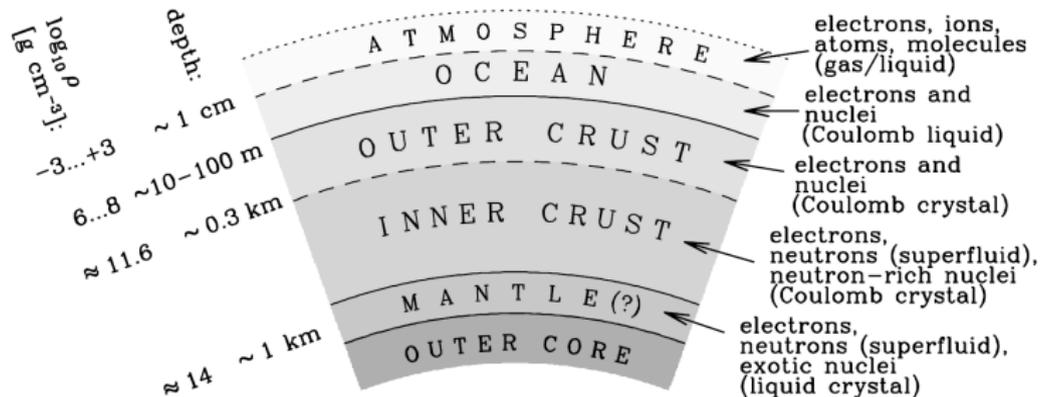
Karpacz, 25 February 2020

## Part 2: Superfluidity and superconductivity in neutron stars

# Neutron stars

With a mass of about  $1 - 2M_{\odot}$  and a radius of about 10 km, neutron stars are the most compact stars in the Universe. Their average density exceeds that of atomic nuclei!

Their interior is stratified into distinct layers:



Haensel, Potekhin, Yakovlev, "Neutron Stars" (Springer, 2007)

Blaschke&Chamel, *Astrophys. Space Sci. Lib.* 457, eds L. Rezzolla, P. Pizzochero, D. I. Jones, N. Rea, I. Vidaña p. 337-400 (Springer, 2018), arXiv:1803.01836

# Electron superconductivity in neutron stars

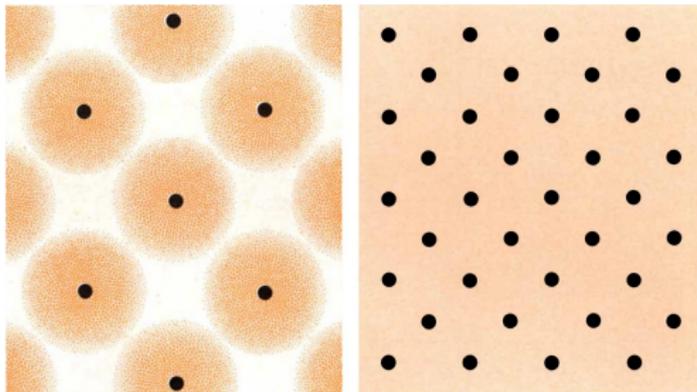
The surface layers of non-accreting neutron stars are mostly composed of iron, the end-product of stellar nucleosynthesis.

**Iron is superconducting** at density  $\rho \simeq 8.2 \text{ g cm}^{-3}$  with  $T_c \simeq 2 \text{ K}$ , much lower than neutron-star surface temperatures.

*Shimizu et al., Nature 412, 316 (2001)*



In deeper layers ( $\rho \gtrsim 10^4 \text{ g cm}^{-3}$ ), atoms are fully ionised by gravity:



*Ruderman, Scientific American 224, 24 (1971)*

## Electron superconductivity in neutron stars

- The critical temperature of a uniform non-relativistic electron gas in a background of ions (jellium) is given by ( $T_{\text{pi}}$  is the plasma temperature,  $v_{\text{Fe}}$  the Fermi velocity,  $a_0 \equiv \hbar^2/m_e e^2$ )

$$T_c = T_{\text{pi}} \exp(-8\hbar v_{\text{Fe}}/\pi e^2) \Rightarrow T_c \propto \exp(-\zeta(\rho/\rho_{\text{ord}})^{1/3})$$

with  $\rho_{\text{ord}} = m_u/(4\pi a_0^3/3)$  and  $\zeta$  some numerical factor.

Therefore  $T_c$  decreases with increasing density.

## Electron superconductivity in neutron stars

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with  $\rho_{\text{ord}} = m_u/(4\pi a_0^3/3)$  and  $\zeta$  some numerical factor.

Therefore  $T_c$  decreases with increasing density.

- At densities above  $\sim 10^6 \text{ g cm}^{-3}$  ( $\sim 10 \text{ m}$  below the surface), electrons become relativistic  $v_{\text{Fe}} \sim c$  so that ( $\alpha = e^2/\hbar c \simeq 1/137$ )

$$T_c = T_{\text{pi}} \exp(-8/\pi\alpha) \sim 0!$$

*Ginzburg, J. Stat. Phys. 1, 3(1969)*

Electrons in neutron stars are not superconducting.

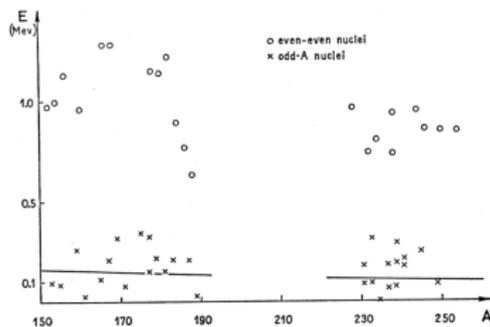
## BCS pairing of nucleons

The implications of the BCS theory (published in January 1957) for atomic nuclei were first discussed by A. Bohr, B. R. Mottelson, and D. Pines during the Summer of 1957.

*D. Pines in "BCS: 50 Years" (World Scientific, 2011), pp.85-105.*

Bohr, Mottelson, and Pines speculated that nuclear pairing might explain the **energy gap** in the excitation spectra of nuclei.

*Phys. Rev. 110, 936 (1958)*



They also anticipated that nuclear pairing could explain **odd-even mass staggering**, and the **reduced moments of inertia** of nuclei.

There is however a fundamental difference between nuclei and electrons in solids: **nuclei contain a small number of particles.**

*"the present data are insufficient to indicate the limiting value for the gap in a hypothetical infinitely large nucleus."* Bohr, Mottelson, Pines.

## Superfluidity and superconductivity in neutron stars

In the 1960's, several superconductors were known but only  $^4\text{He}$  was found to be superfluid (superfluidity of  $^3\text{He}$  was discovered at the beginning of the 1970's).



Bogoliubov, who developed a microscopic theory of superfluidity and superconductivity, was the first to explore its application to nuclear matter.

*Dokl. Ak. nauk SSSR 119, 52 (1958).*

Neutron-star superfluidity was predicted by Arkady Migdal in 1959, and first studied by Ginzburg & Kirzhnits in 1964 **before the discovery of pulsars** in 1967.

*Migdal, Nucl. Phys. 13, 655 (1959)*

*Ginzburg & Kirzhnits, Zh. Eksp. Teor. Fiz. 47, 2006 (1964)*

# Neutron pairing channels

Below  $\sim 10^{10}$  K, neutrons form pairs and condense into a superfluid phase

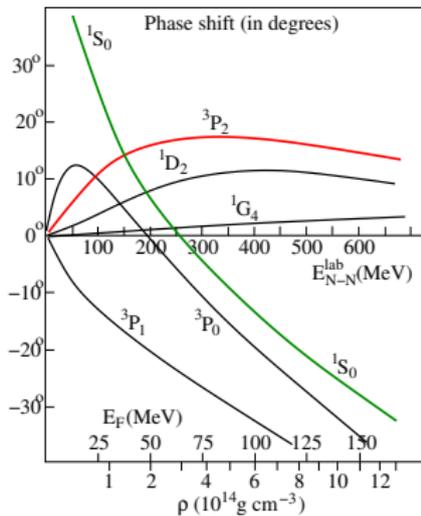
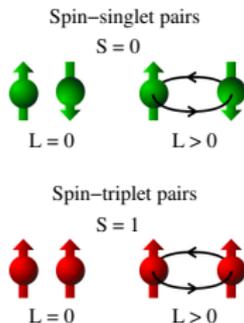
Pairing ( ${}^{2S+1}L_J$ ) if  $\delta > 0$

Gezerlis, Pethick, Schwenk in

*Novel Superfluids, Vol.2,*

Bennemann&Ketterson (Oxford

University Press, 2014), Chap. 22



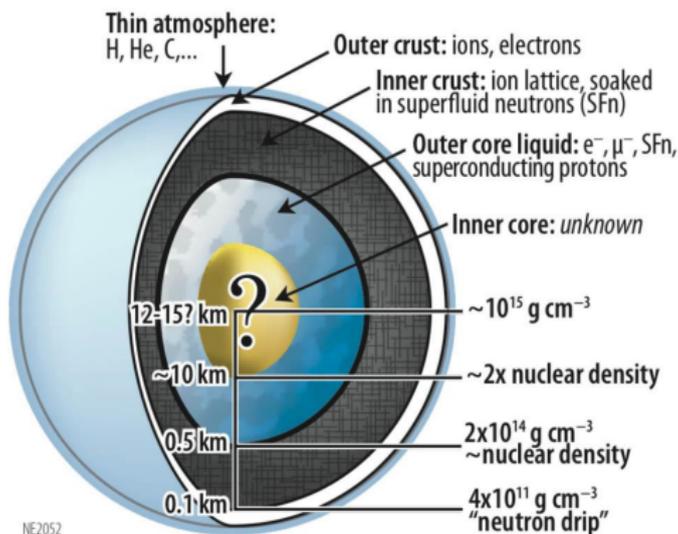
Microscopic calculations in homogeneous nuclear matter:

- diagrammatic methods
- variational methods
- quantum Monte Carlo methods.

For a recent review: *Sedrakian & Clark, Eur. Phys. J. A55, 167 (2019)*

# Anatomy of a neutron star

The interior of a neutron star is thought to contain different kinds of superfluids and superconductors in the crust and in the core:



For an overview: *Chamel, J. Astrophys. Astr. 38, 43 (2017)*

## Pairing in neutron star cores

The interior of a neutron star is not only made of neutrons, but consists of protons, leptons, hyperons, and possibly mesons, and even deconfined quarks.

Possible phases:

- $^1S_0$  and  $^3PF_2$  proton pairing
- neutron-proton pairing
- hyperon-hyperon pairing ( $^1S_0 \Lambda\Lambda$ )
- hyperon-nucleon pairing ( $^1S_0 n\Lambda$ ,  $^1S_0 n\Sigma^-$ ,  $^3SD_1 n\Sigma^-$ )
- quark pairing

Although  $^1S_0$  proton superconductivity is well established, the other superfluid/superconducting phases have been much less studied.

# Pairing in neutron star crusts

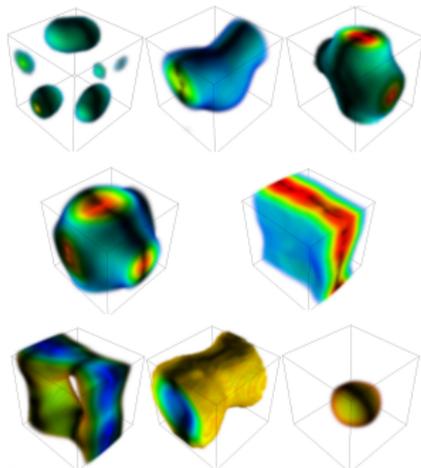
Because of spatial inhomogeneities, fully microscopic calculations in neutron-star crusts are currently not feasible.

Phenomenological approaches:

- local density approximation
- semi-classical methods  
(e.g. Thomas-Fermi approach)
- self-consistent “mean-field” methods  
(density functional theory)
- beyond “mean-field” methods.

## **Additional complication:**

Unlike terrestrial superconductors, the composition and the structure of the neutron-star crust are not known a priori!



# Bogoliubov-De Gennes theory for nucleons

Superfluid neutrons in neutron-star crusts are analogous to free electrons in inhomogeneous superconducting materials

Neutrons are thus describable by the Bogoliubov-De Gennes theory known as **Hartree-Fock-Bogoliubov** equations in nuclear physics:

$$\begin{pmatrix} h(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta(\mathbf{r})^* & -h(\mathbf{r})^* \end{pmatrix} \begin{pmatrix} \Psi_1(\mathbf{r}) \\ \Psi_2(\mathbf{r}) \end{pmatrix} = E \begin{pmatrix} \Psi_1(\mathbf{r}) \\ \Psi_2(\mathbf{r}) \end{pmatrix}$$

- $h(\mathbf{r}) \equiv -\nabla \cdot B(\mathbf{r})\nabla + U(\mathbf{r}) - i\mathbf{W}(\mathbf{r}) \cdot \nabla \times \boldsymbol{\sigma} - \frac{i}{2} \{I(\mathbf{r}), \nabla\} + \dots - \mu$  is the single-particle Hamiltonian with  $\mu$  the chemical potential,
- $\Delta(\mathbf{r})$  is the pairing potential.

These equations are **highly nonlinear** since  $h(\mathbf{r})$  and  $\Delta(\mathbf{r})$  are determined by the set of occupied wave functions  $\{\Psi_1(\mathbf{r}); \Psi_2(\mathbf{r})\}$ .

*Margueron & Sandulescu, in "Neutron Star Crust" (Nova Science Publisher, 2012)*

*Dobaczewski & Nazarewicz, in "50 Years of Nuclear BCS" (World Scientific Publishing, 2013), pp.40-60; Chamel, Goriely, Pearson, ibid. pp.284-296.*

*Bulgac, Phys. Status Solidi B256, 1800592 (2019).*

# Neutron superfluidity in a crystalline crust

Superfluid neutrons in neutron-star crusts are analogous to free electrons in inhomogeneous superconducting materials

## Floquet-Bloch theorem

*I found to my delight that the wave differed from the plane wave of free electrons only by a periodic modulation.*

*Bloch, Physics Today 29 (1976), 23-27.*



The wave functions satisfy

$$\Psi_{1\alpha\mathbf{k}}(\mathbf{r} + \boldsymbol{\ell}) = e^{i\mathbf{k}\cdot\boldsymbol{\ell}}\Psi_{1\alpha\mathbf{k}}(\mathbf{r})$$

$$\Psi_{2\alpha\mathbf{k}}(\mathbf{r} + \boldsymbol{\ell}) = e^{i\mathbf{k}\cdot\boldsymbol{\ell}}\Psi_{2\alpha\mathbf{k}}(\mathbf{r})$$

for any lattice vector  $\boldsymbol{\ell}$ .

- band index  $\alpha$ : rotational symmetry around each lattice site
- wave vector  $\mathbf{k}$ : translational symmetry of the crystal.

## Band theory

By symmetry, the crystal can be partitioned into identical primitive cells. The HFB equations need to be solved only inside one cell.

# Band theory

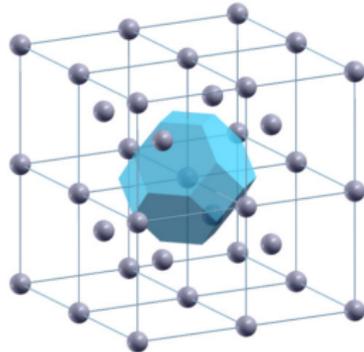
By symmetry, the crystal can be partitioned into identical primitive cells. The HFB equations need to be solved only inside one cell.

- The shape of the cell depends on the crystal symmetry

The **Wigner-Seitz cell** reflects the local symmetry.

The boundary conditions are fixed by the Floquet-Bloch theorem

$$\Psi_{\alpha\mathbf{k}}(\mathbf{r} + \boldsymbol{\ell}) = e^{i\mathbf{k}\cdot\boldsymbol{\ell}}\Psi_{\alpha\mathbf{k}}(\mathbf{r})$$



# Band theory

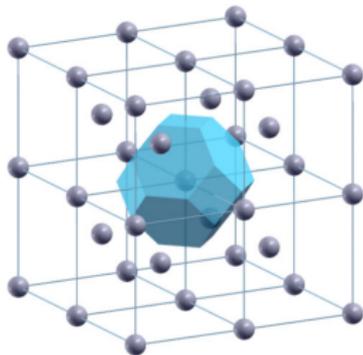
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$$\Psi_{\alpha\mathbf{k}}(\mathbf{r} + \boldsymbol{\ell}) = e^{i\mathbf{k}\cdot\boldsymbol{\ell}}\Psi_{\alpha\mathbf{k}}(\mathbf{r})$$



- $\mathbf{k}$  can be restricted to the **first Brillouin zone** (primitive cell of the reciprocal lattice) since for any reciprocal lattice vector  $\mathbf{K}$

$$\Psi_{\alpha\mathbf{k}+\mathbf{K}}(\mathbf{r}) = \Psi_{\alpha\mathbf{k}}(\mathbf{r})$$

## From HFB to multi-band BCS theory

Due to **proximity effects**,  $\Delta(\mathbf{r})$  varies smoothly in the densest layers:  
 $\Psi_{1\alpha\mathbf{k}} \approx U_{\alpha\mathbf{k}}\varphi_{\alpha\mathbf{k}}$ ,  $\Psi_{2\alpha\mathbf{k}} \approx V_{\alpha\mathbf{k}}\varphi_{\alpha\mathbf{k}}$ , where  $h(\mathbf{r})\varphi_{\alpha\mathbf{k}}(\mathbf{r}) = \varepsilon_{\alpha\mathbf{k}}\varphi_{\alpha\mathbf{k}}(\mathbf{r})$ .

The HFB equations reduce to the **multiband BCS gap equations**:

$$\Delta_{\alpha\mathbf{k}} = -\frac{1}{2} \sum_{\beta} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \bar{v}_{\alpha\mathbf{k}\alpha-\mathbf{k}\beta\mathbf{k}'\beta-\mathbf{k}'}^{\text{pair}} \frac{\Delta_{\beta\mathbf{k}'}}{E_{\beta\mathbf{k}'}} \tanh \frac{E_{\beta\mathbf{k}'}}{2k_{\text{B}}T}$$

$$\bar{v}_{\alpha\mathbf{k}\beta\mathbf{k}'}^{\text{pair}} = \int d^3r v^{\pi}[n_n(\mathbf{r}), n_p(\mathbf{r})] |\varphi_{\alpha\mathbf{k}}(\mathbf{r})|^2 |\varphi_{\beta\mathbf{k}'}(\mathbf{r})|^2$$

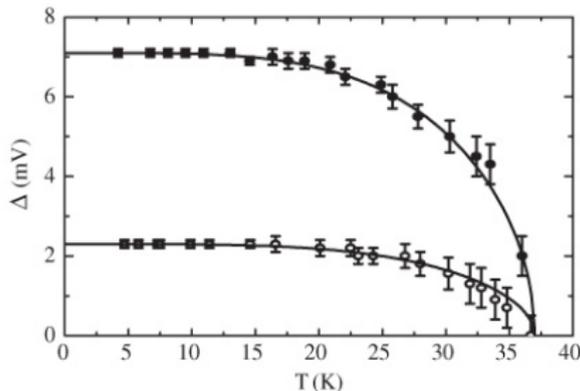
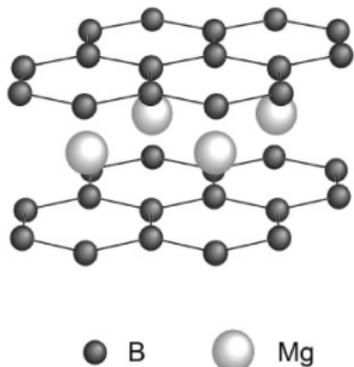
where  $E_{\alpha\mathbf{k}} = \sqrt{(\varepsilon_{\alpha\mathbf{k}} - \mu)^2 + \Delta_{\alpha\mathbf{k}}^2}$  and  $v^{\pi}[n_n(\mathbf{r}), n_p(\mathbf{r})]$  is an effective pairing interaction. The HFB solutions are

$$U_{\alpha\mathbf{k}} = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\varepsilon_{\alpha\mathbf{k}} - \mu}{E_{\alpha\mathbf{k}}}}, \quad V_{\alpha\mathbf{k}} = -\frac{1}{\sqrt{2}} \sqrt{1 - \frac{\varepsilon_{\alpha\mathbf{k}} - \mu}{E_{\alpha\mathbf{k}}}}$$

## Multi-band BCS superconductors

Multi-band superconductors were first studied in 1959 but clear evidence were found only in 2001 with the discovery of  $\text{MgB}_2$ .

Electrons in different bands feel different pairing interactions leading to different pairing gaps:

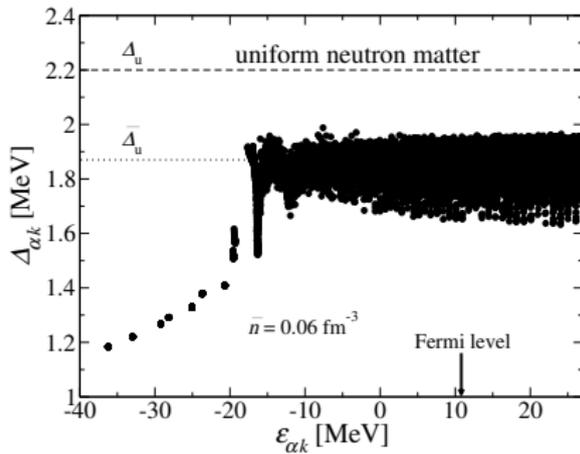
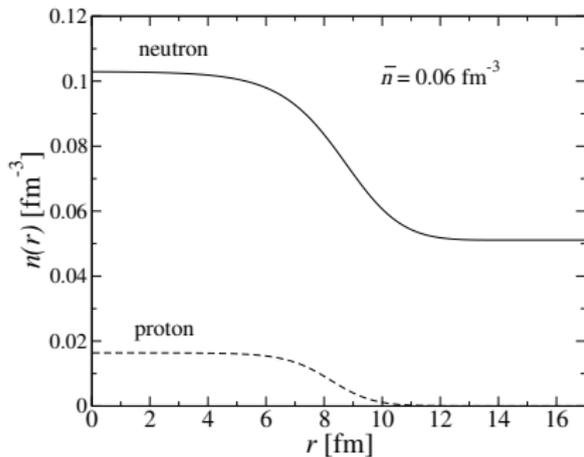


X. X. Xi, *Rep. Prog. Phys.* 71, 116501 (2008)

In the crust of a neutron star, the number of bands involved is  $\sim 10^2 - 10^3$  due to strong attractive nuclear pairing interactions!

# Multi-band BCS neutron superfluid

Wigner-Seitz cell with  $Z = 40$ ,  $N = 1220$  (body-centered cubic lattice)

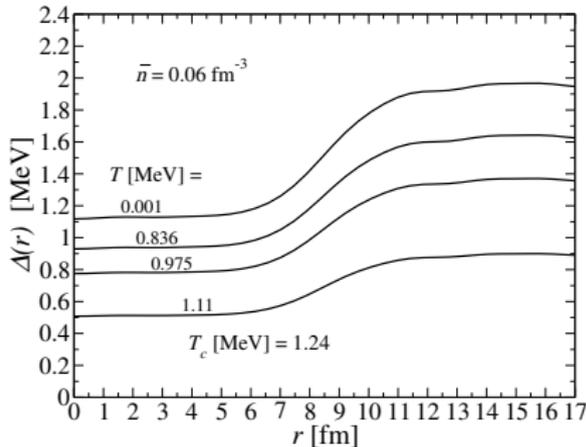
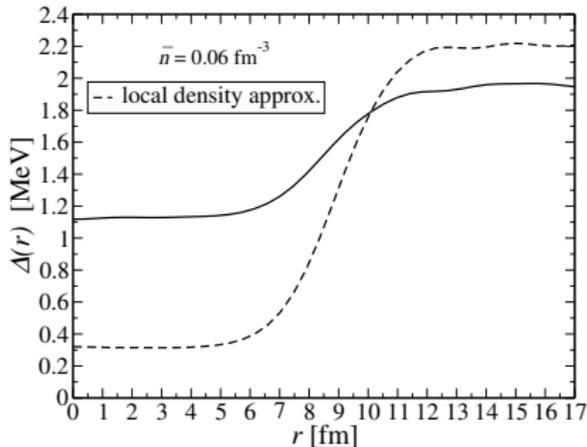


Chamel et al., *Phys.Rev.C*81,045804 (2010)

- Nuclear clusters lower the gap by  $\sim 10 - 20\%$
- $\Delta_{\alpha\mathbf{k}}(T)/\Delta_{\alpha\mathbf{k}}(0)$  is given by the same function of  $T$  for all bands
- The critical temperature is given by the BCS relation  
 $T_c \simeq 0.567 \bar{\Delta}_u$

# Multi-band BCS neutron superfluid

Wigner-Seitz cell with  $Z = 40$ ,  $N = 1220$  (body-centered cubic lattice)



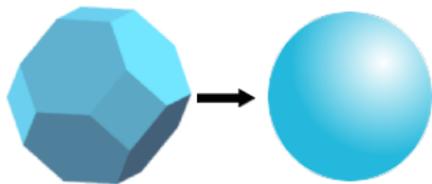
*Chamel et al., Phys.Rev.C81,045804 (2010)*

- Both bound and unbound neutrons contribute to superfluidity
- Superfluidity permeates clusters (loosely bound Cooper pairs)
- The superfluid becomes homogeneous as  $T$  approaches  $T_c$

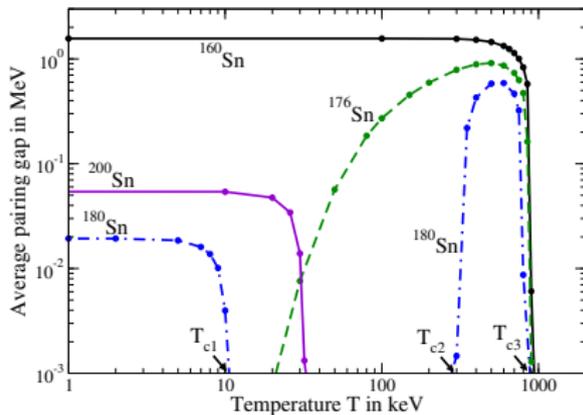
# Pairing in the shallow layers of neutron-star crusts

In the shallow layers, the BCS approximation is not very accurate. But solving the full 3D HFB equations is computationally very expensive.

Wigner-Seitz approximation in solid-state physics (1933)



first adopted by Negele&Vautherin in nuclear physics (1973)



*Margueron&Khan, PRC86, 065801(2012).*

Pairing in the crust and in pure neutron matter are very different:

- reentrance phenomenon
- existence of several critical temperatures!

These features cannot be explained by the simple BCS theory.

# Pairing in the shallow layers of neutron-star crusts

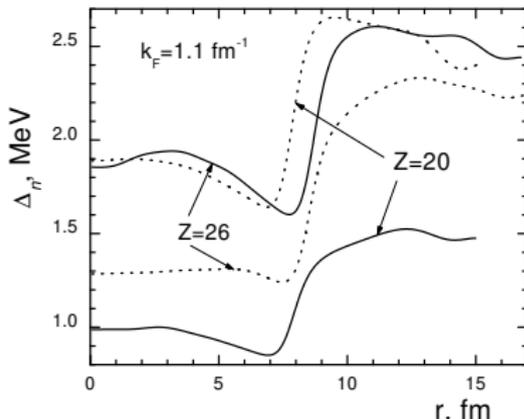
Medium effects and collective excitations (Anderson-Bogoliubov phonons) have been also studied within this approximation.

*Grasso et al., Nucl. Phys. A807, 1(2008); Baroni et al., Phys. Rev. C82, 015807(2010);*

*Inakura&Matsuo, Phys. Rev. C99, 045801 (2019)*

## Problems:

- nonuniqueness of the boundary conditions
- spurious shell effects  $\propto 1/R^2$
- static configurations



*Baldo et al., Nucl. Phys. A775, 235 (2006); Chamel et al., Phys. Rev. C 75, 055806 (2007)*

*Margueron & Sandulescu, in "Neutron Star Crust" (Nova Science Publisher, 2012)*

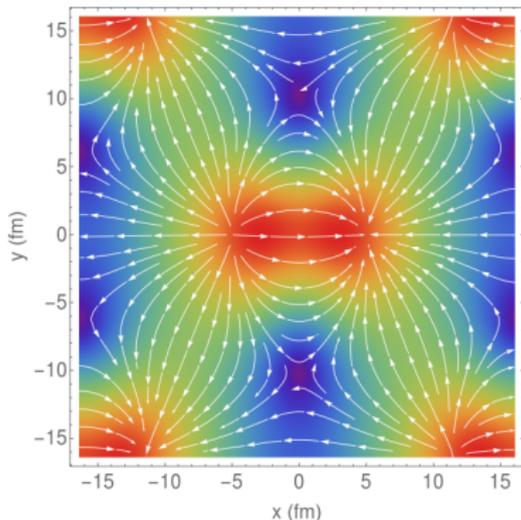
*Pastore et al., J. Phys. G44, 094003 (2017)*

The Wigner-Seitz approach is unreliable at densities  $\bar{n} \gtrsim 0.02 \text{ fm}^{-3}$  and cannot describe neutron flows throughout the crust

## Flow of neutrons in neutron-star crusts

The neutron flow was studied in the **strong coupling limit** adopting a purely **classical hydrodynamical** treatment.

- Superfluid “velocity”:  $\mathbf{v}_{sn} = \frac{\hbar}{2m_n} \nabla\Phi \equiv \frac{\pi_n}{m_n}$
- Incompressible superfluid flow:  $\nabla \cdot \mathbf{v}_{sn} = 0$
- Spherical clusters (obstacles) with sharp surfaces.



Classical potential flow  $\Delta\Phi = 0$

The neutron mass current is

$$\begin{aligned} \mathbf{j}_n &\equiv n_n m_n \mathbf{v}_n = \frac{1}{V_{\text{cell}}} \int_{\text{cell}} n_n(\mathbf{r}) \nabla\Phi(\mathbf{r}) \\ &= n_n^s m_n \mathbf{v}_{sn} \end{aligned}$$

$n_n^s$  is the **superfluid density**

$\mathbf{v}_n$  is the true velocity

*Martin&Urban, Phys.Rev.C94, 065801(2016)*

# Classical potential flow past obstacles

Permeability of the clusters:

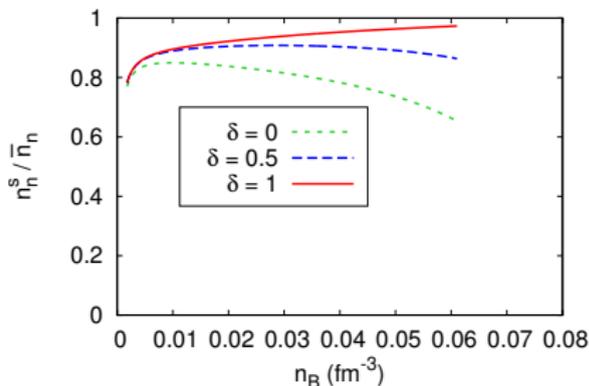
$\delta = 0$  no superfluidity,

$\delta = 1$  superfluidity everywhere.

*Martin&Urban, Phys. Rev. C94, 065801(2016)*

**Added perturbations from different clusters are negligible.**

*Epstein, ApJ333, 880 (1988)*



The potential flow past a single cluster can be solved analytically:

$$\frac{n_n^s}{n_n} = 1 + 3 \frac{\mathcal{V}_{cl}}{\mathcal{V}_{cell}} \frac{\delta - \gamma}{\delta + 2\gamma} \Rightarrow 1 - \frac{3}{2} \frac{\mathcal{V}_{cl}}{\mathcal{V}_{cell}} \leq \frac{n_n^s}{n_n} \leq 1 + 3 \frac{\mathcal{V}_{cl}}{\mathcal{V}_{cell}}$$

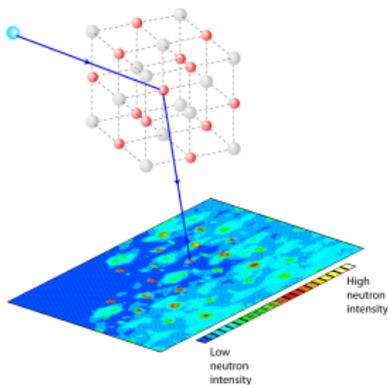
*Magierski&Bulgac, Act. Phys. Pol. B35, 1203(2004); Magierski, IJMPE13, 371(2004)*

*Sedrakian, Astrophys. Spa. Sci. 236, 267(1996); Epstein, ApJ333, 880 (1988)*

The superflow is found to be only **weakly perturbed** by clusters. However, the strong coupling regime is usually not reached.

# Bragg scattering of neutrons in neutron-star crusts

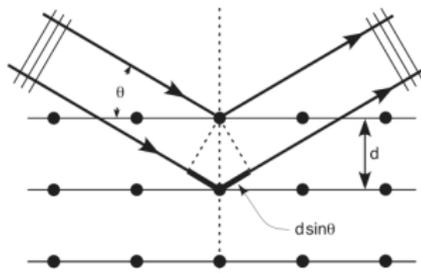
Microscopically, neutrons in the crust do not flow freely because they are diffracted by the nuclear lattice.



Neutrons can be diffracted by a crystal. This is routinely used to explore the structure of materials. **Diffraction means no flow!**

Bragg's law: a neutron with wavevector  $k > \pi/d$  can be **coherently scattered** if  $d \sin \theta = N\pi/k$ , where  $N = 0, 1, 2, \dots$

In neutron stars, **neutrons are highly degenerate** and have momenta up to  $k_F > \pi/d$  (except in the shallowest layers). Bragg scattering should be taken into account!



## Neutron “conduction” from the band theory

In the BCS regime, the neutron superfluid density is given

$$n_n^s = \frac{m_n}{24\pi^3\hbar^2} \sum_{\alpha} \int |\nabla_{\mathbf{k}} \varepsilon_{\alpha\mathbf{k}}|^2 \frac{\Delta_{\alpha\mathbf{k}}}{E_{\alpha\mathbf{k}}} d^3k \quad E_{\alpha\mathbf{k}} = \sqrt{(\varepsilon_{\alpha\mathbf{k}} - \mu)^2 + \Delta_{\alpha\mathbf{k}}^2}$$

In the weak coupling limit  $\Delta_{\alpha\mathbf{k}} \ll \mu$ ,

$$n_n^s \approx \frac{m_n}{24\pi^3\hbar^2} \sum_{\alpha} \int |\nabla_{\mathbf{k}} \varepsilon_{\alpha\mathbf{k}}| dS_F^{(\alpha)}$$

### “Neutronics” of neutron-star crusts

The neutron conduction depends on the shape of the Fermi surface:

- spherical Fermi surface  $n_n^s = n_n$   
neutrons are free: **metal**
- no Fermi surface (band gap)  $n_n^s = 0$   
neutrons cannot propagate: **insulator**.

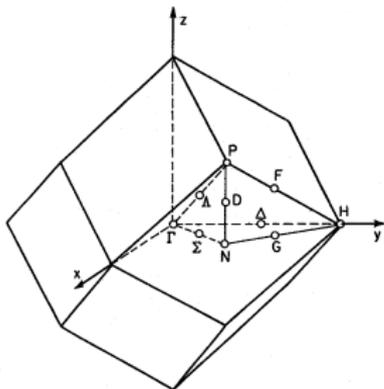
Recent review:

*Chamel, J. Low Temp. Phys. 189, 328 (2017)*

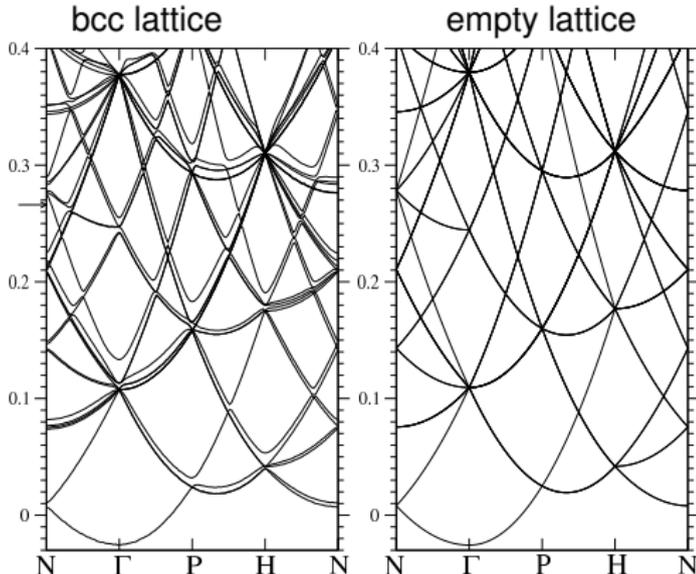
## Neutron band structure: shallow region

Neutron band structure (s.p. energy in MeV vs  $\mathbf{k}$ ) in a body-centered cubic (bcc) lattice at  $\bar{n} = 0.0003 \text{ fm}^{-3}$  ( $Z = 50, A = 200$ ):

First Brillouin zone:



Chamel, *Phys.Rev.C*85,035801(2012)

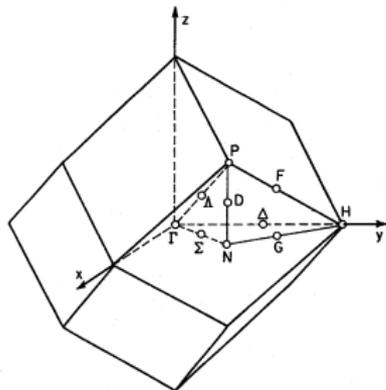


The band structure is similar that of free neutrons: 83% of unbound neutrons can flow freely (“metallic state”).

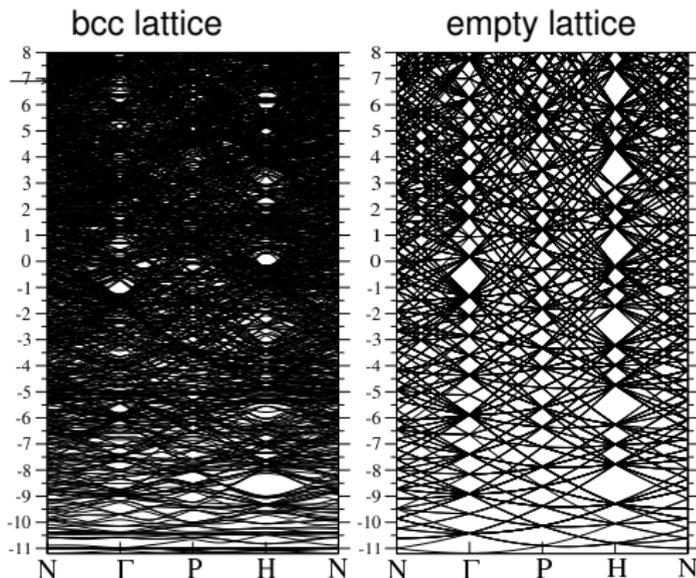
## Neutron band structure: intermediate region

Neutron band structure (s.p. energy in MeV vs  $\mathbf{k}$ ) in a body-centered cubic (bcc) lattice at  $\bar{n} = 0.03 \text{ fm}^{-3}$  ( $Z = 40, A = 1590$ ):

First Brillouin zone:



Chamel, *Phys. Rev. C* 85, 035801 (2012)

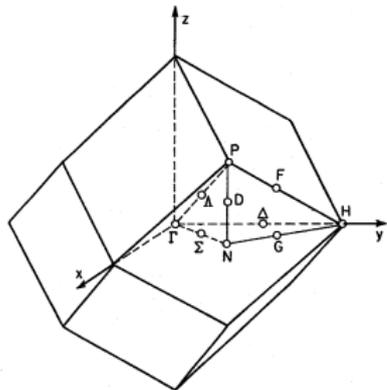


The band structure is very different from that of free neutrons: only 7% of unbound neutrons can flow freely (almost “insulating”).

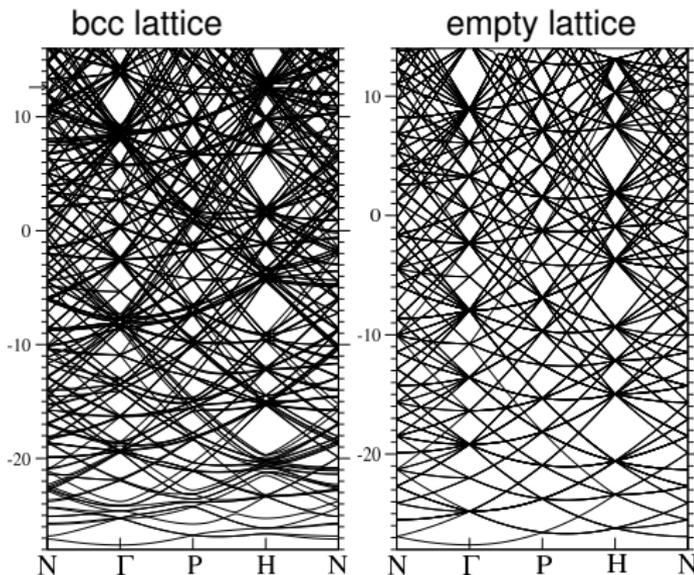
## Neutron band structure: deep region

Neutron band structure (s.p. energy in MeV vs  $\mathbf{k}$ ) in a body-centered cubic (bcc) lattice at  $\bar{n} = 0.08 \text{ fm}^{-3}$  ( $Z = 20, A = 665$ ):

First Brillouin zone:



*Chamel, Phys. Rev. C85, 035801 (2012)*



The band structure is similar to that of free neutrons: 65% of unbound neutrons can flow freely (almost “insulating”).

## Suppression of band structure effects by pairing?

By solving numerically the HFB equations at  $\bar{n} = 0.03 \text{ fm}^{-3}$ , Watanabe&Pethick have found that the superflow is much less perturbed:  $n_n^s/n_n \sim 60 - 70\%$  instead of  $\sim 7\%$ .

*Watanabe&Pethick, PRL 119, 062701 (2017)*

But various approximations were made:

- 3D body-centered cubic lattice replaced by a **1D lattice**
- no effective mass  $B(\mathbf{r}) = \hbar^2/(2m_n)$
- pairing potential  $\Delta(\mathbf{r})$  not solved self-consistently but fixed
- **Fourier components of  $U(\mathbf{r})$  contribute independently**
- numerical extraction of  $n_n^s$  from second derivatives of the energy

$n_n^s/n_n \sim 6\%$  was obtained in 2005 from 3D band calculations (no pairing) with a different crust model.

*Chamel, Nucl. Phys. A747, 109 (2005)*

## Role of pairing further examined

- full **3D band-structure** calculations
- same crust model as in 2012
- pairing included in the **BCS approximation**
- **analytical** extraction of  $n_n^S$

$\Delta$ (MeV)	$\Delta/\varepsilon_F$	$n_n^S/n_n$ (%)
3.09	0.169	7.86
2.15	0.117	7.73
1.49	0.0813	7.62
1.04	0.0567	7.55
0.721	0.0393	7.54
0.501	0.0273	7.56
0.348	0.0190	7.61
0.242	0.0132	7.67
0.168	0.00917	7.73
0.117	0.00638	7.77
0	0	7.79

bcc lattice with spacing 47.3 fm  
1550 neutrons in the Wigner-Seitz cell  
 $35 \times 35 \times 35$  points ( $\delta r \sim 0.7$  fm)  
1650 bands (half without pairing)  
integrations with 1360 special  $\mathbf{k}$  points  
(65280  $\mathbf{k}$  points in the first Brillouin zone)  
 $\Rightarrow 2 \times 10^6$  s.p. wavefunctions  
*Chamel, in prep.*

Including pairing is computationally very costly, but results are essentially the same as those obtained without.

## Mutual entrainment in superfluid mixtures

Superfluids in mixtures such as  $^3\text{He}$ - $^4\text{He}$  can be **mutually entrained** despite the absence of viscous drag.

*Andreev & Bashkin, Sov. Phys. JETP 42, 164 (1975)*

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Similarly in neutron-star cores, neutron and protons are coupled:

$$\begin{aligned}\mathbf{j}_n &\equiv \rho_n \mathbf{v}_n = \rho_{nn} \mathbf{v}_{sn} + \rho_{np} \mathbf{v}_{sp} \\ \mathbf{j}_p &\equiv \rho_p \mathbf{v}_p = \rho_{pn} \mathbf{v}_{sn} + \rho_{pp} \mathbf{v}_{sp}\end{aligned}$$

The “superfluid velocities”  $\mathbf{v}_{sx}$  represent the hydrodynamic part of the corresponding momenta  $\mathbf{p}_{sx} \equiv m_x \mathbf{v}_{sx} \equiv \boldsymbol{\pi}_{sx} - (q_x/c)\mathbf{A}$ .

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Galilean invariance:

$$\mathbf{j}_n + \mathbf{j}_p = m_n \mathbf{v}_{sn} + m_p \mathbf{v}_{sp}$$

$$\rho_n = \rho_{nn} + \rho_{np}$$

$$\rho_p = \rho_{pn} + \rho_{pp}$$

$$\rho_{np} = \rho_{pn} \text{ (energy)}$$

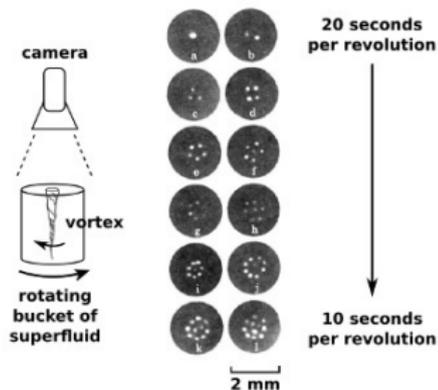
Entrainment arises from strong neutron-proton interactions:

- Fermi liquid theory
- density functional theory

# Quantized vortices and fluxoids in neutron stars

A rotating superfluid neutron star is threaded by **neutron vortices**, each carrying an angular momentum  $\hbar$ .

Surface density of neutron vortices:  
 $n_V(\text{cm}^{-2}) \sim 6 \times 10^5 / P(10 \text{ ms})$ .



*Yarmchuk, Gordon, Packard, PRL43, 214 (1979)*

A neutron star is also threaded by **proton fluxoids**, each carrying a flux  $\Phi_0 = hc/(2e)$  (assuming type II superconductivity).

Surface density of proton fluxoids:  $n_\Phi(\text{cm}^{-2}) \sim 5 \times 10^{18} / B(10^{12} \text{ G})$ .

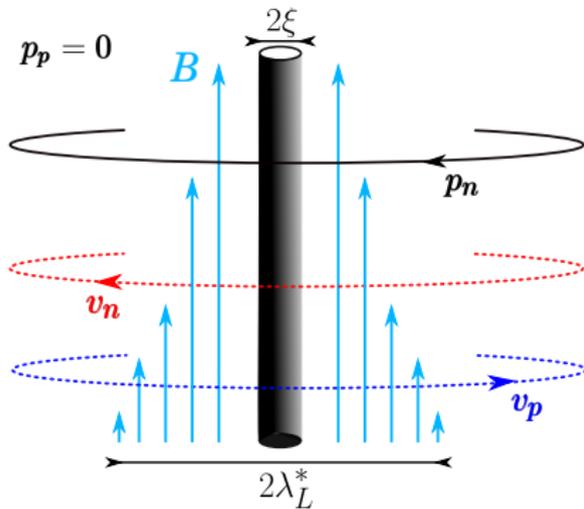
In a neutron star,  $n_\Phi \sim 10^{13} - 10^{14} n_V$  but type I superconductivity is not excluded (superconductivity may be neither of type I nor II).

*Sedrakian & Clark, in "Pairing in Fermionic Systems", (World Scientific, 2006)*

# Entrainment and dissipation in neutron-star cores

Due to entrainment effects, neutron vortices carry a **fractional magnetic quantum flux!**

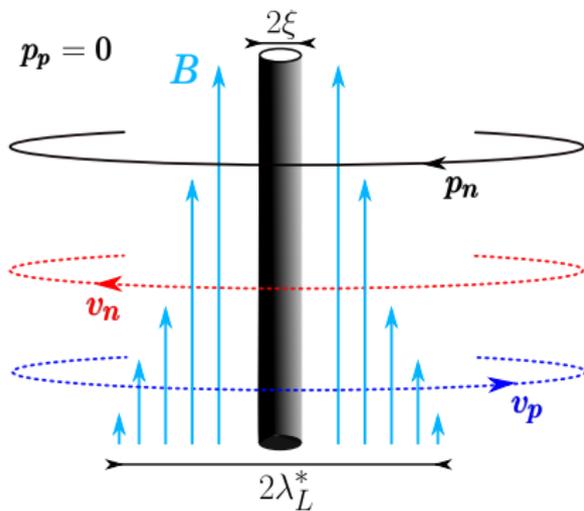
*Sedrakyan&Shakhabasyan, Astrofizika 8 (1972), 557; 16 (1980), 727*



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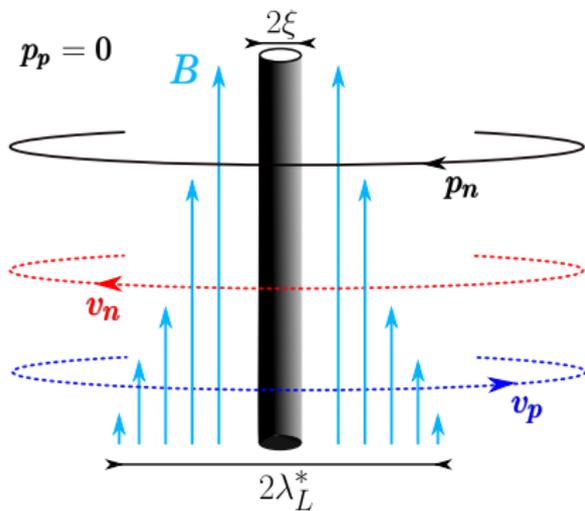


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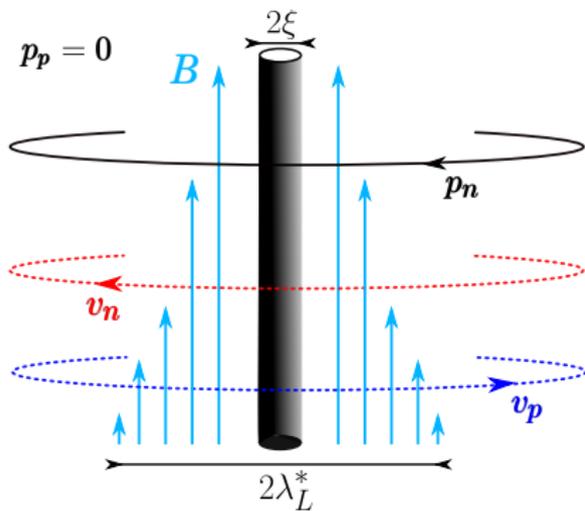


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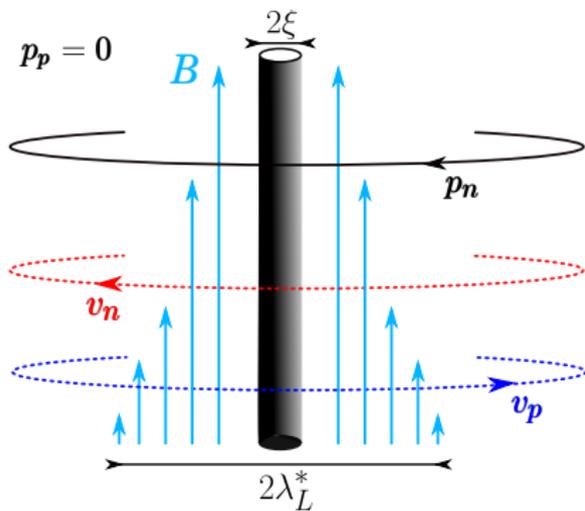


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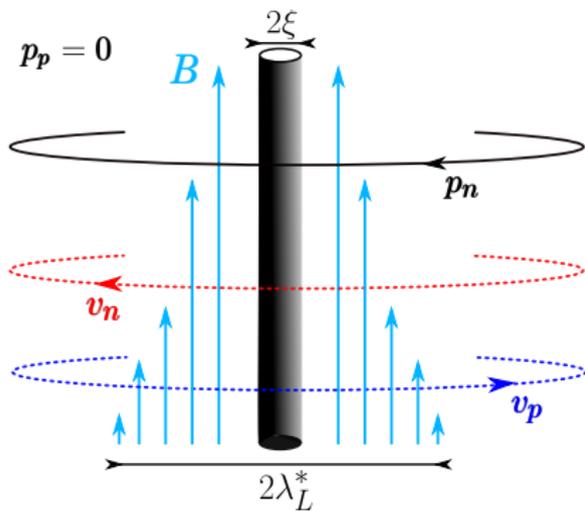
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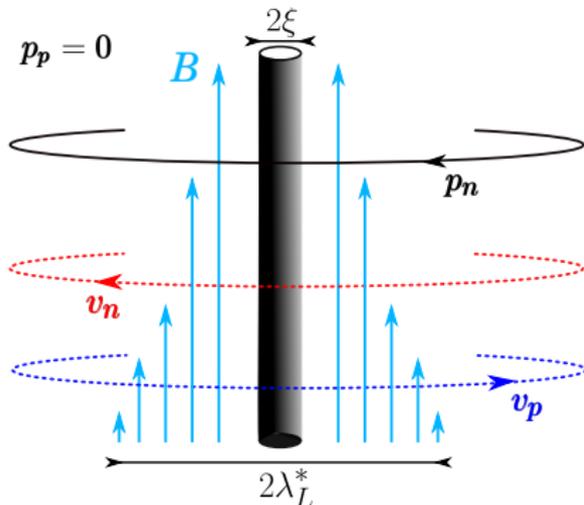
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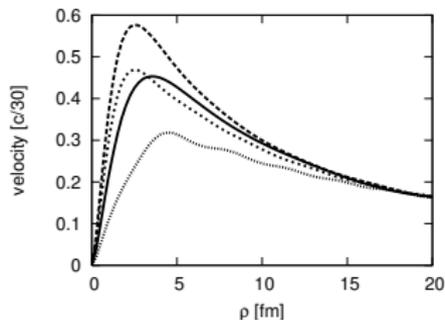
Electrons scattering off the magnetic field of the vortices lead to a **friction force** between the core superfluid and charged particles

*Alpar, Langer, Sauls, ApJ282 (1984) 533*

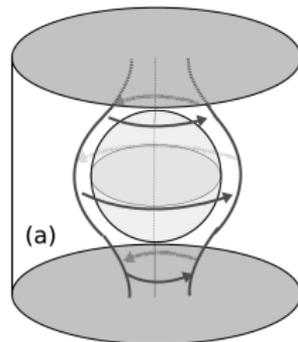
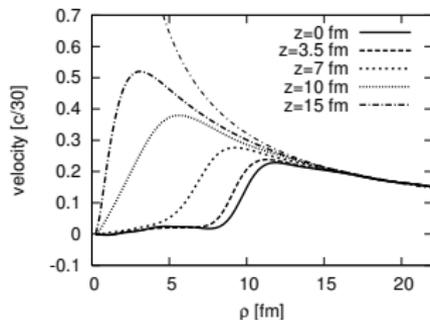
# Pinning of vortices in neutron-star crust

Neutron superfluid vortices can pin to nuclei in the crust:

single vortex



single vortex + cluster



*Avogadro et al, Nucl.Phys.A811,378(2008)*

Microscopic calculations of pinning forces:

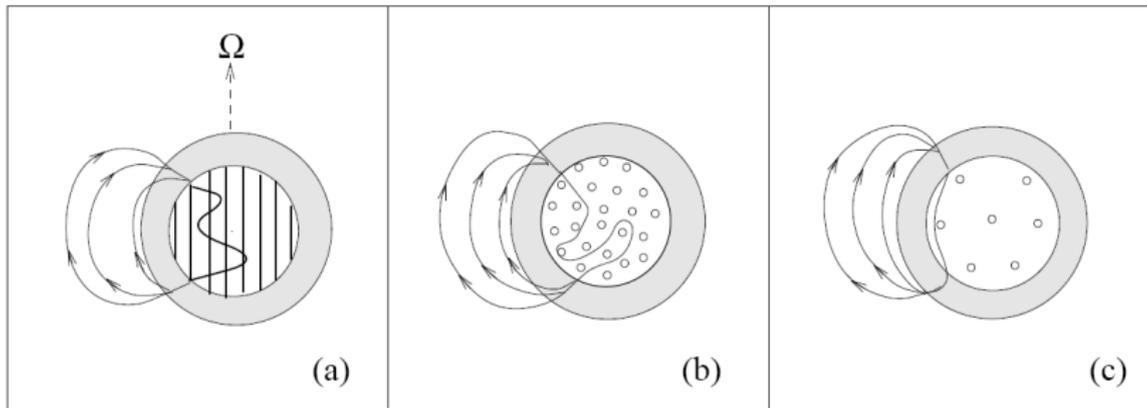
- local density approximation and semi-classical methods
- density functional theory.

Pinning depends on the structure of the crust, on the rigidity of the lines and on the vortex dynamics.

*Wlazlowski et al., PRL 117, 232701 (2016)*

## Pinning of vortices in neutron-star core

Neutron vortices can pin to proton fluxoids in the core. The interaction between vortices and fluxoids leads to movement of crustal plates.



*Ruderman, Astrophys. Space Sci. 357, 353 (2009)*

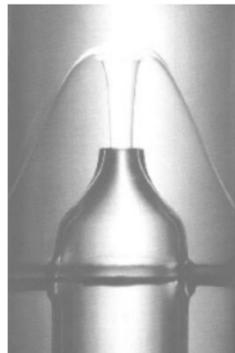
The evolution of the pulsar spin and magnetic field are intimately related to superfluidity and superconductivity

# Superfluid models of neutron stars

Superfluid helium cannot be described using classical hydrodynamics.

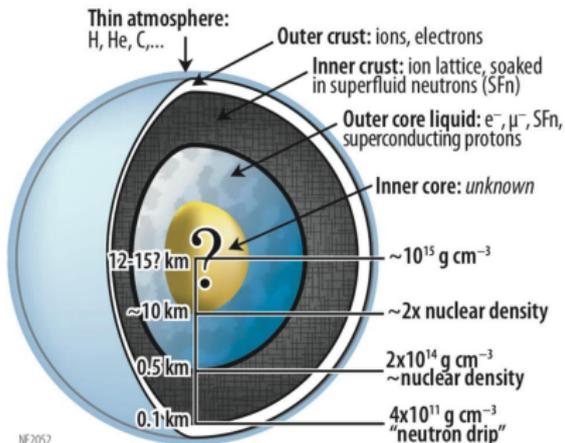
Two distinct dynamical components coexist:

- a superfluid, which carries no heat,
- a normal viscous fluid.



Similarly, neutron stars contain at least **two distinct components**:

- a neutron superfluid in the crust and in the core,
- a plasma of charged particles (ions in the crust, protons and leptons in the core).



# General relativity and neutron stars

Neutron stars should be described using general relativity.



PSR J1846-0258 (Chandra)

Compactness  $\frac{2GM}{Rc^2}$  for various bodies of mass  $M$  and radius  $R$ :

Earth	$10^{-10}$
Sun	$10^{-6}$
White dwarf	$10^{-4} - 10^{-3}$
<b>Neutron star</b>	<b><math>\sim 0.2 - 0.4</math></b>
Black hole	1

General relativity leads to additional fluid couplings (even in the absence of interactions) at large scales due to frame-dragging effects!

*B. Carter, Ann. Phys. 95, 53 (1975); Sourie et al., MNRAS 464, 4641(2017)*

# Relativistic superfluid hydrodynamics



credit: M. Lorenzo

An elegant **variational formalism** for describing relativistic superfluid mixtures was developed by Brandon Carter.

*Carter in "Relativistic fluid dynamics" (Springer-Verlag, 1989), pp.1-64*

*Carter, Lect. Notes Phys. 578 (Springer, 2001), 54.*

For a pedagogical introduction, see

*Gourgoulhon, EAS Publications Series 21 (2006), 43.*

It was later extended to superfluid (magneto)elastohydrodynamics to describe the superfluid crust of (magnetised) neutron stars

*Carter & Samuelsson, Class.Quant. Grav.23, 5367 (2006).*

- This formalism based on an action principle relies on the use of **differential forms and Elie Cartan's exterior calculus.**
- It allows for a rigorous and systematic derivation of hydrodynamic equations and conservation laws.

## Exterior calculus in a nut shell

- A **k-form**  $\underline{\omega}$  is a k-times covariant tensor  $\omega_{\mu_1 \dots \mu_k}$
- The **exterior derivative** of a k-form  $\underline{\omega}$  is defined by

$$(\partial \underline{\omega})_{\mu_1 \dots \mu_k} = (k + 1) \partial_{[\mu_1} \omega_{\mu_2 \dots \mu_{k+1}]}$$

[. . .] means full antisymmetrization over spacetime indices

e.g.  $(\partial \underline{\omega})_{\mu} = 2 \partial_{[\mu} \omega_{\nu]} = \partial_{\mu} \omega_{\nu} - \partial_{\nu} \omega_{\mu}$

$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}$  denotes partial derivative

- Closure property:  $\partial \partial \underline{\omega} = 0$
- The **Lie derivative** of a k-form along a 4-vector field  $\vec{u}$  is another k-form defined by  $\mathcal{L}_{\vec{u}} \underline{\omega} = \frac{\partial}{\partial x^1} \underline{\omega}$  in a coordinate system such that  $\vec{u} = (1, 0, 0, 0)$ .
- Cartan's formula:  $\mathcal{L}_{\vec{u}} \underline{\omega} = \vec{u} \cdot \partial \underline{\omega} + \partial(\vec{u} \cdot \underline{\omega})$

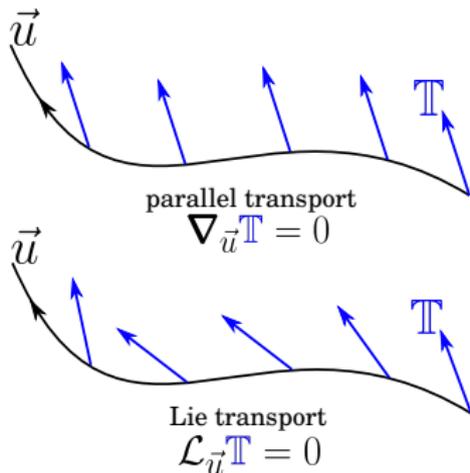
Unlike the covariant derivative  $\nabla$ , exterior and Lie derivatives **are independent of the spacetime structure.**

## Covariant vs Lie derivatives

The covariant derivative requires the specification of a **connection**: that of Levi-Civita in general relativity. Such a connection does not exist in Newtonian theory (no spacetime metric!).

The connection defines a **parallel transport** of a tensor  $\mathbb{T}$  along a (4-)vector field  $\vec{u}$ :

$$\nabla_{\vec{u}}\mathbb{T} = u^\mu \nabla_\mu \mathbb{T} = 0$$



## Relativistic superfluid hydrodynamics

Carter's formalism relies on the **action integral**  $\mathcal{A} = \int \Lambda\{n_x^\mu\} d\mathcal{M}^{(4)}$ .

The **Lagrangian density** or “**master function**”  $\Lambda$  depends on the **4-currents**  $n_x^\mu = n_x u_x^\mu$  of the different (super)fluids.

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Considering variations of the fluid particle trajectories  $\delta n_x^\mu$  such that  $\delta\mathcal{A} = 0$  yields  $n_x^\mu \varpi_{\mu\nu}^x + \pi_\nu^x \nabla_\mu n_x^\mu = f_\nu^x$

**4-momentum (1-form)**

$$\pi_\mu^x = \frac{\partial \Lambda}{\partial n_x^\mu}$$

**4-vorticity (2-form)**

$$\varpi_{\mu\nu}^x = 2\nabla_{[\mu} \pi_{\nu]}^x = \nabla_\mu \pi_\nu^x - \nabla_\nu \pi_\mu^x$$

**4-force density (1 form)**

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**4-force density (1 form)**

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- $\pi_\mu^x$  and  $n_x^\mu$  are mathematically different objects: the former is a *covector*, while the latter is a *vector*.
- $\varpi_{\mu\nu}^x$  is defined by the **exterior derivative** of the momentum  $\pi_\mu^x$

# Covariant nonrelativistic superfluid hydrodynamics

The formalism was adapted to the more intricate Newtonian theory.

- Because Carter's formalism relies on exterior calculus, the equations take the **same form**:

$$n_x^\mu \varpi_{\mu\nu}^x + \pi_\nu^x \nabla_\mu n_x^\mu = f_\nu^x$$

- The difference lies in the underlying spacetime structure (absence of a spacetime metric, Galilean gauge symmetry)

*Carter&Chamel, Int.J.Mod.Phys.D13,291(2004); ibid. D14,717(2005); D14,749(2005)*

This formalism was extended to magneto-elastohydrodynamics:

*Carter, Chachoua & Chamel, Gen.Rel.Grav.38, 83 (2006)*

*Carter & Chachoua, Int.J.Mod.Phys.D15, 1329 (2006).*

Why a fully 4D covariant formulation?

- direct comparison with relativistic theory
- matching between local (nonrelativistic) and global dynamics
- conservation laws and identities can be more easily derived using mathematical concepts from differential geometry!

# Noether identities and stress-energy density tensor

Using Noether identities leads to the general expression of the **stress-energy density tensor** of any superfluid mixture

$$T_{\nu}^{\mu} = \Psi \delta_{\nu}^{\mu} + \sum_{\mathbf{x}} n_{\mathbf{x}}^{\mu} \pi_{\nu}^{\mathbf{x}}$$

where  $\Psi$  is a **generalised pressure**

$$\Psi = \Lambda - \sum_{\mathbf{x}} n_{\mathbf{x}}^{\mu} \pi_{\mu}^{\mathbf{x}}$$

In general,  $\Psi$  depends on the currents  $n_{\mathbf{x}}^{\mu}$ .

The **4-force density** acting on the fluids is

$$f_{\nu} = \nabla_{\mu} T_{\nu}^{\mu}$$



The above expressions are formally valid for any spacetime!

## Example: relativistic perfect fluid

Let us consider a single relativistic perfect fluid at  $T = 0$ . The Lagrangian density is  $\Lambda = -\rho c^2$  with  $\rho$  the mass-energy density.

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- The **4-current** is  $n^\mu = nu^\mu$

$n$  is the particle number density

$u^\mu \equiv \frac{dx^\mu}{d\tau}$  is the 4-velocity with  $\tau$  the proper time.

By definition  $g_{\mu\nu} dx^\mu dx^\nu = -c^2 d\tau^2$  therefore  $u^\nu u_\nu = -c^2$

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- The variation of the Lagrangian density is

$$\delta\Lambda = -c^2 \frac{d\rho}{dn} \delta n = -\mu \delta n$$

where  $\mu$  is the **chemical potential**.

$$n^2 c^2 = -n_\nu n^\nu \Rightarrow 2c^2 n \delta n = -2n_\nu \delta n^\nu$$

$$\Rightarrow \delta\Lambda = \frac{\mu}{c^2} u_\nu \delta n^\nu$$

## Example: relativistic perfect fluid

$$\delta\Lambda = \frac{\mu}{c^2} u_\nu \delta n^\nu$$

The **4-momentum** is thus given by  $\pi_\nu = \frac{\partial\Lambda}{\partial n^\nu} = \frac{\mu}{c^2} u_\nu$ .

NB: the 4-momentum is not merely given by  $\pi_\nu = m u_\nu$ . Massless particles can thus carry a momentum (e.g. radiation pressure).

The **generalised pressure** reduces to the ordinary pressure

$$\Psi \equiv \Lambda - n^\mu \pi_\mu = -\rho c^2 + n\mu = P$$

The **stress-energy density tensor** is

$$T^\mu_\nu \equiv \Psi \delta^\mu_\nu + n^\mu \pi_\nu = P \delta^\mu_\nu + \left( \rho + \frac{P}{c^2} \right) u^\mu u_\nu$$

# Kelvin-Helmholtz theorem revisited

In the absence of external forces  $f_\nu = 0$  and for conserved particles  $\nabla_\mu n^\mu = 0$ , the hydrodynamic equations reduce to  $n^\mu \varpi_{\mu\nu} = 0$ .

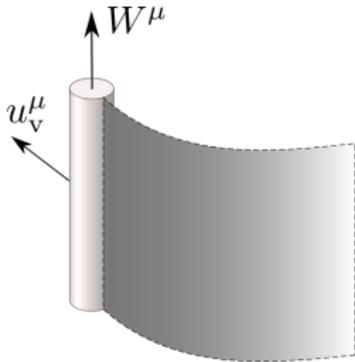


figure by A. Sourie

Kelvin-Helmholtz theorem immediately follows from Cartan's formula  $\mathcal{L}_{\vec{u}} \underline{\varpi} = \vec{u} \cdot \partial \underline{\varpi} + \partial(\vec{u} \cdot \underline{\varpi})$

$$\mathcal{L}_{\vec{u}} \varpi_{\mu\nu} = 3u^\sigma \nabla_{[\sigma} \varpi_{\mu\nu]} - 2\nabla_{[\mu} (\varpi_{\nu]\sigma} u^\sigma) = 0$$

**Vorticity is (Lie) transported by the fluid.**

In Newtonian spacetime, the vorticity conservation reads

$$\frac{dC}{dt} = 0, \quad C = \oint \pi \cdot d\ell$$

For a single fluid,  $\pi = m\mathbf{v}$ . This may no longer be the case for mixtures!

## Killing vectors and Bernoulli theorem

The concept of Killing vectors has been extremely fruitful to analyse symmetries in general relativity, but in hydrodynamics as well!

The existence of **fluid symmetries** translates into the **invariance of the flow by Lie transport** along corresponding Killing vectors:

$$\mathcal{L}_{\vec{k}_a} \pi_\mu = 0 \Rightarrow \nabla_\mu (k_a^\nu \pi_\nu) = \varpi_{\mu\nu} k_a^\nu$$

### Generalised Bernoulli theorem

- $\mathcal{B}_a \equiv k_a^\nu \pi_\nu$  are constants for irrotational flows  $\varpi_{\mu\nu} = 0$ .
- $\mathcal{B}_a$  are constants along the flow lines in the absence of forces:  $n^\nu \nabla_\nu \mathcal{B}_a = f_\nu k_a^\nu = 0$  (parallel transport).

Example: stationary flows are invariant by Lie transport along the time direction meaning that  $\pi_0 = -\left(\frac{1}{2}mv^2 + m\phi + \chi\right)$  is constant.

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Helicity in superfluids was first discussed by Peradzynski in 1990.

*Int. J. Theo. Phys.* 29, 1277 (1990)

# Helicity and vortices

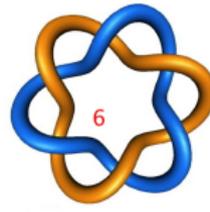
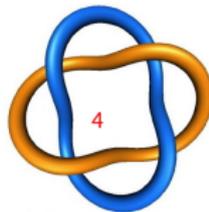
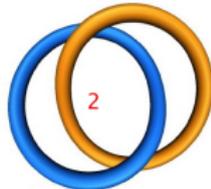
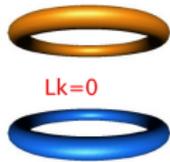
Helicity is a measure of the degree of knottedness of vortex lines.

*Moffatt&Ricca, Proc. R. Soc. Lond. A439, 411 (1992)*

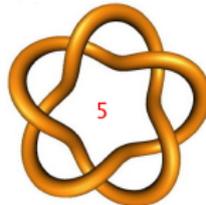
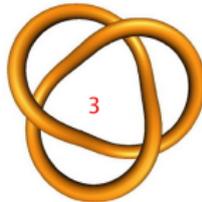
For  $N$  quantized vortices with linking numbers  $L_{\alpha\beta}$

$$\mathcal{H} = \sum_{\alpha,\beta=1}^N L_{\alpha\beta} C_{\alpha} C_{\beta} = h^2 \sum_{\alpha,\beta=1}^N L_{\alpha\beta} \text{ since } C_{\alpha} = C_{\beta} = \oint \boldsymbol{\pi} \cdot d\boldsymbol{\ell} = h$$

Links:



Knots:



## Generalized Kutta-Joukowski theorem

Forces acting on a longitudinally invariant straight vortex in asymptotically uniform and steady superfluid flows with currents  $\bar{n}_x^\nu$ ?

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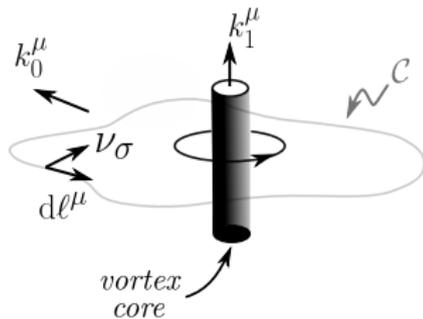
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**Far from the vortex:**

- no forces  $f_\nu^x = 0$
- irrotational flows  $\varpi_{\mu\nu}^x = 0$
- current conservation  $\nabla_\nu n_x^\nu = 0$
- $T_\mu^\sigma = \bar{T}_\mu^\sigma + \delta T_\mu^\sigma$



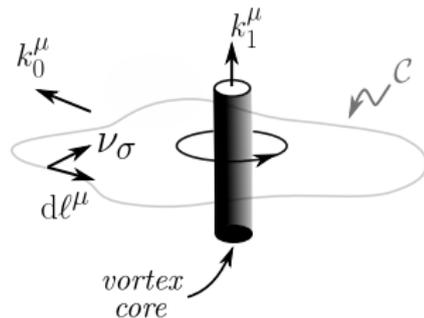
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Force per unit length:  $\mathcal{F}_\mu = \oint_C \nu_\sigma T_\mu^\sigma d\ell$

$$\mathcal{F}_\nu = \oint_C \nu_\sigma \delta T_\mu^\sigma d\ell = \sum_x (\varepsilon_{\sigma\nu\mu\rho} k_0^\mu k_1^\rho \bar{n}_x^\sigma C^x + \bar{\pi}_\nu^x D^x)$$

$$\text{where } C^x = \oint_C \pi_\nu^x d\ell^\nu \text{ and } D^x = \oint_C \bar{n}_x^\mu \nu_\mu d\ell$$

Carter, Langlois, Prix, *Vortices in unconventional superconductors and superfluids* (2002, Springer), pp 167–173

# Generalized Kutta-Joukowski theorem

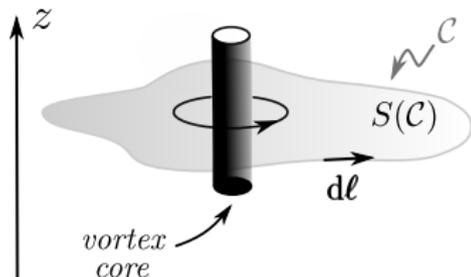
In Newtonian spacetime (in the vortex frame):

$$\mathcal{F} = \sum_x (\bar{n}_x \mathcal{C}^x \bar{\mathbf{v}}_x \times \hat{\mathbf{z}} + \bar{\pi}^x \mathcal{D}^x)$$

$$\mathcal{C}^x = \oint_c \pi^x \cdot d\ell$$

$$\mathcal{D}^x = - \iint_{S(c)} \nabla \cdot (n_x \mathbf{v}_x) dS$$

*Carter&Chamel, Int.J.Mod.Phys.D14,717(2005)*



- The first term is a **Magnus force** with the **momentum circulation**:  $\mathcal{C}^x = h$  for a superfluid
- The second term is a **transfusive force** arising from chemical/nuclear reactions: if particles are conserved  $\mathcal{D}^x = 0$

## Application to vortex motion in neutron-star cores

Force per unit length acting on a neutron vortex to which  $N_p$  proton fluxoids are pinned?

Combining the nuclear, electromagnetic, and electron contributions to the stress-energy tensor, the force (in the vortex frame) reads

$$\mathcal{F} = -\bar{\rho}_n \kappa_n \hat{\mathbf{z}} \times \bar{\mathbf{v}}_n - \bar{\rho}_p N_p \kappa_p \hat{\mathbf{z}} \times \bar{\mathbf{v}}_p + D_e \bar{\mathbf{v}}_p$$

$\kappa_{n/p} = h/(2m_{n/p})$  are the quanta of circulation

$D_e$  accounts for electron-scattering off the magnetic field of the lines

*Sourie&Chamel, MNRAS 493, 382 (2020)*

Limiting cases previously considered:

- single neutron vortex  $N_p = 0$ ,  $D_e(\Phi_*)$   
*Alpar, Langer, Sauls, ApJ282 (1984) 533*
- single proton fluxoid  $\kappa_n = 0$ ,  $N_p = 1$ , and  $D_e \neq 0$   
*Gusakov, MNRAS 485, 4936 (2019)*

## Application to vortex motion in neutron-star cores

The motion of a vortex line is determined by Newton's law

$$m_L \frac{\partial \mathbf{v}_L}{\partial t} = \mathcal{F}.$$

In the stationary regime, the vortex velocity is given by

$$\mathbf{v}_L = \bar{\mathbf{v}}_n + \mathcal{B} \hat{\mathbf{z}} \times (\bar{\mathbf{v}}_p - \bar{\mathbf{v}}_n) - \mathcal{B}' \hat{\mathbf{z}} \times \hat{\mathbf{z}} \times (\bar{\mathbf{v}}_p - \bar{\mathbf{v}}_n)$$

with **mutual-friction coefficients** ( $\xi \equiv D_e / (\bar{\rho}_n \kappa_n)$  and  $X \equiv \bar{n}_p N_p / \bar{n}_n$ )

$$\mathcal{B} = \frac{\xi}{\xi^2 + (1 + X)^2} \quad \text{and} \quad 1 - \mathcal{B}' = \frac{1 + X}{\xi^2 + (1 + X)^2}$$

*Sourie&Chamel, MNRAS 493, 382 (2020)*

The onset of **superfluid turbulence** is thought to be governed by the parameter  $\mathcal{B} / (1 - \mathcal{B}') > 1$ .

Potentially  $N_p$  may be as large as  $\sim 10^{13}$ . Pinning may thus play an important role.

## Superfluidity condition at different scales

- **On a scale  $\ell \ll$  intervortex spacing  $d_v$**

The superfluid is “irrotational”  $\varpi_{\mu\nu}^x = 0$

$$\Rightarrow \pi_{\mu}^x = \hbar \nabla_{\mu} \phi^x$$

where  $\phi^x$  is the quantum phase of the (boson) condensate.

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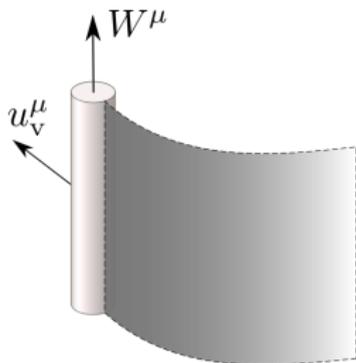
- **On a scale  $\ell \gg d_v$**

The superfluid essentially rotates as a rigid body  $\langle \varpi_{\mu\nu}^x \rangle \neq 0$ .

If dissipation is weak and helicity is conserved  $\nabla_{\mu} \langle \eta_x^{\mu} \rangle = 0$

$$\Rightarrow \varepsilon^{\mu\nu\rho\sigma} \langle \varpi_{\mu\nu}^x \rangle \langle \varpi_{\rho\sigma}^x \rangle = 0$$

The evolution of vorticity spans a 2D surface in spacetime defined by two vectors  $u_{\nu}^{\mu}$  and  $W^{\mu}$  such that  $\mathcal{L}_{\vec{u}_{\nu}} \langle \varpi_{\mu\nu}^x \rangle = 0$  and  $\mathcal{L}_{\vec{W}} \langle \varpi_{\mu\nu}^x \rangle = 0$



# Relativistic superfluid models of neutron stars

Minimal model of cold superfluid neutron stars:

- a neutron superfluid
- a “proton” fluid (nuclei in the crust, protons, leptons)

Corresponding Lagrangian in the limit of small relative currents:

$$\Lambda = -\rho(n_n, n_p)c^2 + \lambda_1(n_n, n_p)(x^2 - n_n n_p) + \dots$$

$$\text{with } x^2 c^2 = -g_{\mu\nu} n_n^\mu n_p^\nu, \quad n_n^2 c^2 = -g_{\mu\nu} n_n^\mu n_n^\nu, \quad n_p^2 c^2 = -g_{\mu\nu} n_p^\mu n_p^\nu.$$

The microscopic physics is embedded in  $\rho(n_n, n_p)$ ,  $\lambda_1(n_n, n_p) \dots$

The gravitational field is described by the Einstein-Hilbert Lagrangian density  $\Lambda_{\text{EH}} = \frac{c^4}{16\pi G} R$ , where  $R$  is the Ricci scalar.

*Chamel, MNRAS 388, 737 (2008)*

## Pulsar frequency glitches

Pulsars are spinning very rapidly with **extremely stable periods**:

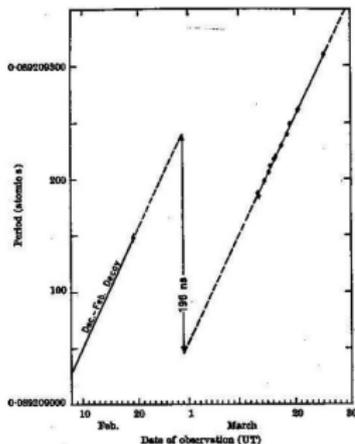
$\dot{P} \gtrsim 10^{-21}$ , outperforming the best atomic clocks.

Milner et al., PRL 123, 173201 (2019)

Still, some pulsars have been found to suddenly spin up. So far, 555 glitches have been detected in 190 pulsars.

<http://www.jb.man.ac.uk/pulsar/glitches.html>

Recent review: Haskell&Melatos, Int. J. Mod. Phys. D 24, 1530008 (2015)



The first glitch was detected in Vela:

The rotational frequency had increased by  $\Delta\Omega/\Omega \simeq 2 \times 10^{-6}$ .

The increase in the spin-down rate was even larger  $\Delta\dot{\Omega}/\dot{\Omega} \simeq 7 \times 10^{-3}$ .

Radhakrishnan& Manchester, Nature 222, 228 (April 1969); Reichley& Downs, ibid. 229

## Speculations about the origin of the Vela glitch

Melvin Ruderman proposed that the glitch is the manifestation of **crustquakes** (spin-down induced crustal stress).

*Ruderman, Nature 223, 597 (August 1969).*



As David Pines pointed out in 1999, the Vela quake would be a cataclysmic event since this would correspond to an **Earthquake of magnitude 17 in which the entire surface is shifted by 15m!**

*Pulsar Timing, General Relativity and the Internal Structure of Neutron Stars, Edited by Z. Arzoumanian, F. Van der Hooft, and E. P. J. van den Heuvel. Amsterdam, The Netherlands, 1999, p. 199.*

## Crustquakes

According to Ruderman's theory, glitches should be rare: no such event should be observed again in Vela in a human lifetime.

*Smoluchowski, PRL 24, 923 (1970).*

*Baym&Pines, Ann. Phys. 66, 816 (1971).*

Ruderman realized that the observation of the first glitch would thus have been an unlikely coincidence!



Freeman Dyson speculated that glitches could be more frequent if the crustal stress is induced by the accumulation of materials from volcanoes.

*Dyson, Nature 223, 486 (August 1969).*

## Further observations and speculations

In the fall of 1971, a **second glitch occurred in Vela** thus ruling out Ruderman's crustquake theory. Other theories were proposed:

- accretion
- volcanic activity
- corequakes
- planetary perturbations
- magnetospheric instabilities.

But none of them was really convincing.

*see, e.g., Pines, Shaham & Ruderman, IAU proceedings 53, 189 (1974).*

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After **one year** (vs  $\sim 10^{-23}$  s for typical nuclear time scale!), the spin-down rate relaxed to the value it had before the glitch, providing very strong evidence for superfluidity.

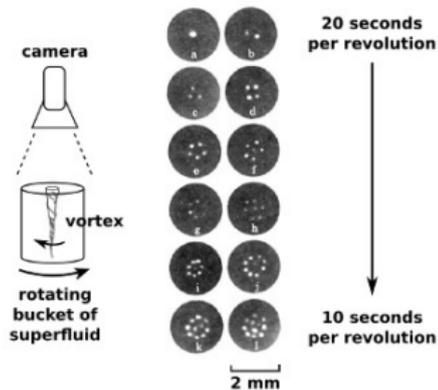
*Baym, Pethick, Pines, Nature 224, 673 (November 1969)*

In 1972, Packard suggested that glitches are related to the **metastability** of the superfluid (*PRL 28, 1080*).

## Vortex dynamics in neutron stars

A rotating superfluid is threaded by **quantized vortex lines**, each of which carries an angular momentum  $\hbar$ .

Similarly, a rotating neutron star is threaded by  $\sim 10^{18}$  vortices, as pointed out by Ginzburg & Kirzhnits in 1964.



*Yarmchuk et al., PRL43, 214 (1979).*

In 1975, it was proposed that giant glitches are triggered by the sudden **unpinning of vortices** in neutron-star crust.

*Anderson&Itoh, Nature 256, 25 (1975)*

This scenario found support from **laboratory experiments** on He II.

*J. S. Tsakadze & S. J. Tsakadze, J. Low Temp. Phys. 39, 649 (1980).*

Postglitch relaxation can be explained by **vortex creep**.

*Pines & Alpar, Nature 316, 27(1985)*

## Two-component model of giant glitches

Giant glitches are usually interpreted as sudden transfers of angular momentum between the superfluid and the “crust”.

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- Non-dissipative entrainment

$$\begin{aligned} J_s &= I_{ss} \Omega_s + I_{sc} \Omega & I_s &= I_{ss} + I_{sc} \\ J_c &= I_{cs} \Omega_s + I_{cc} \Omega & I_c &= I_{cs} + I_{cc} \end{aligned}$$

where  $I_{sc} = I_{cs}$

## Giant glitches and the superfluid inertia

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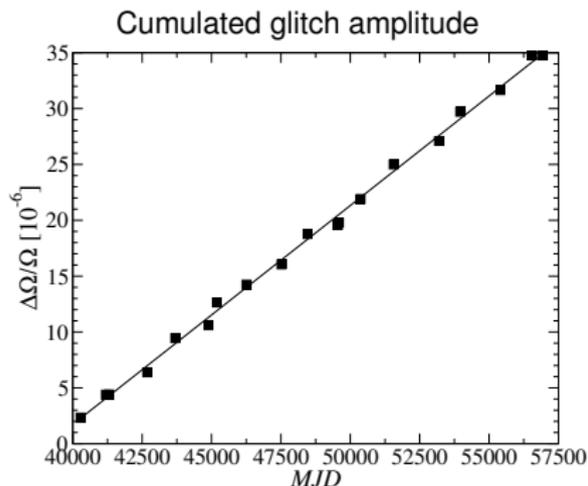
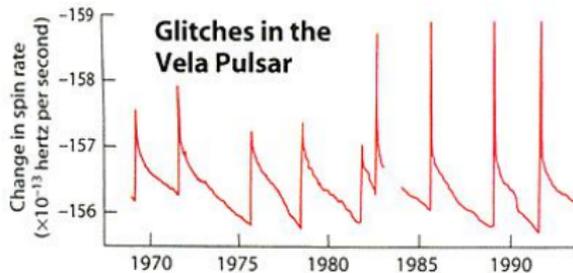
$$\frac{(I_s)^2}{I_{ss} I} \geq \mathcal{G}, \quad \mathcal{G} = 2\tau_c A_g$$

The parameter  $\mathcal{G}$  can be inferred from pulsar timing:

- $\tau_c = \frac{\Omega}{2|\dot{\Omega}|}$  is the characteristic age of the pulsar
- $A_g = \frac{1}{t} \sum_i \frac{\Delta \Omega_i}{\Omega}$  is the glitch activity.

# Vela pulsar glitch constraint

Since 1969, many glitches have been regularly detected. The latest one occurred in December 2016.



A linear fit of  $\frac{\Delta\Omega}{\Omega}$  vs  $t$  yields

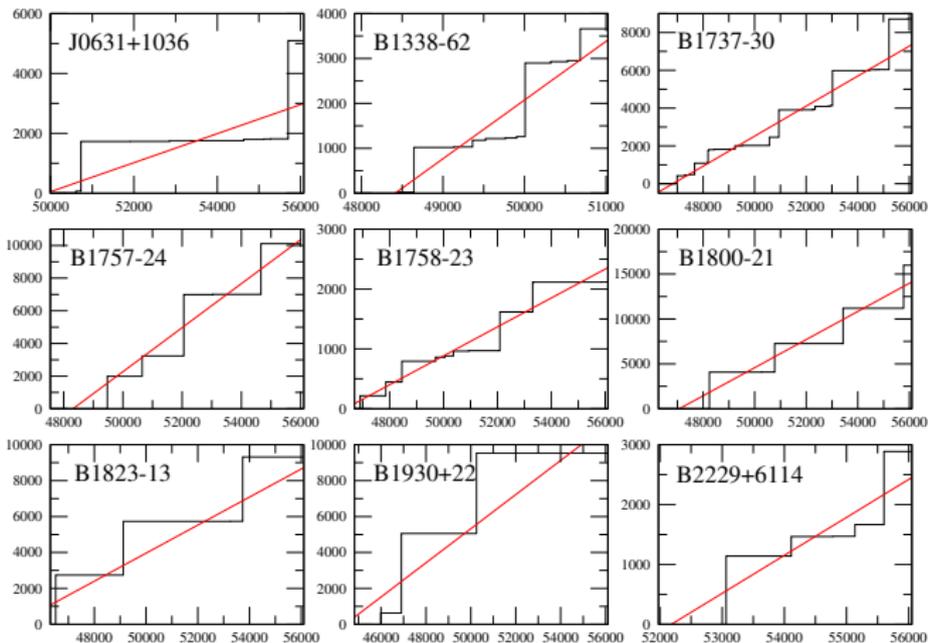
$$A_g \simeq 2.25 \times 10^{-14} \text{ s}^{-1}$$

Assuming  $|I_{sc}|$  is small, the low  $\mathcal{G} = 2\tau_c A_g \simeq 1.62\%$  suggests that only the superfluid in the crust is involved in accordance with the strong friction in the core.

*Link et al., PRL 83, 3362 (1999)*

## Vela like glitches

This interpretation is supported by other glitching pulsars:



$\mathcal{G} = 2\tau_c A_g$  is of the same order as for Vela.

*Andersson et al. PRL 109, 241103.*

## Crust angular momentum reservoir

$I_{SS}/I_S$  depends on the **physics of neutron-star crusts**.

Using the thin-crust approximation

$$\frac{I_{SS}}{I_{\text{crust}}} \approx \frac{1}{P_{\text{cc}}} \int_{P_{\text{drip}}}^{P_{\text{cc}}} \frac{n_n(P)^2}{\bar{n}(P)n_n^S(P)} dP, \quad \frac{I_S}{I_{\text{crust}}} \approx \frac{1}{P_{\text{cc}}} \int_{P_{\text{drip}}}^{P_{\text{cc}}} \frac{n_n(P)}{\bar{n}(P)} dP.$$

where  $P_{\text{drip}}$  is the pressure at the neutron-drip transition, and  $P_{\text{cc}}$  the pressure at the crust-core interface, we found  $I_{SS} \approx 4.6 I_{\text{crust}}$  and  $I_S \approx 0.89 I_{\text{crust}}$  leading to  $I_{SS}/I_S \approx 5.1$ .

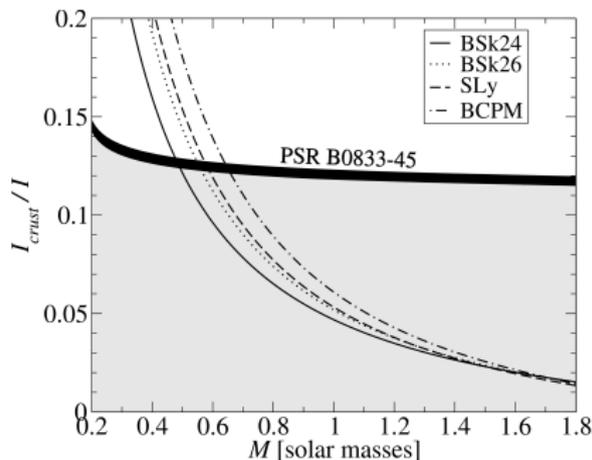
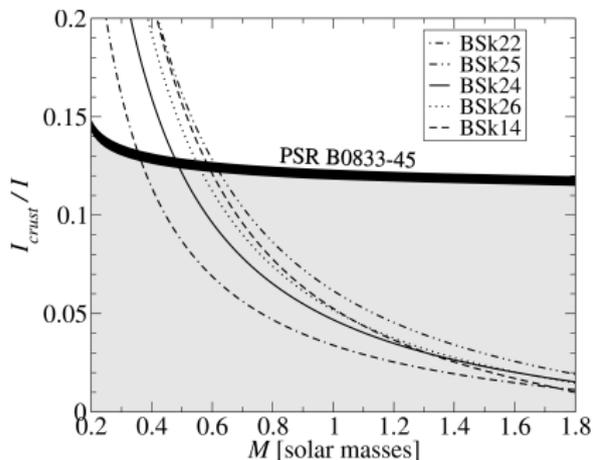
The Vela glitch constraint thus becomes  $\frac{I_S}{I} \geq 8.3\%$ , or  $\frac{I_{\text{crust}}}{I} \geq 9.3\%$

The superfluid in the crust of a neutron star with a mass  $M > M_{\odot}$  (as indicated by supernova simulations and known neutron-star masses) does not carry enough angular momentum to explain glitches!

*Andersson et al., PRL 109, 241103; Chamel, PRL 110, 011101 (2013)*

## Nuclear uncertainties

We have calculated  $I_S$  and  $I_{SS}$  in the slow-rotation approximation for different realistic models of neutron stars:



The inferred mass of Vela is at most  $M \sim 0.66M_{\odot}$ , corresponding to central baryon densities  $\bar{n} \approx 0.23 - 0.33 \text{ fm}^{-3}$ .

At such densities, the equation of state is fairly well constrained by laboratory experiments.

*Delsate et al., Phys. Rev. D 94, 023008 (2016)*

## Core-induced glitches ?

- If glitches were due to corequakes, the energy released should have been seen in X-rays  
*Ruderman, ApJ 203, 213 (1976)*
- If protons form a type II superconductor, the crust-core coupling time could be much longer: the core could contribute to glitches.  
*Sedrakian et al., ApJ 447, 324 (1995)*
- With the core superfluid, giant glitches can be explained even if crustal superfluidity is suppressed  
*Montoli et al., MNRAS 492, 4837 (2020)*
- Angular momentum reservoir?
  - Vortex-interface interactions  
*Sedrakian&Corde, MNRAS 307, (1999)*
  - Extent of superfluid regions  
*Ho et al., Science Adv. 1, e1500578 (2015)*
  - Vortex pinning to fluxoids  
*Ruderman, Astrophys. Space Sci.357, 353 (2009)*  
*Gügercinoğlu & Alpar, ApJ 788, L11 (2014)*

## Glitch rise

Timing of the Crab and Vela pulsars have recently revealed very peculiar evolutions of their spin frequency during the rise of a glitch.

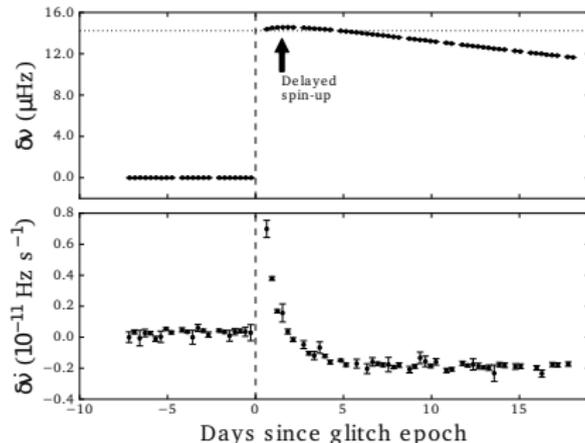
# Glitch rise

Timing of the Crab and Vela pulsars have recently revealed very peculiar evolutions of their spin frequency during the rise of a glitch.

- Analyses of a Vela glitch in 2016 suggest a rotational-frequency **overshoot and a fast relaxation** ( $\sim$  min) following the glitch.

*Ashton, Lasky, Graber, Palfreyman, Nature Astronomy 3, 1143 (2019)*

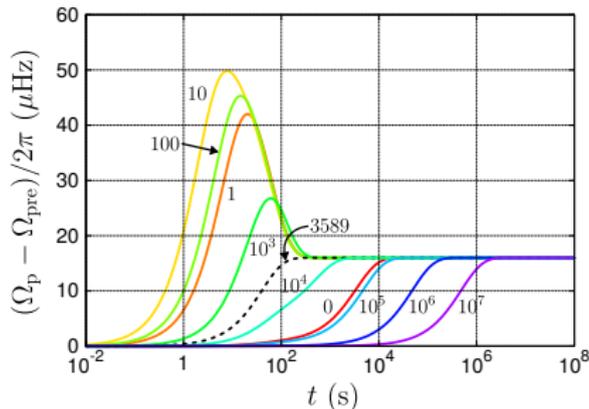
- A **delayed spin-up** has been detected in the 1989, 1996 and 2017 Crab glitches.



*Shaw et al., MNRAS, 478, 3832 (2018)*

## Role of vortex pinning to fluxoids

These differences can be interpreted from the interactions between superfluid vortices and proton fluxoids in neutron-star cores.



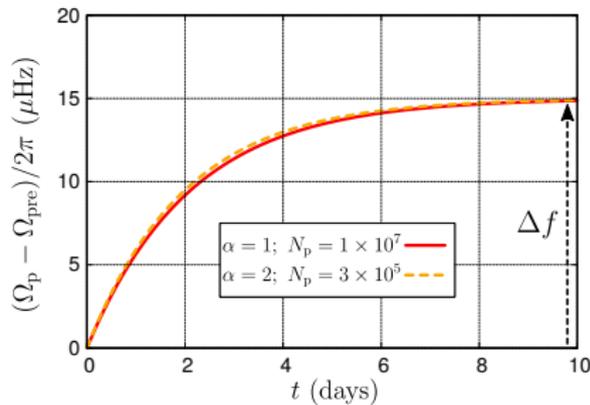
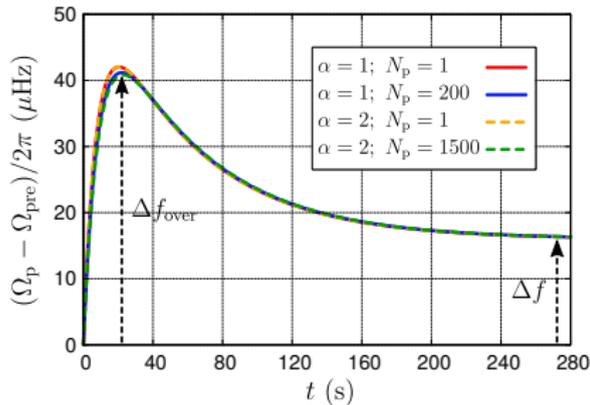
The number  $N_p$  of fluxoids attached to vortices turns out to be a key parameter governing the global dynamics of the star:

- $N_p < N_p^{\text{crit}}$ : overshoot  $\Delta\Omega_{\text{over}} < \Delta\Omega / (1 - I_n^{\text{free}} / I)$ ,
- $N_p < N_p^{\text{crit}}$ : smooth spin-up on a longer timescale.

*Sourie&Chamel, MNRAS 493, L98 (2020)*

# Role of vortex pinning to fluxoids

The behavior of Vela and Crab glitches can be reproduced:



However, this neutron-star model remains very simplified:

- Newtonian approach
- physical reason for different  $N_p$  remains to be investigated
- $N_p^{\text{crit}}$  depends on poorly-known mutual friction.

Alternative explanations:

*Haskell et al., MNRAS 481, L146 (2018)*

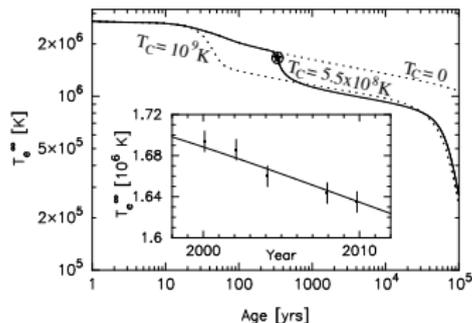
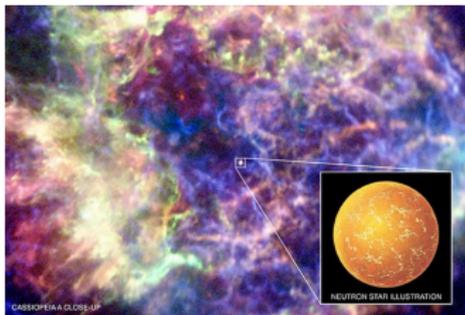
*Gügercinoğlu&Alpar, MNRAS 488, 2275 (2019)*

# Neutron-star cooling

Apart from glitches, other independent observations support the existence of neutron-star superfluids:

- Observations of Cassiopeia A provide strong evidence for neutron-star core superfluidity.

*Page et al., PRL 106, 081101; Shternin et al., MNRAS 412, L108.*



- Observations of quasi-persistent soft X-ray transients provide evidence for neutron-star crust superfluidity.

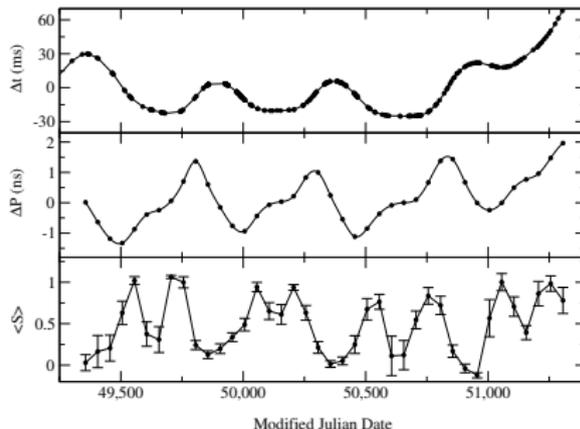
*Shternin et al., Mon. Not. R. Astron. Soc. 382(2007), L43.*

*Brown and Cumming, ApJ 698 (2009), 1020.*

# Neutron star precession

**Long-term cyclical variations** of order months to years have been reported in a few neutron stars: Her X-1 (accreting neutron star), the Crab pulsar, PSR 1828–11, PSR B1642–03, PSR B0959–54 and RX J0720.4–3125.

Example: Time of arrival residuals, period residuals, and shape parameter for PSR 1828–11  
*Stairs et al., Nature 406(2000),484.*



These variations have been interpreted as the signature of **neutron star precession**.

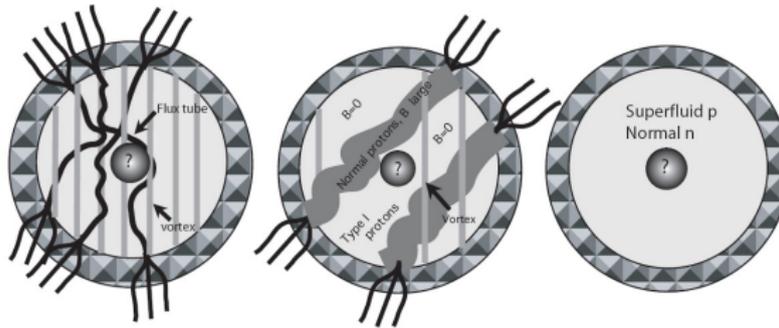
# Precession and superfluidity

For a non-superfluid star with deformation  $\epsilon = \Delta I / I$ ,

$$P_{\text{prec}} = \frac{P}{\epsilon} \gg P$$

For a superfluid star with pinned vortices

$$P_{\text{prec}} = \frac{I_{\text{pin}}}{I} P \ll P$$



*Link, Astrophys. Space Sci.308,435 (2007)*

Observations of precession could thus shed light on superfluidity. On the other hand, precession may trigger instabilities that could unpin vortices.

*Glampedakis, Andersson, Jones, PRL 100, 081101 (2008).*

## Asteroseismology of neutron stars

The presence of superfluids and superconductors in neutron stars leads to the existence of new oscillations modes.

In the simplest two-fluid model, there exists two families of modes:

- "normal" modes (comoving fluids)
- "superfluid" modes (countermoving fluids)

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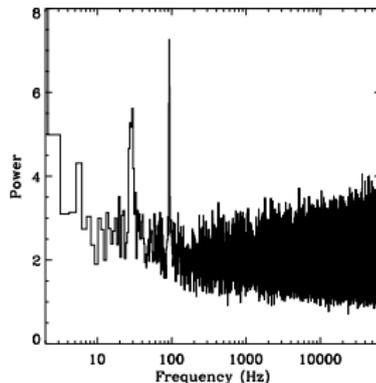
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- "normal" modes (comoving fluids)
- "superfluid" modes (countermoving fluids)

Quasiperiodic oscillations (QPOs) have been detected in the X-ray flux of giant flares from a few soft gamma-ray repeaters.

Example: SGR 1806–20

*Strohmayer&Watts, ApJ653,593 (2006)*

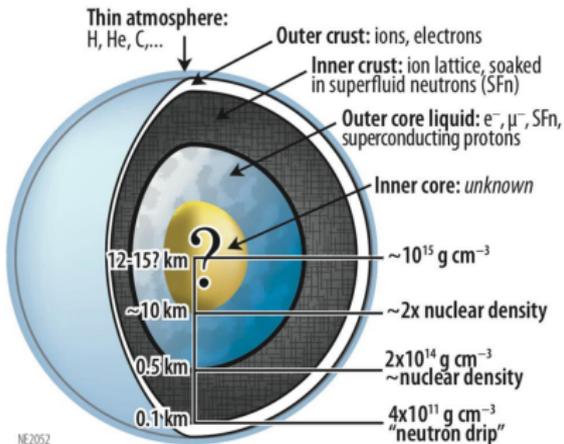


These QPOs are thought to be the signatures of **superfluid magneto-elastic oscillations** but detailed models are still lacking.

*Gabler et al., PRL 111, 211102 (2013).*

# Summary

The huge gravity of neutron stars produces the highest- $T_c$  ( $\sim 10^{10}$  K) superfluids and superconductors known in the Universe.



Neutron-star superfluidity and superconductivity are supported by independent observations (pulsar glitches, cooling).

Prospects: gravitational-wave asteroseismology (3d generation).

However, many aspects still remain poorly understood.

The main challenge is to relate the local nonrelativistic dynamics of vortices and fluxoids at the nuclear scale ( $\sim 10 \text{ fm} = 10^{-14} \text{ m}$ ) to the global general-relativistic dynamics of the star ( $\sim 10 \text{ km}$ ).