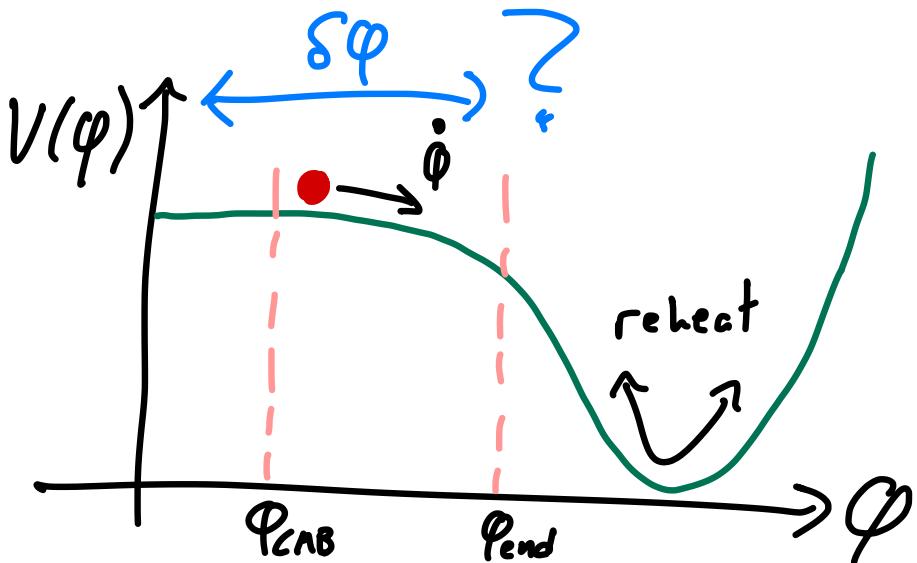


Large Deviations in the Early Universe

Tim Cohen
CERN/EPFL/UOregon

with
Dan Green
Akhil Premkumar



BBSM
August 17, 2023

This talk

EFT
Validity

BBSM?

Eternal
inflation

from
rare
fluctuations

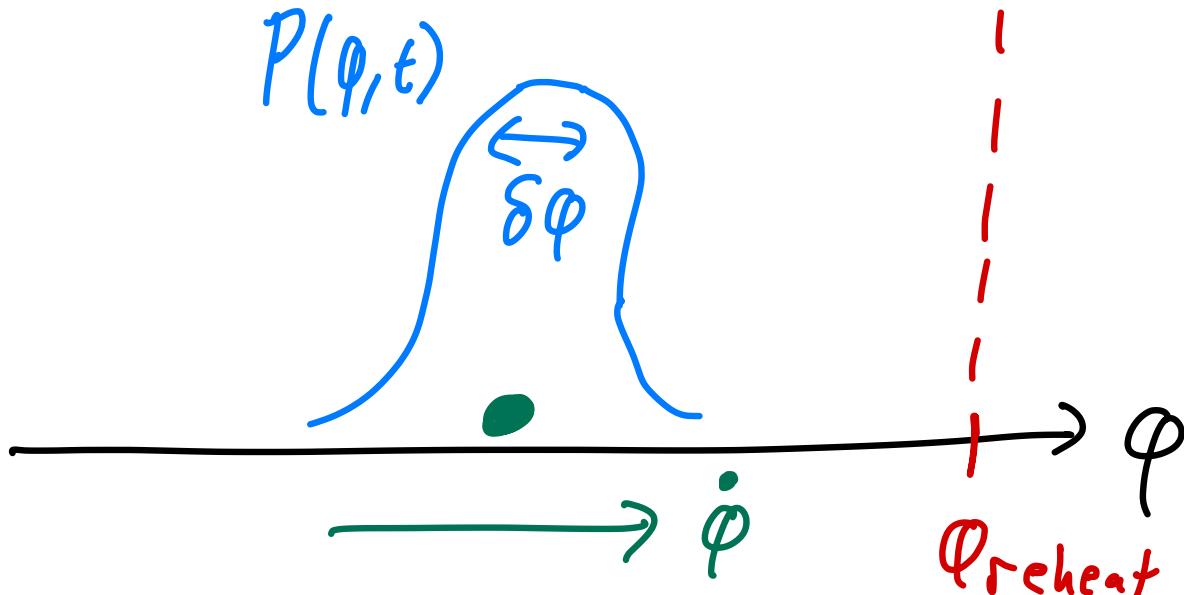
Primordial
inflation

non-Gaussianity

Didn't
Paolo
already
solve this
problem?!?



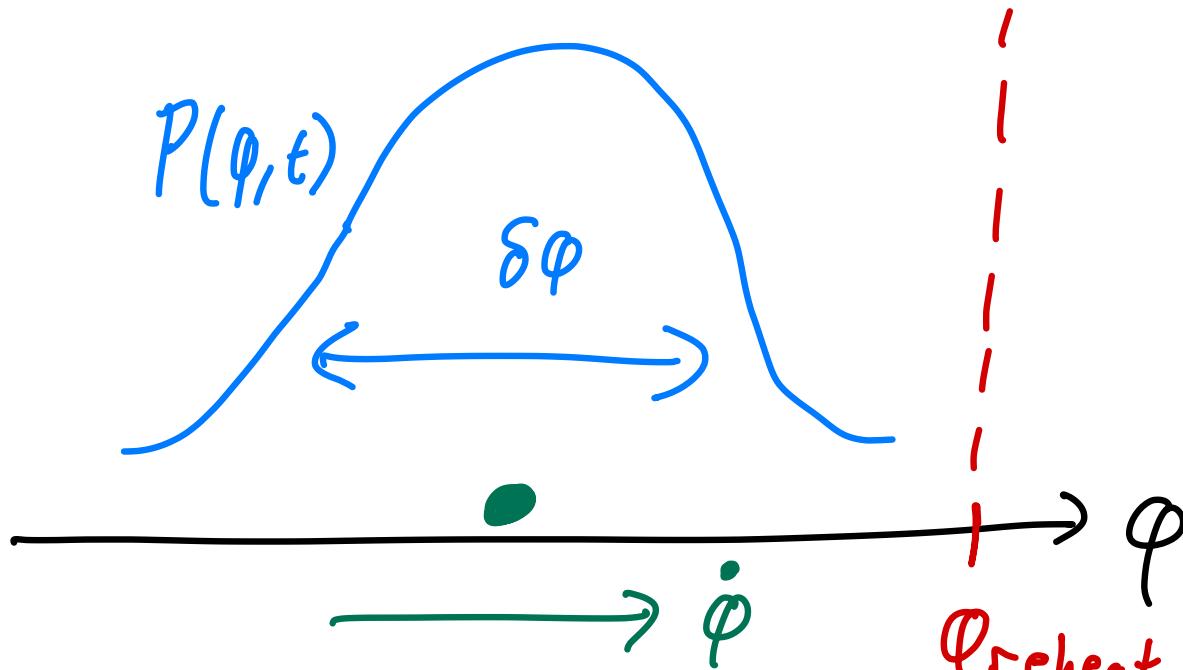
Inflaton Fluctuations



$$(\Delta\varphi)_{\text{classical}} > (\Delta\varphi)_{\text{noise}}$$

Small $\delta\varphi \Rightarrow$ universal end to inflation

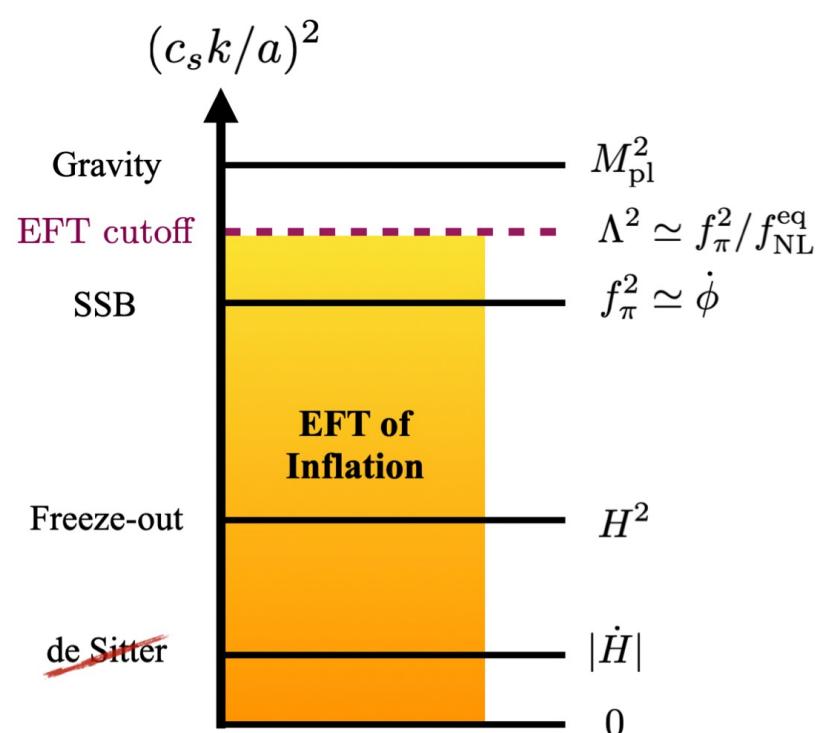
Eternal Inflation



$$(\Delta\phi)_{\text{classical}} \sim (\Delta\phi)_{\text{noise}}$$

Some patches never reheat!

EFT of Inflation



Gaussian

$$\Delta t \sim 1/H$$

$$(\Delta \varphi)_{\text{cl}} \sim \dot{\varphi}/H = f_\pi^2/H$$

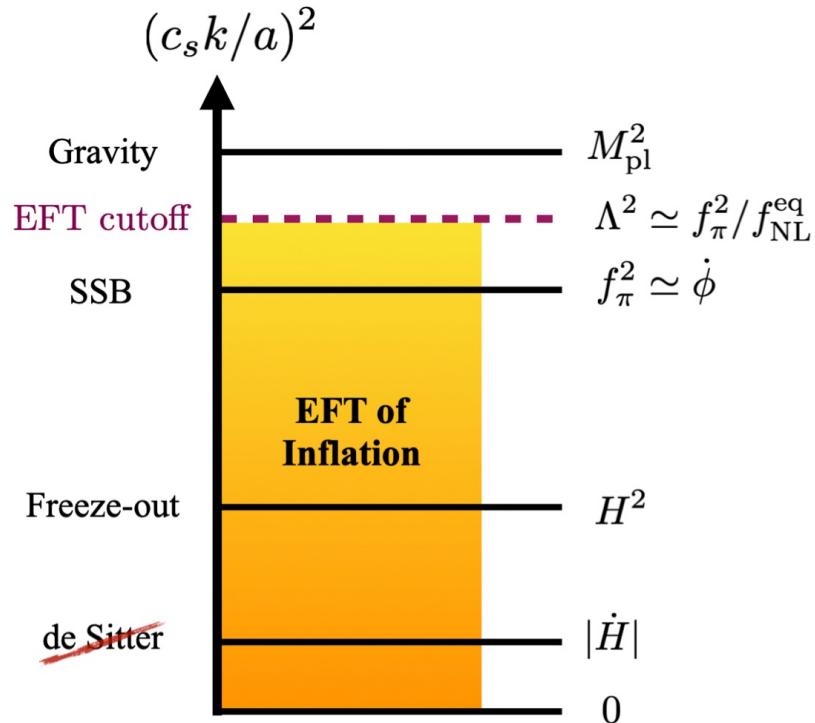
$$(\Delta \varphi)_{\text{noise}} \sim H$$

Eternal Inflation

Cheung, Creminelli, Fitzpatrick, Kaplan, Senatore
arXiv: 0790.0293

$$\Rightarrow f_\pi \sim H$$

EFT of Inflation



Non-Gaussian tails

$$\Delta t \sim 1/H$$

$$(\Delta \phi)_{\text{cl}} \sim \dot{\phi}/H \sim f_\pi^2/H$$

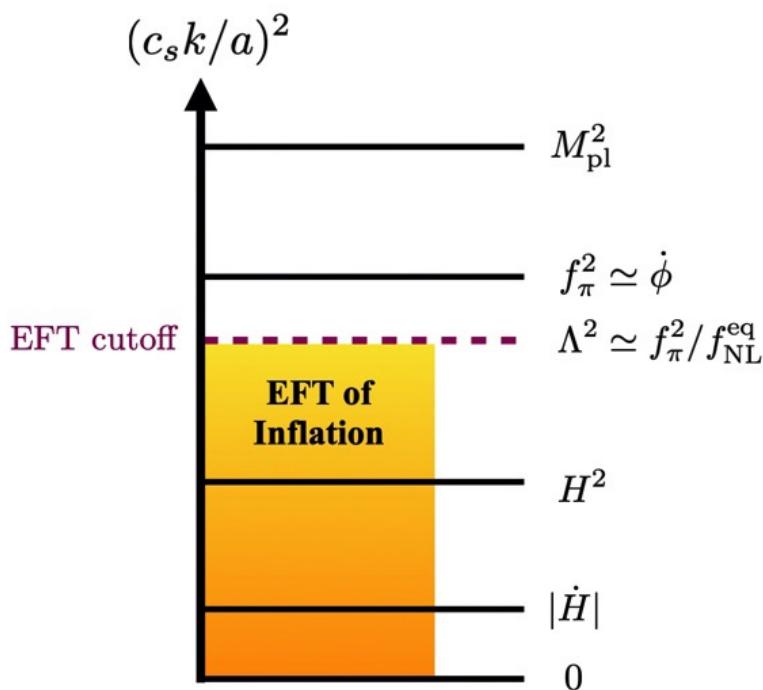
$$(\Delta \phi)_{\text{tail}} \sim f_\pi^2/H$$

Eternal Inflation
 $\Rightarrow f_\pi \gg H$

Energy scale associated with tail:

$$(\dot{\phi})_{\text{tail}} \sim H (\Delta \phi)_{\text{tail}} \sim f_\pi^2$$

EFT of Inflation: $\mathcal{L}_\pi < 1$



$$f_{\text{NL}}^{\text{eq}} \sim f_\pi^2 / \Lambda^2$$

Current constraint:

$$f_{\text{NL}}^{\text{eq}} = -26 \pm 47$$

$$\Rightarrow \Lambda^2 \ll f_\pi^2 \text{ allowed}$$

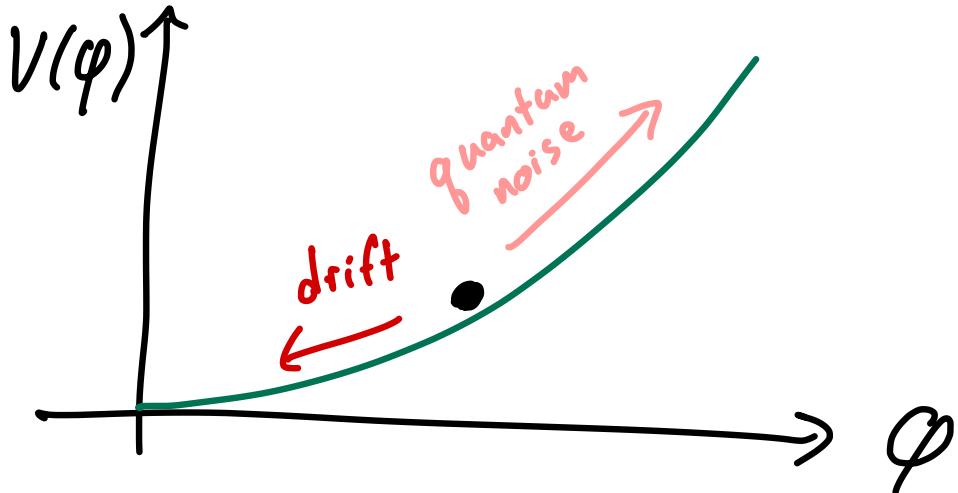
Only need $\Lambda > H$
for EFT consistency

Energy Scale associated with tail:
 $(\dot{\phi})_{\text{tail}} \sim H (\Delta\phi)_{\text{tail}} \sim f_\pi^2 \gg \Lambda^2$

Can we reliably
compute tail
in these models??

Starobinsky's Stochastic Inflation

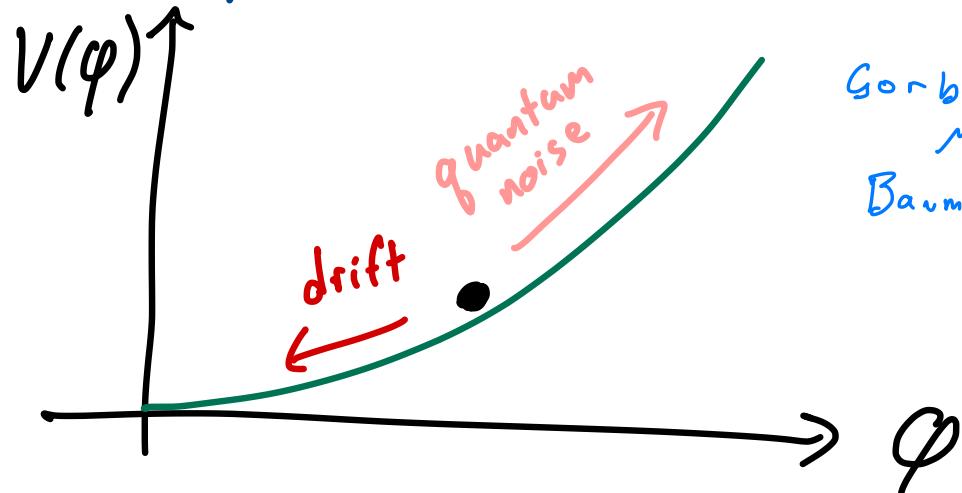
Massless scalar field in dS



\Rightarrow Fokker - Planck equation :

$$\frac{\partial}{\partial t} P(\phi, t) = \frac{H^3}{8\pi^2} \frac{\partial^2}{\partial \phi^2} P(\phi, t) + \frac{1}{3H} \frac{\partial}{\partial \phi} [V'(\phi) P(\phi, t)]$$

Starobinsky's Stochastic Inflation



See also
Gorbenko & Senatore
Mirbabayi;
Baumgardt & Sundrum

Gaussian noise

Tree-level potential

Systematic Corrections?

Leading Order Probability distribution $P(\phi, t)$

$$\frac{\partial}{\partial t} P(\phi, t) = \underbrace{\frac{H^3}{8\pi^2} \frac{\partial^2}{\partial \phi^2} P(\phi, t)}_{\text{noise}} + \underbrace{\frac{1}{3H} \frac{\partial}{\partial \phi} [V'(\phi) P(\phi, t)]}_{\text{drift}}$$

$$H = \text{Hubble} \quad \text{and} \quad V' = \frac{\partial V}{\partial \phi}$$

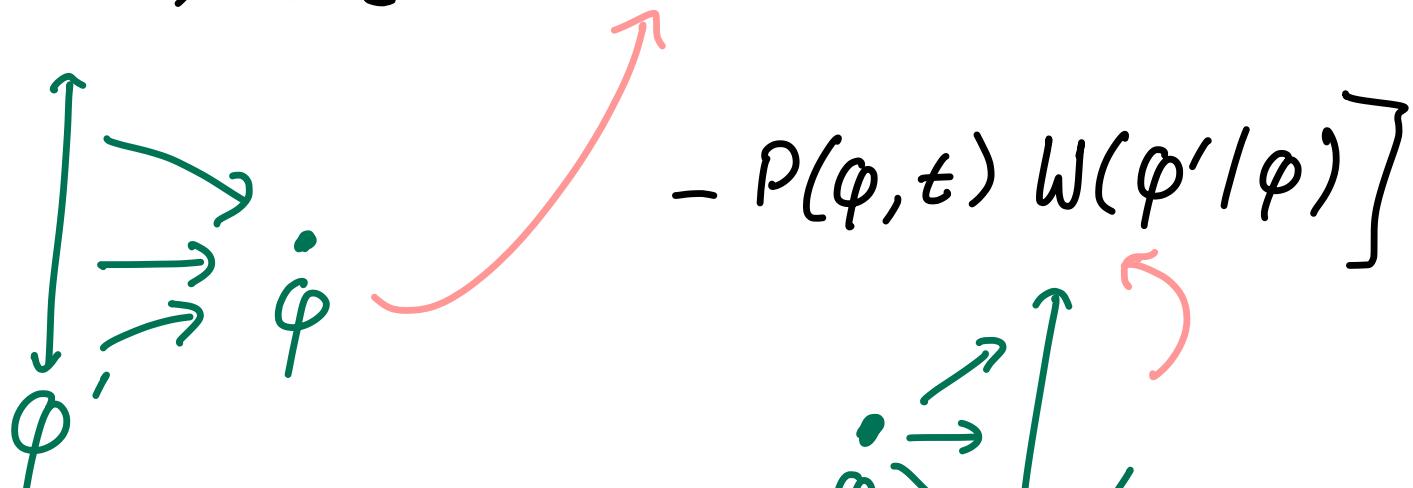
Fixed point solution ($\partial P / \partial t = 0$)

$$P_{\text{eq}} \sim \exp(-8\pi V / 3H^4)$$

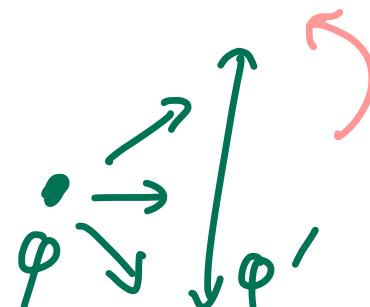
Beyond Leading Order

Assume "Markovian" fluctuations (no memory)

$$\frac{\partial}{\partial t} P(\varphi, t) = \int d\varphi' [P(\varphi'; t) W(\varphi'|\varphi)]$$



$W(\varphi|\varphi')$ is transition rate $\dot{\varphi}' \rightarrow \dot{\varphi}$



Beyond Leading Order

Perform Kramers-Moyal "local" expansion

$$\frac{\partial}{\partial t} P(\varphi, t) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \varphi^n} \mathcal{L}_n(\varphi) P(\varphi, t)$$

w/ $\mathcal{L}_n(\varphi) = \int d\Delta\varphi (-\Delta\varphi)^n W(\varphi + \Delta\varphi / \Delta\varphi)$

Beyond Leading Order

$$\frac{\partial}{\partial t} P(\varphi, t) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \varphi^n} \mathcal{L}_n(\varphi) P(\varphi, t)$$

$\mathcal{L}_n(\varphi)$ has polynomial expansion

$$\mathcal{L}_n(\varphi) = \sum_{m=0}^{\infty} \frac{1}{m!} \mathcal{L}_n^{(m)} \varphi^m$$

LO Stochastic Inflation

$$V = \sum_l \frac{1}{l!} C_l \varphi^l \Rightarrow \mathcal{L}_1^{(m)} = \frac{1}{3H} C_{m+1} \quad \left| \quad \mathcal{L}_2^{(0)} = \frac{H^3}{8\pi^2}$$

Beyond Leading Order

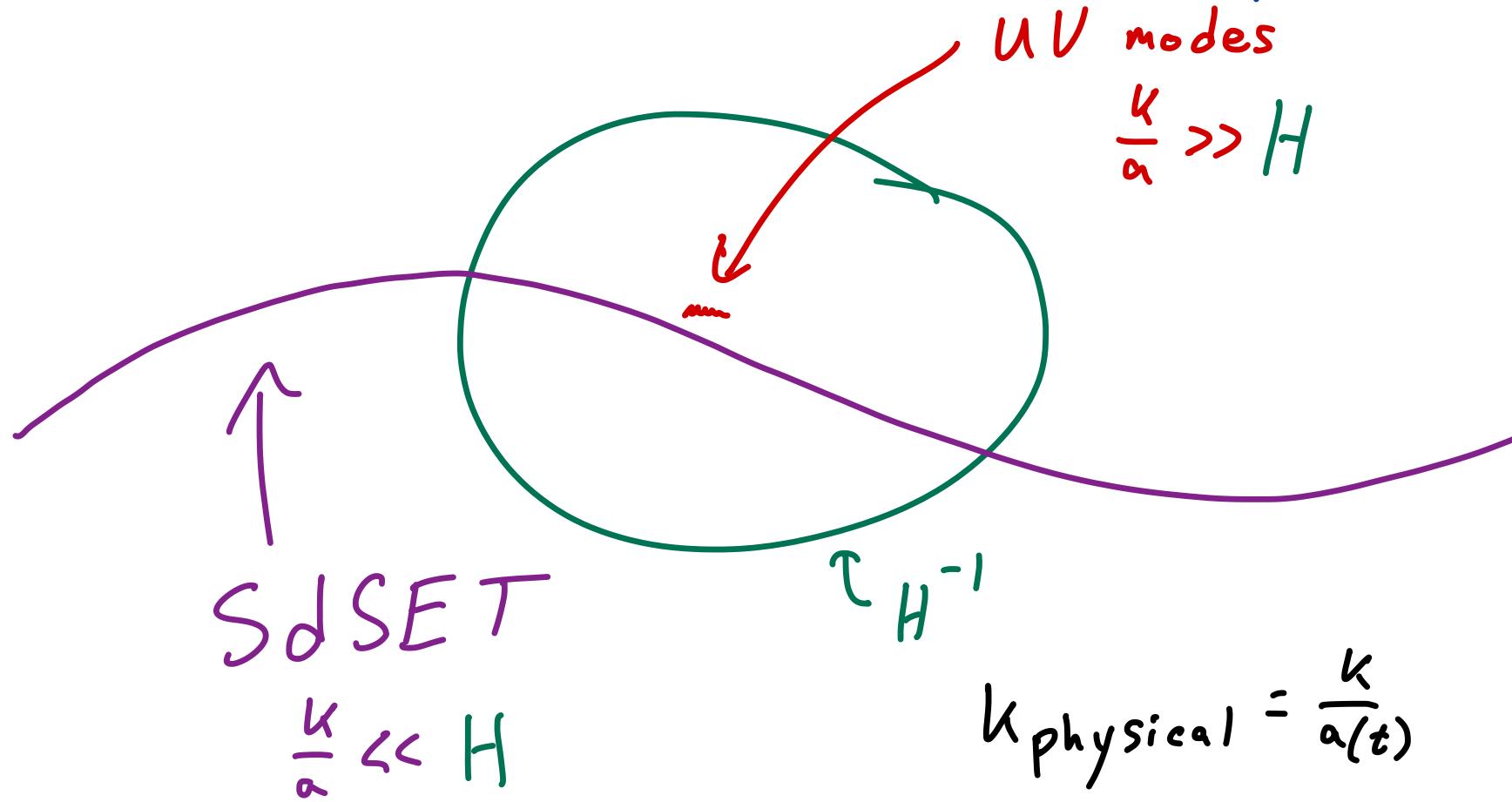
Generic structure

$$\frac{\partial}{\partial t} P(\phi, t) = \sum_{n=2}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \phi^n} \left[\sum_{m=0}^{\infty} \frac{1}{m!} \mathcal{R}_n^{(m)} \phi^m P(\phi, t) \right]$$

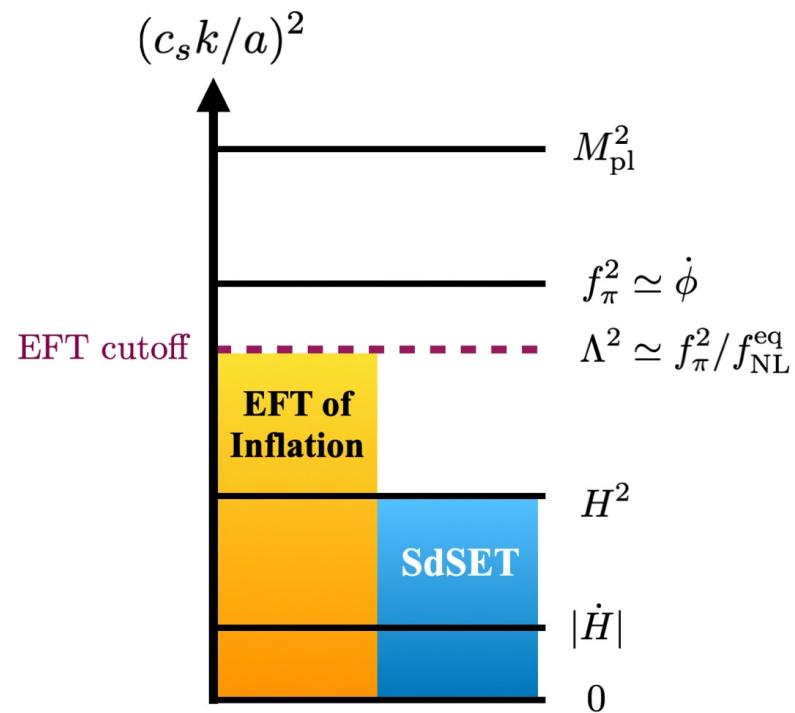
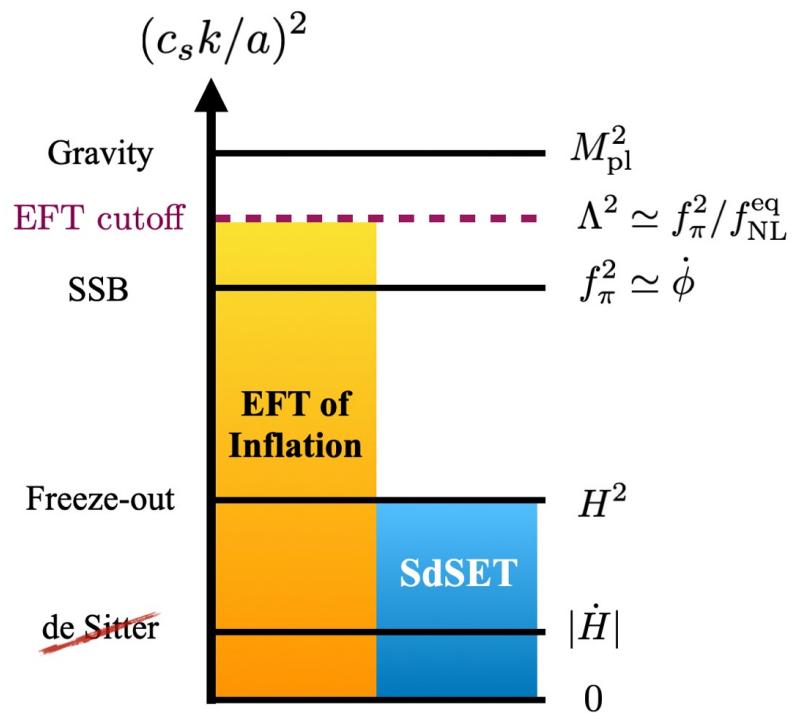
J
Higher order noise

$$+ \frac{1}{3H} \frac{\partial}{\partial \phi} [V'(\phi) P(\phi, t)]$$

Soft de Sitter Effective Theory



Soft de Sitter Effective Theory



SdSET Fields

Two IR degrees of freedom

- "Growing" mode $\phi_+ \leftarrow$ Correlators of interest
- "Decaying" mode ϕ_-

w/ $\phi_s = H \left((\alpha H)^{-\alpha} \phi_+ + (\alpha H)^{-\beta} \phi_- \right)$

Time dependence factorizes \downarrow

Defining SdSET

- DOF φ_+ and φ_-
 - Power counting $\sim \frac{k}{\Lambda_{uv}}$ w/ $\Lambda_{uv} = aH$
 - Symmetries
 - (1) "spacetime"
 - (2) "reparametrization"
 - Initial conditions
- * Very close analogy w/ Heavy Quark EFT

Light Scalars in dS

Composite operators

$$\mathcal{O}_n = \phi_+^n \sim (\kappa/aH)^{n\alpha} \rightarrow O(1)$$

RG mixing expected

Contract any two legs

$$\langle \mathcal{O}_n \dots \rangle \supset \langle \mathcal{O}_{n-z} \dots \rangle \binom{n}{z} \frac{C_\alpha^2}{z} \int \frac{d^3 p}{(2\pi)^3} \frac{H^{2-2\alpha}}{p^{3-2\alpha}}$$

Light Scalars in dS

$\int \frac{d^3 p}{(2\pi)^3} \frac{H^{2-2\alpha}}{p^{3-2\alpha}}$ is scaleless and diverges as $\alpha \rightarrow 0$

Isolate UV divergence

$$P^z \rightarrow P^z + \overline{K}_{IR}^2$$

$$\langle \theta_n \dots \rangle > \langle \theta_{n-z} \dots \rangle \binom{n}{z} \frac{c_\alpha^z}{4\pi^2} \left(-\frac{1}{z\alpha} - \gamma_E - \log \frac{\alpha H}{\overline{K}_{IR}} \right)$$

Dynamical RG \Leftrightarrow Stochastic Inflation

Resum time dependent logs:

$$\frac{\partial}{\partial t} \langle O_n \dots \rangle = -\frac{n}{3} \sum_{m>1} \frac{c_m}{m!} \langle O_{n-1}, O_m \dots \rangle + \frac{n(n-1)}{8\pi^2} \langle O_{n-2} \dots \rangle$$

(Starobinsky; Starobinsky, Yokoyama)

Is equivalent to a Fokker-Planck eq

for $P(\phi, t)$ w/ $\langle \phi^n \rangle = \int d\phi P(\phi, t) \phi^n$ (Baumgart + Sundrum)

Stochastic Inflation for Inflation
Work with scalar metric fluctuation

$$\zeta \simeq H(t - \varphi/f_{\pi}^2)$$

Has non-linearly realized shift symmetry

$$\Rightarrow \frac{\partial}{\partial t} P(\zeta, t) = \sum_{n \geq 2} (-1)^n \frac{\gamma_n}{n!} \frac{\partial^n}{\partial \zeta^n} P(\zeta, t)$$

H=1

Determine γ_n by computing operator mixing

$$\frac{\partial}{\partial t} \langle \zeta^n \rangle = \sum_n \gamma_n \binom{n}{n} \langle \zeta^{n-n} \rangle$$

Non-Gaussian Corrections

$$\gamma_2 : \langle \zeta^2 \rangle = \int \frac{d^3 k}{(2\pi)^3} \underbrace{\langle \zeta(\vec{k}) \zeta(\vec{k}') \rangle}_{\frac{J_\zeta}{k^3} (2\pi)^3 \delta^3(\vec{k} - \vec{k}')} = 2\gamma_2 \log \frac{aH}{k}$$

IR \rightarrow

$$\gamma_3 : \langle \zeta^3 \rangle = \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{(2\pi)^9} \underbrace{\langle \zeta(k_1) \zeta(k_2) \zeta(k_3) \rangle}_{\text{Two contributions}}$$

$$\gamma_3 = \frac{J_\zeta^2}{32\pi^4} \left(\left(1 - \frac{1}{c_s^2}\right) \left(9 + c_s^2\right) + \frac{c_3}{c_s^2} \right)$$

Two contributions

$\zeta^3 \quad \downarrow$
 $\zeta \partial_i \zeta \partial^i \zeta$

Non-Gaussian Corrections

$$\frac{\partial}{\partial t} P(\zeta, t) = \frac{\gamma_2}{2} \frac{\partial^2}{\partial \zeta^2} P(\zeta, t) - \frac{\gamma_3}{3!} \frac{\partial^3}{\partial \zeta^3} P(\zeta, t)$$

Assuming P is nearly Gaussian:

$$\langle \zeta \zeta \rangle \sim D_\zeta$$

$$\zeta \sim D_\zeta^{1/2}$$

$$\partial_\zeta \sim D_\zeta^{-1/2}$$

$$D_\zeta \sim \frac{H^4}{f_\pi^4}$$

\Rightarrow

$$\gamma_2 \frac{\partial^2}{\partial \zeta^2} \sim O(1)$$

$$\gamma_3 \frac{\partial^3}{\partial \zeta^3} \sim \frac{D_\zeta^{1/2}}{C_S^2} \underset{\Delta = f_\pi C_S}{\underset{\uparrow}{\sim}} \frac{H^2}{\Lambda^2} \ll 1$$

Gaussian Distribution

Solve $\frac{\partial}{\partial t} P_G(\xi, t) = \frac{1}{4\pi^2} \frac{\partial^2}{\partial \xi^2} P_G(\xi, t)$

with boundary condition

$$P_G(\xi, \xi_i; t=0) = \delta(\xi - \xi_i) \quad \text{and} \quad P_G(\varphi_r[\xi]) = 0$$

$$\Rightarrow P_G(\varphi[\xi] < 0, \xi_i; t)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 t}} \left[e^{-\frac{(\xi - \xi_i)^2}{2\sigma^2 t}} - e^{-\frac{4\xi_i}{2\sigma^2} e^{-\frac{(\xi + \xi_i)^2}{2\sigma^2 t}}} \right]$$

$$\sigma^2 = 2\tau_2 = \frac{D_s}{2\pi}$$

Onset of Eternal Inflation

Probability of reheating at time t :

$$P_{R,G}(t) = -\frac{d}{dt} \int_{-\infty}^0 d\varphi P_G(\varphi; t) \sim e^{-t/\sigma^2}$$

Volume of reheating surface:

$$\langle V \rangle_G = L^3 \int_0^\infty dt e^{3t} P_{R,G}(t) \simeq L^3 \int_0^\infty dt e^{t(3 - 1/\sigma^2)}$$

$$\Rightarrow \langle V \rangle_G \rightarrow \infty \quad \text{when} \quad \sigma^2 = \frac{\Delta g}{2\pi^2} > \frac{1}{6} \Rightarrow \text{Eternal inflation}$$

Non-Gaussian Distribution

Solve $\frac{\partial}{\partial t} P_{NG}(\xi, t) = \left(\frac{D_\xi}{8\pi^2} \frac{\partial^2}{\partial \xi^2} - \gamma_3 \frac{\partial^3}{\partial \xi^3} \right) P_{NG}(\xi, t)$

Boundary conditions: $P_{NG}(\xi, \tau_i; t=0) = \delta(\xi - \xi_i)$
 $P_{NG}(q_r[\xi]) = 0$

Solution: $P_{NG} = \exp \left(- \frac{\gamma_3 t}{3!} \frac{\partial^3}{\partial \xi^3} \right) P_\xi + \text{images}$

Non-Gaussian Onset of Eternal Inflation

Probability of reheating at time t :

$$P_{R, NG}(t) \sim \exp\left[-t\left(\frac{1}{2\sigma^2} - \frac{\gamma_3}{3!} \frac{1}{\sigma^6}\right)\right]$$

Volume of reheating surface:

$$\langle V \rangle_{NG} = L^3 \int_0^\infty dt e^{3t} P_{R, NG}(t)$$

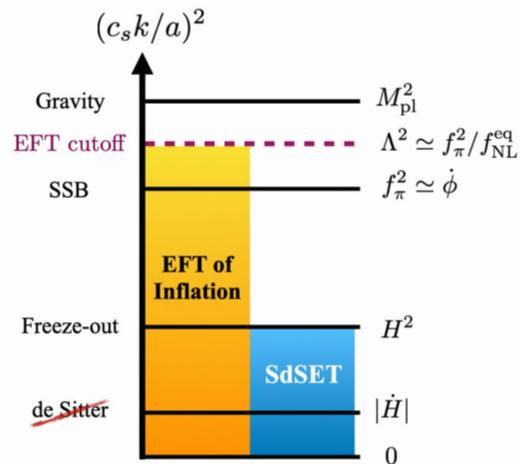
$$\langle V \rangle_{NG} \rightarrow \infty \text{ when } \frac{1}{2\sigma^2} - \frac{\gamma_3}{3!} \frac{1}{\sigma^6} < 3$$

$$\Rightarrow \frac{1}{2} - \frac{1}{48} \frac{f\pi^2}{A^2} \left((c_s^2 - 1)(9 + 3c_s^2) + c_3 \right) < \frac{3D\xi}{2\pi^2}$$

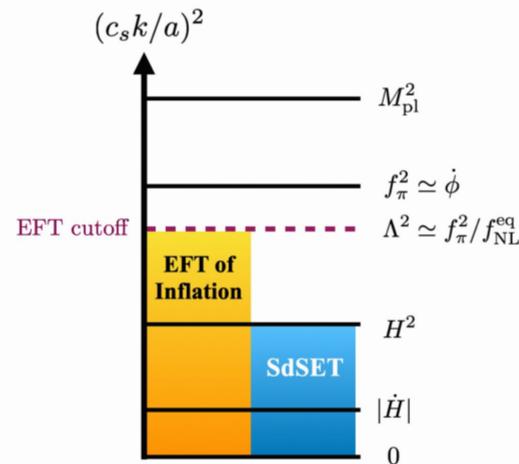
Interpretation

$$\frac{1}{2} - \frac{1}{48} \frac{f_\pi^2}{\Lambda^2} \left((c_s^2 - 1) (9 + 3c_s^2) + c_3 \right) < \frac{3D_\xi}{2\pi^2}$$

Gaussian
non-Gaussian



$\Lambda \gg f_\pi$

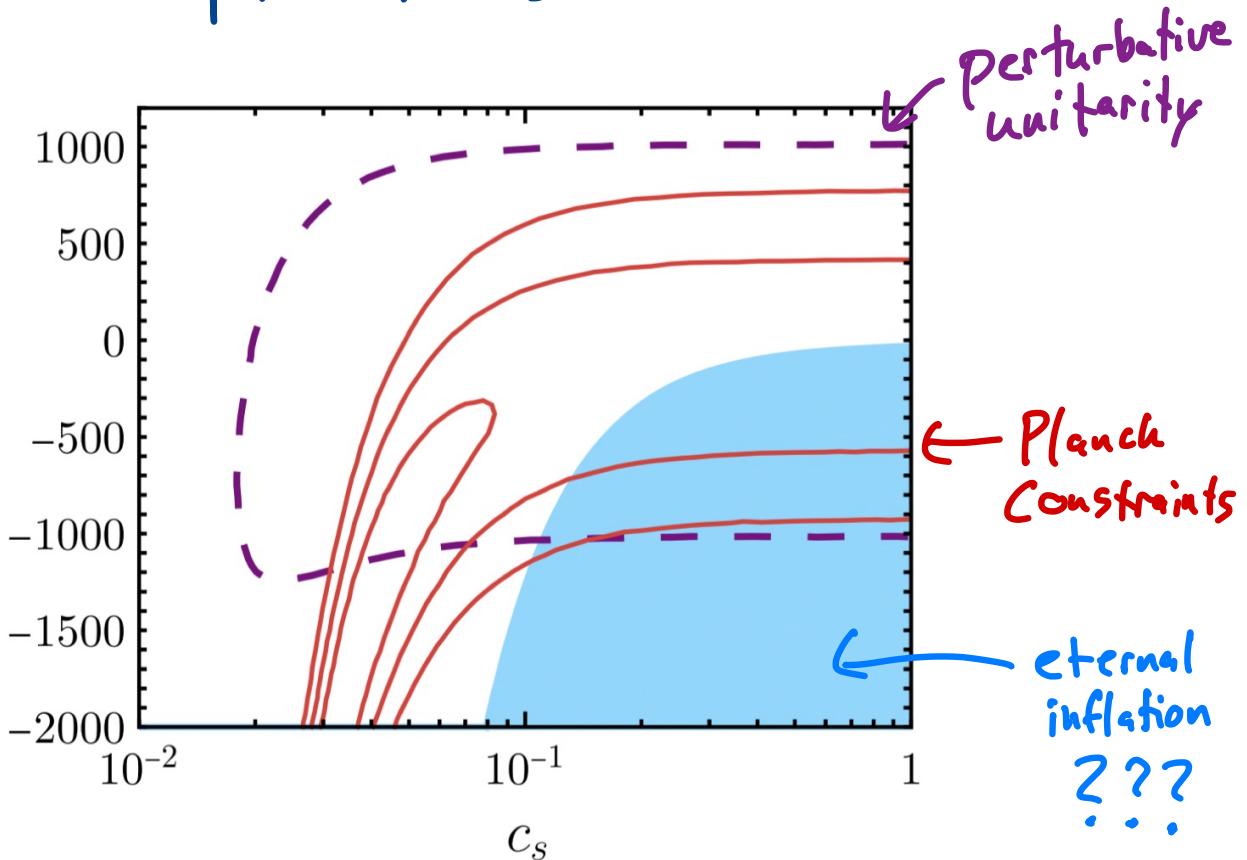


$\Lambda \ll f_\pi$

Correction is out of control

Implications

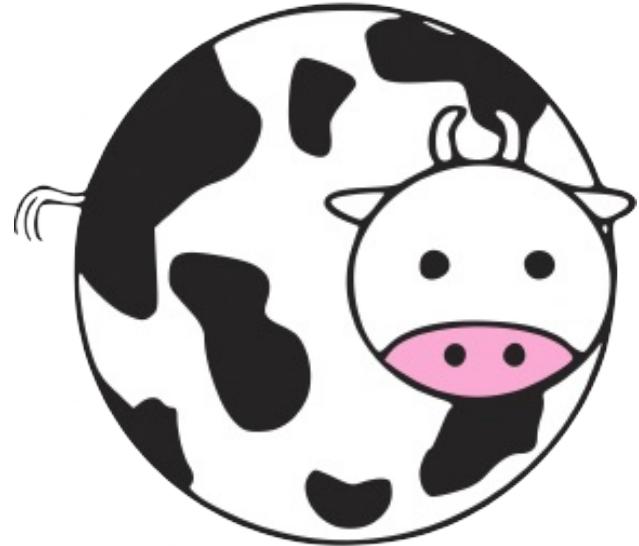
$$-\frac{4}{3} \frac{c_3}{c_s^2} - \frac{1}{2} \frac{(1-c_s^2)^2}{c_s^2}$$





Do these models
really predict
eternal inflation??

Or is there
a surprising
breakdown of EFT??



Random Walks and RG

Toy models of fluctuating fields

Independent and identically distributed random variable X .

1) Gaussian

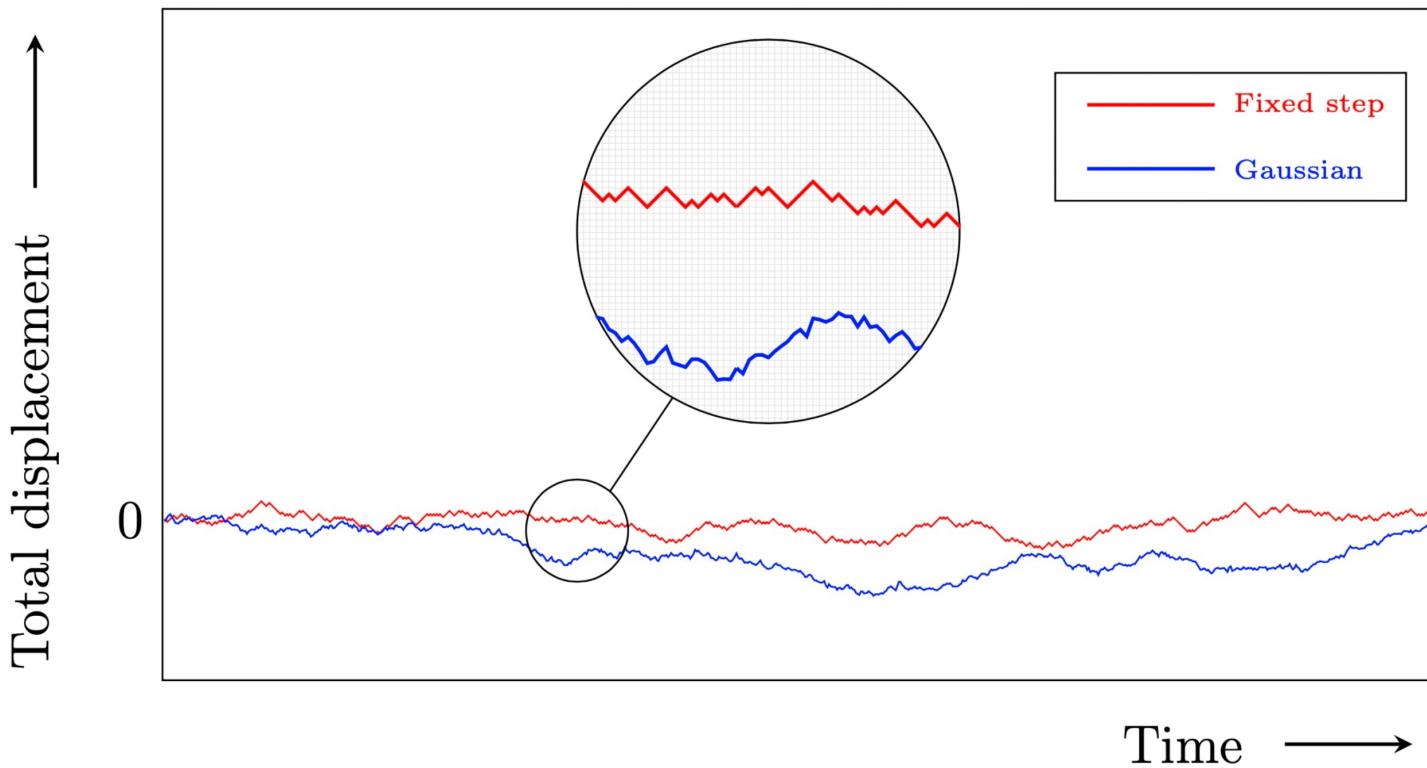
$$P_g(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$$

2) Fixed step

$$P_f(x) = \frac{1}{Z} \delta(|x| - 1)$$

$$\langle x \rangle = 0 \quad \text{and} \quad \langle x^2 \rangle - \langle x \rangle^2 = 1$$

Same Coarse Grained Predictions



Compute Displacement

How far does walker move in N steps?

$$\bar{X} = \sum_{i=1}^N x_i \Rightarrow P(\bar{X}) = \int \prod_{i=1}^N dx_i p(x_i) \delta(\bar{X} - \sum_{i=1}^N x_i)$$

$$\int \frac{dk}{2\pi} e^{-ik(\bar{X} - \sum x_i)}$$

$$\Rightarrow P(\bar{X}) = \int \frac{dk}{2\pi} e^{-ik\bar{X}} \int \prod_{i=1}^N dx_i p(x_i) e^{ikx_i}$$

$$\langle e^{ikx} \rangle^N$$

Large Step Limit

1) Gaussian: $\langle e^{ikx} \rangle = e^{-k^2/2}$

$$\Rightarrow P_g(x) = (2\pi N)^{-1/2} \exp(-x^2/2N)$$

2) Fixed Step: $P_f(x) = \frac{N!}{\left(\frac{N-x}{2}\right)! \left(\frac{N+x}{2}\right)!} 2^{-N}$

Combinatorics

Stirling's approx $\Rightarrow P_f(x) \approx \exp(-x^2/2N - x^4/12N^3 + \dots)$

Gaussians!

Central Limit Theorem

Assume expansion $\log \langle e^{ikx} \rangle = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \langle x^n \rangle_G$

✓ $\langle x \rangle_G = \langle x \rangle$, $\langle x^2 \rangle_G = \langle x^2 \rangle - \langle x \rangle^2$, ...
 $= \sigma_0^2$ connected correlators

$$P(\Sigma) = \int \frac{dk}{2\pi} e^{-ik\Sigma} \exp\left(ikN - \frac{1}{2} N k^2 \sigma_0^2 + \frac{i^3}{3!} N k^3 \langle x^3 \rangle_G + \dots\right)$$

If integral support dominated by Gaussian:

$$k \lesssim 1/\sqrt{N} \sigma_0$$

Central Limit Theorem

Gaussian Support: $k \lesssim 1/\sqrt{N} \sigma_0$

$$P(\bar{x}) = \int \frac{dk}{2\pi} e^{-ik\bar{x}} \exp\left(ikN - \frac{1}{2} N k^2 \sigma_0^2 + \frac{i^3}{3!} N k^3 \langle x^3 \rangle_{\zeta} + \dots\right)$$

$\cancel{\sqrt{N}}$ $\cancel{O(1)}$ $\cancel{1/\sqrt{N}}$

Rescale (coarse grain): $k \rightarrow k/\sqrt{N}$, $\bar{x} \rightarrow \sqrt{N}\bar{x}$

$$\Rightarrow P(\bar{x}) \rightarrow \frac{1}{\sqrt{N}} \int \frac{dk}{2\pi} \exp\left(ik(\sqrt{N}\langle x \rangle - \bar{x}) - \frac{1}{2} k^2 \sigma_0^2 + O\left(\frac{1}{\sqrt{N}}\right)\right)$$

$$\cong \exp\left(-\bar{x} - \sqrt{N}\langle x \rangle\right)^2 / 2\sigma_0^2$$

Central Limit Theorem is RG

"UV insensitive"

In $N \rightarrow \infty$ limit:

$\langle x \rangle$: relevant

σ_0 : marginal

$\langle x^n \rangle_G$ w/ $n > 2$: irrelevant

Large Deviation Principle

Consider probability of finding random walker distance beyond N after N steps:

$$P(\bar{X} > N) = ?$$

Expect UV dependence:

Fixed step walker: $P_f(\bar{X} > N) = 0$

Gaussian walker: $P_g(\bar{X} > N) \neq 0$

Large Deviation Principle

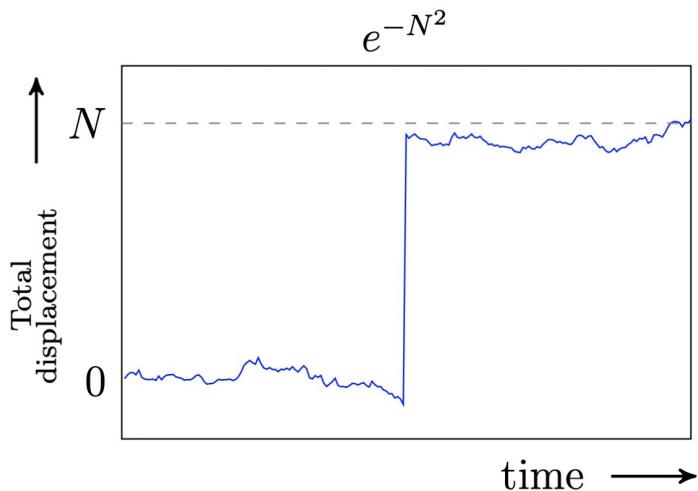
Compute $\langle e^{\theta X} \rangle$ w/ $\theta > 0$

Naively expect $\langle e^{\theta X} \rangle \sim e^{\theta \langle X \rangle}$

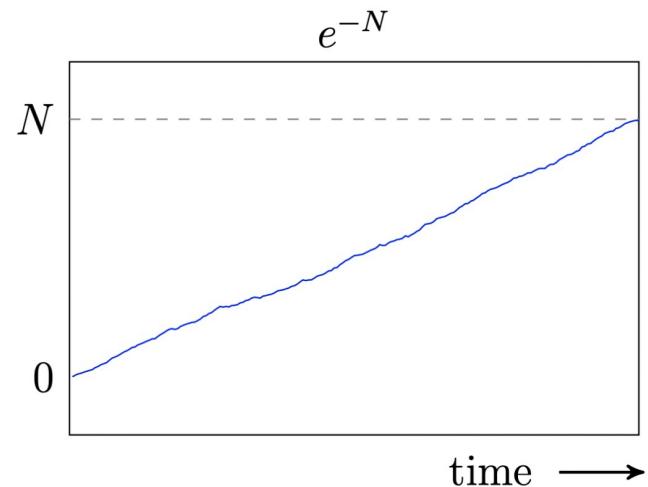
But $\langle e^{\theta X} \rangle = \langle e^{\theta x} \rangle^N = e^{N\theta^2/2}$

Central limit theorem fails!

Large Deviation Principle



↑
CLT



↑
LDP

Large Deviation Principle

Define "sample mean" $\tilde{\bar{X}} = \bar{X}/N$

A distribution satisfies the LDP if

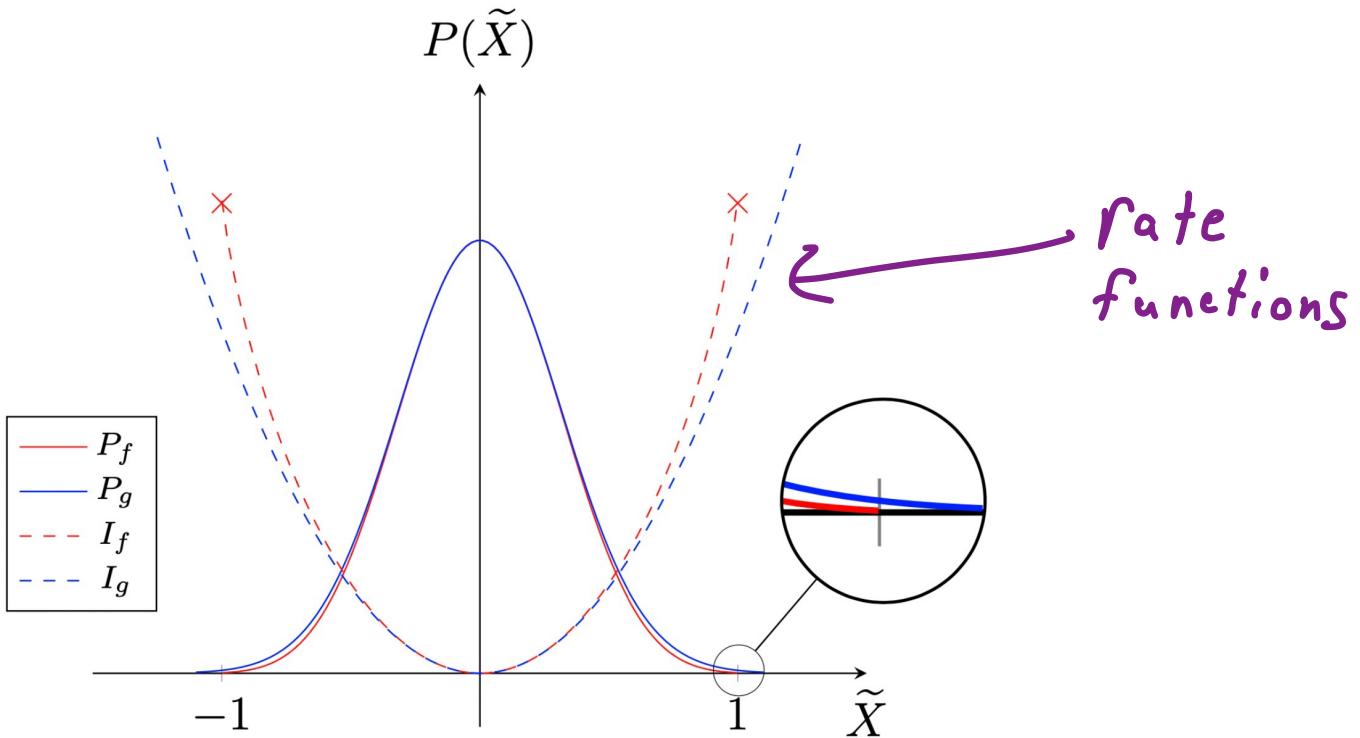
$$P(\bar{X}) \simeq \exp(-NI(\tilde{\bar{X}}))$$

where $I(\tilde{\bar{X}})$ is the "rate function"

$$I_g(\tilde{\bar{X}}) = \tilde{\bar{X}}^2/2$$

$$I_f(\tilde{\bar{X}}) = \frac{1}{2} \left[(1-\tilde{\bar{X}}) \ln(1-\tilde{\bar{X}}) + (1+\tilde{\bar{X}}) \ln(1+\tilde{\bar{X}}) \right]$$

Comparing Tails



Cramér's Theorem

Distribution of \tilde{X} satisfies LDP w/

$$I(\tilde{X}) = \sup_{\theta} [\theta \tilde{X} - \lambda(\theta)] \quad \text{w/} \quad \lambda(\theta) = \ln \langle e^{\theta X} \rangle$$

Proof: Assume LDP holds $P(\tilde{X}) \approx e^{-NI(\tilde{X})}$

Then $\langle e^{\theta X} \rangle = \langle e^{N\theta \tilde{X}} \rangle \approx \int d\tilde{X} e^{N(\theta \tilde{X} - I(\tilde{X}))}$

Saddle point $\xrightarrow{\curvearrowleft} \simeq e^{N \sup_{\tilde{X}} [\theta \tilde{X} - I(\tilde{X})]}$

Cramér's Theorem

So we have $\langle e^{\theta \bar{X}} \rangle \simeq e^{N \sup_{\tilde{X}} [\theta \tilde{X} - I(\tilde{X})]}$

Also $\langle e^{\theta \bar{X}} \rangle = (e^{\theta \bar{X}})^N = e^{N \lambda(\theta)}$

$$\Rightarrow \lambda(\theta) = \sup_{\tilde{X}} [\theta \tilde{X} - I(\tilde{X})]$$

Legendre transform: $I(\tilde{X}) = \sup_{\theta} [\theta \tilde{X} - \lambda(\theta)]$

LDP \Rightarrow New saddle!

From LDP \rightarrow CLT

If $I(\tilde{x})$ is convex and has
single global minimum \tilde{x}_0

$$\Rightarrow P(\tilde{x}) \approx \exp\left(-\frac{1}{2}NI''(\tilde{x}_0)(\tilde{x} - \tilde{x}_0)^2\right)$$

Good approximation for small deviations
 \rightarrow Central Limit Theorem

Eternal Inflation and the LPD

Fokker - Planck for Gaussian theory

$$\Rightarrow P_G \sim \exp(-\xi^2/2\sigma^2 t)$$

$$\Rightarrow \text{Typical fluctuations } \xi \sim \sqrt{t}$$

Eternal inflation dominated by $\xi \sim t$

Eternal Inflation and the LPD

Eternal inflation dominated by $\xi \sim t$

General solution to F-P eq:

$$P = \exp\left(-\frac{\partial^n}{\partial \xi^n}\right) P_S$$

Write $P = \int \frac{dk}{2\pi} e^{-ik\xi} p(k)$

w/ $p(k) = \left(\exp\left(-k^2 \frac{\sigma^2}{2} t + \sum_{n \geq 2} (ik)^n \frac{x_n}{n!} t\right)\right)$

$$\equiv \langle \exp(ik\xi) \rangle$$

Eternal Inflation and the LPD

When $\xi = \alpha t$ for some constant α

as $t \rightarrow \infty$ steepest descents \Rightarrow

$$P \approx \exp(-t I(\alpha))$$

w/ $I(\alpha) = (-i\alpha k_*(\alpha) - k_*^2(\alpha) \frac{\sigma^2}{2})$

$$+ \sum_{n \geq 2} (ik_*(\alpha))^n \frac{\gamma_n}{n!}$$

Cramér's
Thm!

and

$$\left(i \sum (ik_*(\alpha))^n \frac{\gamma_{n+1}}{n!} \right) - k_* \sigma^2 = i\alpha$$

Eternal Inflation is UV Sensitive

Phase transition to eternal inflation

probes the tail of the distribution

Sensitive to new saddle of the path integral

Breakdown of EFT of Inflation

$$\alpha f_\pi^2 > \Lambda^2$$

Outlook

Fluctuations of fields in dS governed
by RG in SdSET

Typical fluctuations governed by

coarse grained RG \Rightarrow CLT

Rare fluctuations governed by LDP

\Rightarrow UV sensitive