

The classical equations of motion of quantised gauge theories

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Classical physics asserts

Maxwell

$$\partial_\mu F^{\mu\nu} = J^\nu$$

Einstein

$$G^{\mu\nu} = T^{\mu\nu}$$

Quantum physics described by Schrodinger equation

$$i \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle$$

From this equation classical physics follows...

Quantum physics described by Schrodinger equation

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$$\begin{aligned}\partial_t \langle \hat{\phi} \rangle &= i \left\langle [\hat{H}, \hat{\phi}] \right\rangle = \left\langle \frac{\partial \hat{H}}{\partial \hat{\pi}} \right\rangle \\ \partial_t \langle \hat{\pi} \rangle &= i \left\langle [\hat{H}, \hat{\pi}] \right\rangle = - \left\langle \frac{\partial \hat{H}}{\partial \hat{\phi}} \right\rangle\end{aligned}$$

..in expectation value

A subtlety for gauge theories

$$\left\langle \frac{\delta S}{\delta \phi_i} \right\rangle = \left\langle \frac{\delta \mathcal{L}}{\delta \phi_i} - \partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_i)} \right\rangle = 0$$

Fewer d.o.f. than fields

- 1. Fewer 2nd order equations for evolution**
- 2. Additional constraint equations on the dof**

Compare

$$\partial_\mu F^{\mu\nu} = J^\nu$$

$$i\frac{\partial}{\partial t}|\psi\rangle = H_{EM}|\psi\rangle$$

$$G^{\mu\nu} = T^{\mu\nu}$$

$$i\frac{\partial}{\partial t}|\psi\rangle = H_{GR}|\psi\rangle$$

**Both dynamics and
Constraints:**

$$\partial_\mu F^{\mu 0} = J^0$$

Dynamics only



& Initial conditions

$$G^{0\mu} = T^{0\mu}$$

$$|\psi(0)\rangle$$

Compare

$$\partial_\mu F^{\mu\nu} = J^\nu$$

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**Both dynamics and
Constraints:**

Gauss' law $\nabla \cdot \mathbf{E} = \rho_{ch}$

Dynamics only



& Initial conditions

**e.g. First
Friedman**

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

$$|\psi(0)\rangle$$

What do we believe?

$$G^{\mu\nu} = T^{\mu\nu}$$

$$i\frac{\partial}{\partial t}|\psi\rangle = H_{GR}|\psi\rangle$$

**Full set of
classical
equations**

**Schrodinger
equation**

Initial conditions $|\psi(0)\rangle$

What do we believe?

$$G^{\mu\nu} = T^{\mu\nu}$$

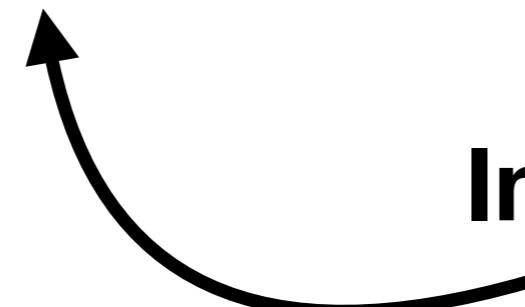
$$i\frac{\partial}{\partial t}|\psi\rangle = H_{GR}|\psi\rangle$$

**Full set of
classical
equations**

**Schrodinger
equation**
Most fundamental

Initial conditions $|\psi(0)\rangle$

Why should they be just so?



Summary of the talk

**Quantum mechanics only guarantees
that, in the classical limit**

$$\partial_\mu F^{\mu i} = 0$$

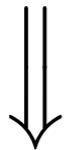
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$$\nabla \cdot \mathbf{E} = \rho_{ch} + \rho_s(\mathbf{x})$$

Shadow charge

Summary of the talk

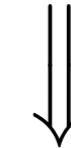
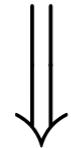
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$$G^{ij} - T^{ij} = 0$$

$$\partial_t (\nabla \cdot \mathbf{E} - \rho_{ch}) = 0$$

$$\nabla_\mu (G^{\mu\nu} - T^{\mu\nu}) = 0$$



$$\nabla \cdot \mathbf{E} = \rho_{ch} + \rho_s(\mathbf{x})$$

$$G^{\mu\nu} = T^{\mu\nu} + T_{aux}^{\mu\nu}$$

Shadow charge

Shadow Matter

$$T_{aux}^{\mu\nu} = \frac{1}{\sqrt{-g}} \begin{pmatrix} \mathbb{H} & \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^3 \\ \mathbb{P}^1 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 0 & 0 & 0 \end{pmatrix}$$

Summary of the talk

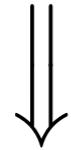
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$$\nabla_\mu (G^{\mu\nu} - T^{\mu\nu}) = 0$$



$$\nabla \cdot \mathbf{E} = \rho_{ch} + \rho_s(\mathbf{x})$$

$$G^{\mu\nu} = T^{\mu\nu} + T_{aux}^{\mu\nu}$$

Shadow charge

Shadow Matter

**Not adding any dof to the theory.
Just the choice of quantum state**

$$T_{aux}^{\mu\nu} = \frac{1}{\sqrt{-g}} \begin{pmatrix} \mathbb{H} & \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^3 \\ \mathbb{P}^1 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 0 & 0 & 0 \end{pmatrix}$$

Familiar from EM

In Weyl gauge

$$A_0 = 0 \quad \Pi_j = \frac{\partial \mathcal{L}_{\mathcal{EM}}}{\partial \dot{A}_j} = -E_j$$

Comm.

$$[\hat{A}_j(\mathbf{x}), \hat{E}_{j'}(\mathbf{x}')] = -i \delta(\mathbf{x} - \mathbf{x}') \delta_{jj'}$$

Ham.

$$\hat{H}_W = \int d^3\mathbf{x} \left(\frac{1}{2} \left(\hat{\mathbf{E}} \cdot \hat{\mathbf{E}} + \hat{\mathbf{B}} \cdot \hat{\mathbf{B}} \right) + \hat{\mathbf{J}} \cdot \hat{\mathbf{A}} + \hat{\mathcal{H}}_J \right)$$

SE

$$i \frac{\partial |\Psi\rangle}{\partial t} = \hat{H}_W |\Psi\rangle$$

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SE

$$i \frac{\partial |\Psi\rangle}{\partial t} = \hat{H}_W |\Psi\rangle$$

Ampère's law follows in expectation value

$$\partial_t \langle \hat{E}^j(\mathbf{x}) \rangle = \langle (\nabla \times \hat{\mathbf{B}})^j(\mathbf{x}) \rangle - \langle \hat{J}^j(\mathbf{x}) \rangle$$

Familiar from EM

In Weyl gauge

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SE

$$i\frac{\partial|\Psi\rangle}{\partial t} = \hat{H}_W|\Psi\rangle$$

Gauss' Law does not

Familiar from EM

Supplement with requirement that physical states satisfy

$$\left(\nabla \cdot \hat{\mathbf{E}} - \hat{J}^0 \right) |\Psi_{EM}\rangle = 0$$

Familiar from EM

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Gauss' law operator commutes with H

Familiar from EM

Supplement with requirement that physical states satisfy

$$\left(\nabla \cdot \hat{\mathbf{E}} - \hat{J}^0 \right) |\Psi_{EM}\rangle = 0$$

In fact, $\hat{G} \equiv \nabla \cdot \hat{\mathbf{E}} - \hat{J}^0$ is the generator of the remaining spatial gauge freedom in the Weyl gauge

The above selects states that are invariant

$$e^{-i \int \alpha(\mathbf{x}) \hat{G}(\mathbf{x})} |\Psi_{EM}\rangle = |\Psi_{EM}\rangle$$

Familiar from EM

Instead consider

$$\left(\nabla \cdot \hat{\mathbf{E}} - \hat{J}^0 \right) |\Psi_{EM}\rangle = J_s^0(\mathbf{x}) |\Psi_{EM}\rangle$$

States transform with an overall phase

$$e^{-i \int \alpha(\mathbf{x}) \hat{G}(\mathbf{x})} |\Psi_{EM}\rangle = e^{-i \int \alpha(\mathbf{x}) J_s^0(\mathbf{x})} |\Psi_{EM}\rangle$$

Still leads to gauge invariant physics.

Familiar from EM

Instead consider

$$\left(\nabla \cdot \hat{\mathbf{E}} - \hat{J}^0 \right) |\Psi_{EM}\rangle = J_s^0(\mathbf{x}) |\Psi_{EM}\rangle$$

This is just like adding a background
classical charge density

$$\tilde{\mathcal{L}}_{\mathcal{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu + A_\mu J_s^\mu + \mathcal{L}_J \quad J_s^\mu = (J_s^0(\mathbf{x}), 0, 0, 0)$$

But, ...

Familiar from EM

Instead consider

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No additional microphysics



Simply a choice of EM quantum state

Minisuperspace

$$g_{\mu\nu} \rightarrow ds^2 = -N(t)^2 dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2)$$



00 component of metric, lapse function

$$S_{ms} = \int dt \sqrt{-g} (M_{pl}^2 R + \mathcal{L}_{matter})$$

$$= \int dt \left(-6M_{pl}^2 \frac{a(t)\dot{a}(t)^2}{N(t)} + \frac{a(t)^3 \dot{\phi}(t)^2}{2N(t)} - N(t)a(t)^3 V(\phi) \right)$$

Minisuperspace

$$S_{ms} = \int dt \left(-6M_{pl}^2 \frac{a(t)\dot{a}(t)^2}{N(t)} + \frac{a(t)^3 \dot{\phi}(t)^2}{2N(t)} - N(t)a(t)^3 V(\phi) \right)$$

The equations of motion naively follow

$$\frac{\delta S_{ms}}{\delta N} = a^3 \left(6M_{pl}^2 \frac{\dot{a}^2}{N^2 a^2} - \frac{\dot{\phi}^2}{2N^2} - V(\phi) \right) = 0$$

$$\frac{\delta S_{ms}}{\delta a} = 3Na^2 \left(4M_{pl}^2 \frac{\ddot{a}}{N^2 a} + 2M_{pl}^2 \frac{\dot{a}^2}{N^2 a^2} - 4M_{pl}^2 \frac{\dot{a}\dot{N}}{N^3 a} + \frac{\dot{\phi}^2}{2N^2} - V(\phi) \right) = 0$$

$$\frac{\delta S_{ms}}{\delta \phi} = -\frac{a^3}{N} \left(\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{\dot{N}\dot{\phi}}{N} + N^2 \frac{\partial V(\phi)}{\partial \phi} \right) = 0$$

Minisuperspace

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But this variation is suspect!

$$\frac{\delta S_{ms}}{\delta N} = a^3 \left(6M_{pl}^2 \frac{\dot{a}}{N^2 a^2} - \frac{\dot{\phi}^2}{2N^2} - V(\phi) \right) = 0$$

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If

$$N(t) \rightarrow N(t) + \delta N(t)$$

Redefine

$$dt' = \left(1 + \frac{\delta N}{N} \right) dt$$

Then

$$S_{ms} [N + \delta N, a, \phi] = S_{ms} [N, a, \phi]$$

Minisuperspace

$$S_{ms} = \int dt \left(-6M_{pl}^2 \frac{a(t)\dot{a}(t)^2}{N(t)} + \frac{a(t)^3\dot{\phi}(t)^2}{2N(t)} - N(t)a(t)^3V(\phi) \right)$$

For the quantum theory

$$\pi_N = \frac{\delta \mathcal{L}}{\delta \dot{N}} = 0 \quad \pi_a = \frac{\delta \mathcal{L}}{\delta \dot{a}} = -12M_{pl}^2 \frac{a\dot{a}}{N} \quad \pi_\phi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \frac{a^3}{N} \dot{\phi}$$

Hamiltonian

$$H = \left[\pi\dot{a} + \pi_\phi\dot{\phi} - \mathcal{L} \right]_{\dot{a}=\dots, \dot{\phi}=\dots} = -\frac{N}{24M_{pl}^2 a} \pi^2 + \frac{N}{2a^3} \pi_\phi^2 + Na^3 V(\phi)$$

$$= -N \frac{\delta S_{ms}}{\delta N} \equiv N\tilde{H}$$

Minisuperspace

Schrodinger equation

$$i\partial_t |\psi\rangle = N(t) \hat{\tilde{H}} |\psi\rangle$$

Just choose N = choice of time coordinate

Only get the ‘a’ hamiltonian eqns in gravitational sector

$$\partial_t \langle \hat{a} \rangle = -i \langle \left[\hat{\tilde{H}}, \hat{a} \right] \rangle$$

$$\partial_t \langle \hat{\pi} \rangle = -i \langle \left[\hat{\tilde{H}}, \hat{\pi} \right] \rangle$$

Minisuperspace

First Friedman $\langle \hat{\tilde{H}} \rangle = 0$ not a consequence of the quantum theory

QM only guarantees

$$\partial_t \langle \hat{\tilde{H}} \rangle = i \langle [\hat{\tilde{H}}, \hat{\tilde{H}}] \rangle = 0$$

i.e.

$$\langle \hat{\tilde{H}} \rangle = \mathbb{H}_0$$

c.f.
Wheeler-DeWitt

$$\hat{H}|\psi\rangle = 0$$

Minisuperspace

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i.e.

$$\langle \hat{\tilde{H}} \rangle = \mathbb{H}_0$$

$$6M_{pl}^2 \frac{\dot{a}^2}{a^2} - \frac{\dot{\phi}^2}{2} - V(\phi) = \frac{\mathbb{H}_0}{a^3}$$

Could be zero,
but generally
contributes like
dark matter

General(er) relativity

$$S = \int d^4x \sqrt{-g} (M_{pl}^2 R + \mathcal{L}_{matter})$$

The equations of motion naively follow

$$\frac{\delta S}{\delta g_{\mu\nu}} = \sqrt{-g} (M_{pl}^2 G^{\mu\nu} - T^{\mu\nu}) = 0$$

For the quantum theory

$$\pi^{ij} \equiv \frac{\delta L_{grav}}{\delta g_{ij,0}}$$

$$\pi^i \equiv \frac{\delta L_{grav}}{\delta g_{0i,0}} = 0$$

$$\pi \equiv \frac{\delta L_{grav}}{\delta g_{00,0}} = 0$$

General(er) relativity

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The corresponding variations are suspect!

‘Gauge fix’

$$g_{00} = -1 \quad g_{0i} = 0$$

General(er) relativity

$$S = \int d^4x \sqrt{-g} (M_{pl}^2 R + \mathcal{L}_{matter})$$

The equations of motion that actually follow from the quantum theory

$$G^{00} = 8\pi G_N T^{00} + 8\pi G_N \frac{\mathbb{H}}{\sqrt{-g}}$$

$$G^{0i} = 8\pi G_N T^{0i} + 8\pi G_N \frac{\mathbb{P}^i}{\sqrt{-g}}$$

$$G^{ij} = 8\pi G_N T^{ij}$$

An auxiliary ‘shadow matter’ tensor

$$T_{\text{aux}}^{\mu\nu} = \frac{1}{\sqrt{-g}} \begin{pmatrix} \mathbb{H} & \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^3 \\ \mathbb{P}^1 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} 0 &= \nabla_\mu T_{\text{aux}}^{\mu\nu} \\ \partial_0 \mathbb{H} &= -\partial_i \mathbb{P}^i \\ \partial_0 (\gamma_{ij} \mathbb{P}^j) &= 0 \end{aligned}$$

Shadow matter in the cosmos

$$ds^2 = -dt^2 + a(t)^2(\delta_{ij} + h_{ij})dx^i dx^j$$

$$h_{ij} = h\delta_{ij} + D_{ij}\eta + (\partial_i w_j + \partial_j w_i) + s_{ij}$$

$$\tilde{D}_{ij} = \hat{k}_i \hat{k}_j - \delta_{ij}/3.$$

Linear perturbation theory of scalars

and shadow matter with $\mathbb{H} = \mathbb{H}_0 = \text{const}$ $\mathbb{P}_i = 0$

Zeroth order

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G_N \frac{\mathbb{H}_0}{a^3}$$

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First order

$$\tilde{\eta} \propto \text{const}$$

$$\tilde{h} \propto a$$

To summarise

Classical physics is a limit of quantum mechanics

Certain classical equations are not guaranteed by the quantum theory

One should consider a broader class of states in EM and GR

Some observations to conclude with



Shadow matter sources linear growth in curvature perturbations, like dark matter



Could have either sign, trivial violation of NEC



The nonlinear regime is important to study, along with detailed cosmological evolution



Inflation dynamically drives to conventional EM/GR

Thanks for listening!

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Linear perturbation theory of scalars

and shadow matter: $\mathbb{H} \equiv \mathbb{H}_0 + \delta\mathbb{H}$ $\mathbb{P}^i = \mathbb{P}_0^i + \delta\mathbb{P}^i$

Zeroth order $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G_N \frac{\mathbb{H}_0}{a^3}$ $\mathbb{H}_0 = \text{const}$

$$\mathbb{P}_0^i = 0$$

Shadow matter in the cosmos

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First order

$$\begin{aligned}\ddot{\tilde{h}} + 3\frac{\dot{a}}{a}\dot{\tilde{h}} - 2\frac{k^2}{a^2}\tilde{\eta} &= 0 \\ \ddot{\tilde{h}} + 3\frac{\dot{a}}{a}\dot{\tilde{h}} + 6\left(\ddot{\tilde{\eta}} + 3\frac{\dot{a}}{a}\dot{\tilde{\eta}}\right) - 2\frac{k^2}{a^2}\tilde{\eta} &= 0\end{aligned}$$

$$\begin{aligned}\frac{\dot{a}}{a}\dot{\tilde{h}} - 2\frac{k^2}{a^2}\tilde{\eta} &= 8\pi G_N \frac{1}{a^3} \left[-\frac{1}{2}\tilde{h}\mathbb{H}_0 + \delta\mathbb{H} \right] \\ -2i\frac{k}{a^2}\dot{\tilde{\eta}} &= 8\pi G_N \frac{\mathbb{P}_{\parallel}}{a^3}.\end{aligned}$$

$$\tilde{\eta} \propto \text{const}$$

$$\tilde{h} \propto a$$

$$\delta\mathbb{H} \propto \text{const} + 1/a^{1/2}$$

$$\delta\mathbb{P}^i \propto 1/a^2$$

Shadow matter in the cosmos

$$ds^2 = -dt^2 + a(t)^2(\delta_{ij} + h_{ij})dx^i dx^j$$

$$h_{ij} = h\delta_{ij} + D_{ij}\eta + (\partial_i w_j + \partial_j w_i) + s_{ij})$$

$$\tilde{D}_{ij} = \hat{k}_i \hat{k}_j - \delta_{ij}/3.$$

Linear perturbation theory of scalars

and shadow matter: $\mathbb{H} \equiv \mathbb{H}_0 + \delta\mathbb{H}$ $\mathbb{P}^i = \mathbb{P}_0^i + \delta\mathbb{P}^i$

First order

$$\ddot{\tilde{h}} + 2\dot{a}\dot{\tilde{h}} - 2k^2\tilde{h} = 0$$

Consistent with

$$\partial_0 \delta\mathbb{H} = -ik\mathbb{P}_{\parallel}$$

$$\partial_0 (a^2 \mathbb{P}_{\parallel}) = 0$$

$$-2i\frac{\kappa}{a^2}\dot{\tilde{\eta}} = 8\pi G_N \frac{\mathbb{P}_{\parallel}}{a^3}.$$

$$\tilde{\eta} \propto \text{const}$$

$$\tilde{h} \propto a$$

$$\delta\mathbb{H} \propto \text{const} + 1/a^{1/2}$$

$$\delta\mathbb{P}^i \propto 1/a^2$$