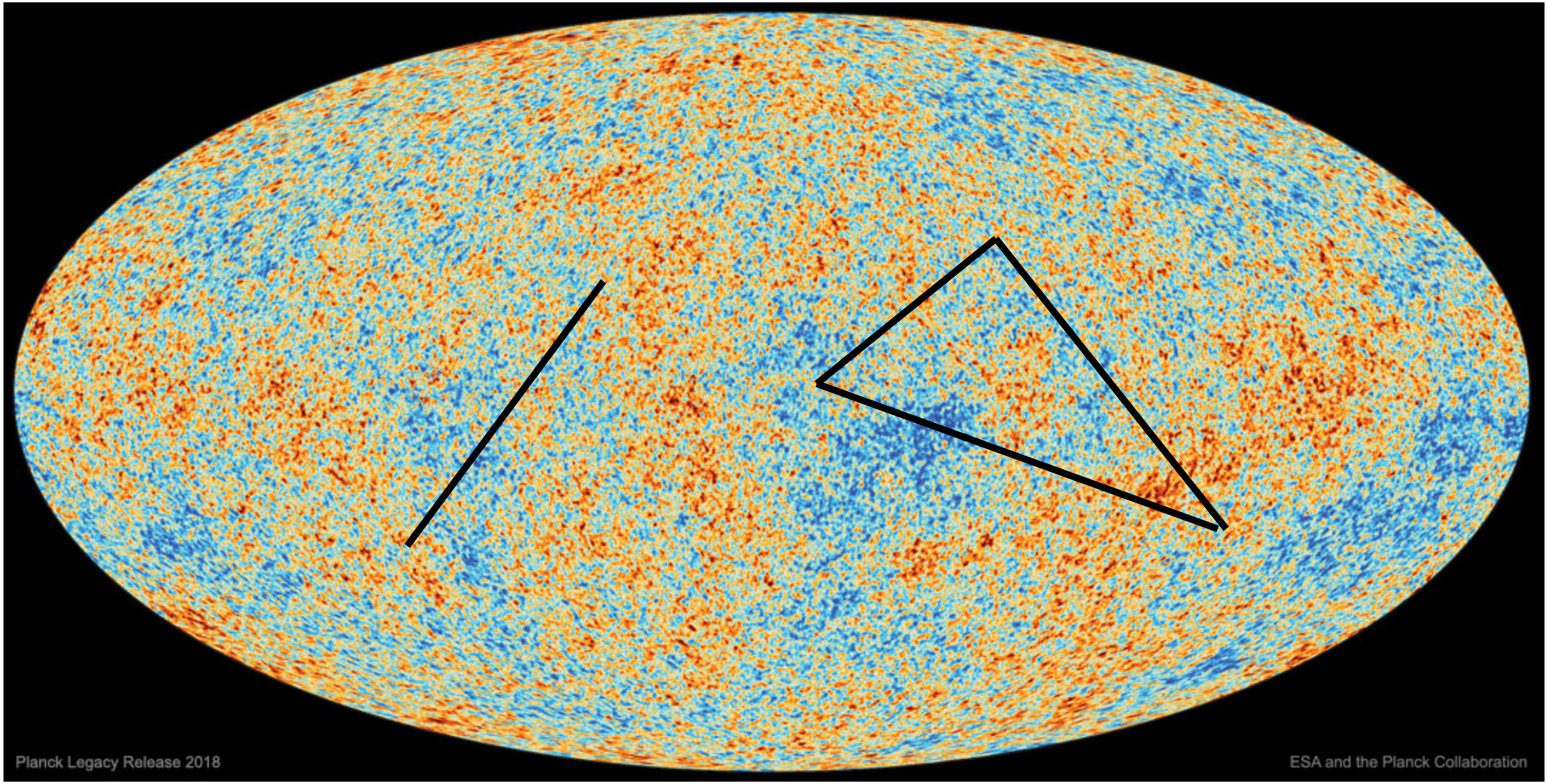
Paolo Creminelli, ICTP (Trieste)

Beyond Perturbation Theory in Inflation

with M. Celoria, G. Tambalo and V. Yingcharoenrat, 2103.09244 (JCAP)
with S. Renaux-Petel, G. Tambalo and V. Yingcharoenrat in progress

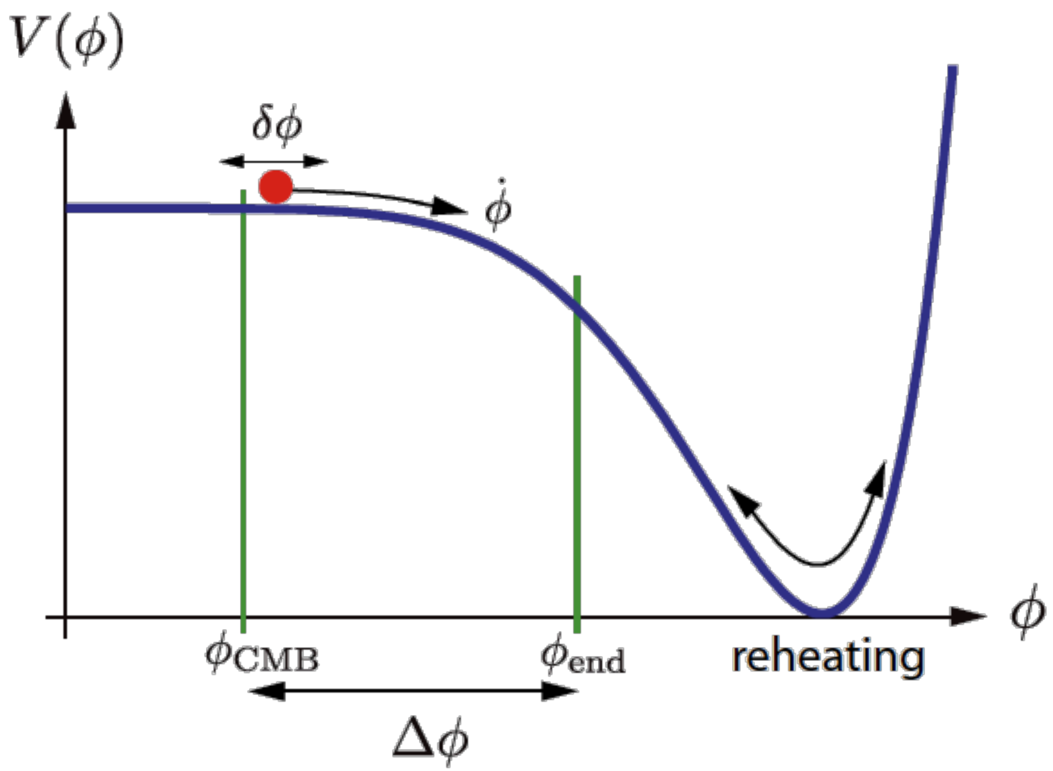
Beyond BSM, Monte Verità: 16.8.23

Gaussian or not ?



$|f_{\rm NL}^{\rm loc}| < 5$ $\qquad$ $|f_{\rm NL}^{\rm equil}| < 40$ $\qquad$ $\times\, 10^{-4}$!!

Slow-roll → Weak coupling → Gaussianity



$$\epsilon = \frac{1}{2} M_P^2 \left(\frac{V'}{V}\right)^2$$

$$\eta = M_P^2 \frac{V''}{V}$$

$$\epsilon, \eta, \ldots \ll 1$$

$\lambda \equiv V^{(4)} \lesssim \mathcal{O}(\epsilon^3, \eta^3)(10^{-5})^2$ Higgs $\lambda \sim 1$ $f_{\rm NL}^{\rm slow-roll} \sim |\eta| \sim 10^{-2}$

Large NG: derivative interactions, multi-field, features, warm inflation, dissipation, different symmetries, alternatives to inflation...

In-In Perturbation theory

Beyond free theory, correlation functions are calculated in PT

$$\langle Q(\eta)\rangle = \langle 0|\bar{T}e^{i\int_{\underline{-\infty(1-i\epsilon)}}^{\eta} H_{\rm int}(\eta')\mathrm{d}\eta'}\; Q^I(\eta)\; Te^{-i\int_{\underline{-\infty(1+i\epsilon)}}^{\eta} H_{\rm int}(\eta'')\mathrm{d}\eta''}\rangle$$

Bunch-Davies vacuum is obtained by a small deformation in Euclidean time in far past

E.g.

$$\frac{1}{2H^2\eta^2}\left[\zeta_c'^2-(\partial_i\zeta_c)^2\right]+\frac{\lambda}{H^4}\zeta_c'^4$$

Since field and derivative are ~ H, expansion is in λ

$$\frac{\langle\zeta\zeta\zeta\rangle}{P_\zeta^{3/2}}\sim f_{\rm NL}P_\zeta^{1/2}\ll 1 \qquad \frac{\langle\zeta\zeta\zeta\zeta\rangle}{P_\zeta^{2}}\sim g_{\rm NL}P_\zeta\sim\lambda\ll 1\ .$$

Experimentally (Planck and LSS) $\lesssim 10^{-3}$

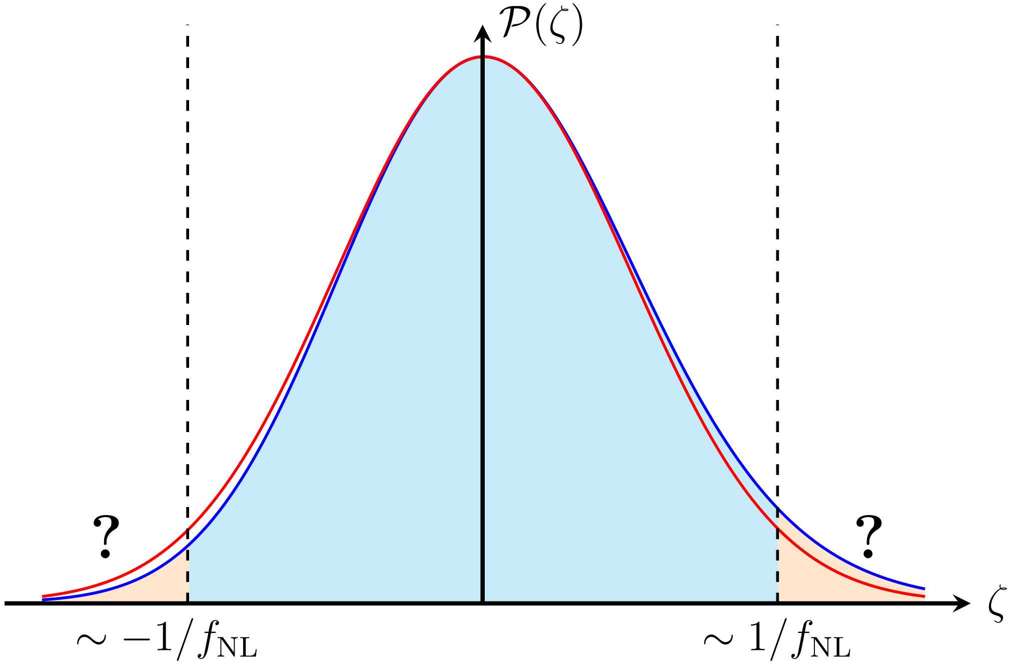
Tails of the distribution

This is ok for correlation functions but

$$P[\zeta] \sim \exp\left[-\frac{\zeta^2}{2P_\zeta} + \frac{\langle\zeta\zeta\zeta\rangle}{P_\zeta^3}\zeta^3 + \frac{\langle\zeta\zeta\zeta\zeta\rangle}{P_\zeta^4}\zeta^4 + \ldots\right] \sim \exp\left[-\frac{\zeta^2}{2P_\zeta}\left(1 + \frac{\langle\zeta\zeta\zeta\rangle}{P_\zeta^2}\zeta + \frac{\langle\zeta\zeta\zeta\zeta\rangle}{P_\zeta^3}\zeta^2 + \ldots\right)\right]$$

$$\frac{\langle\zeta\zeta\zeta\rangle}{P_\zeta^2}\zeta \sim f_{\rm NL}\zeta \qquad \frac{\langle\zeta\zeta\zeta\zeta\rangle}{P_\zeta^3}\zeta^2 \sim g_{\rm NL}\zeta^2 \; .$$

Expansion parameter depends on size of ζ



Motivations

- Black-hole formation is sensitive to $\zeta \sim 1$

$$\hat{\zeta}(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} W(k)\zeta(\vec{k})e^{i\vec{k}\cdot\vec{x}} \qquad \beta(M) = \int_{\zeta_c}^{+\infty} P(\hat{\zeta})d\hat{\zeta}$$

BH mass fraction at formation

Perturbative calculation is ok only for $f_{NL}\,\zeta \sim f_{NL} \ll 1$

Ok if one remains in slow-roll, $f_{NL} \sim O(\varepsilon,\eta)$, but not in general.

E.g. $f_{NL} \sim (1-1/c_s^2)$ in K-inflation models.

- Eternal inflation. Can the tail be relevant?
- Surprise in the data on the tails?
- It is the WFU!

Main idea

Since perturbations are proportional to $\hbar^{1/2}$ looking at unlikely events corresponds to the semiclassical limit $\hbar \to 0$

In this limit one is able to calculate the WFU semiclassically

Anharmonic oscillator

$$V(x) = \hbar\omega\left[\frac{1}{2}\left(\frac{x}{d}\right)^2 + \lambda\left(\frac{x}{d}\right)^4\right] \qquad d \equiv \sqrt{\hbar/m\omega}$$

Usual PT is in λ

PT breaks down for $\lambda\left(\frac{x}{d}\right)^2 \equiv \frac{\bar{x}^2}{2} \sim 1$

Consider the ground-state wavefunction (as in inflation)

One could use WKB, but let us look at Euclidean path-integral

$$\langle x_f|e^{-H\tau/\hbar}|x_i\rangle = \sum_n e^{-E_n\tau/\hbar}\Psi_n(x_f)\Psi_n^*(x_i)$$

$$\Psi_0(x_f)\overline{\Psi}_0(0)\, e^{-E_0\tau_f} = \lim_{\tau_f\to\infty}\int_{x(0)=0}^{x(\tau_f)=x_f} \mathcal{D}x(\tau)\, e^{-S_{\mathrm{E}}[x(\tau)]/\hbar}$$

$$= e^{-S_{\mathrm{E}}[x_{\mathrm{cl}}(\tau)]/\hbar}\int_{x(0)=0}^{x(\tau_f)=x_f} \mathcal{D}y(\tau)\, e^{-\frac{1}{\hbar}\left(\frac{1}{2}\frac{\delta^2 S}{\delta x^2}y^2+\frac{1}{3!}\frac{\delta^3 S}{\delta x^3}y^3+\dots\right)}$$

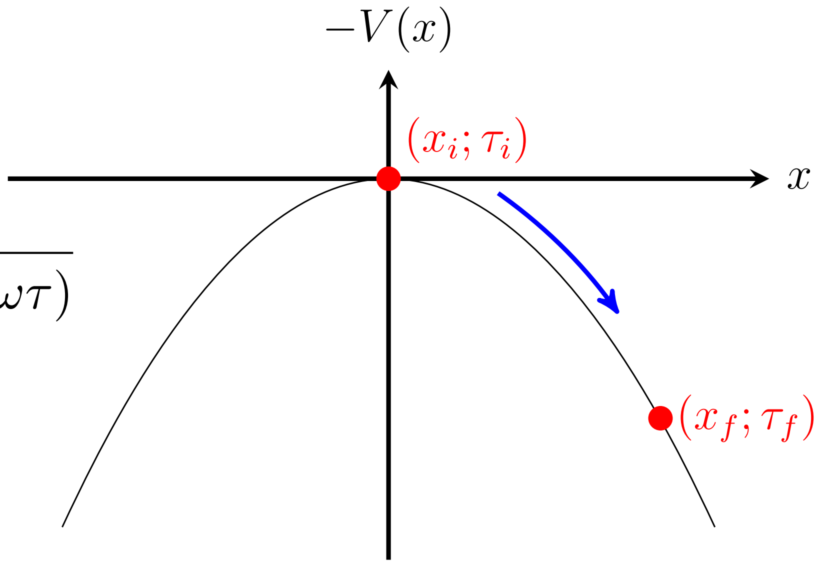
Anharmonic oscillator

$$S_{\rm E}[x(\tau)] = \int d\tau \left(\frac{1}{2}m\dot{x}^2 + V(x)\right)$$

$$\frac{S_{\rm E}[x(\tau)]}{\hbar} = \frac{1}{\hbar}\int d\tau\ m\dot{x}^2 \ = \frac{1}{\hbar}\int_{x_i=0}^{x_f} dx\ \sqrt{2mV} \ = \frac{1}{6\lambda}\left[(1+\bar{x}^2)^{3/2} - 1\right] \qquad \lambda\left(\frac{x}{d}\right)^2 \equiv \frac{\bar{x}^2}{2}$$

$$\Psi_0(x_{\rm f}) = \mathcal{I}(x_{\rm f})e^{-S_{\rm E}[x_{\rm cl}(\tau)]/\hbar}$$

$$x(\tau) = -\frac{d}{\sqrt{2\lambda}\sinh(\omega\tau)}$$



Van Vleck, Pauli, Morette formula:

$$\mathcal{I}(x_f) = \mathcal{N}\sqrt{\frac{m}{2\pi i\hbar v_i v_f \int_0^{x_f}\frac{{\rm d}x'}{v^3(x')}}}$$

$$\Psi_0(\bar{x}) = \mathcal{N}\frac{\exp\left\{-\frac{1}{6\lambda}\left[\left(1+\bar{x}^2\right)^{3/2} - 1\right] - \frac{\omega T}{2}\right\}}{(1+\bar{x}^2)^{1/4}\left(1+\sqrt{1+\bar{x}^2}\right)^{1/2}} \quad + \mathcal{O}(\lambda)F(\bar{x})$$

Gaussian WFU

Maldacena 02

In inflation the wavefunction of the Universe is

$$\Psi[\zeta_0(\vec{x})] = \int_{\mathrm{BD}}^{\zeta_0(\vec{x})} \mathcal{D}\zeta\, e^{iS[\zeta]/\hbar}$$

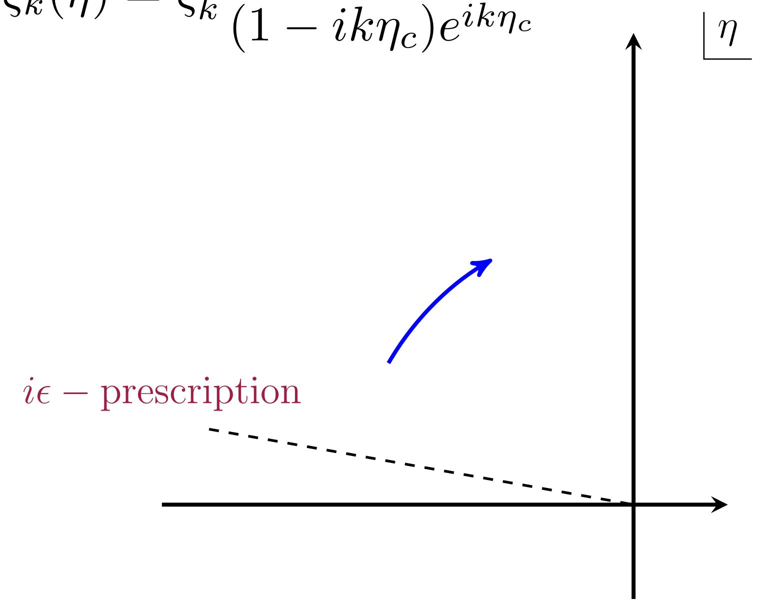
For free theory the saddle solution is

$$\zeta_k(\eta) = \zeta_k^0 \frac{(1 - ik\eta)e^{ik\eta}}{(1 - ik\eta_c)e^{ik\eta_c}}$$

It decays exponentially after iε rotation.

It is complex, since BD boundary condition is not real

$$\zeta_{-\vec{k}} \neq \zeta_{\vec{k}}^*$$

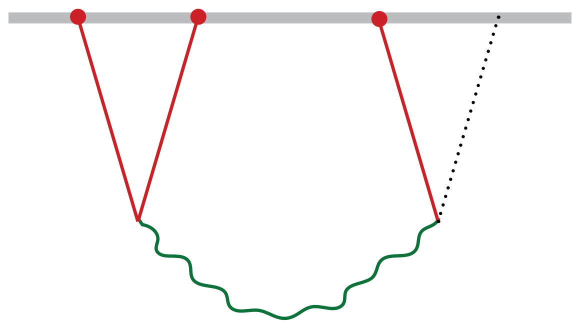


$$iS = i\int \frac{d^3k}{(2\pi)^3}\, \frac{1}{2H^2}\frac{1}{\eta_c^2}\zeta_{-\vec{k}}^0 \partial_\eta \zeta_{\vec{k}}^0\Big|_{\eta=\eta_c} \sim \int \frac{d^3k}{(2\pi)^3}\frac{1}{2H^2}\left[i\frac{k^2}{\eta_c} - k^3 + \dots\right]\zeta_{-\vec{k}}^0\zeta_{\vec{k}}^0$$

Scale-invariant power spectrum

WFU Beyond Gaussianity

Perturbative recipe for WFU is the same as in AdS/CFT: Witten diagrams



On shell action with prescribed boundary conditions at late times

$$\Psi = Exp\left[\frac{1}{2}\int d^3x d^3x' \langle \mathcal{O}(x)\mathcal{O}(x')\rangle f(x) f(x') + \frac{1}{6}\int d^3x d^3x' d^3x'' \langle \mathcal{O}(x)\mathcal{O}(x')\mathcal{O}(x'')\rangle f(x) f(x') f(x'')\right]$$

$\langle \mathcal{O}(x)\mathcal{O}(x')\ldots\rangle$ are the "CFT" correlators

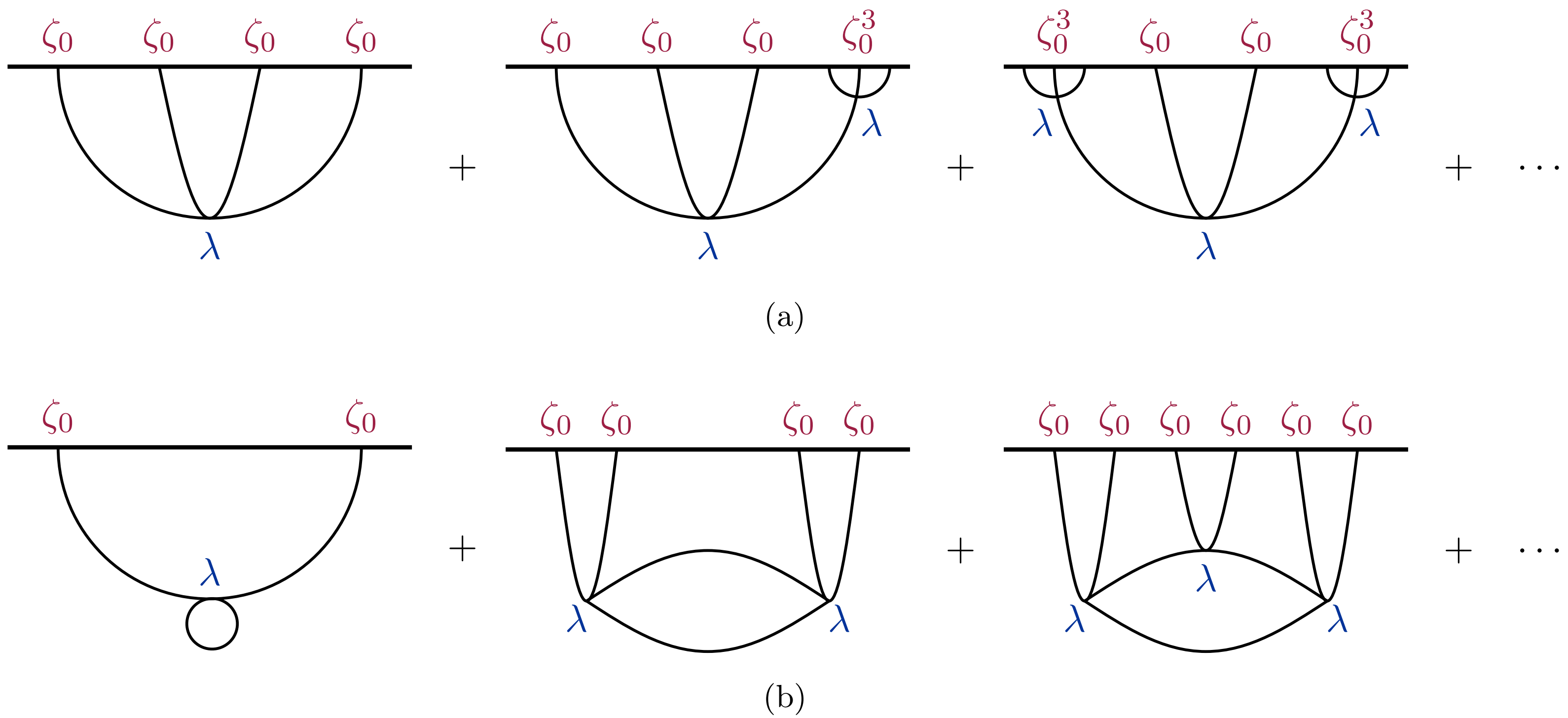
Cosmological correlators:

$$\langle f_{\vec{k}} f_{-\vec{k}}\rangle' = -\frac{1}{2Re\langle \mathcal{O}_{\vec{k}}\mathcal{O}_{-\vec{k}}\rangle'}$$

$$\langle f_{\vec{k}_1} f_{\vec{k}_2} f_{\vec{k}_3}\rangle' = \frac{2Re\langle \mathcal{O}_{\vec{k}_1}\mathcal{O}_{\vec{k}_2}\mathcal{O}_{\vec{k}_3}\rangle'}{\prod_i(-2Re\langle \mathcal{O}_{\vec{k}_i}\mathcal{O}_{-\vec{k}_i}\rangle')}$$

Resumming Witten diagrams

Tree level diagrams are dominant since, at a given order in λ, they have more ζ_0



The selection of diagrams makes sense only for large ζ_0

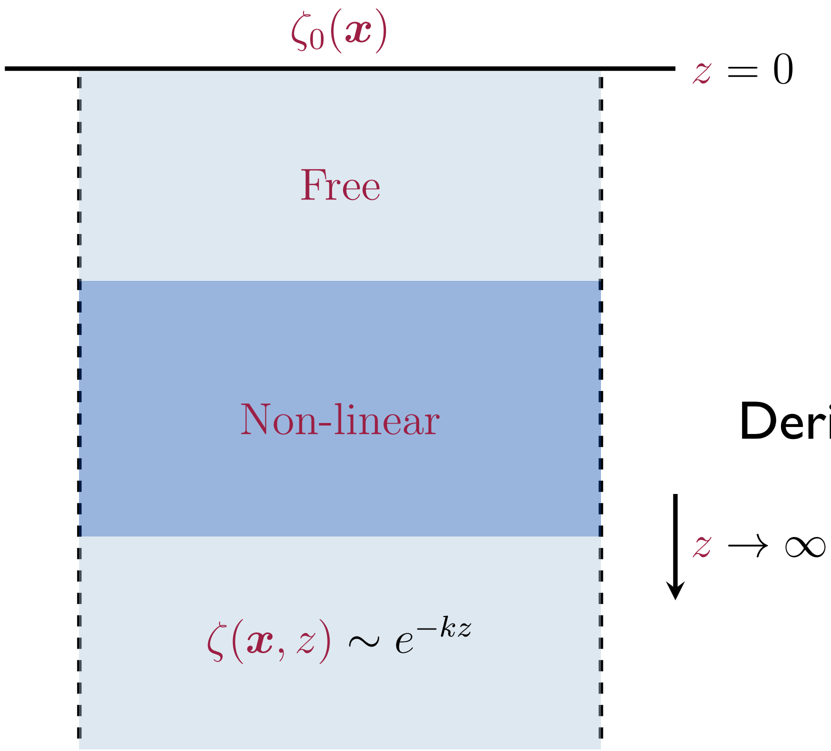
Non-linear WFU

1. Fix boundary conditions at late times ζ_0 and BD in far past
2. Find the classical non-linear solution of the EOM in Euclidean time
3. Calculate the action S and get Ψ

$$\Psi[\zeta_0(\vec{x})] \sim e^{-S/\hbar}$$

It corresponds to resumming all tree-level Witten diagrams

No evolution outside H



Derivative interactions

Exponential decay

Inflation with ζ'⁴ interaction

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}M_{\rm Pl}^2 R + M_{\rm Pl}^2 \dot{H} g^{00} - M_{\rm Pl}^2(3H^2+\dot{H}) + \right.$$
$$\left. + \frac{1}{2}M_2(t)^4(\delta g^{00})^2 + \frac{1}{3!}M_3(t)^4(\delta g^{00})^3 + \underline{\frac{1}{4!}M_4(t)^4(\delta g^{00})^4} + \dots\right]$$

Particular limit of EFTI

Senatore, Zaldarriaga 10

$$S = \int \mathrm{d}^3x\mathrm{d}\eta \left\{ \frac{1}{2\eta^2 P_\zeta}\left[\zeta'^2 - (\partial_i\zeta)^2\right] + \frac{\lambda\zeta'^4}{4! P_\zeta^2} \right\}$$

Euclidean EOM $\quad -\zeta'' + \frac{2}{\eta}\zeta' - \partial_i^2\zeta - \frac{\lambda}{2P_\zeta}\eta^2\zeta'^2\zeta'' = 0$

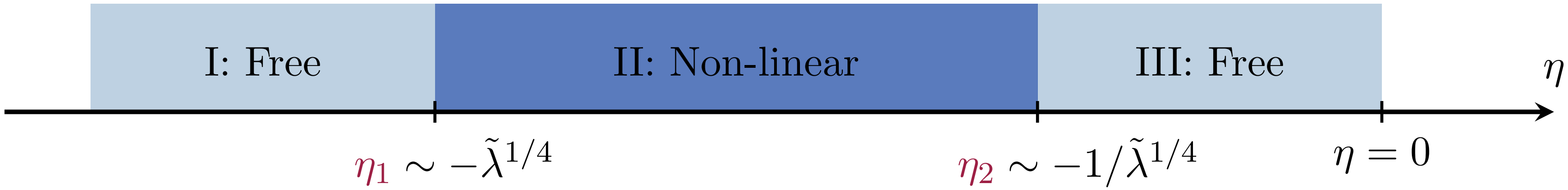
Before getting to the numerical solution of PDE, one can get some intuition reducing to an ODE

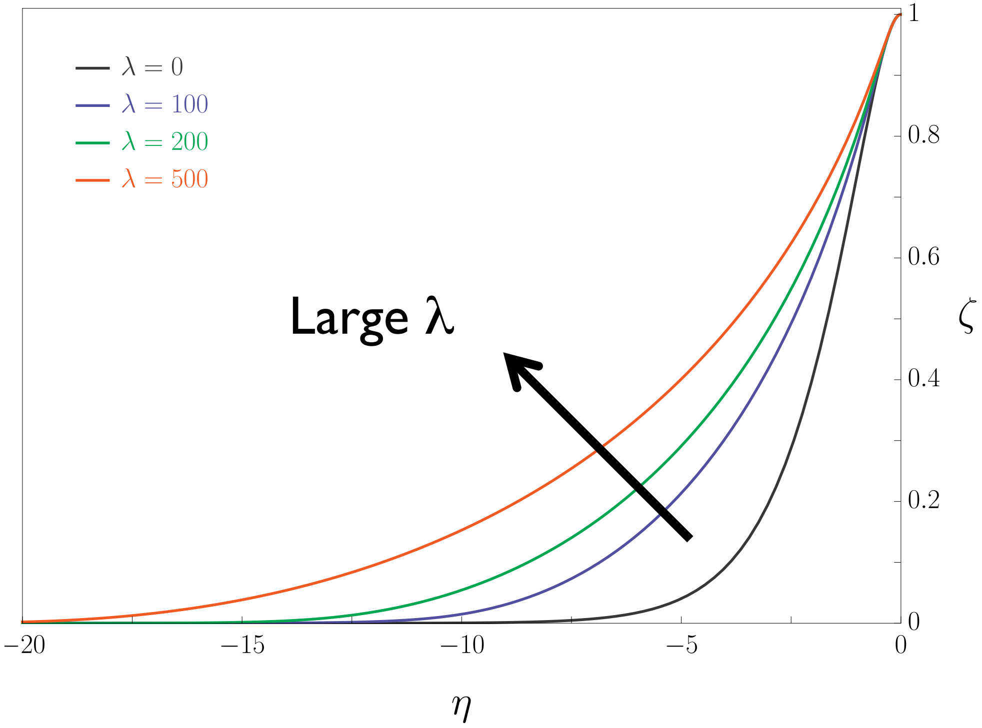
ODE

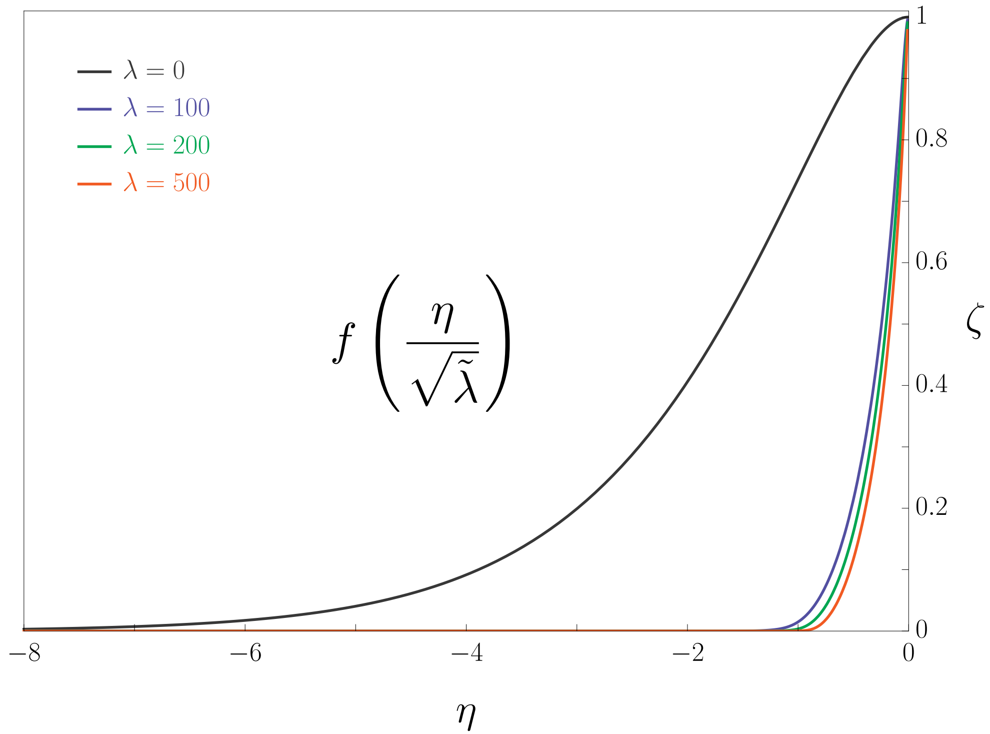
Once you fix a scale in ζ_0 derivative interactions will only affect comparable modes. Reduction to ODE should be O(1) ok

$$-\zeta'' + \frac{2}{\eta}\zeta' + H^2\zeta - \frac{\lambda}{2P_\zeta}\eta^2\zeta'^2\zeta'' = 0$$

$$\tilde{\lambda} \equiv \lambda \frac{\zeta_0^2}{P_\zeta}$$







ODE

Using this scaling one gets the behaviour at large λ

$$S_{\rm ODE} = -\frac{\zeta_0^2}{P_\zeta}\int_{-\infty}^{\eta_{\rm out}} d\eta \left\{\frac{1}{2\eta^2}\left[\zeta'^2+\zeta^2\right]+\frac{\tilde\lambda}{4!}\zeta'^4\right\}$$

Subtract free part to avoid late time divergence

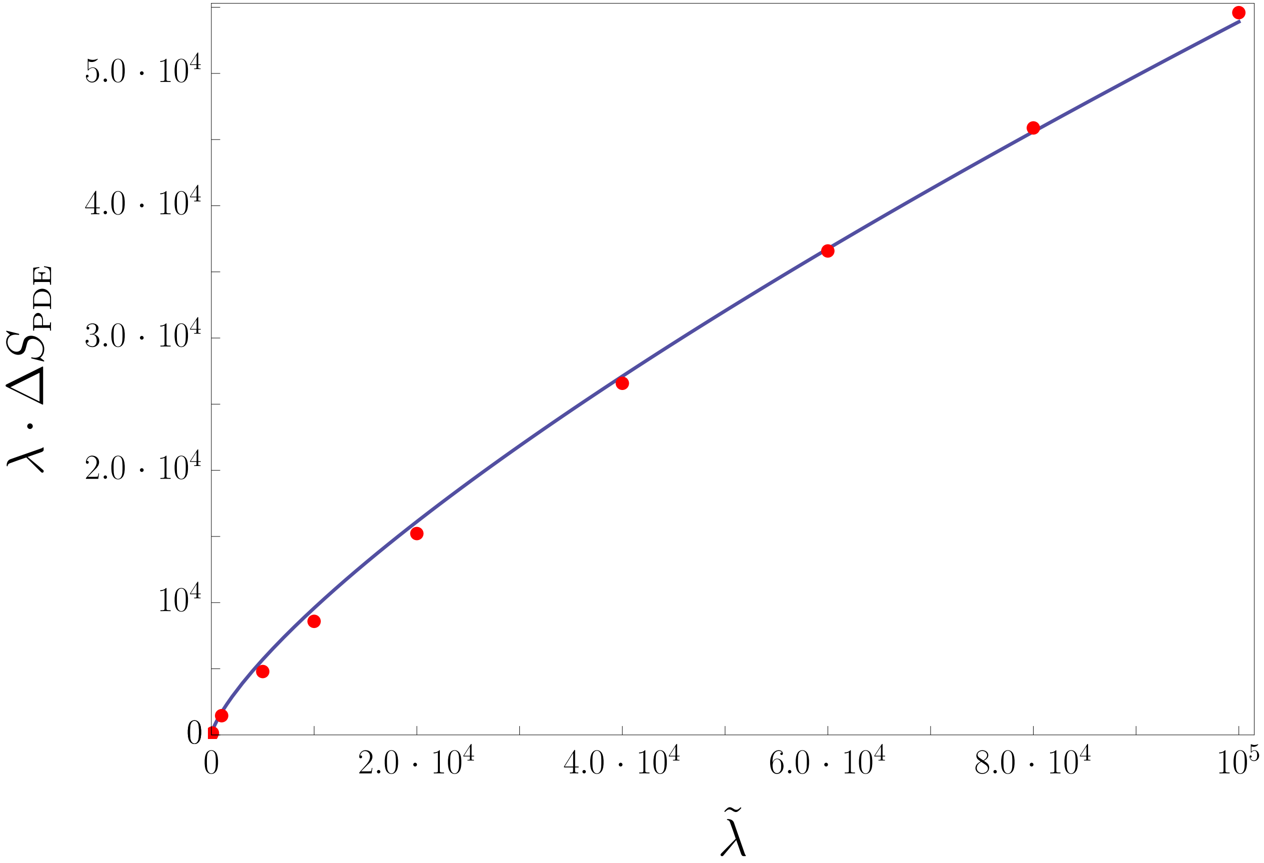
For large ζ_0

$$\Psi \simeq \exp\left[-\frac{1}{\lambda}\left(\frac{\lambda\zeta_0^2}{P_\zeta}\right)^{3/4}\right]$$

Not analytic in λ

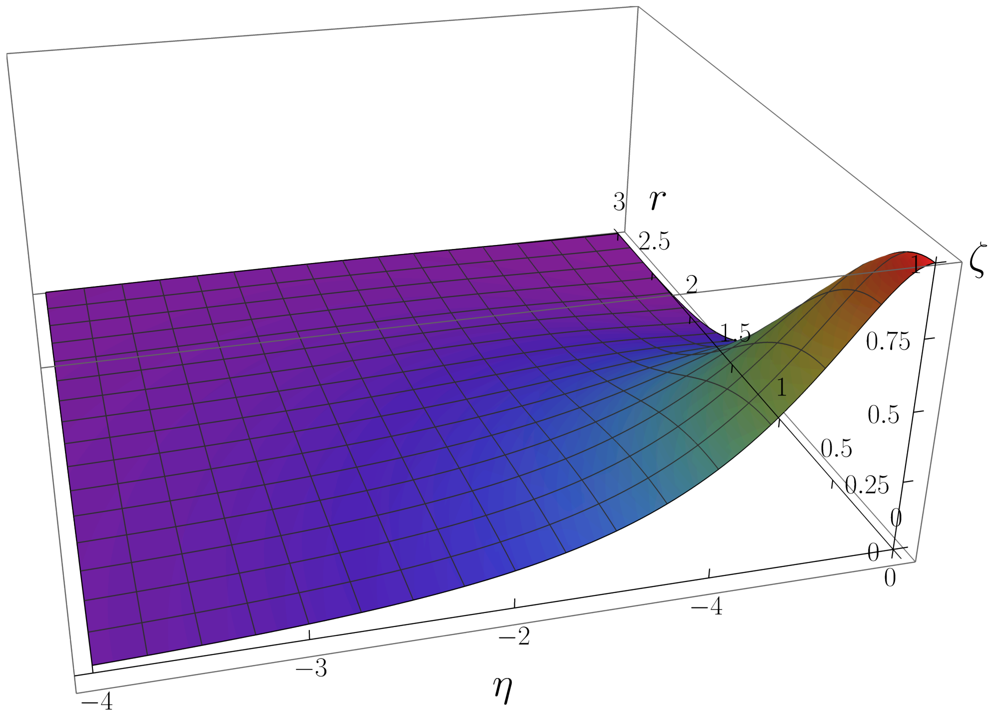
compared with

$$\Psi \simeq \exp\left[-\frac{\zeta_0^2}{2P_\zeta}\right]$$

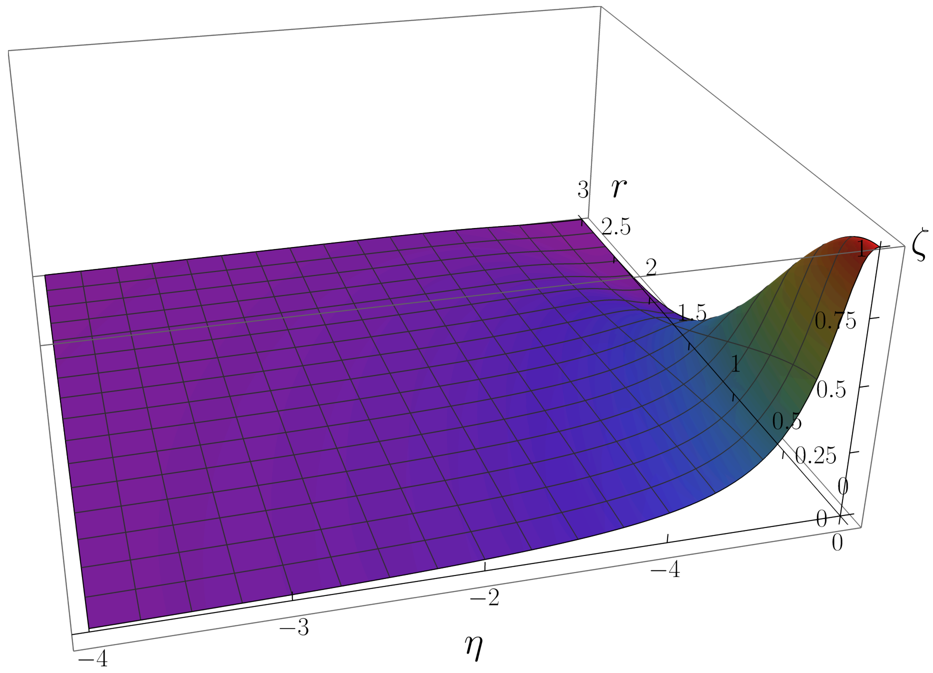


PDE

Qualitatively similar with the same asymptotic scaling in λ



Small λ



Large λ

(One can check to reproduce perturbative result at small λ)

WFU for resonant features

Leblond, Pajer 11

Behbahani, Dymarsky, Mirbabayi, Senatore 11

Focus on $V(\phi) = V_{\rm sr}(\phi) + \Lambda^4 \cos(\phi/f)$

For ε → 0 $S = \int \mathrm{d}^4x\, a(t)^3 M_{\rm Pl}^2 \dot{H}(t+\pi)(\partial_\mu \pi)^2$ $\dot{H}(t) = \dot{H}_\star\left(1 - \tilde{b}\cos(\omega t + \delta)\right)$

$\alpha \equiv \dfrac{\omega}{H_*}$ Non-linearity parameter is $\alpha(\alpha\zeta) = \alpha^2\zeta$

Three simplifications:

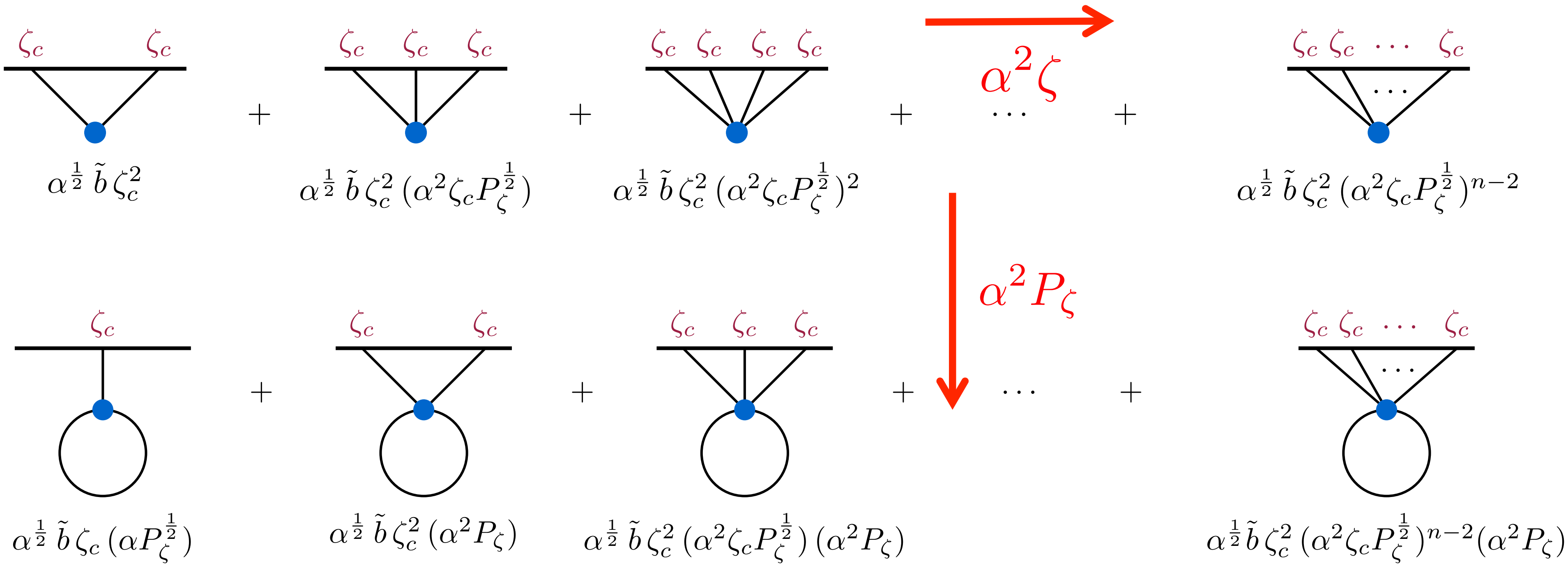
1. Small features, we expand in $\tilde{b}$

Since the action is stationary around EOM we only need $\tilde{b} = 0$ solution

2. $\alpha \gg 1$ Time integral can be done in saddle-point

3. Loops are negligible also for typical fluctuations

Valid also for typical fluctuations



Loops are constrained to be zero at late times: lack one α enhancement
Suppressed by $\alpha^2 P_\zeta$

For typical fluctuations the tree-level expansion corresponds to $\tilde{b}\left(\frac{\omega}{4\pi f}\right)^n$
Higher-order terms suppressed

WFU for resonant features

$$\Psi[\bar{\zeta}] = e^{-S_g} \cdot e^{-\tilde{b}\Delta S_{E,1}}$$

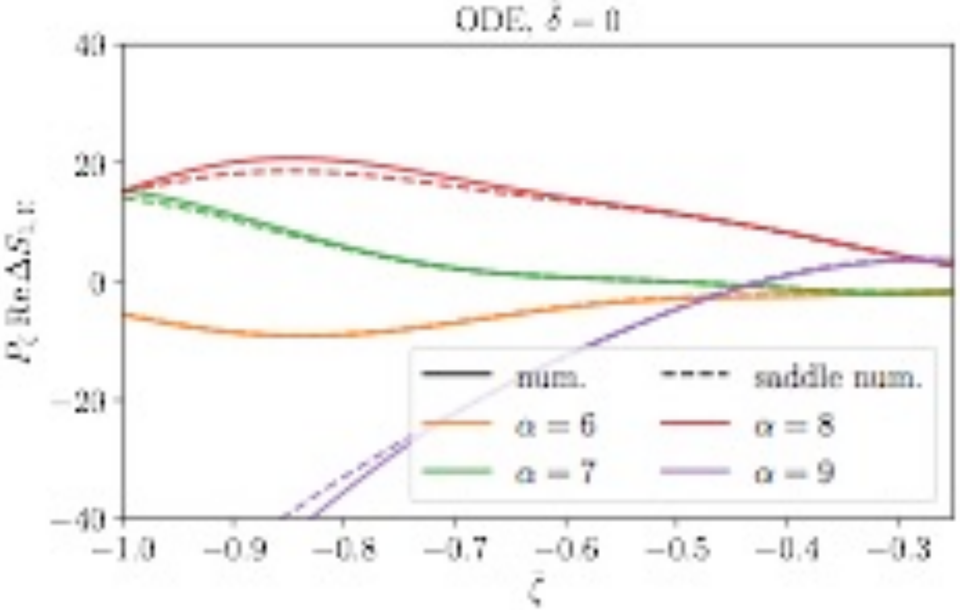
$$\Delta S_{\mathrm{E},1}[\bar{\zeta}] = \int_{-\infty}^{0} \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x}\, \frac{1}{2\tau^2 P_\zeta} \Big\{ [\zeta'^2 + (\partial_i\zeta)^2] \cos\Big(\alpha\,(\log(\tau/\eta_\star) + \zeta) - \tilde{\delta} - i\alpha\pi/2\Big) \\ -(\partial_i\bar{\zeta})^2 \cos\Big(\alpha\,(\log(\tau/\eta_\star) + \bar{\zeta}) - \tilde{\delta} - i\alpha\pi/2\Big) \Big\} \,,$$

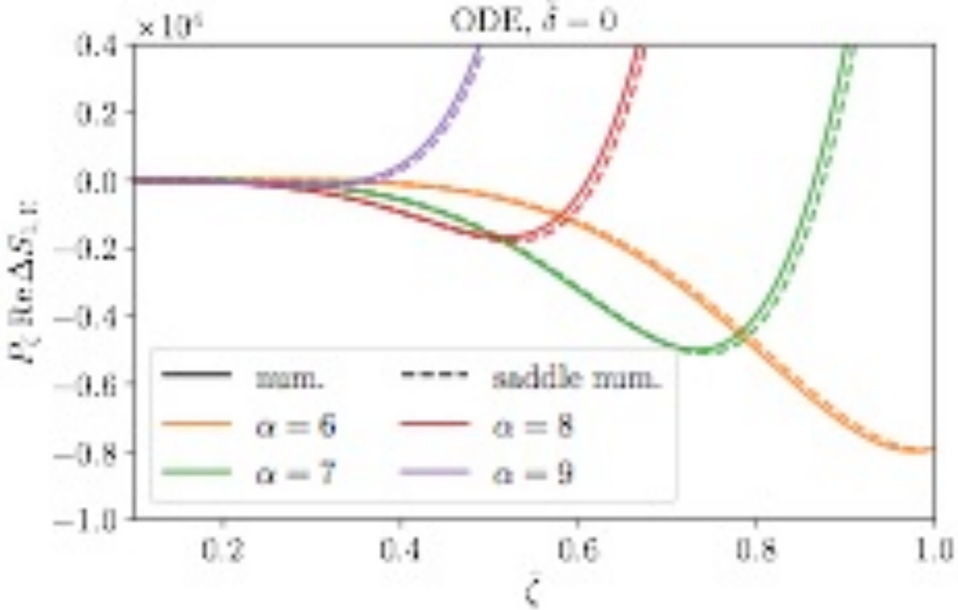
with $\quad \zeta(\tau, \boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \bar{\zeta}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} (1 - k\tau) e^{k\tau}$

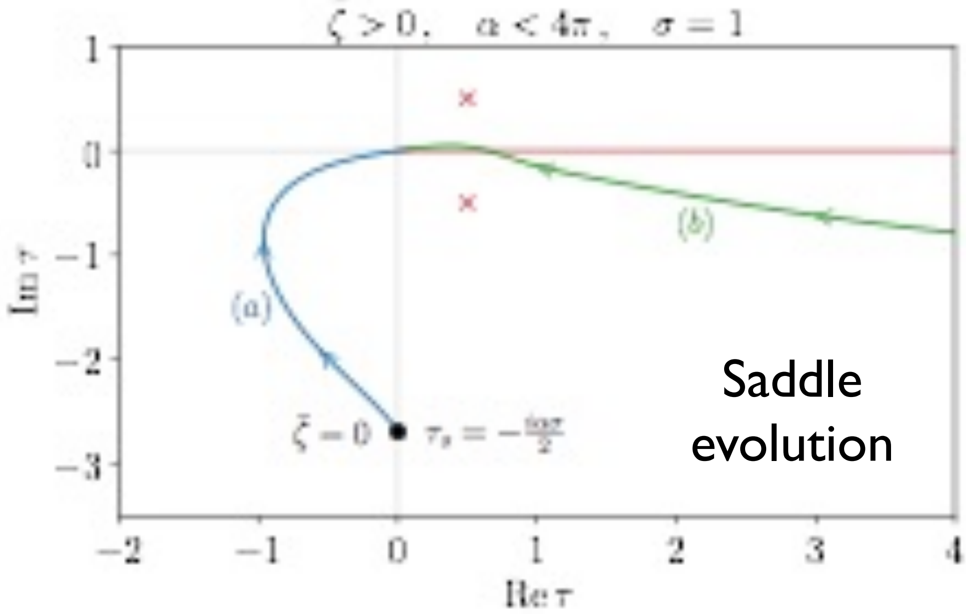
Explicit: convergent integral + no DE to solve

Euclidean rotation ok, exponentially convergent at early times

Results for a single Fourier mode







VERY different for $\zeta > 0$ and $\zeta < 0$

Future

1. Is it possible some info is hidden in the CMB tails?
   For instance features with $\omega/4\pi f \sim 1$

2. More realistic applications to PBHs: threshold, spin, clustering...

3. Generalizations:

   a. Different interactions (doing DBI...)
   b. Slow-roll inflation and eternal inflation
   c. Tensor modes (exact solutions of GWs in dS or numerical GR)

4. What is the connection with large number of legs?