Lattice QCD Three-Quark Potential Analysis using Hyperspherical Variables Redux: Sakumichi & Suganuma's Data and Interpretation

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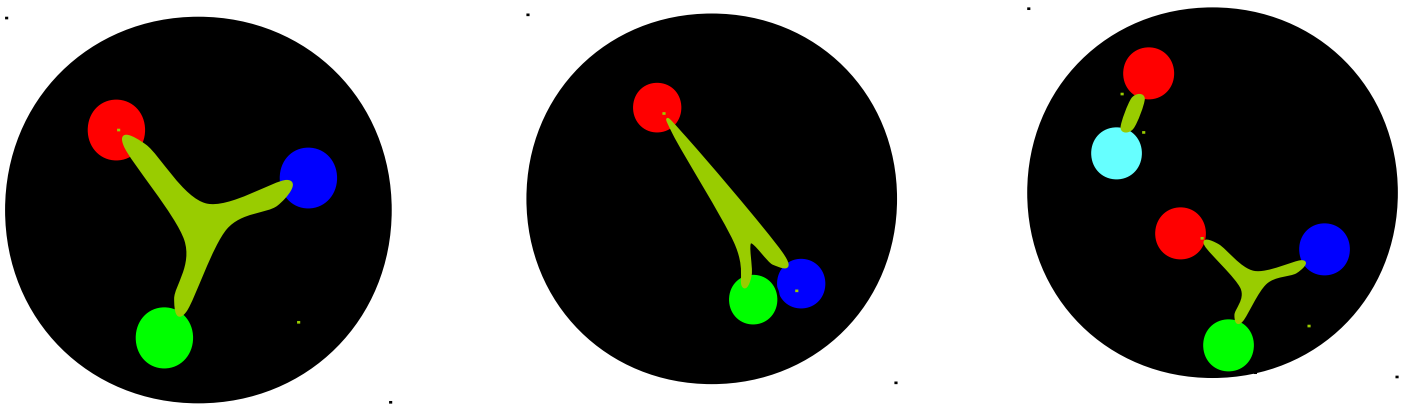
Outline:

- **Motivation: Delta or Y-string confinement in baryons?**
- **Intro to 3-body hyperspherical coordinates**
- **Review of lattice QCD calculations**
- **Lattice QCD data in terms of h.s. variables**
- **Interpretation of results**
- **Summary and Conclusions**

Strings as confinement mechanism?



S-matrix “dual resonance models” (Veneziano, 1969) led to first notions of “hadronic strings”

Advent of QCD (1973) led some (Mandelstam,’tHooft) to talk about possible mechanisms for string formation in QCD, but no proof, as yet.

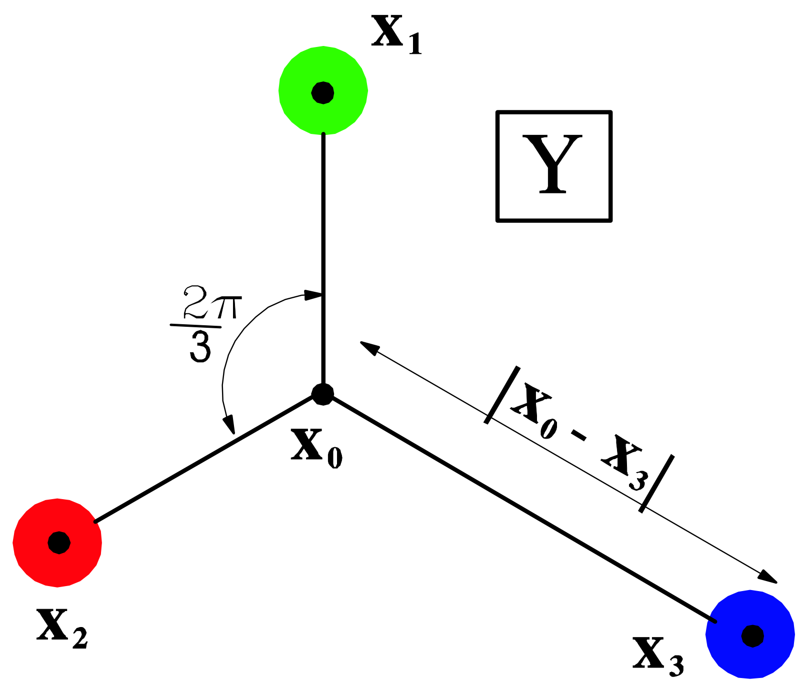
The Y-string

$$\frac{1}{\sigma_Y} V_Y = \sum_{i=1}^{3} |\mathbf{x}_i - \mathbf{x}_{\mathrm{F.T.}}|$$

- Defined as the shortest sum of string lengths; this means that the strings pointing to the three quarks form 120 degree angles at the juncture (Fermat-Torricelli point)
- "Support from lattice QCD"

All these fit analyses support the *Y Ansatz*

Takahashi, Matsufuru, Nemoto & Suganuma, PRL 86, 18 ('01); PRD65, 11409 ('02); Sakumichi & Suganuma, PRD92, 03451 ('15).



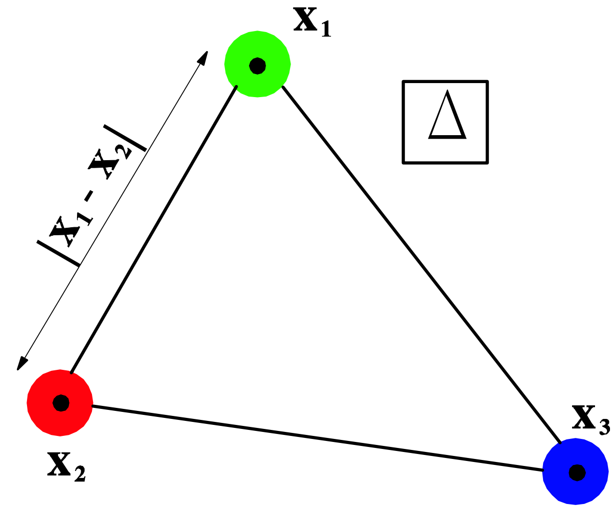
The Δ-string

- Sum of two-body potentials
- Also "supported by Lattice QCD"
- (Alexandrou, deForcrand Tsapalis, PRD65, 054503 ('02))
- How can one distinguish between the Y and Delta string potentials?

  Y. Koma and M. Koma, PRD95, 094513 (2017) reported a new calculation that is neither pure Delta nor pure Y string!

cannot answer here whether or not both functional forms can change continuously depending on the movement of quarks, which should be clarified in future study. It is

$$\frac{1}{\sigma_\Delta} V_\Delta = \sum_{i<j=1}^{3} |\mathbf{x}_i - \mathbf{x}_j|$$

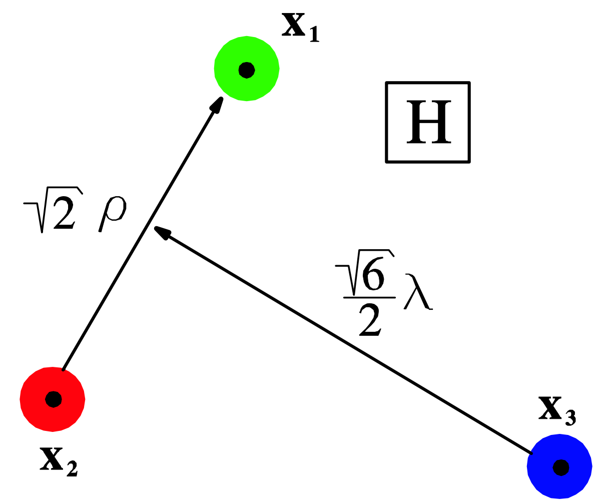


Required Properties of 3-body potentials:

- **Symmetries:**
- **Translations: must be independent of CM variable; may depend only on two relative vectors!**
- **Rotations: must be only a function of 3 scalar products of two relative vectors!**
- **Permutations of 3 particle’s labels: non-trivial implications – see below!**

Jacobi and hyper-spherical variables

- Jacobi relative coordinate vectors
  $(\vec{\rho},\vec{\lambda})$
- They form the basis of two-dimensional representation of permutation group S_3



$$\vec{\rho}=\frac{1}{\sqrt{2}}\left(\vec{r}_1-\vec{r}_2\right)$$

$$\vec{\lambda}=\frac{1}{\sqrt{6}}\left(\vec{r}_1+\vec{r}_2-2\vec{r}_3\right)$$

Hyper-spherical variables

- The hyper-radius R sets the overall size of the triangle.
- Two hyper-angles determine the shape of the triangle, or a point on the shape sphere.

$$R^2 = \vec{\rho}^{\,2} + \vec{\lambda}^{\,2}$$

$$2\chi = \tan^{-1}\left(\frac{2\,\vec{\rho}\cdot\vec{\lambda}}{\vec{\rho}^{\,2} - \vec{\lambda}^{\,2}}\right)$$

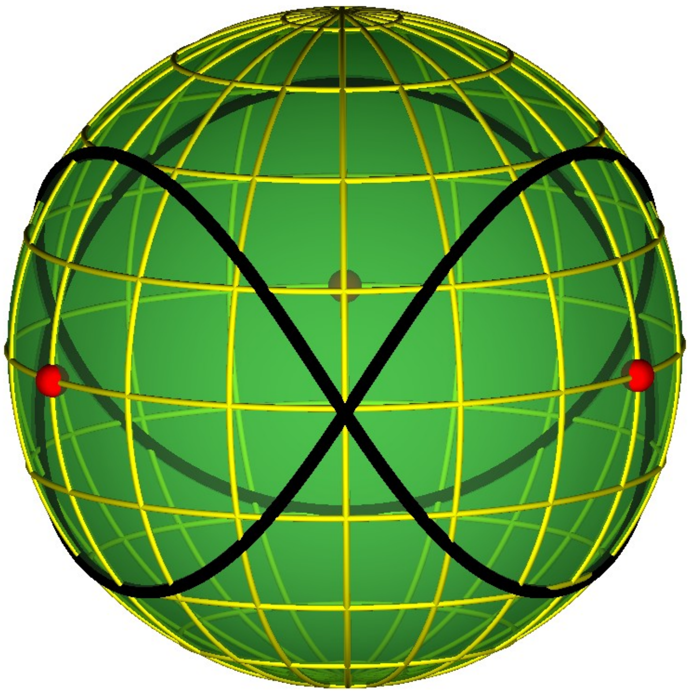
$$\theta = \cos^{-1}\left(\frac{\vec{\rho}\cdot\vec{\lambda}}{\rho\lambda}\right)$$

The shape-space sphere

$$X = \left(\frac{2\vec{\rho}\cdot\vec{\lambda}}{\vec{\rho}^2 + \vec{\lambda}^2} \right)$$

$$Y = \left(\frac{\vec{\rho}^2 - \vec{\lambda}^2}{\vec{\rho}^2 + \vec{\lambda}^2} \right)$$

$$Z = \left(\frac{2\left(\vec{\rho}\times\vec{\lambda}\right)_z}{\vec{\rho}^2 + \vec{\lambda}^2} \right)$$

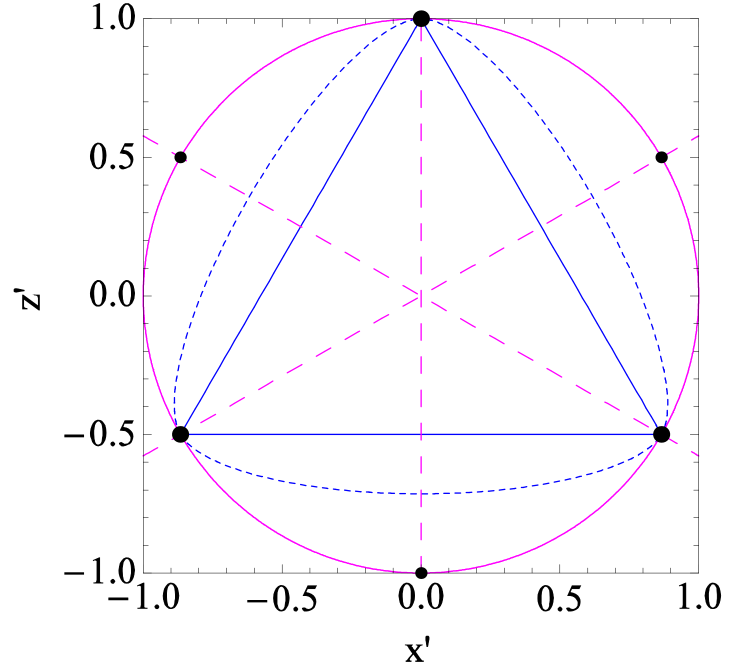


- Define a unit sphere with (X,Y,Z) coordinates
- We shall use a projection from a point infinitely high above the North Pole.
- Red points correspond to collision configurations – opposite (on the equator) to equi-distant collinear ("Euler") configurations.
- The solid black line is the trajectory of "figure-8" periodic orbit.

Permutation-symmetric hyper-angles

- Define the new (permutation-symmetric) hyper-angles by view from “infinity above the North Pole”.

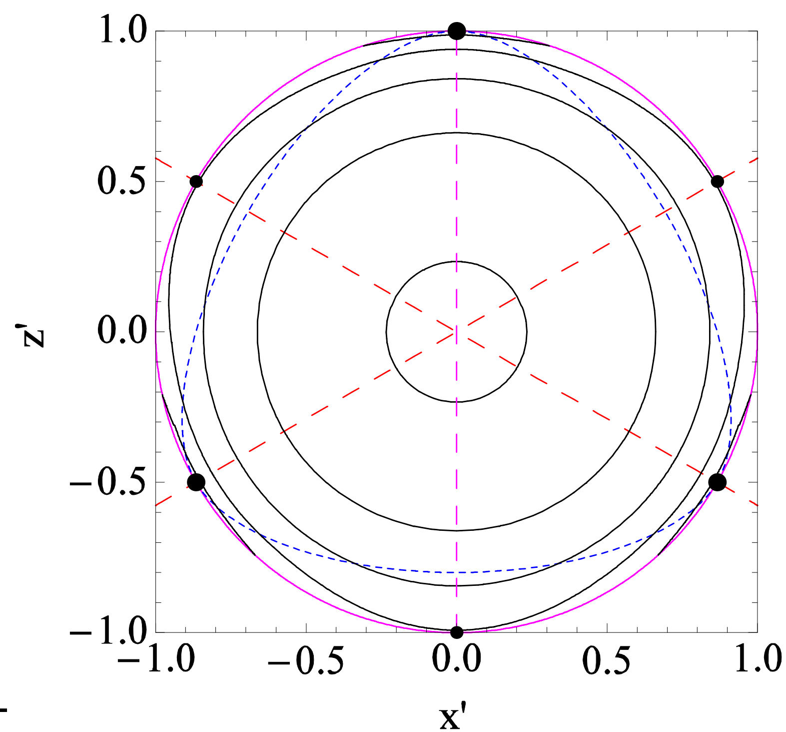
- The discrete symmetry group consisting of three reflections (about the vertical (solid black) and the two slanted (pink dashes) axes) and two rotations through $^{2\pi}\!/_{3}$ that correspond to the permutation group S_3 of three quarks.



The Y-string in terms of new hyper-angles

- The contour plot of the Y-string potential consists of concentric circles (solid black)
- The Y-string potential is axially symmetric under rotations: not a function of the (new) hyperangle $\boldsymbol{\Phi}$

$$V_Y(R,\alpha,\varphi)=\sigma R\sqrt{\frac{3}{2}\left(1+|\cos\alpha|\right)}$$

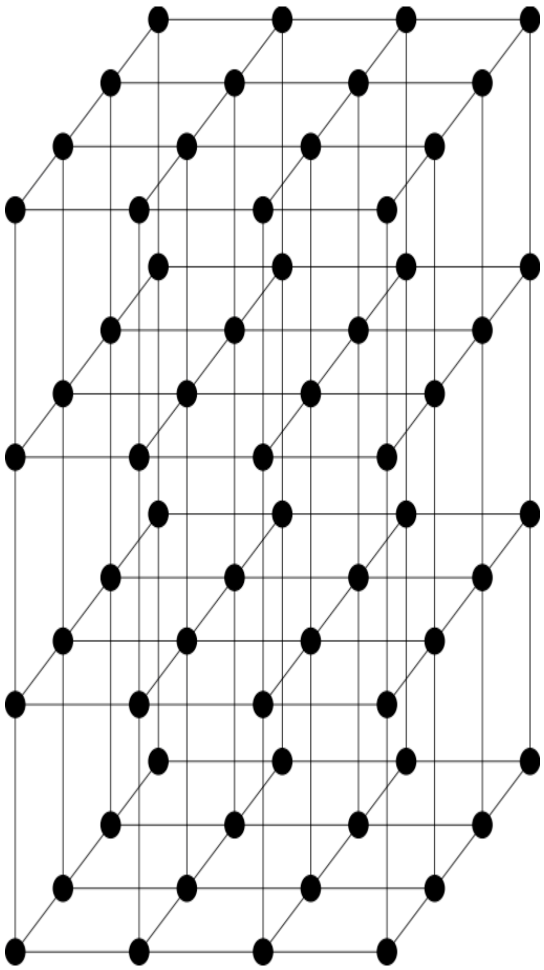


Lattice QCD calculations

1) Takahashi,Matsufuru, Nemoto and Suganuma, PRL86, 18 ('01); PRD65, 11409 (2002) used smaller lattices: 12^3x24, at β =5.7 and 16^3x32, at β=5.8, 6.0.

2) Sakumichi & Suganuma PRD92, 094513 (2015), used a larger lattice: 16^3x32 (1000 gauge config., 101 geometries, Wilson loop) at β = 5.8, and 20^3x32 (2000 gauge config., 211 geometries) at β = 6.0.

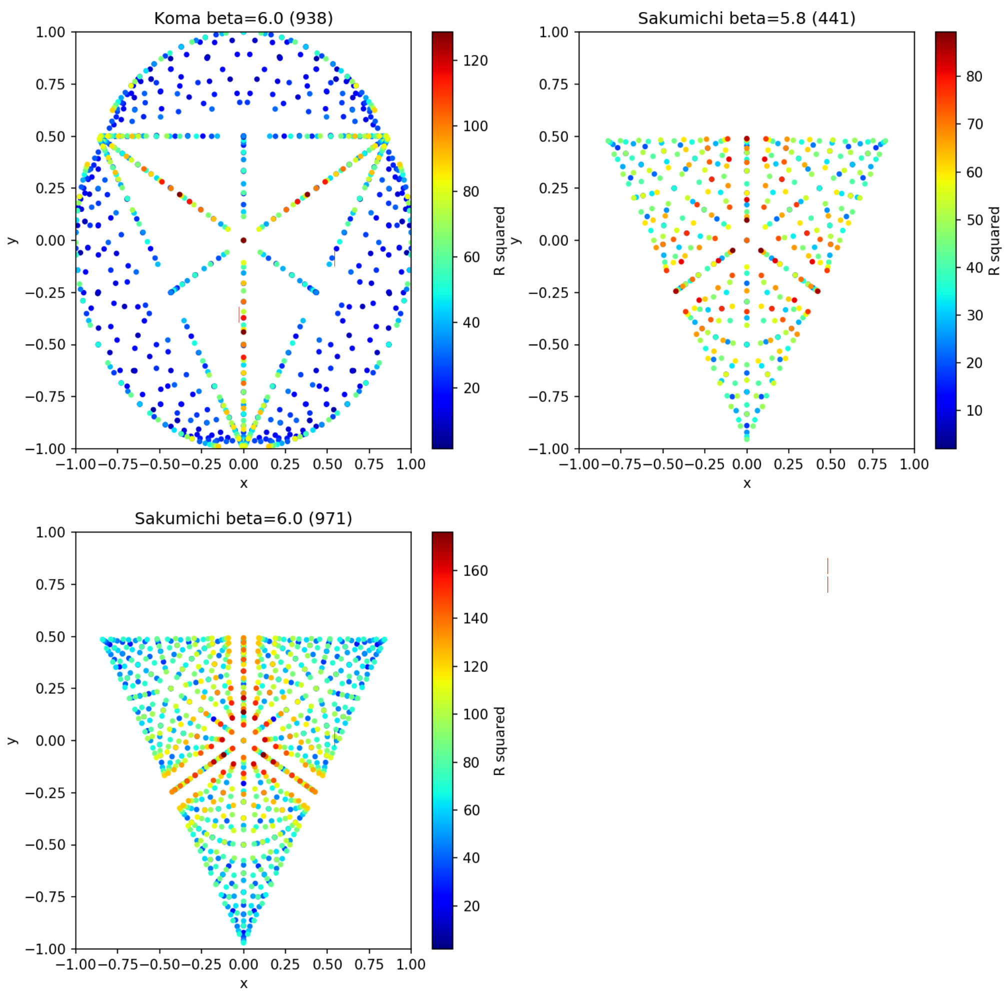
3) Koma & Koma PRD95, 094513 (2017), used the largest lattice: 24^4, (1 gauge config. and 221 geometries, Polyakov loop), at β = 5.85, 6.0, 6.3, actually just 6.0.



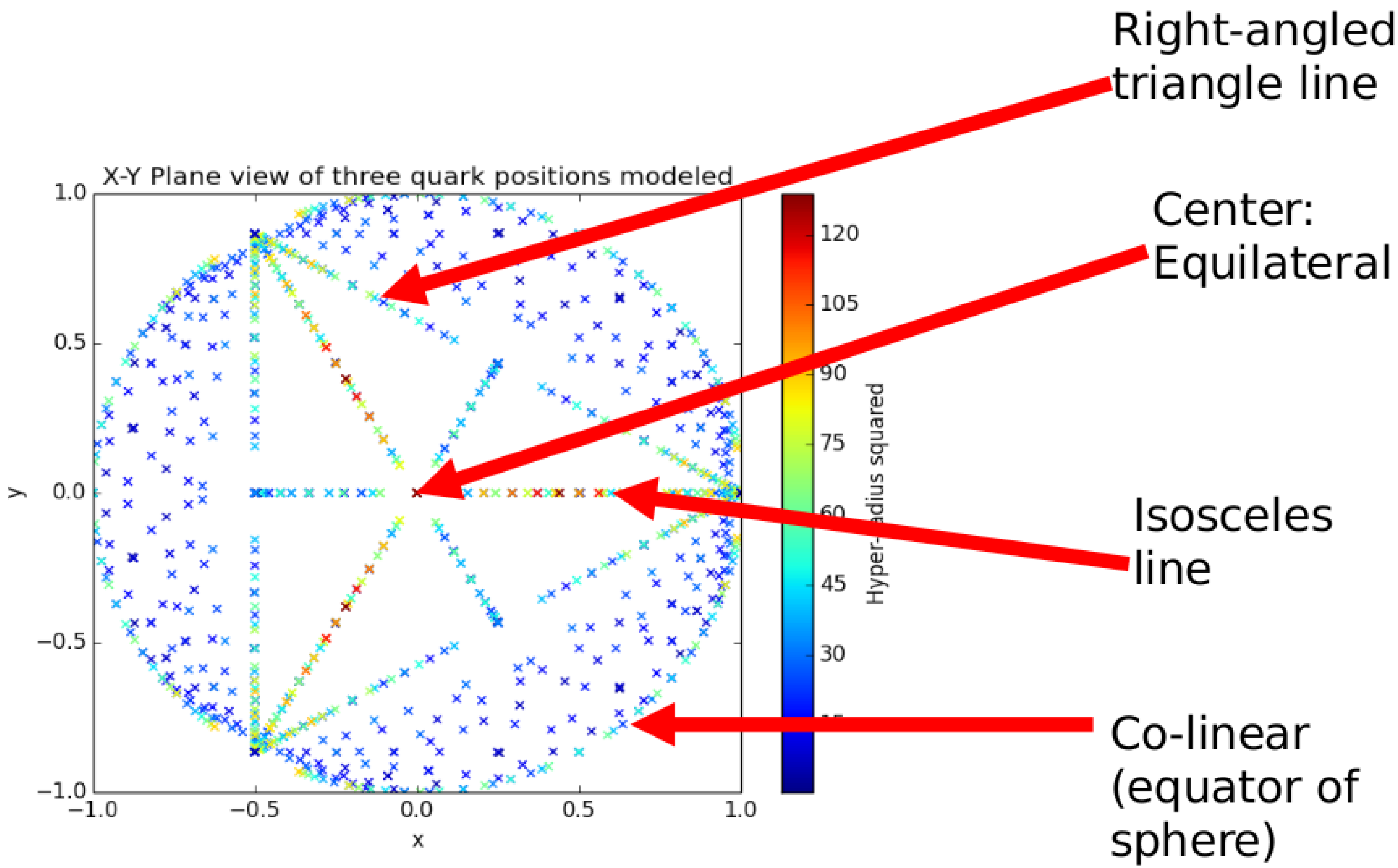
Lattice QCD calculations

- Two of these had been analysed in eQCD 2018: a) Takahashi, Matsufuru, Nemoto and Suganuma, PRL86, 18 ('01); PRD65, 11409 (2002), and b) Koma & Koma PRD95, 094513 (2017).
- In 2018 Sakumichi & Suganuma sent us their unpublished raw data.
- Their choice of 1) mathod (Wilson vs Polyakov loop), 2) lattice size, 3) triangle geometries, 4) number of lattice configurations, 5) values of coupling beta, all differ from previous calculations a) and b). Therefore they must have different statistical and systematic errors. This complicates the comparison.
- Data is presented in lattice units – one can move to physical length units: this change depends on the value of beta and rescales the hyper-radius, but leaves hyper-angles intact. The latter fact can be a check of self-consistency!

Distribution of geometries on the lattice



Koma data – Hyper-radius as a function of hyper-angles:

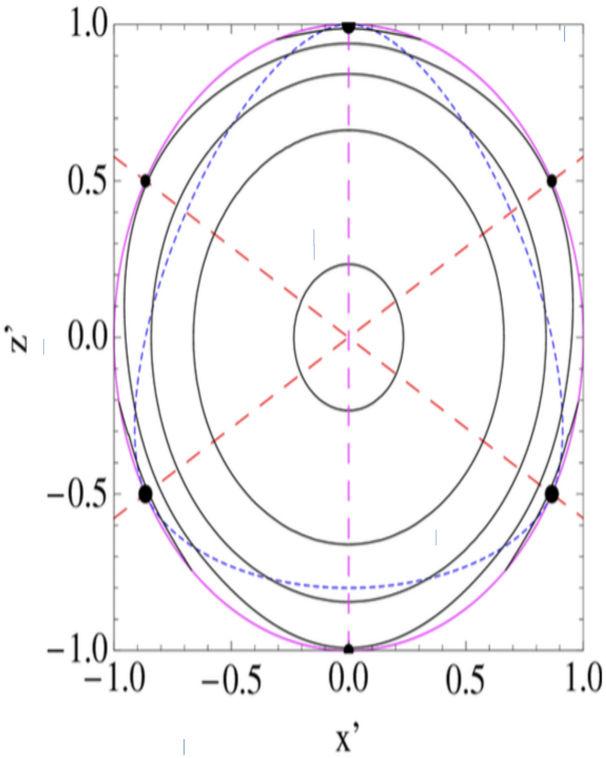


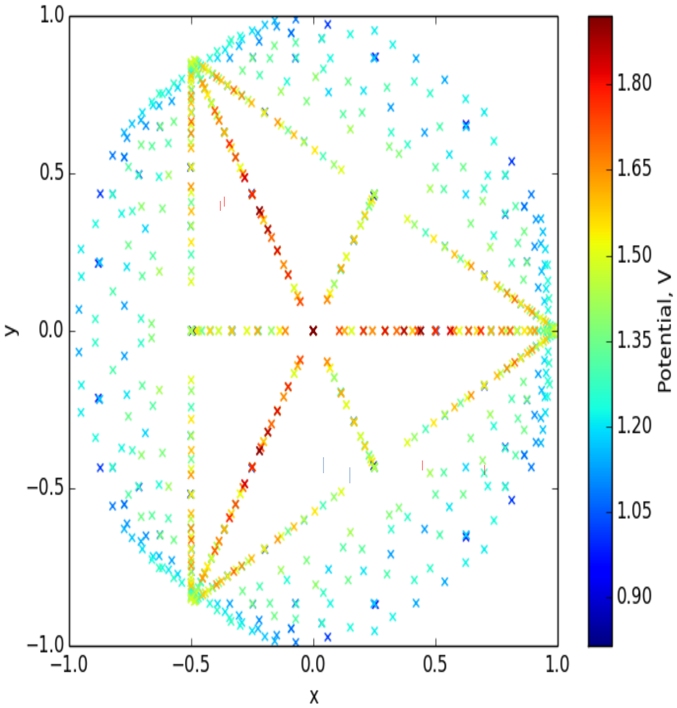
Dynamical symmetry of the Y potential

Y potential on plane:

Equipotential circle

Koma & Koma potential on plane:



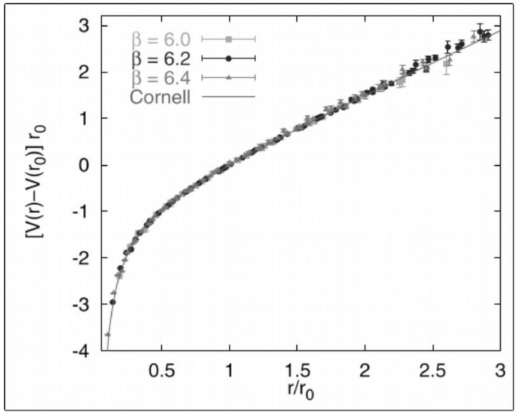


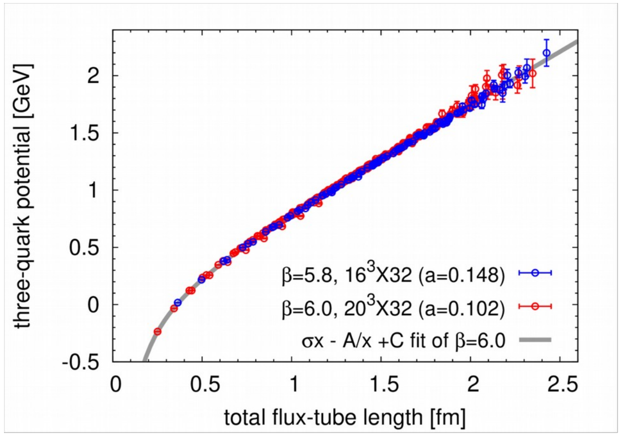
Coulomb interaction 'warps' circles near collision points

No points on circles!

Rescaling of the potential to physical units & hyper-angles

- Lattice QCD potentials are calculated at a fixed lattice spacing *a* and coupling constant beta.
- Change from any value of lattice spacing a and beta to unique physical spacing by a (complicated but well known) procedure of 2-body potential renormalization – see fig. 1
- Same procedure can be applied to three-body potentials - see fig. 2
- Main difference is that there are 3 variables for 3-bodies: Only hyperradius changes under rescaling – the two hyperangles do not!





Analysis of lattice QCD data: the basic Ansatz

$$V = \frac{-A}{R} + BR + C$$

Lattice data potential
QCD Coulomb interaction – QCD coupling constant alpha_C unknown
Confining potential – string tension sigma and shape dependence unknown
Additive constant C unknown

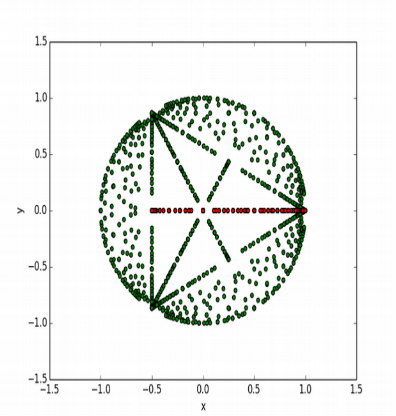
Aim: to remove QCD Coulomb interaction and the hyper-radial dependence of confining potential - thus isolate the hyper-angular dependence of confining potential

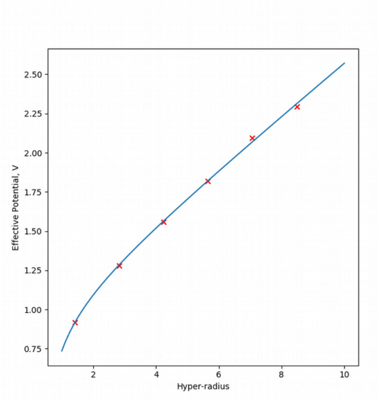
We have multiple data points with different R at these two configurations:

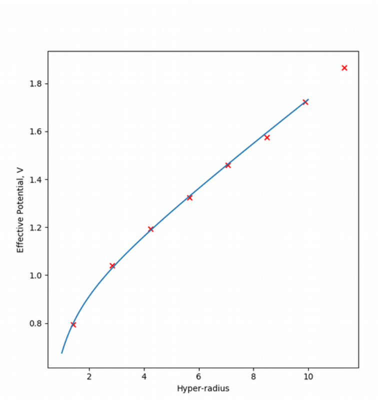
1) equilateral,

2) right-angled isosceles.

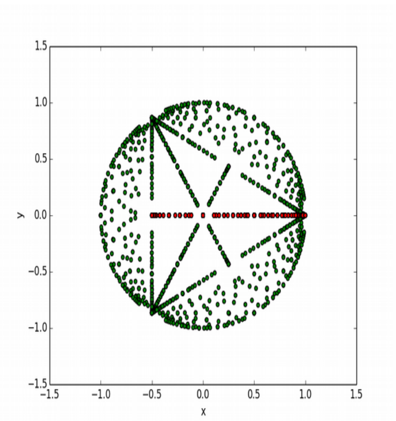
Fits to Sakumichi data (2015) at two values of beta

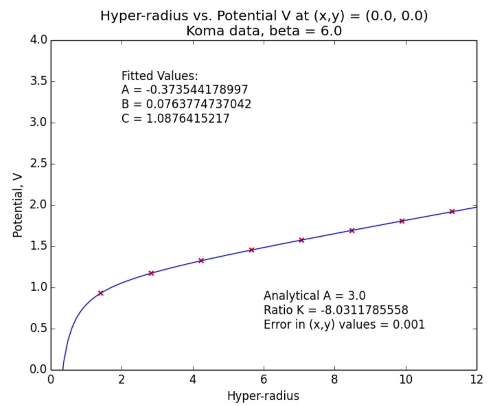


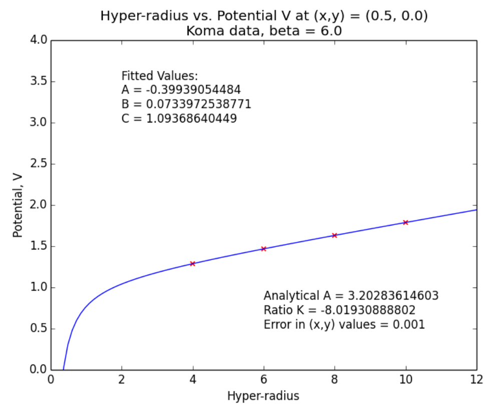




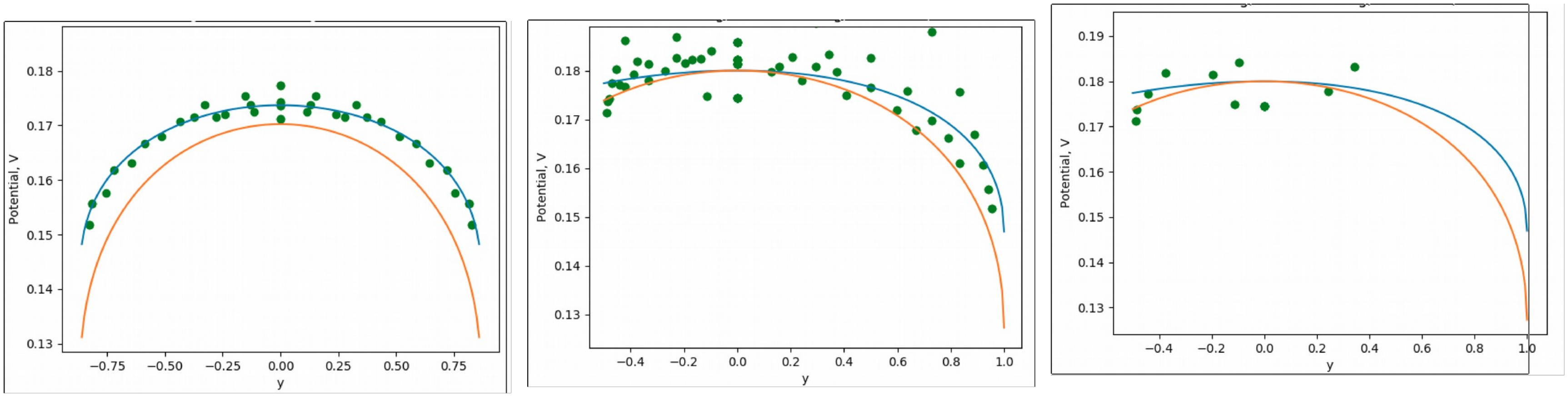
Fits to Koma data (2017) at two points on the shape sphere





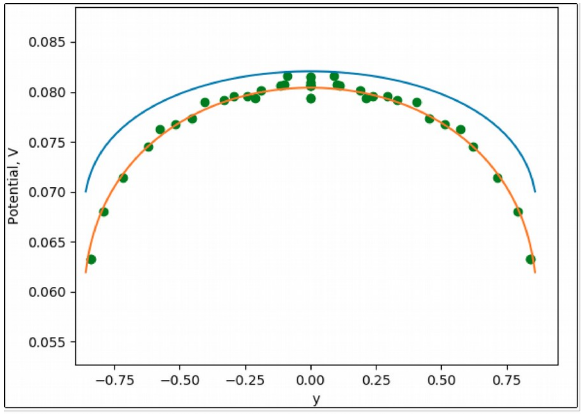


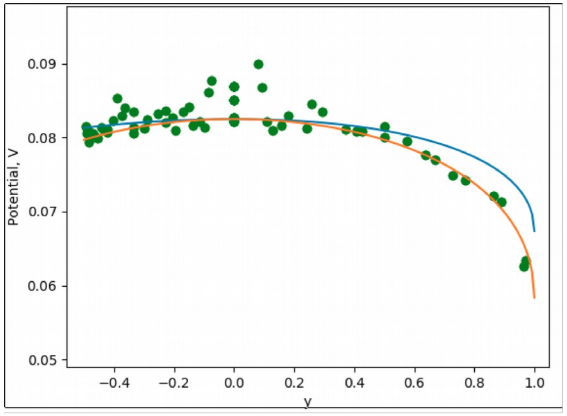
Sakumichi (2015) results (beta=5.8)



Agreement with Delta string for 90deg triangles, but huge dispersion for isosceles

Sakumichi (2015) results (beta = 6.0)





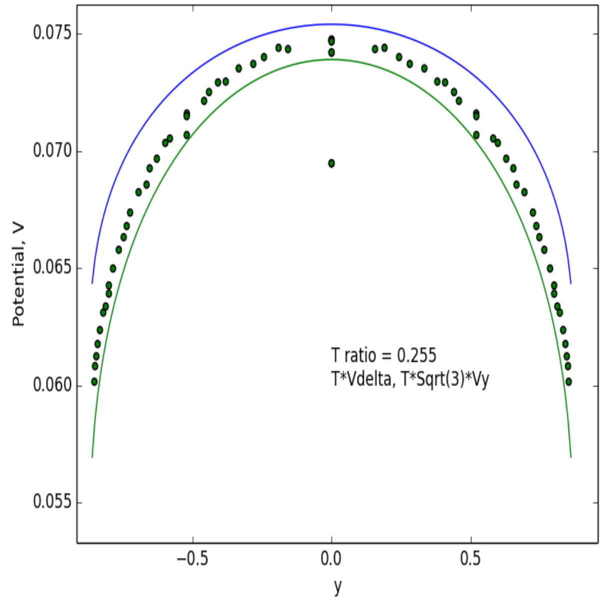
Agreement with Y string for 90 deg triangles

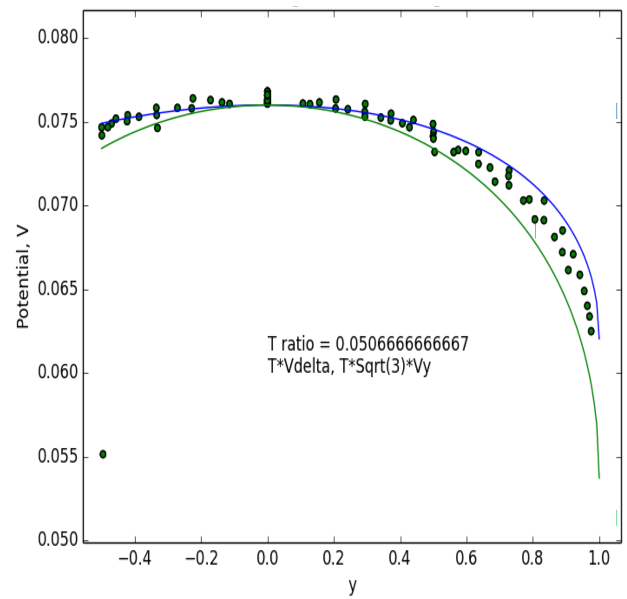
Possible agreement with Y string (huge dispersion)

Remember that hyper-angular dependences must not depend on beta (scaling)! This discrepancy between different beta values is a sign of (underestimated) error bars.

Koma (2017) results

Y **Delta**

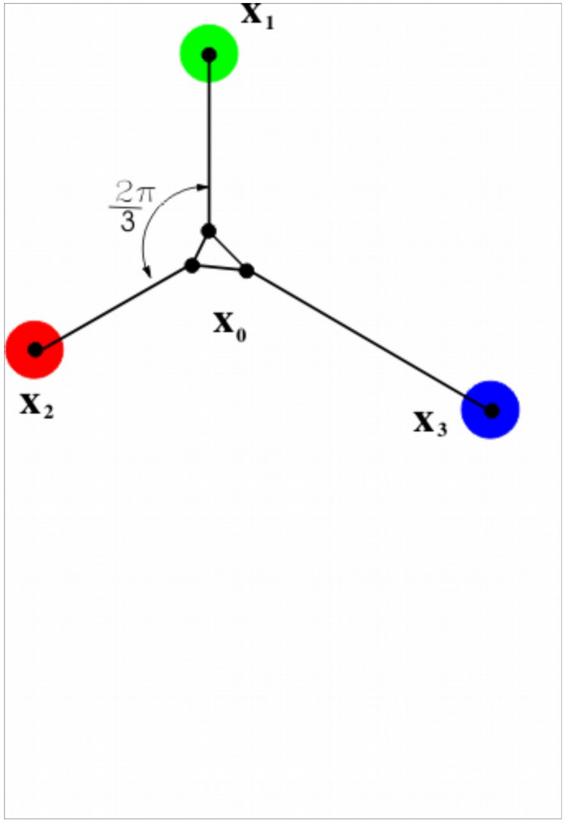




Isosceles and 90 deg. Triangles are both between Y & Delta!

What kind of string would reproduce such a potential?

- The Δ and Y-strings have very different topologies - why are their potentials so close together?
- (Many) people have suggested a Δ-Y hybrid configuration to allow for a (smooth) transition from one to another.
- But, this configuration is forbidden by elementary geometrical arguments!



There is a "numerical identity" [V.D., T. Sato and M. Šuvakov, PRD 80, 054501 (2009)]:

$$2\sum_{i=1}^{3}|\mathbf{x}_i-\mathbf{x}_{\mathrm{F.T.}}|-\sum_{i=1}^{3}|\mathbf{x}_\mathrm{i}-\mathbf{x}_{\mathrm{CM}}| = \frac{1}{\sqrt{3}}\sum_{i<j}^{3}|\mathbf{x}_i-\mathbf{x}_j|\pm\mathcal{O}(0.1\%)$$

This leads to a relation between the Delta and a linear combo of two types of Y-strings: the Fermat-Torricelli (Y) and the barycenter (CM) junction ones.

$$\frac{1}{\sigma_Y}V_{\mathrm{Y}} = \frac{1}{2}\left(\frac{1}{\sigma_{CM}}V_{\mathrm{CM}} + \frac{1}{\sqrt{3}}\frac{1}{\sigma_{\Delta}}V_{\Delta}\right) \pm \mathcal{O}(0.1\%)$$

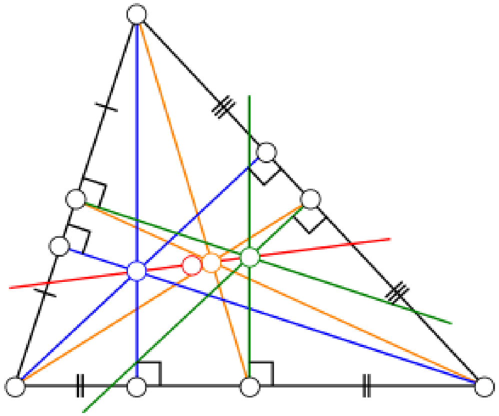
The resulting string has the topology of Y-string, but the junction is at a triangle center different from the Fermat-Torricelli one ...

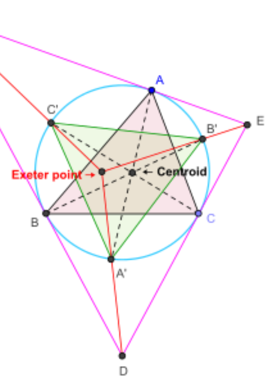
Triangle centers and the Euler line

1) Euler line exists for any non-equilateral triangle, (otherwise it shrinks to a point)

2) Euler line passes through (all) triangle centers, such as orthocenter, circumcenter, centroid, Fermat-Torricelli point etc.,

3) All of these points are defined in a permutation-symmetric manner.





Triangle centers and the Euler line

1) Euler line defines a 2-dimensional (unit) vector.

2) This vector is permutation-symmetric.

3) Any point on the Euler line, i.e., triangle center can be expressed in terms of one parameter (alpha) and defines a (new) 3-string potential.

4) We use alpha to parametrize all such 3-string potentials.

$$\begin{aligned}
\Delta \mathbf{x}_0 &= \mathbf{x}_0^{\mathrm{Fermat-Toricelli}} - \mathbf{x}_0^{\mathrm{barycenter}} \\
&= \frac{l_1 l_2 \mathbf{x}_3 + l_1 l_3 \mathbf{x}_2 + l_2 l_3 \mathbf{x}_1}{l_1 l_2 + l_1 l_3 + l_2 l_3} - \frac{1}{\sqrt{3}}\mathbf{R} \\
&= \frac{1}{2 l_Y^2}\sqrt{\frac{3}{2}}\left(\mathbf{V} + \frac{\mathbf{W}\times\mathbf{A}}{|\boldsymbol{\rho}\times\boldsymbol{\lambda}|}\right)
\end{aligned}$$

$$\begin{aligned}
\mathbf{V} &= \boldsymbol{\lambda}\left(\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2\right) - 2\boldsymbol{\rho}(\boldsymbol{\rho}\cdot\boldsymbol{\lambda}) \\
\mathbf{A} &= \boldsymbol{\rho}\left(\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2\right) + 2\boldsymbol{\lambda}(\boldsymbol{\rho}\cdot\boldsymbol{\lambda}) \\
\mathbf{W} &= \boldsymbol{\rho}\times\boldsymbol{\lambda}.
\end{aligned}$$

$$\boxed{\mathbf{x}_0(\alpha) = \mathbf{x}_{\mathrm{barycenter}} + \alpha\Delta\mathbf{x}_{F.T.} = \frac{1}{\sqrt{3}}\mathbf{R} + \alpha\Delta\mathbf{x}_0}$$

QCD flux-tubes in baryons I

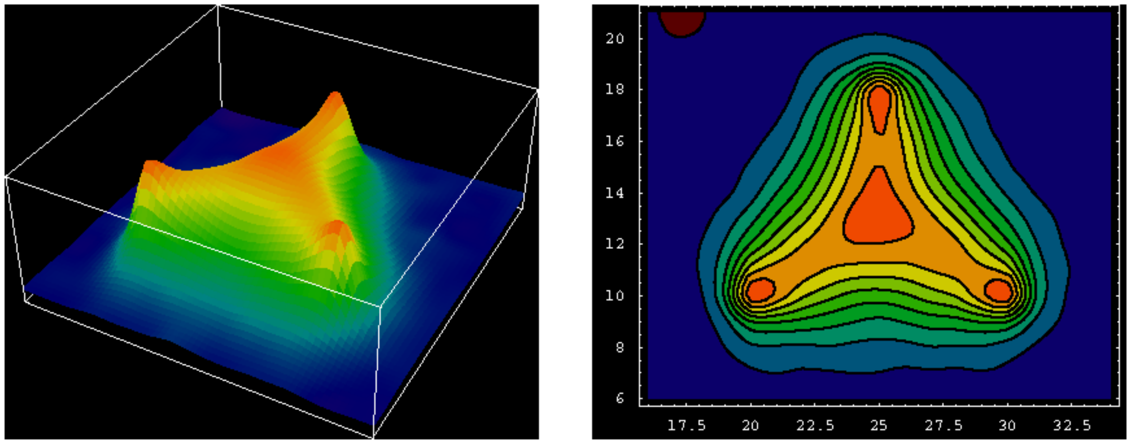
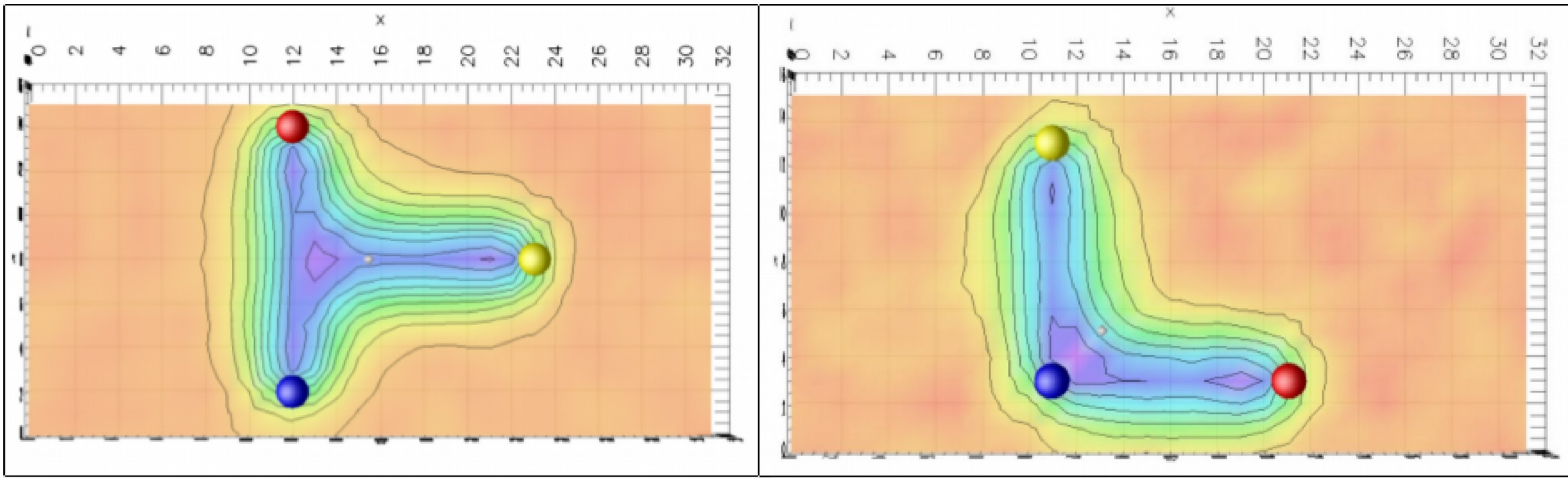


Figure 2. The flux-tube profile in the spatially-fixed 3Q system, in the MA projected QCD.[6] The distance between the junction and each quark is about 0.5 fm.

- Color flux-tube profiles in lattice QCD, Takahashi, Ichie and Suganuma, ("Wako 2003", p. 470-474), see also Bornyakov et al. PRD70,054506 (2004)
- Looks like Y-string – can we check this quantitatively?

The junction is shifted away from the Fermat-Torricelli point which leads to new types of junctions, “T-” and “L-”junction



- Color flux-tube profiles in lattice QCD, Bissey et al. PRD76, 114512 (2007).
- Lattice QCD leads to an L- or T-string, not a Y-string which is consistent with lattice QCD 3-body potential!

Summary

- Analyzed lattice QCD results of Sakumichi & Suganuma (2015) and of Koma & Koma (2017) in terms of hyper-spherical coordinates.
- Their choices of shapes and sizes of triangles do not allow a direct test of the O(2) dynamical symmetry of the Y-string.
- The totality of evidence leads to the conclusion that the 3-quark potential lies between Y- and the Delta string ones.
- This leads to a new interpretation in terms of 3 flux tubes ("strings") joined at a non-Fermat-Torricelli junction, in qualitative agreement with Bissey et al.'s visual observations.

Open questions and Outlook

- If true, the shift of junction must have consequences in 3-quark and multi-quark hadrons.
- Need better lattice calculation(s) to “nail down” the precise position of shifted junction.
- Multiquark confinement ought to be influenced by the shift of junction – lattice to the rescue again?
- For baryon spectra in the quark model with Y- or Delta potential, see Igor Salom’s talk tomorrow afternoon

James Leech (2017); Miho and Yoshiaki Koma (2017)

Toru Sato (2007)

Milovan Šuvakov (2008)

Toru Takahashi (2001) and Hideo Matsufuru (2005)

Publications

- **V.D., T. Sato and M. Šuvakov, Eur. J. Phys. C 62, 383 (2009); Phys. Rev. D 80, 054501 (2009).**
- **V.D. and I. Salom, Nucl.Phys. B 920, 521 (2017); Phys. Lett. A 380, 1904 (2016); J. Math. Phys. 55, 082105 (2014); Acta Phys. Polon. Supp. 6, 905 (2013).**
- **M. Šuvakov and V.D., Phys. Rev. E 83, 056603 (2011), PRL 110, 114301 (2013); with Ana Hudomal PRL 113, 101102 (2014)**

The Y-string potential

- The minimal Y-string length (potential) contains two square-roots: complicated
- How to discriminate between Delta and Y on the lattice?
- For answer, see **V.D., T. Sato and M. Šuvakov,** PRD 80,054501(2009)

$$L_{\min} = \left[\frac{1}{2}(a^2+b^2+c^2) + \frac{\sqrt{3}}{2} \times \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}\right]^{1/2},$$

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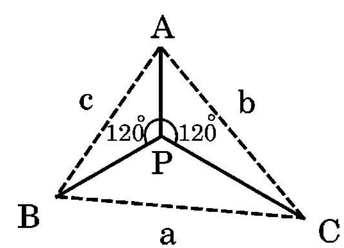
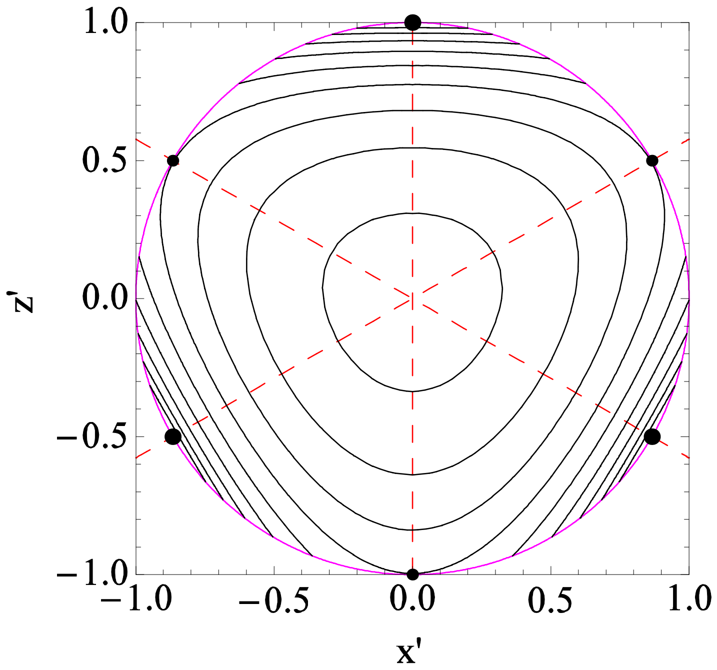


FIG. 1. The flux-tube configuration of the 3Q system with the minimal value of the total flux-tube length. There appears a physical junction linking the three flux tubes at the Fermat point P.

The Delta-string potential & hyper-angles

- This O(2) is not a symmetry of sums of two-body potentials, like the Delta string.
- The Delta-string potential has only three (discrete) reflection symmetries
- Other sums of two-body potentials - e.g. Coulomb - have the same symmetry as Delta and similar contours
- This difference between the Delta and Y-strings is small, but ought to be detectable on the lattice!



Analysis of lattice data

There are only two such configurations in both data sets: 1) equilateral, 2) right-angled isosceles.

The potential takes the form: $V = \frac{-A}{R} + BR + C$

Do a least-square fit for constants A, B, C at each point.

(x,y)	A_{fitted}	B_{fitted}	C_{fitted}	K	δK
(0, 0)	-0.37354418	0.0763775	1.0876415	-8.0311786	-0.07%
(0, -0.5)	-0.3993905	0.07339725	1.0936864	-8.019309	+0.07%

The percentage difference in K is much larger for the Takahashi data sets: for $\beta = 5.8$ and $\beta = 6.0$ data sets they are 22.5 % and 18.1 %, respectively.

Hyper-angular analysis of lattice data

Having done a least-square fit for constants A, B, C, at two different geometries, define K, the ratio of fitted (Coulomb) A and theoretical A – so as to remove the Coulomb potential from the lattice total.

$$\boxed{K = \frac{A_{analytical}}{A_{fitted}} = \frac{1}{\alpha_S}}$$

$$\boxed{\frac{A_K(\alpha,\phi)}{R} = \frac{1}{K}\left(\frac{1}{a(\alpha,\phi,R)} + \frac{1}{b(\alpha,\phi,R)} + \frac{1}{c(\alpha,\phi,R)}\right)}$$

If K is constant, then one may define the hyper-angular part of the confining potential:

$$V^* = \frac{1}{R}\left(V + \frac{A}{R} - C\right) = B$$

Which is independent of hyper-radius.

Estimate of error bars

Koma & Koma estimated statistical error in equilateral geometry as <0.8%

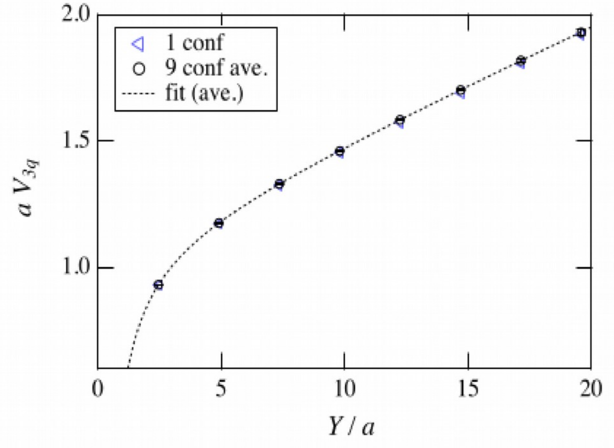


FIG. 6. The three-quark potentials of the equilateral triangle geometries at $\beta = 6.00$ obtained from one gauge configuration and from the average of 9 gauge configurations as a function of Y. The dotted line represents the fit curve to the averaged potential.

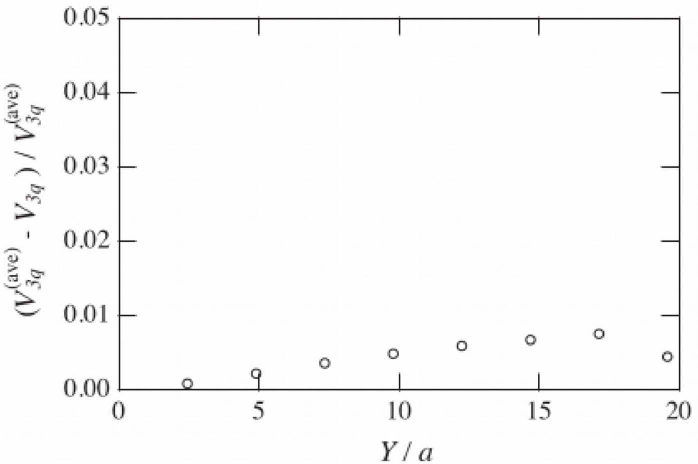


FIG. 7. The relative error between the two potentials in Fig. 6, $(V_{3q}^{(\mathrm{ave})} - V_{3q})/V_{3q}^{(\mathrm{ave})}$.

Estimate of systematic error bars

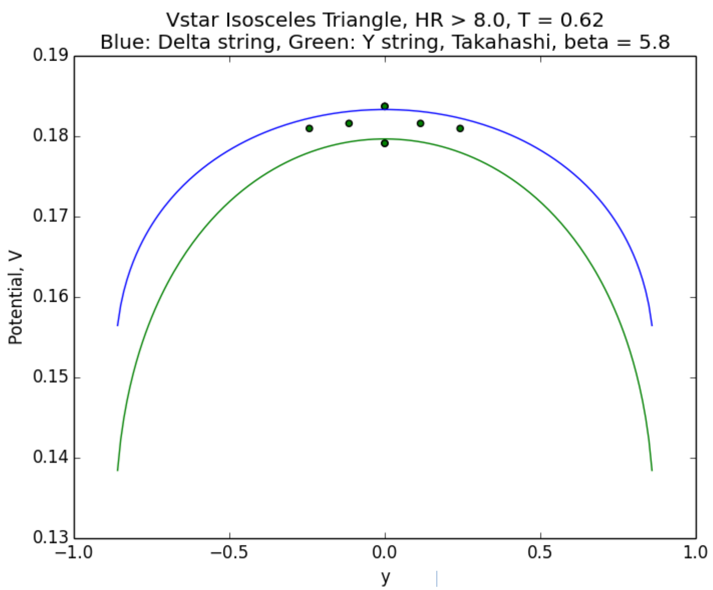
Koma & Koma noticed significant variation of Coulomb coupling "constant" up to 26%

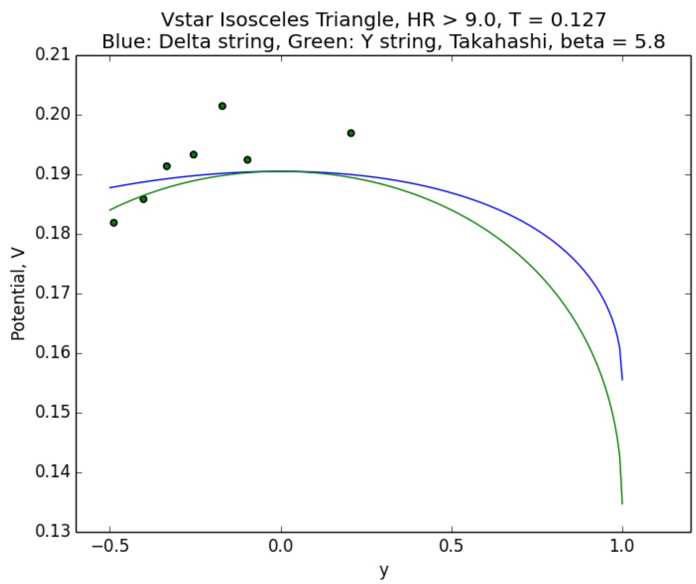
figure). We find that $A_{3q}^{(\mathrm{R})}$ is significantly smaller than $A_{q\bar{q}}/2 = 0.170$ about 26%, namely,

$$A_{3q}^{(\mathrm{R})} = \frac{A_{q\bar{q}}}{2}(1 - 0.259),$$

This variation is on a lattice at a single value of beta! How can this be? Is our Coulomb+constant+ confinement Ansatz valid?

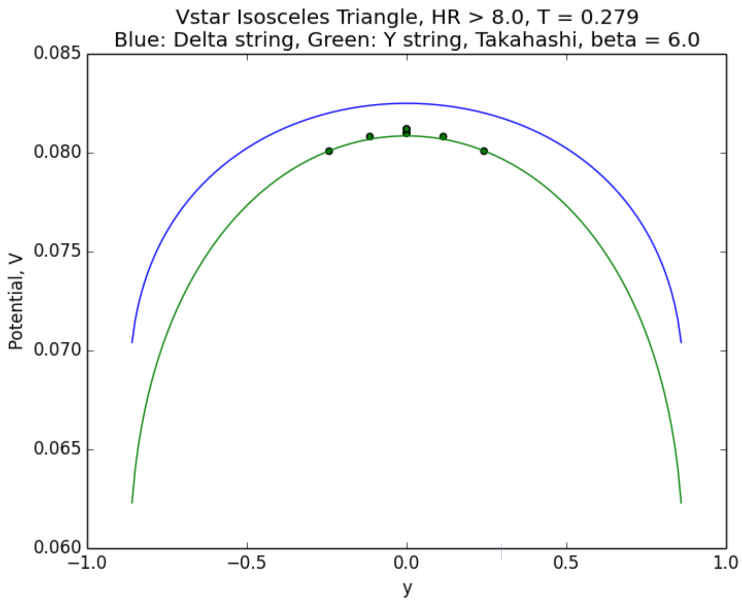
Takahashi (2002) results (beta=5.8)

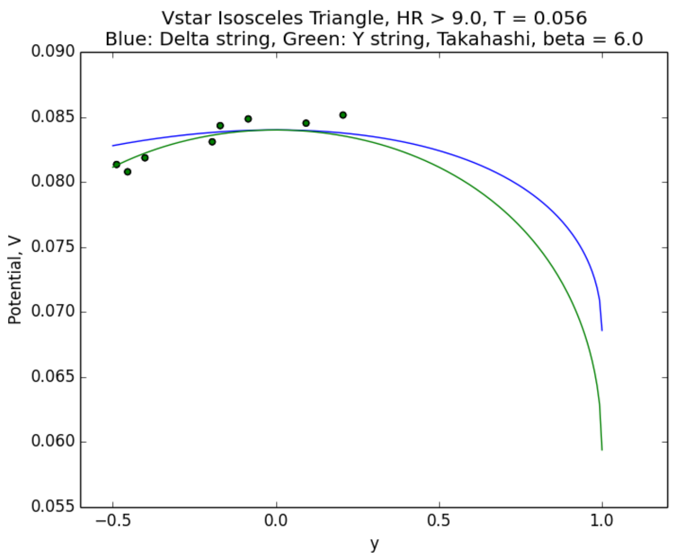




Disagreement with both Y- and Delta string!
(huge dispersion on the isosceles line)

Takahashi (2002) results (beta = 6.0)





The only example of agreement with Y string

Remember that hyper-angular dependences must not depend on beta (scaling)! This discrepancy is a sign of underestimated error bars.