NNLO mixed EW-QCD corrections to single vector boson production

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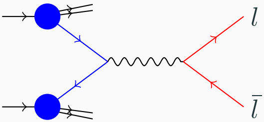
The need for accuracy

The Standard Model is the most successful theory in describing the elementary particles and their fundamental interactions, due to combined effort from both the magnificent experiments like Tevatron, HERA, LHC etc. and precise theory predictions, namely perturbative calculations.

Tevatron	LHC	FCC/ILC
the top quark	the Higgs boson	BSM physics?!
fundamental laws $\sim 10\%$	fundamental laws $\sim 5\%$	more precision!
agreement with NLO theory predictions	agreement with NNLO theory predictions	More precise theory predictions needed!

Drell-Yan



✓ **One of the standard candle processes**

- Large cross section and clean experimental signature - important for detector calibration and constraining parton distribution functions

✓ **Precise predictions for electroweak parameter**

- W boson mass, $\sin^2\theta^l_{eff}$ ...

✓ **New physics potential**

- Many BSM scenarios with same final states - W', Z', KK modes *etc.*

Present status and next goals

Threshold corrections at N^3LO	[Ahmed, Mahakhud, Rana, Ravindran]
Threshold corrections at N^3LO	[Li, von Manteuffel, Schabinger, Zhu]
Threshold resummation at N^3LL	[Catani, Cieri, de Florian, Ferrera, Grazzini]
Mixed QCD-QED corrections	[de Florian, Der, Fabre]
Approximated mixed QCD-EW corrections	[Dittmaier, Huss, Schwinn]
Master integrals for two-loop virtual	[Bonciani, Di Vita, Mastrolia, Schubert]
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Complete N^3LO	Mixed QCD$\otimes$EW
per mille contributions	*per mille* contributions

1. NLO EW effects are large for W mass measurements. Hence, one needs to include mixed QCD$\otimes$EW corrections while aiming for 10 MeV precision.
2. Because of initial-final interaction in case of EW correction, it becomes more important in the tail of the distribution (Enhancement by Sudakov logarithms at large invariant mass of the lepton pair).
3. The appearance of photon induced processes effects the PDFs.

Goal

Based on whether the hadronic production of the vector boson can be factorized from the leptonic decay, the mixed QCD$\otimes$EW corrections can be classified into two distinct categories: the factorizable and non-factorizable contributions.

In this work, we start with the factorizable contribution and consider the production of a on-shell Z boson, specially the quark initiated channel.

Notation

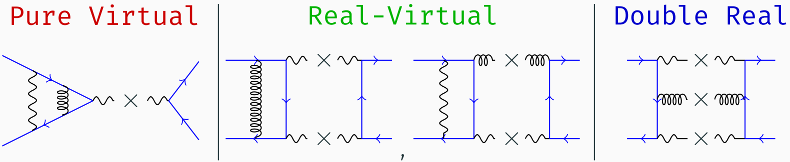
$$\sigma_{tot}(z) = \sum_{i,j \in q,\bar{q},g,\gamma} \int \mathrm{d}x_1 \mathrm{d}x_2 \; f_i(x_1,\mu_F) f_j(x_2,\mu_F) \sigma_{ij}(z,\varepsilon,\mu_F)$$

In the full QCD-EW SM, we have a double expansion of the partonic cross sections in the electromagnetic and strong coupling constants, α and α_s, respectively:

$$\begin{aligned} \sigma_{ij}(z) &= \sigma_{ij}^{(0)} \sum_{m,n=0}^{\infty} \alpha^m \alpha_s^n \; \sigma_{ij}^{(m,n)}(z) \\ &= \sigma_{ij}^{(0)} \Big[\sigma_{ij}^{(0,0)}(z) + \alpha_s \sigma_{ij}^{(1,0)}(z) + \alpha \sigma_{ij}^{(0,1)}(z) \\ &\qquad + \alpha_s^2 \sigma_{ij}^{(2,0)}(z) + \alpha \alpha_s \sigma_{ij}^{(1,1)}(z) + \alpha^2 \sigma_{ij}^{(0,2)}(z) + \cdots \Big] \end{aligned}$$

In this talk, we present the computational details of $\sigma_{q\bar{q}}^{(1,1)}(z)$.

NNLO contributions



For different vector bosons, the contribution can be organized into four types

- QCD⊗QED : γ propagator in the loop / emission of γ
- EW1 : single Z propagator in the loop
- EW2 : single W propagator in the loop
- EW3 : Contributions with WWZ vertex

Emission of massive boson is infrared finite, hence, is treated as separate process.

gauge invariant and finite : QCD⊗QED, EW1, EW2+EW3

The generic procedure

$d = 4 - 2\epsilon$

- Diagrammatic approach -> QGRAF [Nogueira] to generate diagrams
- FORM [Vermaseren] for algebraic manipulation :

  Lorentz, Dirac and Color [Ritbergen, Schellekens, Vermaseren] *algebra*
- Reverse unitarity : phase-space integrals to loop integrals

$$\delta(k^2 - m^2) \sim \frac{1}{2\pi i}\left(\frac{1}{k^2 - m^2 + i0} - \frac{1}{k^2 - m^2 - i0}\right)$$

- Decomposition of the dot products to obtain scalar integrals

$$\frac{2l.p}{l^2(l-p)^2} = \frac{l^2 - (l-p)^2 + p^2}{l^2(l-p)^2} = \frac{1}{(l-p)^2} - \frac{1}{l^2} + \frac{p^2}{l^2(l-p)^2}$$

- Identity relations among scalar integrals : *IBPs, LIs & SRs*
- Algebraic linear system of equations relating the integrals

  ⇓ LiteRed

  Master integrals (MIs)
- Computation of MIs : *Differential eqns.*

The method of differential equations

$$\frac{m^2}{q^2} = z$$

The integral is a function of d, q^2 and m^2.

$$J(1,1,1,0,1,0,1) = \mathcal{N} \int \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d} \frac{1}{l_1^2 l_2^2 ((l_1 - l_2)^2 - m^2)(l_1 - q)^2 (l_2 - q)^2}$$
$$\equiv f(d,\, q^2,\, m^2) \equiv f(d,\, z)$$

The idea is to obtain a differential eqn. for the integral $w.r.t.$ z and solve it.

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$$\frac{d}{dz} J_i = \text{some combinations of integrals}$$
$$\Downarrow \text{ IBP identities}$$
$$= \sum_j c_{ij} J_j$$

c_{ij}'s are rational function of d and z.

The method of differential equations

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The idea is to obtain a differential eqn. for the integral $w.r.t.$ z and solve it.

$$d_z \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}$$

$$d_z \mathbb{J} = \mathbb{A}(d,\, z) \mathbb{J}$$

The method of differential equations

$$\frac{m^2}{q^2} = z$$

The integral is a function of d, q^2 and m^2.

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To solve such a system, it would be best to organize it in such a way that it diagonalizes, or at least it takes a block-triangular form.

Let's consider an example

$$\frac{d}{dz}\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} + \begin{pmatrix} R_1(\epsilon, z) \\ R_2(\epsilon, z) \\ R_3(\epsilon, z) \end{pmatrix},$$

$$\begin{aligned} c_{11} &= \frac{\left(7 + 6z + 7z^2 - 2d(1 + z + z^2)\right)}{z(1+z)^2}\,, \\ c_{12} &= \frac{(-4+d)(-10+3d)}{2(-3+d)^2(1+z)^2}\,, \\ c_{13} &= \frac{\left(d^2(15+8z+15z^2) + 8(20+9z+20z^2) - 2d(49+24z+49z^2)\right)}{4(-3+d)^2 z(1+z)^2}\,, \ldots \end{aligned}$$

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$$c_{11} = \frac{(7 + 6z + 7z^2 - 2d(1 + z + z^2))}{z(1 + z)^2},$$

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$$c_{13} = \frac{(d^2(15 + 8z + 15z^2) + 8(20 + 9z + 20z^2) - 2d(49 + 24z + 49z^2))}{4(-3 + d)^2 z(1 + z)^2}, \ldots$$

For massive case, the integrals can have, at max, a quadratic pole in ϵ.

$$J_i = \frac{1}{\epsilon^2} J_i^{-2} + \frac{1}{\epsilon} J_i^{-1} + J_i^0 + \epsilon J_i^1 + \cdots$$

Series expansion and compare each order of ϵ!

Let's consider an example

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Each order in ϵ-expansion gives a much simpler form

$$\frac{d}{dz}\begin{pmatrix} J_1^{-2} \\ J_2^{-2} \\ J_3^{-2} \end{pmatrix} = \begin{bmatrix} \frac{1}{z} + \frac{2}{1-z} & 0 & \frac{1}{1+z} - \frac{2}{z} - \frac{3}{1-z} \\ -\frac{1}{z} + \frac{2}{1+z} & \frac{1}{1+z} - \frac{1}{z} - \frac{1}{1-z} & \frac{1}{z} - \frac{2}{1+z} \\ \frac{1}{z} + \frac{2}{1-z} & 0 & \frac{1}{1+z} - \frac{2}{z} - \frac{3}{1-z} \end{bmatrix} \begin{pmatrix} J_1^{-2} \\ J_2^{-2} \\ J_3^{-2} \end{pmatrix} + \begin{pmatrix} R_1^{-2}(z) \\ R_2^{-2}(z) \\ R_3^{-2}(z) \end{pmatrix},$$

The general form of expansion

$$\frac{d}{dz}J_n(z,\epsilon) = \mathcal{C}_{nm}(z,\epsilon)J_n(z,\epsilon) + \mathcal{R}_n(z,\epsilon)$$

Expand in ϵ

$$J_n(z,\epsilon) = \sum_{k=-2}^{\infty} J_n^{(k)}(z)\epsilon^k$$

$$\mathcal{C}_n(z,\epsilon) = \sum_{k=0}^{\infty} \mathcal{C}_n^{(k)}(z)\epsilon^k$$

$$\mathcal{R}_n(z,\epsilon) = \sum_{k=-2}^{\infty} \mathcal{R}_n^{(k)}(z)\epsilon^k$$

$$\frac{d}{dz}J_n^{(k)}(z) = \mathcal{C}_{nm}^{(0)}(z)J_n^{(k)}(z) + \sum_{p=1}^{k+2}\mathcal{C}_{nm}^{(p)}(z)J_n^{(k-p)}(z) + \mathcal{R}_n^{(k)}(z)$$

Algorithm

- It boils down to solving a system of linear first order diff. eqns.
- First step is to reduce the system to a higher order eqn in a single unknown
- Start with the leading pole (ϵ^{-2}) - find the homogeneous solution and best uncoupling procedure - solve for the nonhomogeneous part using the method of variation of constant
- Structure of homogeneous part is same at each order in ϵ-expansion
- Hence the homogeneous solutions and uncoupling procedure are similar for each order
- Now at each order in ϵ, find the nonhomogeneous parts keeping the uncoupling structure fixed
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We use

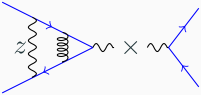
`Sigma` [Schneider] and `HarmonicSums` [Ablinger, Blümlein, Schneider]

The results are obtained in terms of HPLs and Cyclotomic HPLs.

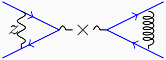
Computational details for QCD⊕QED

- Photon being massless, it has same topology as pure QCD.
- The loop and phase-space master integrals are known from Higgs calculations.
- We have used the method of differential equations to compute them.
- To obtain the finite partonic cross-section, we need to perform subtraction of collinear singularities through mass factorization. Thanks to the recent calculation by de Florian *et al.*, the splitting functions are available in this order. [de Florian, Sborlini, Rodrigo]
- Finally, we cross-check ours with the available results which has been obtained using Abelianization. [de Florian, Der, Fabre]

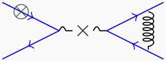
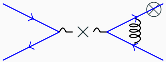
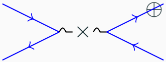
Computing EW1 : virtual



- The integrals are proportional to $(m_Z^2)^n$, hence method of differential equation can not be applied. Hence, to compute the integrals, we consider production of a off-shell Z boson. Now we use the variable $x = -s/m_Z^2$ to differentiate and finally take the limit $x \to -1$ to obtain the on-shell solution. [Aglietti, Bonciani]
- There are cross-contributions of the kind



- UV counter-terms :

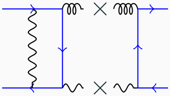
 ,  , 

$\otimes$: one-loop Z corrections to quark wave function

$\oplus$: two-loop mixed QCD-Z corrections to quark wave function

After adding all contributions, we obtain UV finite virtual piece which is still infrared divergent.

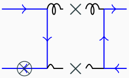
Computing EW1 : real-virtual



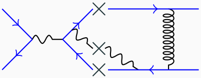
- Method of reverse unitarity

$$\delta(k^2 - m^2) \sim \frac{1}{2\pi i}\left(\frac{1}{k^2 - m^2 + i0} - \frac{1}{k^2 - m^2 - i0}\right)$$

  transform all phase-space integrals to loop integrals and then we can use all the techniques like IBP identities and method of differential equations.
- $z = m_Z^2/s$ is used as the variable for differential equations.
- The differential equations are then solved using the method described earlier. The threshold limit ($z \to 1$) of the integrals are used as the boundary values.
- UV counter-terms :

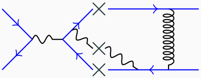


Computing EW1 : double-real



- Similar procedure as real-virtual. However, because of particular color flow, only mixed channels (s with t/u) contribute.
- On the other hand, Z being massive, it is free of infrared singularities.

Computing EW1 : double-real



- Similar procedure as real-virtual. However, because of particular color flow, only mixed channels (s with t/u) contribute.
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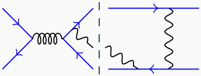
Contains elliptic integrals



The topology generates a set of three integrals which factorize as $\{2 \oplus 1\}$. The homogeneous part was solved in [Aglietti, Bonciani, Grassi, Remiddi].

One can obtain a compact solution in terms of elliptic functions. However, for numerical evaluation, we anyway need the expansion and hence we solve them by expanding the differential equations near threshold up to sufficient order.

Computing EW1 : final

Finally, to obtain the finite partonic cross section for the ’EW1’ type contributions, we add all the virtual, real-virtual and double-real contributions with appropriate UV counter-terms. According to KLN theorem, all soft and final state collinear singularities cancel, leaving only initial state collinear singularities.

The initial state collinear singularities are removed by mass factorization.

$$\Gamma_{qq}^{(0)} \times F_Z^{(0,1)}$$

We find the absence of ’pure EW’ contribution to the splitting function.

The finite cross section is constituted by

$$\ln 2, \zeta_2, \zeta_3, \mathrm{Li}_4(1/2), H[\_, z], H[\_, \_, z], H[\_, \_, \_, z]$$

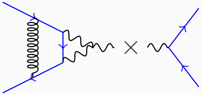
Computing EW2

We expand m_W around m_Z

$$m_W = m_Z - \delta_m$$

and then the topology of EW2 contributions becomes similar to EW1. To improve the accuracy, this expansion is performed up to sufficient order.

Computing EW3 : virtual



- In this case also, we consider first off-shell production of Z boson and then consider the Landau variable x for the parameter in differential equations.

$$\frac{s}{m^2} = -\frac{(1-x)^2}{x}$$

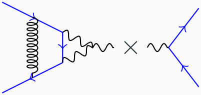
The solution space is spanned by following alphabet

$$\left\{ \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{1}{1-x+x^2}, \frac{x}{1-x+x^2} \right\}.$$

Finally, the on-shell solution is obtained by taking the limit $x \to (-1)^{1/3}$.

New constants! During this procedure, we obtain constants of the type H[__, $(-1)^{1/3}$] up to weight 4 in terms of [Henn, Smirnov, Smirnov].

Computing EW3 : virtual



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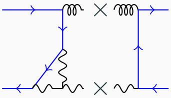
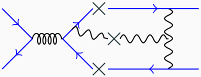
- For the WWZ vertex, we consider the *background field gauge*. The reason is that in this case, the vertex and UV counter-terms (two point functions), are separately UV finite.

Computing EW3 : real-virtual



- Similar procedure as EW1, except that in this case square root letter appears in the alphabet along with the square root of unity. It implies that when we try to rationalize one, the other one takes quartic form.
- We use a smart partial transformation of variable, where needed to deal with this situation.

Computing EW3 : double-real



Same as earlier.

Computing EW3 : final

Combining again all the relevant contributions, along with the mass factorization term, we finally obtain a UV divergent contribution. Only once we combine two contributions EW2 and EW3, we obtain UV finite-ness.

The constants appear in the final result

$$\ln 2, \zeta_2, \zeta_3, \mathrm{Li}_4(1/2), \mathrm{GI}[r2], \mathrm{GI}[0, r2], \mathrm{GI}[0, 1, r4]$$

with $H[\_\_, z]$ up to weight 3 with square root letters.

Other minor points

Treatment of γ_5 : *As all the γ_5 appear in a single quark line, we can safely use the naive anti-commutation rule.*

UV counter terms : *The UV counter terms get contributions from two-point functions with massive propagator insertion which we have obtained up to required accuracy.*

Summarizing

- The mixed NNLO QCD-EW contributions to Drell-Yan production are much sought for. We make an advancement by obtaining analytic results for on-shell Z boson production in quark initiated channel.
- We have computed analytically 12 different matrix element squared purely at two-loop level. Additionally there are one-loop contributions. Combining them to obtain a finite partonic cross section is non-trivial and acts as a strong check on our calculation.
- The method of reverse unitarity allows us to use the techniques (IBP, DE) of loop calculation for the phase-space integrals.
- The solutions are obtained mostly in terms of harmonic poly-logarithms (HPL) and special constants (MZV and cyclotomic HPL at 1). The contributions from elliptic functions are obtained as expansion near threshold.
- We also perform a parallel independent computation to cross check.

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Thank you for your attention!