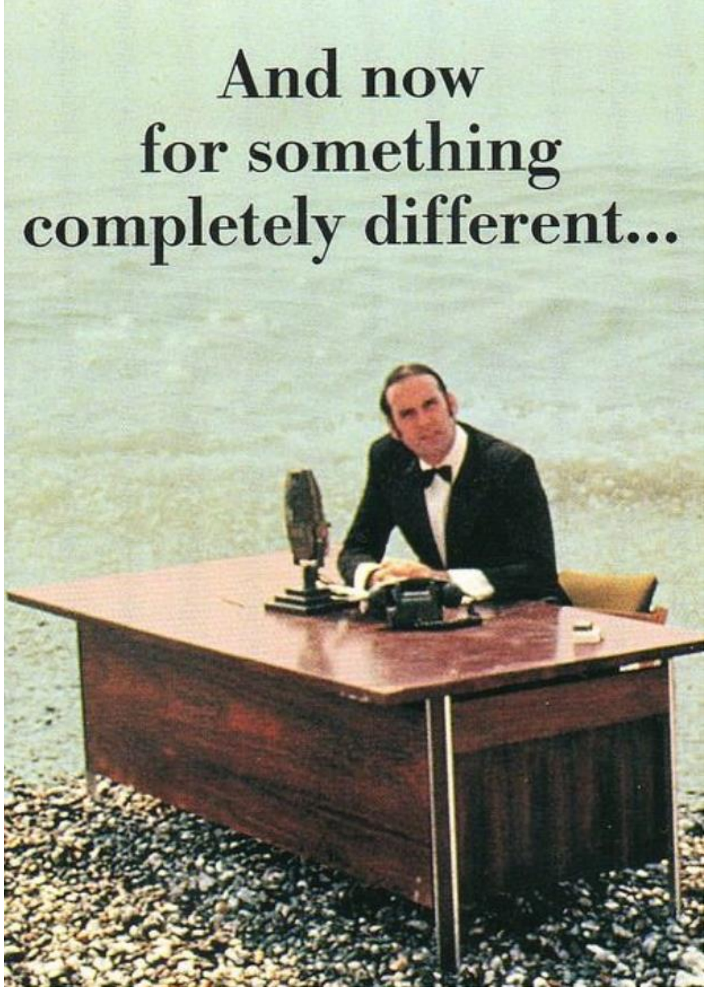
Excited QCD 2020

Excited QCD 2020



O(6) harmonics in the three-heavy-quark problem

I. Salom and V. Dmitrašinović

Institute of Physics,
University of Belgrade

Outline

- Why would we care about SO(6) harmonics?
- How do we treat QM 2-particle problem?
- How we want to treat QM 3-particle problem?
- What should be 3-particle analogs of sp. harmonics?
- How do we find them?
- How to apply them?
  - calculate matrix elements
  - help us solve Schrodinger’s equation
  - even treat some relativistic cases

Why should we care?

- There are a lot of reasons to care for ordinary SO(3) spherical harmonics, yet their importance stems from QM two-particle problem
- SO(6) harmonics are to three-body problem what SO(3) harmonics are to two-body problem
- Everybody knows SO(3) sp. harmonics, yet most have not heard of SO(6) harmonics!?

Solving two particle problems

- Typical example – Hydrogen atom
- Using center-of-mass reference system where a single 3-dim vector determines position
- Split wave function into radial and angular parts
- Using basis of spherical harmonics for the angular wave function (essential)!
- Knowledge of matrix elements required for perturbative corrections/transitions

Goal in 3-particle case

- Use c.m. system, reducing number of fr. deg. from 9 to 6
- Split the problem into radial and hyper-angular parts
- Solve angular part by decomposition to (hyper)spherical harmonics!
- Additional requirements/wanted properties:
  - Harmonics provide manifest permutation and rotation properties
  - Account for certain special dynamical symmetries
- Be able to evaluate matrix elements (integrals of 3 h.s. harmonics)
- $\Rightarrow$ applications: three quark systems, molecular physics, atomic physics (helium atom), positronium ion…

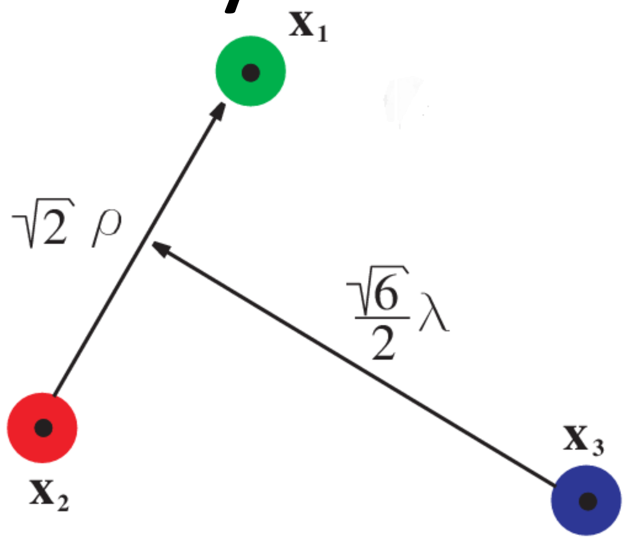
Center-of-mass system

- Jacobi coordinates:

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{x_1} - \mathbf{x_2}),$$

$$\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{x_1} + \mathbf{x_2} - 2\mathbf{x_3})$$

In the case of different masses coordinates are more complicated



$$x_\mu = (\rho_1, \rho_2, \rho_3, \lambda_1, \lambda_2, \lambda_3), \qquad \mu = 1, 2, 3, 4, 5, 6.$$

- Non-relativistic energy – SO(6) invariant:

$$T = \frac{m}{2}\left(\dot{\boldsymbol{\rho}}^2 + \dot{\boldsymbol{\lambda}}^2\right) = \sum_{\mu=1}^{6} \frac{m}{2}\left(\dot{x}_\mu\right)^2$$

$$K_{\mu\nu} \equiv i(x_\mu \partial_\nu - x_\nu \partial_\mu) | \mu, \nu = 1, \ldots 6$$

$$R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2} = \sqrt{\sum_\mu x_\mu^2}$$

⇨

$$T = \frac{m}{2}\dot{R}^2 + \frac{K_{\mu\nu}^2}{2mR^2}$$

6 dim spherical harmonics = ???

- Let us recall a few facts about standard 3D s.h.
  - Functions on sphere: $\mathcal{Y}^J_m$

UIR of $SO(3)$

UIR of $SO(2) \subset SO(3)$

$$\boldsymbol{J}^2 \mathcal{Y}^J_m = J(J+1)\mathcal{Y}^J_m \qquad J_3 \mathcal{Y}^J_m = m\mathcal{Y}^J_m$$

  - Orthogonal:

$$\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \mathcal{Y}^J_m \mathcal{Y}^{J'*}_{m'} d\Omega = \delta_{JJ'}\delta mm'$$

$$\boldsymbol{\nabla}^2 \mathcal{P}^J_m = 0$$

  - E.g.:

$$Y_1^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta \quad = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot\frac{(x-iy)}{r}$$

$$Y_1^{0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\cos\theta \quad = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\frac{z}{r}$$

$$Y_1^{1}(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta \quad = \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\cdot\frac{(x+iy)}{r}$$

$$Y_2^{-2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta \quad = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x-iy)^2}{r^2}$$

$$Y_2^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot\cos\theta \quad = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x-iy)z}{r^2}$$

$$Y_2^{0}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot(3\cos^2\theta-1) \quad = \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot\frac{(2z^2-x^2-y^2)}{r^2}$$

$$Y_2^{1}(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot\cos\theta \quad = \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x+iy)z}{r^2}$$

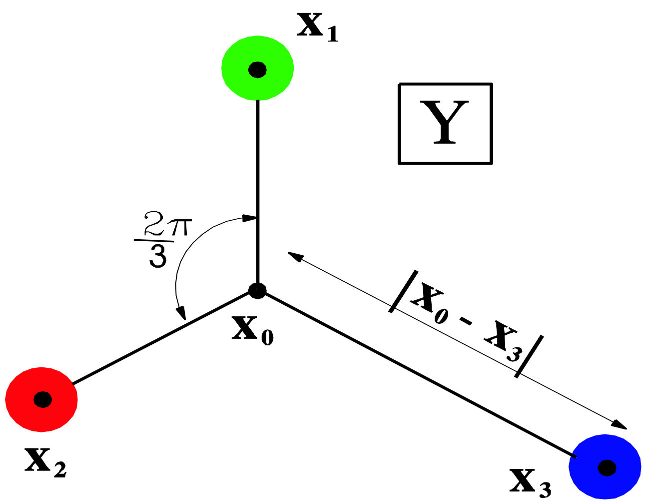
$$Y_2^{2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta \quad = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x+iy)^2}{r^2}$$

D-dim hyper-spherical harmonics

- Intuitively: natural basis for functions on D-dim sphere
- Homogenous harmonic polynomials (obeying Laplace eq.) of order *K* in Cartesian coordinates x_μ restricted to unit sphere
- Harmonics of order *K* are further **labeled** by appropriate quantum numbers, usually related to SO(D) subgroups
- For 3-particles, there are many wrong but only one symmetrically/mathematically proper way to choose labels!

U $$\boxed{[Q, \boldsymbol{\rho} \times \boldsymbol{\lambda}] = 0}$$ $SO(6)$

- Co $Q \equiv ($ … $= 0$ $1, 2, 3.$



Y-string potential = the shortest sum of string lengths ← function of triangle area

$J_{ij} \equiv$ … $),$

Q_i … $,$

SO(3) rotations

U(3)

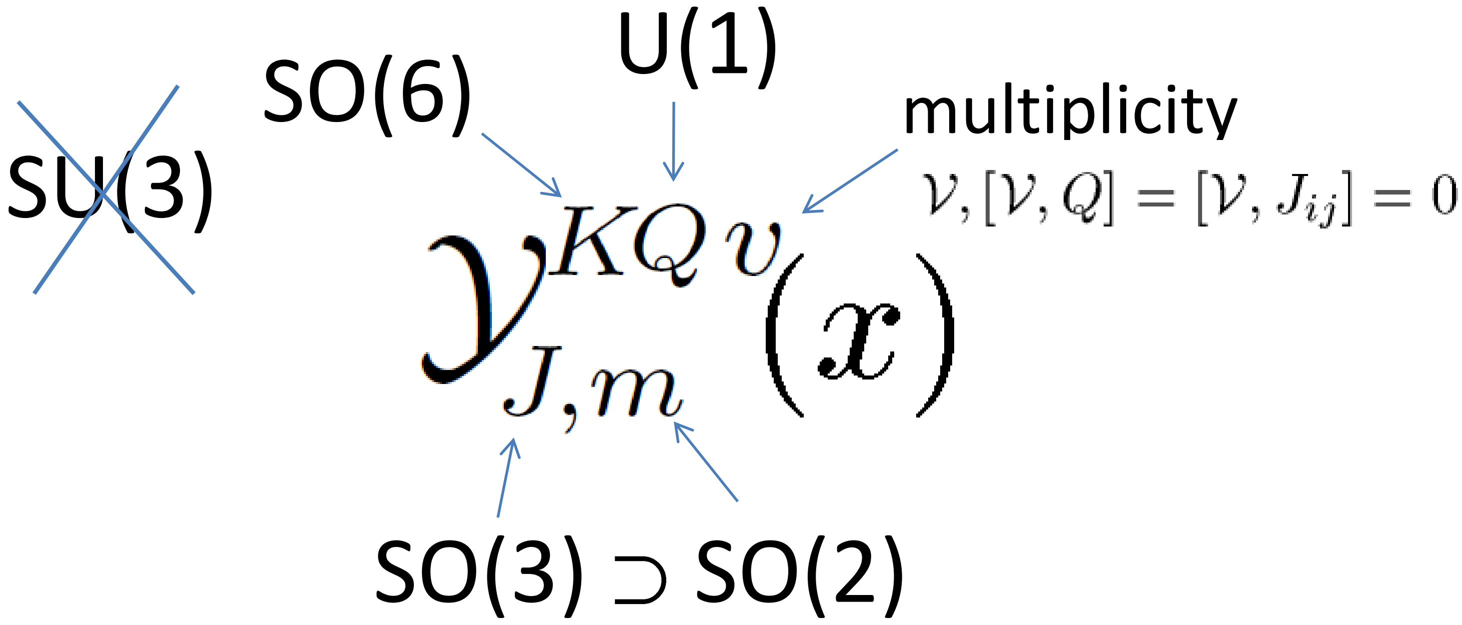
SO(6)

$$\Delta J_{ij} \equiv i(X_i^+ \frac{\partial}{\partial X_j^-} + X_i^- \frac{\partial}{\partial X_j^+} - X_j^+ \frac{\partial}{\partial X_i^-} - X_j^- \frac{\partial}{\partial X_i^+}),$$

$$W_{ij} \equiv X_i^+ \frac{\partial}{\partial X_j^-} - X_i^- \frac{\partial}{\partial X_j^+} - X_j^+ \frac{\partial}{\partial X_i^-} + X_j^- \frac{\partial}{\partial X_i^+}.$$

Quantum numbers

- Labels of SO(6) hyper-spherical harmonics



~~SU(3)~~

SO(6)

U(1)

multiplicity

$\mathcal{V}, [\mathcal{V}, Q] = [\mathcal{V}, J_{ij}] = 0$

$$\mathcal{Y}^{KQ\,v}_{J,m}(x)$$

SO(3) ⊃ SO(2)

$$U(1) \otimes SO(3)_{rot} \subset U(3) \subset SO(6)$$

Long way to the explicit expressions…

SKIP

- Building blocks – two SO(3) vectors $\boldsymbol{X}^+$ and $\boldsymbol{X}^-$
- Start from polynomials sharp in Q:

$$\mathcal{P}^{d_+d_-}_{J_+m_+J_-m_-}(X) = (\boldsymbol{X}^+\cdot\boldsymbol{X}^+)^{\frac{d_+-J_+}{2}}\tilde{y}^{J_+}_{3,m_+}(X^+)(\boldsymbol{X}^-\cdot\boldsymbol{X}^-)^{\frac{d_--J_-}{2}}\tilde{y}^{J_-}_{3,m_-}(X^-)$$

- Define “core polynomials” sharp in J, m and Q:

$$\mathcal{P}^{\overline{K}Q}_{(J_+J_-)J,m}(X) \equiv \sum_{m_+,m_-} C^{J_+J_-J}_{m_+m_-m}\mathcal{P}^{\frac{\overline{K}+Q}{2}\frac{\overline{K}-Q}{2}}_{J_+m_+J_-m_-}(X)$$

- Make them harmonic by finding ortho-complement w.r.t. polynomials with lesser K, *i.e.:*

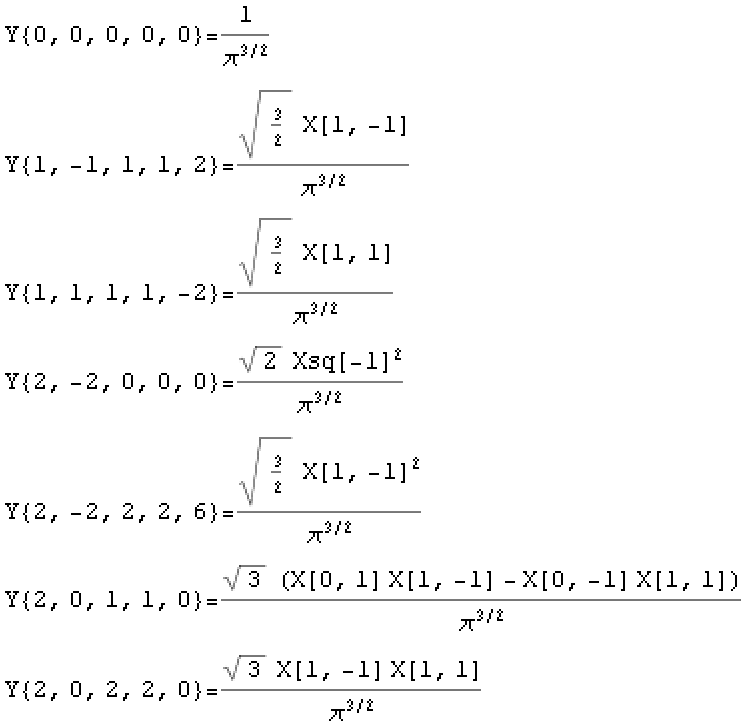
$$\mathcal{P}\mathcal{H}^{\overline{K}Q}_{(J_+J_-)J,m}(X) = \mathcal{P}^{\overline{K}Q}_{(J_+J_-)J,m}(X) - \sum_a c_a R^{\overline{K}-K_a}\mathcal{P}_a(X),$$

- Finally, remove remaining degeneracy, i.e. introduce multiplicity label.

After all that we can…

…explicitly calculate the harmonics e.g. in Wolfram Mathematica…

$\mathcal{Y}\{K, Q, J, m, \nu\}$



$$\mathrm{Y}\{0,0,0,0,0\}=\frac{1}{\pi^{3/2}}$$

$$\mathrm{Y}\{1,-1,1,1,2\}=\frac{\sqrt{\frac{3}{2}}\,\mathrm{X}[1,-1]}{\pi^{3/2}}$$

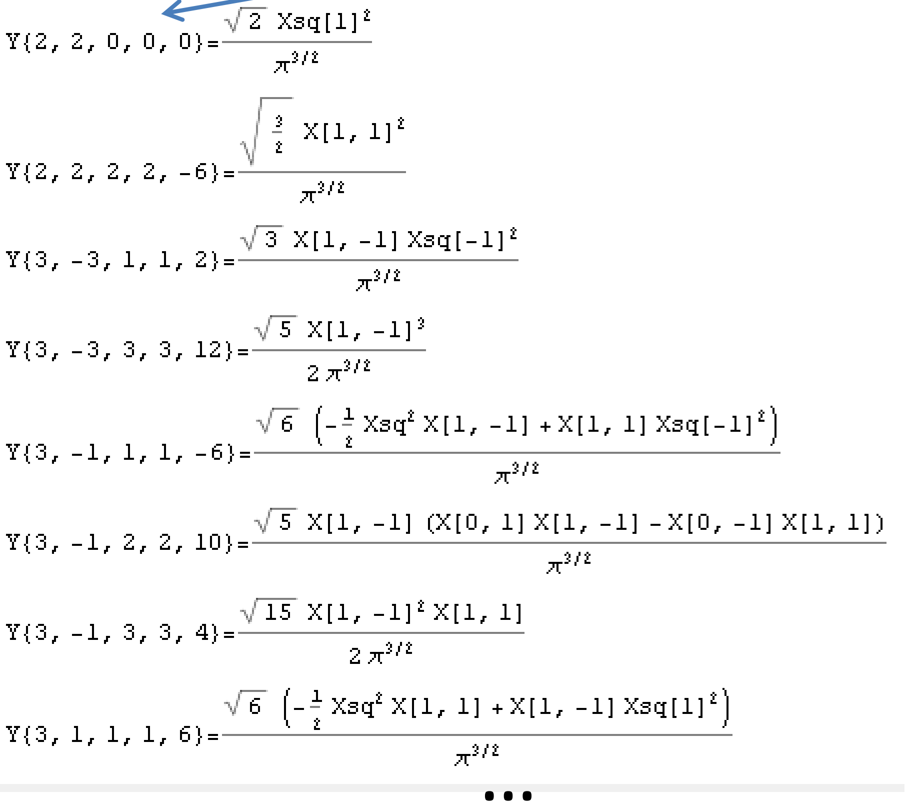
$$\mathrm{Y}\{1,1,1,1,-2\}=\frac{\sqrt{\frac{3}{2}}\,\mathrm{X}[1,1]}{\pi^{3/2}}$$

$$\mathrm{Y}\{2,-2,0,0,0\}=\frac{\sqrt{2}\,\mathrm{Xsq}[-1]^2}{\pi^{3/2}}$$

$$\mathrm{Y}\{2,-2,2,2,6\}=\frac{\sqrt{\frac{3}{2}}\,\mathrm{X}[1,-1]^2}{\pi^{3/2}}$$

$$\mathrm{Y}\{2,0,1,1,0\}=\frac{\sqrt{3}\,(\mathrm{X}[0,1]\,\mathrm{X}[1,-1]-\mathrm{X}[0,-1]\,\mathrm{X}[1,1])}{\pi^{3/2}}$$

$$\mathrm{Y}\{2,0,2,2,0\}=\frac{\sqrt{3}\,\mathrm{X}[1,-1]\,\mathrm{X}[1,1]}{\pi^{3/2}}$$



$$\mathrm{Y}\{2,2,0,0,0\}=\frac{\sqrt{2}\,\mathrm{Xsq}[1]^2}{\pi^{3/2}}$$

$$\mathrm{Y}\{2,2,2,2,-6\}=\frac{\sqrt{\frac{3}{2}}\,\mathrm{X}[1,1]^2}{\pi^{3/2}}$$

$$\mathrm{Y}\{3,-3,1,1,2\}=\frac{\sqrt{3}\,\mathrm{X}[1,-1]\,\mathrm{Xsq}[-1]^2}{\pi^{3/2}}$$

$$\mathrm{Y}\{3,-3,3,3,12\}=\frac{\sqrt{5}\,\mathrm{X}[1,-1]^3}{2\,\pi^{3/2}}$$

$$\mathrm{Y}\{3,-1,1,1,-6\}=\frac{\sqrt{6}\left(-\frac{1}{2}\,\mathrm{Xsq}^2\,\mathrm{X}[1,-1]+\mathrm{X}[1,1]\,\mathrm{Xsq}[-1]^2\right)}{\pi^{3/2}}$$

$$\mathrm{Y}\{3,-1,2,2,10\}=\frac{\sqrt{5}\,\mathrm{X}[1,-1]\,(\mathrm{X}[0,1]\,\mathrm{X}[1,-1]-\mathrm{X}[0,-1]\,\mathrm{X}[1,1])}{\pi^{3/2}}$$

$$\mathrm{Y}\{3,-1,3,3,4\}=\frac{\sqrt{15}\,\mathrm{X}[1,-1]^2\,\mathrm{X}[1,1]}{2\,\pi^{3/2}}$$

$$\mathrm{Y}\{3,1,1,1,6\}=\frac{\sqrt{6}\left(-\frac{1}{2}\,\mathrm{Xsq}^2\,\mathrm{X}[1,1]+\mathrm{X}[1,-1]\,\mathrm{Xsq}[1]^2\right)}{\pi^{3/2}}$$

…

Nice permutation properties:

$$\mathcal{Y}^{K|Q|v}_{J,m,\pm}(\boldsymbol{\lambda},\boldsymbol{\rho}) \equiv \frac{1}{\sqrt{2}}\left(\mathcal{Y}^{K|Q|v}_{J,m}(\boldsymbol{\lambda},\boldsymbol{\rho}) \pm (-1)^{K-J}\mathcal{Y}^{K,-|Q|,v'}_{J,m}(\boldsymbol{\lambda},\boldsymbol{\rho})\right)$$

1) Transposition $\mathcal{T}_{12}$ is pure sign: $\mathcal{Y}^{K|Q|v}_{J,m,\pm}(\boldsymbol{\lambda},\boldsymbol{\rho}) \rightarrow \pm\mathcal{Y}^{K|Q|v}_{J,m,\pm}(\boldsymbol{\lambda},\boldsymbol{\rho})$

2) $Q \not\equiv 0 \pmod 3 \Rightarrow \mathcal{Y}^{K|Q|v}_{J,m,\pm}(\boldsymbol{\lambda},\boldsymbol{\rho}) \in \text{mixed representation M}$

3) $Q \equiv 0 \pmod 3 \Rightarrow \mathcal{Y}^{K|Q|v}_{J,m,+}(\boldsymbol{\lambda},\boldsymbol{\rho}) \in \text{symmetric representation S}$

$\mathcal{Y}^{K|Q|v}_{J,m,-}(\boldsymbol{\lambda},\boldsymbol{\rho}) \in \text{antisymmetric representation A}$

$$S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset O(6)$$

Now we can solve Schrodinger eq. by h.s. harmonics decomposition

- Decompose pot. energy $V(R,\alpha,\phi) = V(R)V(\alpha,\phi)$ into h.s.h:

$$V(\alpha,\phi) = \sum_{\mathrm{K},Q}^{\infty} v_{\mathrm{K},Q}^{3-\mathrm{body}} \mathcal{Y}_{00}^{\mathrm{K}Q\nu}(\alpha,\phi) \qquad v_{\mathrm{K},Q}^{3-\mathrm{body}} = \int \mathcal{Y}_{00}^{\mathrm{K}Q\nu *}(\Omega_5) V(\alpha,\phi) d\Omega_{(5)}$$

- Schrodinger equation –> coupled d.e. in $\psi_{[\mathrm{m}]}^{\mathrm{K}}(R)$:

$$-\frac{1}{2\mu}\left[\frac{d^2}{dR^2} + \frac{5}{R}\frac{d}{dR} - \frac{\mathrm{K}(\mathrm{K}+4)}{R^2} + 2\mu E\right] \psi_{[\mathrm{m}]}^{\mathrm{K}}(R) + V_{\mathrm{eff.}}(R) \sum_{\mathrm{K}',[\mathrm{m}']} C_{[\mathrm{m}]\,[\mathrm{m}']}^{\mathrm{K}\ \ \mathrm{K}'} \psi_{[\mathrm{m}']}^{\mathrm{K}'}(R) = 0$$

where:

Matrix elements

$$C_{[\mathrm{m}'']\,[\mathrm{m}']}^{\mathrm{K}''\ \mathrm{K}'} = \delta_{\mathrm{K}'',\mathrm{K}'} \delta_{[\mathrm{m}''],[\mathrm{m}']} + \pi\sqrt{\pi} \sum_{\mathrm{K}>0,Q}^{\infty} \frac{v_{\mathrm{K},Q}^{3-\mathrm{body}}}{v_{00}^{3-\mathrm{body}}} \langle \mathcal{Y}_{[\mathrm{m}'']}^{\mathrm{K}''}(\Omega_5) | \mathcal{Y}_{00}^{\mathrm{K}Qv}(\alpha,\phi) | \mathcal{Y}_{[\mathrm{m}']}^{\mathrm{K}'}(\Omega_5) \rangle$$

We can evaluate matrix elements!

$$\langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5) | \mathcal{Y}^{\mathrm{K}Qv}_{00}(\alpha, \phi) | \mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5) \rangle = \frac{1}{\sqrt{\pi^3}} \sqrt{\frac{\dim(\mathrm{K}, Q)\dim(\mathrm{K}', Q')}{\dim(\mathrm{K}'', Q'')}}$$

$$\times\, \delta_{L', L''} \delta_{m', m''} C_r{}^{\{\mathrm{K}, Q\}}_{\{0,0\}}{}^{\{\mathrm{K}', Q'\}}_{\{L', v'\}}{}^{\{\mathrm{K}'', Q''\}}_{\{L'', v''\}} C^{\{\mathrm{K}, Q\}}_{0_H}{}^{\{\mathrm{K}', Q'\}}_{0_H}{}^{\{\mathrm{K}'', Q''\}}_{0_H},$$

where:

$$C^{\{\mathrm{K}_1, Q_1\}}_{0_H}{}^{\{\mathrm{K}_2, Q_2\}}_{0_H}{}^{\{\mathrm{K}, Q\}}_{0_H} = \Bigg(A_0^{\mathrm{K}_1, Q} A_0^{\mathrm{K}_2, Q} A_0^{\mathrm{K}, Q} \sqrt{\frac{\pi^3 \dim(\mathrm{K}, Q)}{\dim(\mathrm{K}_1, Q_1)\dim(\mathrm{K}_2, Q_2)}}$$

$$\times \sum_{\mathrm{K}_1' = |Q_1|, |Q_1|+2, \ldots}^{\mathrm{K}_1} \sum_{\mathrm{K}_2' = |Q_2|, |Q_2|+2, \ldots}^{\mathrm{K}_2} \sum_{\mathrm{K}' = |Q|, |Q|+2, \ldots}^{\mathrm{K}} \Pi_{\mathrm{K}_1'}^{\mathrm{K}_1, Q_1} \Pi_{\mathrm{K}_2'}^{\mathrm{K}_2, Q_2} \Pi_{\mathrm{K}'}^{\mathrm{K}, -Q}$$

$$\times \frac{2\pi^3}{(\frac{\mathrm{K}_1' + \mathrm{K}_2' + \mathrm{K}'}{2} + 1)(\frac{\mathrm{K}_1' + \mathrm{K}_2' + \mathrm{K}'}{2} + 2)} \delta_{Q_1 + Q_2, Q} \Bigg)^{\frac{1}{2}}$$

$$\Pi_{\mathrm{K}'}^{\mathrm{K}, Q} = \prod_{\mathrm{K}'' = |Q|, |Q|+2, \ldots}^{\mathrm{K}' - 2} \left(1 - \frac{(\mathrm{K} + 2)^2 - Q^2}{(\mathrm{K}'' + 2)^2 - Q^2} \right)$$

$$A_0^{\mathrm{K}, Q} = (-1)^{\frac{\mathrm{K} - |Q|}{2}} \left(\sum_{\mathrm{K}_1, \mathrm{K}_2 = |Q|, |Q|+2, \ldots}^{\mathrm{K}} \Pi_{\mathrm{K}_1}^{\mathrm{K}, Q} \Pi_{\mathrm{K}_2}^{\mathrm{K}, Q} \frac{2\pi^3}{(\frac{\mathrm{K}_1 + \mathrm{K}_2}{2} + 1)(\frac{\mathrm{K}_1 + \mathrm{K}_2}{2} + 2)} \right)^{-\frac{1}{2}}$$

Realistic potentials for identical particles have only few harmonics!

$$v^{3-\text{body}}_{\text{K},Q} = \int \mathcal{Y}^{\text{K},Q,\nu *}_{0,0}(\Omega_5)\, V_{3-\text{body}}(\alpha,\phi)\; d\Omega_{(5)}$$

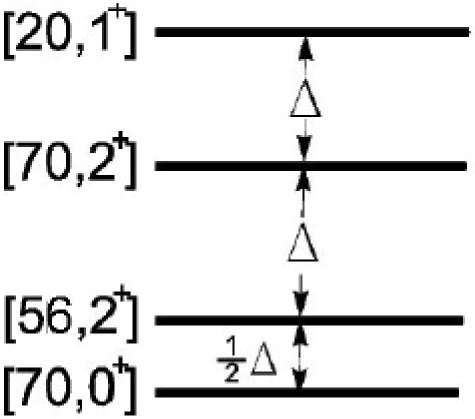
4 out of 2366 possible K<11 !

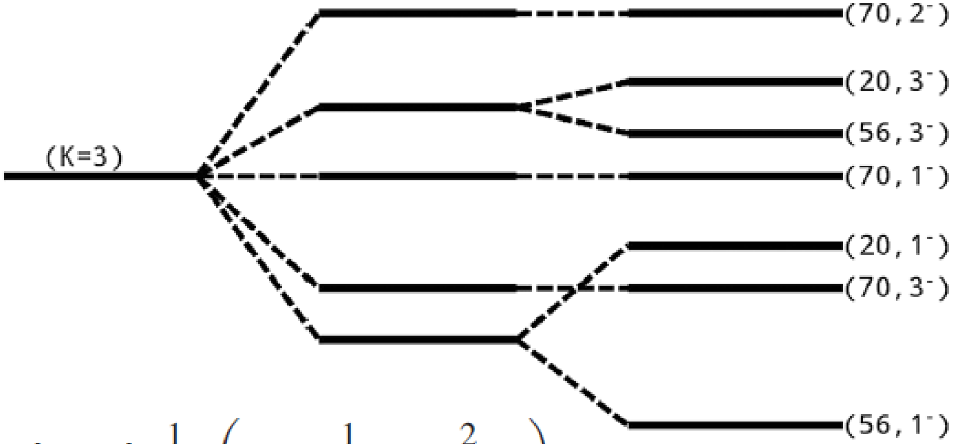
(K,Q)	$v^{\text{Y}}_{\text{K}Q}$	$v^{\Delta}_{\text{K}Q}$	$v^{\text{Coulomb}}_{\text{K}Q}$
(0,0)	8.18	16.04	20.04
(4,0)	-0.44	-0.44	2.95
(6,±6)	0	-0.14	1.88
(8,0)	-0.09	-0.06	1.49
$\sum \frac{(v^{3-\text{body}}_{\text{K},Q})^2}{(\int (V_{3-\text{body}})^2\, d\Omega_{(5)})}$	99%	99%	94%

- Energy and ordering of the states depend only on a few coefficients!

State orderings

State ordering for K=2 and K=3:





$$[20,1^+]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{1}{\sqrt{3}}v_{40}\right)$$

$$[70,0^+]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{\sqrt{3}}v_{40}\right)$$

$$[70,2^+]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{1}{5\sqrt{3}}v_{40}\right)$$

$$[56,2^+]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{\sqrt{3}}{5}v_{40}\right),$$

$$[20,1^-]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{\sqrt{3}}v_{40}-\frac{2}{7}v_{66}\right)$$

$$[56,1^-]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{\sqrt{3}}v_{40}+\frac{2}{7}v_{66}\right)$$

$$[70,1^-]\,\frac{1}{\pi\sqrt{\pi}}(v_{00})$$

$$[70,2^-]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{1}{\sqrt{3}}v_{40}\right)$$

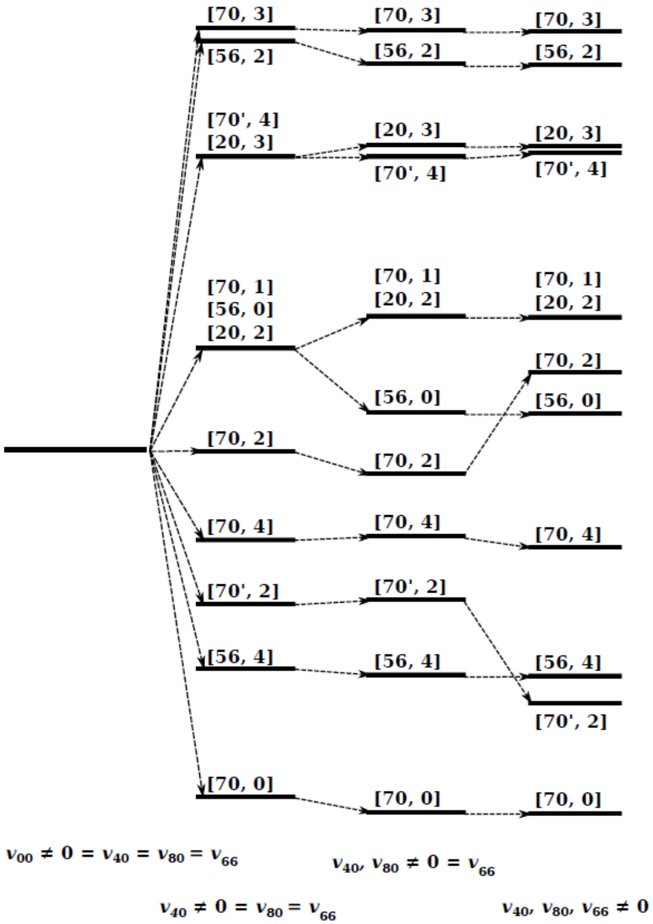
$$[70,3^-]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{1}{\sqrt{3}}v_{40}\right)$$

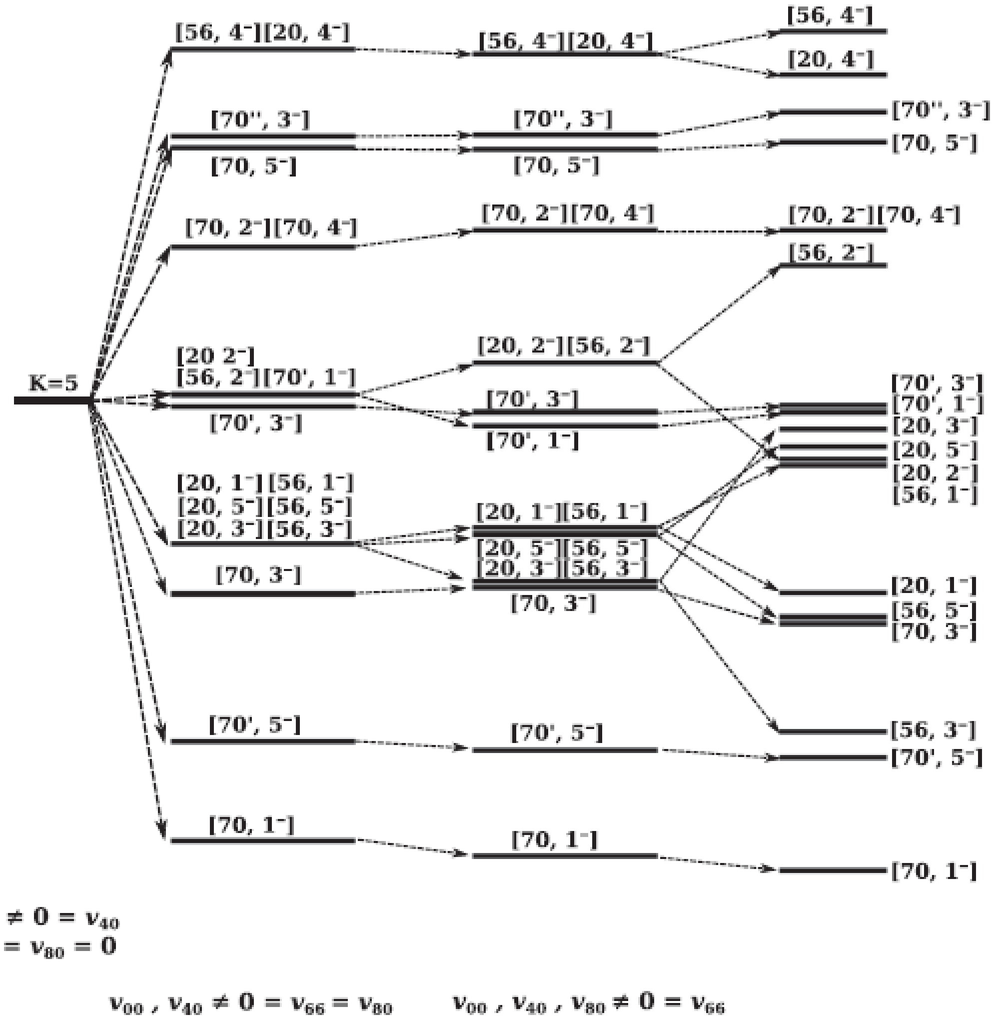
$$[20,3^-]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{\sqrt{3}}{7}v_{40}-v_{66}\right)$$

$$[56,3^-]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{\sqrt{3}}{7}v_{40}+v_{66}\right),$$

State orderings

Delta potential state ordering for K=4 and K=5:





We can also treat some relativistic cases!

- Semi-relativistic three-quark Hamiltonian:

$$H \;=\; \sum_a \sqrt{m_a^2 + \mathbf{p}_i^2} + V_{3b}\left(|\boldsymbol{\rho}|, |\boldsymbol{\lambda}|, \boldsymbol{\rho}\cdot\boldsymbol{\lambda}\right)$$

- Harmonic oscillator potential:

$$V_{3b}\left(|\boldsymbol{\rho}|, |\boldsymbol{\lambda}|, \boldsymbol{\rho}\cdot\boldsymbol{\lambda}\right) = V_{\mathrm{HO}} = \frac{k}{2}\left(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2\right)$$

- Not too realistic (nor covariant) but good as a toy model and basis for perturbation calculus.
- In momentum picture we get a common form of Schrodinger’s equation!

Ultrarelativistic case

- In CM frame, due to $\sum \mathbf{p}_i = 0$ we can use Jacobi coordinates for momenta!
- Ultrarelativistic limit: $\tilde{T} = \sum_{i=1}^{3} |\mathbf{p}_i|$
- Almost becomes Delta pot.: $V = \frac{1}{\sqrt{3}} V_{\Delta}(\rho \leftrightarrow \lambda)$

(K, Q)	$v_{KQ}(\mathrm{Y-string})$	$v_{KQ}(\Delta)$	$v_{KQ}(\mathrm{urHO})$	$v_{KQ}(\mathrm{Coulomb})$	$v_{KQ}(\mathrm{Log})$
(0,0)	8.22	16.04	16.04	20.04	-6.58
(4,0)	-0.398	- 0.445	- 0.445	2.93	-1.21
(6,±6)	-0.027	- 0.14	0.14	1.88	-0.56
(8,0)	-0.064	- 0.04	- 0.04	1.41	-0.33
(12,0)	-0.01	0	0	0	-0.17

To sum up:

- O(6) h.s.h are to three-particle problem what ordinary s.h. are to two-particle problem
- proper labels are $\mathcal{Y}^{K|Q|v}_{J,m,\pm}$
- tables of explicit expressions available
- matrix elements available
- accounting for only few terms effectively solves Schrodinger equation
- help differentiating Δ and Y

Talk based on:

V. Dmitrašinović, Igor Salom, “O(6) algebraic theory of three nonrelativistic quarks bound by spin-independent interactions”, PHYSICAL REVIEW D 97, 094011 (2018), Pages 094011-1-

Igor Salom, V. Dmitrašinović, “Permutation-symmetric three-particle hyper-spherical harmonics based on the $\mathrm{S3} \otimes \mathrm{SO(3)rot} \subset \mathrm{O(2)}\otimes\mathrm{SO(3)rot} \subset \mathrm{U(3)}\rtimes\mathrm{S2} \subset \mathrm{O(6)}$ subgroup chain”, Nuclear Physics B, Volume 920, July 2017, Pages 521-564, ISSN 0550-3213,

Igor Salom and Veljko Dmitrašinović, "O(6) algebraic approach to three bound identical particles in the hyperspherical adiabatic representation", Physics Letters A, Volume 380, Issues 22–23, 20 May 2016, Pages 1904–1911, doi:10.1016/j.physleta.2016.04.008

Veljko Dmitrasinovic and Igor Salom, "SO(4) algebraic approach to the three-body bound state problem in two dimensions", J. Math. Phys. 55, 082105 (2014), DOI: 10.1063/1.4891399

Thank you

Excited QCD 2020

O(6) harmonics in the three-heavy-quark problem

I. Salom and V. Dmitrašinović

Institute of Physics, University of Belgrade

Long way to the explicit expressions…

- Building blocks – two SO(3) vectors $\boldsymbol{X}^+$ and $\boldsymbol{X}^-$
- Start from polynomials sharp in Q:

$$\mathcal{P}^{d_+ d_-}_{J_+ m_+ J_- m_-}(X) = \left(\boldsymbol{X}^+ \cdot \boldsymbol{X}^+\right)^{\frac{d_+ - J_+}{2}} \tilde{\mathcal{Y}}^{J_+}_{3,m_+}(X^+) \left(\boldsymbol{X}^- \cdot \boldsymbol{X}^-\right)^{\frac{d_- - J_-}{2}} \tilde{\mathcal{Y}}^{J_-}_{3,m_-}(X^-)$$

- Define “core polynomials” sharp in J, m and Q:

$$\mathcal{P}^{\overline{K}Q}_{(J_+J_-)J,m}(X) \equiv \sum_{m_+,m_-} C^{J_+ J_- J}_{m_+ m_- m} \mathcal{P}^{\frac{\overline{K}+Q}{2}\frac{\overline{K}-Q}{2}}_{J_+ m_+ J_- m_-}(X)$$

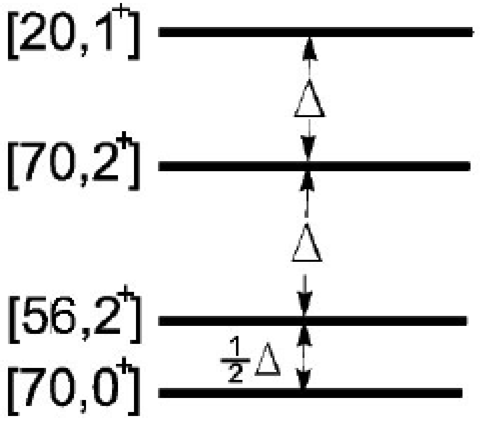
- Make them harmonic by finding ortho-complement w.r.t. polynomials with lesser *K, i.e.:*

$$\mathcal{P}_{\mathcal{H}}{}^{\overline{K}Q}_{(J_+J_-)J,m}(X) = \mathcal{P}^{\overline{K}Q}_{(J_+J_-)J,m}(X) - \sum_a c_a R^{\overline{K}-K_a} \mathcal{P}_a(X),$$

- Finally, remove remaining degeneracy, i.e. introduce multiplicity label.

State orderings

Fixed state ordering for K=2:



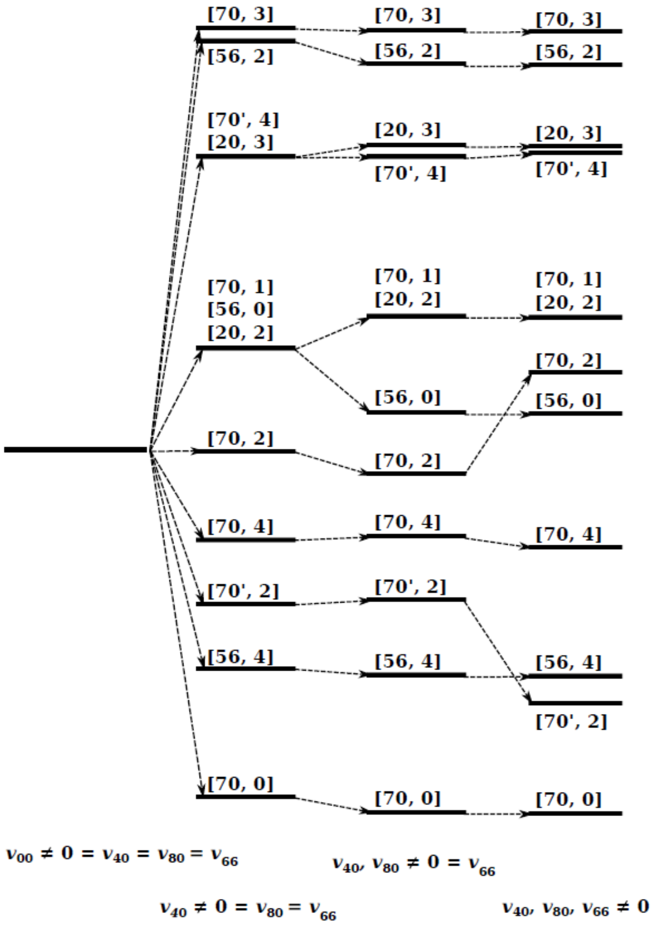
$$[20,1^+]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{1}{\sqrt{3}}v_{40}\right)$$

$$[70,0^+]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{\sqrt{3}}v_{40}\right)$$

$$[70,2^+]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{1}{5\sqrt{3}}v_{40}\right)$$

$$[56,2^+]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{\sqrt{3}}{5}v_{40}\right),$$

Delta potential state ordering for K=4:



Particle permutations

- Transformations are easily inferred since:

$$\begin{aligned}
\mathcal{T}_{12}: &\quad \boldsymbol{\lambda} \to \boldsymbol{\lambda}, \quad \boldsymbol{\rho} \to -\boldsymbol{\rho}, \\
\mathcal{T}_{23}: &\quad \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} + \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} + \frac{\sqrt{3}}{2}\boldsymbol{\lambda}, \\
\mathcal{T}_{31}: &\quad \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} - \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} - \frac{\sqrt{3}}{2}\boldsymbol{\lambda}.
\end{aligned}$$

$$\mathcal{T}_{12}: X_i^{\pm} \to X_i^{\mp}, \quad \mathcal{T}_{23}: X_i^{\pm} \to e^{\pm\frac{2i\pi}{3}} X_i^{\mp}, \quad \mathcal{T}_{31}: X_i^{\pm} \to e^{\mp\frac{2i\pi}{3}} X_i^{\mp}.$$

$$\mathcal{T}_{ab}: Q \to -Q, K \to K, J_{ij} \to J_{ij}, \nu \to \pm\nu$$

In practice we need matrix elements!

- E.g. potential energy term in Schr. eq. turns into matrix elements of form:

$$\langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5)|\mathcal{Y}^{\mathrm{K}Qv}_{00}(\alpha,\phi)|\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle = \int \mathcal{Y}^{*\mathrm{K}}_{[m]}(\Omega_5)\mathcal{Y}^{\mathrm{K}_1}_{[m_1]}(\Omega_5)\mathcal{Y}^{\mathrm{K}_2}_{[m_2]}(\Omega_5)d\Omega_5$$

- In principle these can be calculated using formula:

$$\int_\Omega \frac{1}{R^6} x_1^{m_1} x_2^{m_2} \cdots x_6^{m_6} d\Omega = 2\frac{\prod_{\mu=1}^{6} \frac{1+(-1)^{m_\mu}}{2}\Gamma(\frac{m_\mu+1}{2})}{\Gamma(3+\sum_\mu m_\mu)}$$

- Pros: integral of any number of h.s. harmonics can be evaluated
- Cons: requires prior calculation of explicit h.s. harmonics expressions, is not fast and is not a **closed form**!

A different group-theoretical viewpoint

- A h.s. harmonic of a compact Lie group *G* on $\mathcal{M}$ is a function that transforms as a basis vector *m* of UIR *L*:

$$g : \mathcal{Y}_m^L(\Omega) \to \sum_{m'} D_{m'm}^L(g)\mathcal{Y}_{m'}^L(\Omega), \qquad g \in G, \Omega \in \mathcal{M}$$

where $D_{m'm}^L(g) = \left\langle {L \atop m'} \middle| D(g) \middle| {L \atop m} \right\rangle$ is a Wigner D-function

- This is already satisfied by (conj.) D-functions on entire G:

$$g' : D_{mk}^{*L}(g) \to D_{mk}^{*L}(g'^{-1}g) = \sum_{m'} D_{mm'}^{*L}{}^{-1}(g')D_{m'k}^{*L}(g) = \sum_{m'} D_{m'm}^{L}(g')D_{m'k}^{*L}(g)$$

- What about more common homogeneous spaces?

H.s. harmonics = Wigner D-functions

- Stabilizer of a point: $H_\Omega \subset G$, $H_\Omega \cdot \Omega = \Omega \;\Rightarrow \mathcal{M} = G/H$
- Choose *H* invariant vector $\left|{}^{L}_{0_H}\right\rangle$:

$$D(h)\left|{}^{L}_{0_H}\right\rangle = \left|{}^{L}_{0_H}\right\rangle, \forall h \in H$$

- Wigner D-function $D^{*L}_{m0_H}(g)$ becomes function on G/H

$$D^{*L}_{m0_H}(g) = D^{*L}_{m0_H}(g(\Omega)h) = \left\langle{}^{L}_{m}\right| D(g(\Omega))D(h)\left|{}^{L}_{0_H}\right\rangle = D^{*L}_{m0_H}(g(\Omega)) \equiv D^{*L}_{m0_H}(\Omega)$$

- After normalization, this is the h.s. harmonic function:

$$\mathcal{Y}^{L}_{m}(\Omega) = \sqrt{\frac{dim(L)}{V_{\mathcal{M}}}} D^{*L}_{m0_H}(\Omega)$$

An important direct consequence:

- Integral of three h.s. harmonics always turns into Clebsch-Gordan coefficients:

$$
\begin{aligned}
&\int\limits_{\mathcal{M}} \mathcal{Y}^{*L}_{m}(\Omega)\mathcal{Y}^{L_1}_{m_1}(\Omega)\mathcal{Y}^{L_2}_{m_2}(\Omega)d\Omega \\
&= \sqrt{\frac{\dim(L)\dim(L_1)\dim(L_2)}{V^3_{\mathcal{M}}}}\int\limits_{\mathcal{M}} D^{L}_{m0_H}(\Omega)D^{*L_1}_{m_1 0_H}(\Omega)D^{*L_2}_{m_2 0_H}(\Omega)d\Omega \\
&= \frac{1}{V_H}\sqrt{\frac{\dim(L)\dim(L_1)\dim(L_2)}{V^3_{\mathcal{M}}}}\int\limits_{G} D^{L}_{m0_H}(g)D^{*L_1}_{m_1 0_H}(g)D^{*L_2}_{m_2 0_H}(g)dg \\
&= \frac{1}{\sqrt{V_{\mathcal{M}}}}\sqrt{\frac{\dim(L_1)\dim(L_2)}{\dim(L)}}\,C^{L_1\;L_2\;L}_{m_1\;m_2\;m}C^{L_1\;L_2\;L}_{0_H\;0_H\;0_H},
\end{aligned}
$$

Back to three particles

- Stabilizer subgroup = SO(5), hyper sphere SO(6)/SO(5)
- The integral turns into SO(6) CG coefficients:

$$\int_{\mathcal{M}} \mathcal{Y}^{*\mathrm{K}}_{[m]}(\Omega_5)\,\mathcal{Y}^{\mathrm{K}_1}_{[m_1]}(\Omega_5)\,\mathcal{Y}^{\mathrm{K}_2}_{[m_2]}(\Omega_5)\,d\Omega_5$$

$$= \frac{1}{\sqrt{V_{\mathcal{M}}}}\sqrt{\frac{\dim(\mathrm{K}_1)\dim(\mathrm{K}_2)}{\dim(\mathrm{K})}}\, C^{\mathrm{K}_1\ \ \mathrm{K}_2\ \ \mathrm{K}}_{[m_1]\,[m_2]\,[m]}\, C^{\mathrm{K}_1\ \ \mathrm{K}_2\ \ \mathrm{K}}_{[0_H]\ [0_H]\ [0_H]}$$

SO(5) subgroup
invariant vector

- Problem: values of SO(6) CG coefficients?

But these are also functions on SU(3)/SU(2) !

- This is seen by considering U(3) action on complex coordinates X^+ and noting isometry subgroup U(2)
- Analogous formula with SU(3) CG coefficients is also valid!

$$\int\limits_{\mathcal{M}} \mathcal{Y}^{*\mathrm{K}Qv}_{L,m}(X)\mathcal{Y}^{\mathrm{K}_1Q_1v_1}_{L_1,m_1}(X)\mathcal{Y}^{\mathrm{K}_2Q_2v_2}_{L_2,m_2}(X)dX^3 =$$

$$= \frac{1}{\sqrt{V_{\mathcal{M}}}}\sqrt{\frac{\dim(\mathrm{K}_1,Q_1)\dim(\mathrm{K}_2,Q_2)}{\dim(\mathrm{K},Q)}}\, C^{\{\mathrm{K}_1,Q_1\}\ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{\{L_1,m_1,v_1\}\{L_2,m_2,v_2\}\{L,m,v\}}\, C^{\{\mathrm{K}_1,Q_1\}\ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{0_H\ \ \ \ 0_H\ \ \ \ 0_H}$$

SU(2) subgroup invariant vector

- SU(3) CG coefficients are available!

For the potential energy matrix elements:

$$\langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5)|\mathcal{Y}^{\mathrm{K}Qv}_{00}(\alpha,\phi)|\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle = \frac{1}{\sqrt{\pi^3}}\sqrt{\frac{\dim(\mathrm{K},Q)\dim(\mathrm{K}',Q')}{\dim(\mathrm{K}'',Q'')}}$$

$$\times\, \delta_{L',L''}\delta_{m',m''}\, C_r{}^{\{\mathrm{K},Q\}\ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\{0,0\}\ \ \{L',v'\}\ \ \{L'',v''\}}\, C^{\{\mathrm{K},Q\}\ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\ 0_H\qquad 0_H\qquad 0_H},$$

where:

$$C^{\{\mathrm{K}_1,Q_1\}\ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{\ 0_H\qquad 0_H\qquad 0_H} = \Big(A_0^{\mathrm{K}_1,Q}A_0^{\mathrm{K}_2,Q}A_0^{\mathrm{K},Q}\sqrt{\frac{\pi^3\dim(\mathrm{K},Q)}{\dim(\mathrm{K}_1,Q_1)\dim(\mathrm{K}_2,Q_2)}}$$

$$\times \sum_{\mathrm{K}_1'=|Q_1|,|Q_1|+2,\ldots}^{\mathrm{K}_1}\ \sum_{\mathrm{K}_2'=|Q_2|,|Q_2|+2,\ldots}^{\mathrm{K}_2}\ \sum_{\mathrm{K}'=|Q|,|Q|+2,\ldots}^{\mathrm{K}} \Pi^{\mathrm{K}_1,Q_1}_{\mathrm{K}_1'}\Pi^{\mathrm{K}_2,Q_2}_{\mathrm{K}_2'}\Pi^{\mathrm{K},-Q}_{\mathrm{K}'}$$

$$\times \frac{2\pi^3}{(\frac{\mathrm{K}_1'+\mathrm{K}_2'+\mathrm{K}'}{2}+1)(\frac{\mathrm{K}_1'+\mathrm{K}_2'+\mathrm{K}'}{2}+2)}\delta_{Q_1+Q_2,Q}\Big)^{\frac{1}{2}}$$

$$\Pi^{\mathrm{K},Q}_{\mathrm{K}'} = \prod_{\mathrm{K}''=|Q|,|Q|+2,\ldots}^{\mathrm{K}'-2}\left(1-\frac{(\mathrm{K}+2)^2-Q^2}{(\mathrm{K}''+2)^2-Q^2}\right)$$

$$A_0^{\mathrm{K},Q} = (-1)^{\frac{\mathrm{K}-|Q|}{2}}\left(\sum_{\mathrm{K}_1,\mathrm{K}_2=|Q|,|Q|+2,\ldots}^{\mathrm{K}} \Pi^{\mathrm{K},Q}_{\mathrm{K}_1}\Pi^{\mathrm{K},Q}_{\mathrm{K}_2}\frac{2\pi^3}{(\frac{\mathrm{K}_1+\mathrm{K}_2}{2}+1)(\frac{\mathrm{K}_1+\mathrm{K}_2}{2}+2)}\right)^{-\frac{1}{2}}$$

Now we can solve problems by h.s. harmonics decomposition

- Schrodinger equation – coupled d.e. in $\psi^{\mathrm{K}}_{[\mathrm{m}]}(R)$:

$$- \ \frac{1}{2\mu}\left[\frac{d^2}{dR^2}+\frac{5}{R}\frac{d}{dR}-\frac{\mathrm{K}(\mathrm{K}+4)}{R^2}+2\mu E\right]\psi^{\mathrm{K}}_{[\mathrm{m}]}(R)+V_{\mathrm{eff.}}(R)\sum_{\mathrm{K}',[\mathrm{m}']}C^{\mathrm{K}\ \ \mathrm{K}'}_{[\mathrm{m}][\mathrm{m}']}\psi^{\mathrm{K}'}_{[\mathrm{m}']}(R)=0$$

where:

$$C^{\mathrm{K}''\ \mathrm{K}'}_{[\mathrm{m}''][\mathrm{m}']}=\delta_{\mathrm{K}'',\mathrm{K}'}\delta_{[\mathrm{m}''],[\mathrm{m}']}+\pi\sqrt{\pi}\sum_{\mathrm{K}>0,Q}^{\infty}\frac{v^{3-\mathrm{body}}_{\mathrm{K},Q}}{v^{3-\mathrm{body}}_{00}}\langle\mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5)|\mathcal{Y}^{\mathrm{K}Qv}_{00}(\alpha,\phi)|\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle$$

Realistic potentials for identical particles have only few harmonics!

$$v^{3-\text{body}}_{\mathrm{K},Q} = \int \mathcal{Y}^{\mathrm{K},Q,\nu *}_{0,0}(\Omega_5)\, V_{3-\text{body}}(\alpha,\phi)\; d\Omega_{(5}$$

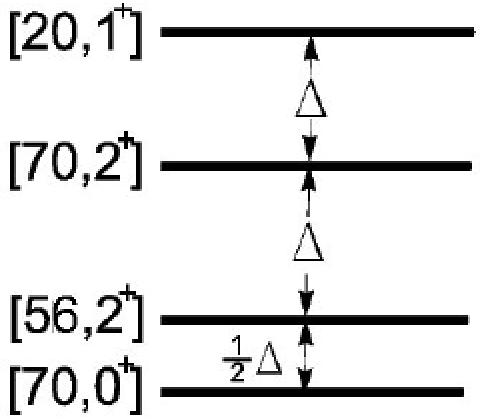
4 out of 2366 possible K<11 !

(K, Q)	$v^{\mathrm{Y}}_{\mathrm{K}Q}$	$v^{\Delta}_{\mathrm{K}Q}$	$v^{\text{Coulomb}}_{\mathrm{K}Q}$
(0,0)	8.18	16.04	20.04
(4,0)	-0.44	-0.44	2.95
(6,±6)	0	-0.14	1.88
(8,0)	-0.09	-0.06	1.49
$\sum \frac{(v^{3-\text{body}}_{\mathrm{K},Q})^2}{(\int (V_{3-\text{body}})^2\, d\Omega_{(5)})}$	99%	99%	94%

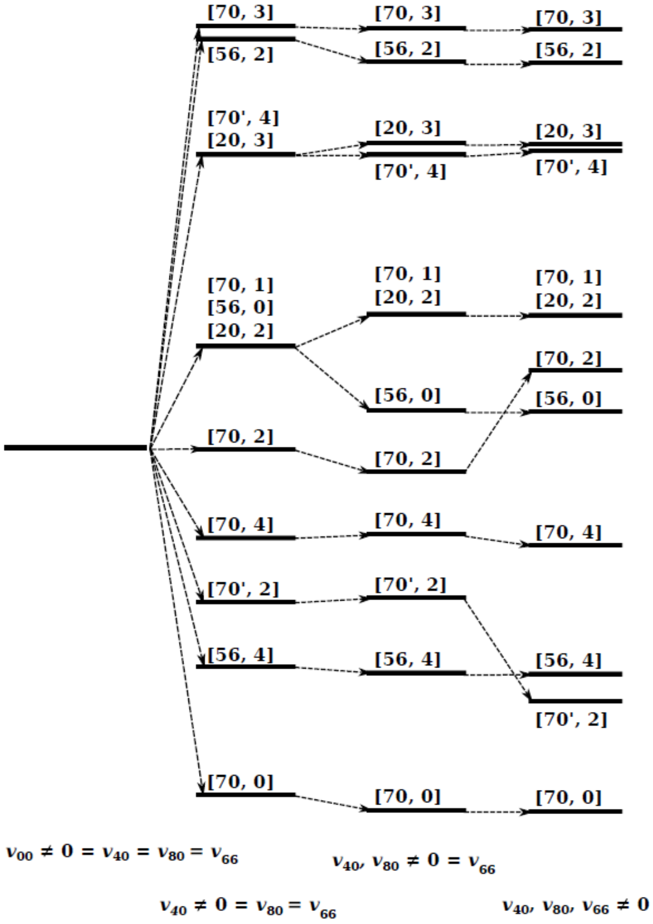
- Energy and ordering of the states depend only on a few coefficients!

State orderings

Fixed state ordering for K=2:



Delta potential state ordering for K=4:



9th MATHEMATICAL PHYSICS MEETING:
School and Conference on Modern Mathematical Physics

18 - 23 September 2017, Belgrade, Serbia

MPHYS9

www.mphys9.ipb.ac.rs

M∩Φ

Thank you

Hyper-spherical coordinates

- Triangle shape-space parameters:

$$R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}$$

$$\hat{\boldsymbol{n}} \;=\; (\boldsymbol{n}_1^{'}, \boldsymbol{n}_2^{'}, \boldsymbol{n}_3^{'}) = \left(\frac{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2}{R^2}, \frac{2\boldsymbol{\rho} \cdot \boldsymbol{\lambda}}{R^2}, \frac{2(\boldsymbol{\lambda} \times \boldsymbol{\rho})_3}{R^2} \right)$$

Smith-Iwai Choice of angles

$$(\sin \alpha)^2 \;=\; \left(\boldsymbol{n}_1^{'2} + \boldsymbol{n}_2^{'2} \right) \;=\; 1 - \left(\frac{2\boldsymbol{\rho} \times \boldsymbol{\lambda}}{R^2} \right)^2$$

$$\phi \;=\; -\tan^{-1} \left(\frac{n_2^{'}}{n_1^{'}} \right) = \tan^{-1} \left(\frac{2\boldsymbol{\rho} \cdot \boldsymbol{\lambda}}{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2} \right)$$

- Plus angles that fix the position/orientation of the triangle plane (some $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$, $\boldsymbol{\Phi}_3$)

I - Case of planar motion

- 4 c.m. degrees of freedom - Jacobi coordinates:

$$x_\mu = (\rho_1, \rho_2, \lambda_1, \lambda_2), \qquad \mu = 1, 2, 3, 4.$$

- or spherically R, α, *φ* and **Φ** ← conjugated to overall angular momentum

- Hyper-angular momenta – so(4) algebra:

$$K_{\mu\nu} \equiv i(x_\mu \partial_\nu - x_\nu \partial_\mu) | \mu, \nu = 1, \dots 4$$

$$T = \frac{m}{2}\dot{R}^2 + \frac{K_{\mu\nu}^2}{2mR^2}$$

Decomposition:

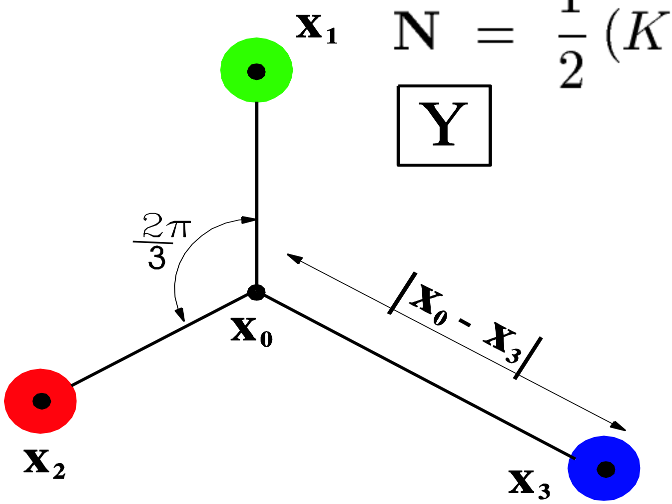
$$so(4) \supset \overset{J_1}{so(3)} \oplus \overset{J_2}{so(3)} \supset \overset{m_1}{so(2)} \oplus \overset{m_2}{so(2)}$$

$$K_{\mu\nu} = \begin{pmatrix} 0 & K_{12} & K_{13} & K_{14} \\ -K_{12} & 0 & K_{23} & K_{24} \\ -K_{13} & -K_{23} & 0 & K_{34} \\ -K_{14} & -K_{24} & -K_{34} & 0 \end{pmatrix} \qquad \begin{pmatrix} \rho_1 \\ \rho_2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$L = L_\rho + L_\lambda$$

$$\mathbf{M} = \frac{1}{2}(\boxed{K_{12} + K_{34}}, K_{23} + K_{14}, K_{31} + K_{24})$$

$$\mathbf{N} = \frac{1}{2}(K_{12} - K_{34}, K_{23} - K_{14}, \boxed{K_{31} - K_{24}}) = Q$$



$$\begin{aligned} [\mathbf{M}^i, \mathbf{M}^j] &= i\varepsilon^{ijk}\mathbf{M}^k \\ [\mathbf{N}^i, \mathbf{N}^j] &= i\varepsilon^{ijk}\mathbf{N}^k \\ [\mathbf{M}^i, \mathbf{N}^j] &= 0 \end{aligned}$$

$$\boxed{[Q, \boldsymbol{\rho} \times \boldsymbol{\lambda}] = 0}$$

Y-string potential = the shortest sum of string lengths ← function of triangle area

Hyper-spherical harmonics

- Labeled by *K*, *L* and *Q*: $\mathcal{Y}^{K}_{L,Q}(\alpha, \phi, \Phi)$

$J_1 = J_2 = K/2 \rightarrow$

$$\left| \begin{matrix} J_1 & J_2 \\ m_1 & m_2 \end{matrix} \right\rangle = |J_1 m_1\rangle \otimes |J_2 m_2\rangle \qquad so(4) \supset so(3) \oplus so(3) \supset so(2) \oplus so(2)$$

- Functions coincide with SO(3) Wigner D-functions:

$$\mathcal{Y}^{K}_{L,Q}(\alpha, \phi, \Phi) = \frac{\sqrt{1+K}}{\sqrt{2}\pi} \mathcal{D}^{K/2}_{Q,-L/2}(-\phi, \alpha, 2\Phi)$$

- Interactions preserve value of *L* (rotational invariance) and some even preserve Q (area dependant like the Y-string three-quark potential)

Calculations now become much simpler…

- We decompose potential energy into hyper-spherical harmonics and split the problem into radial and angular parts:

$$-\frac{1}{2m}\left[\frac{d^2}{dR^2}+\frac{3}{R}\frac{d}{dR}-\frac{K(K+2)}{R^2}+2mE\right]\psi_c(R)+V_{\text{eff.}}(R)\sum_{c'}C_{c,c'}\psi_{c'}(R)=0$$

$$V_{3-\text{body}}\left(\alpha,\phi\right)=\sqrt{\frac{\pi}{2}}\sum_{K_I,Q_I}^{\infty}v_{K_IQ_I}^{3-\text{body}}\mathcal{Y}_{0Q_I}^{K_I}(\alpha,\phi,\Phi)$$

$$C_{[K'],[K]} \;=\; \delta_{[K'],[K]}+\sum_{K_I>0,Q_I}^{\infty}\left(\frac{v_{K_IQ_I}^{3-\text{body}}}{v_{00}^{3-\text{body}}}\right)\sqrt{\frac{(K'+1)(K_I+1)}{(K+1)}}\,C_{\frac{K_I}{2}0,\frac{K'}{2}\frac{L'}{2}}^{\frac{K}{2}\frac{L}{2}}\;C_{\frac{K_I}{2}Q_I,\frac{K'}{2}Q'}^{\frac{K}{2}Q}$$

II - Case of 3D motion

- 6 c.m. degrees of freedom - Jacobi coordinates:

$$x_\mu = (\rho_1, \rho_2, \rho_3, \lambda_1, \lambda_2, \lambda_3), \qquad \mu = 1, 2, 3, 4, 5, 6.$$

- or spherically R, α, φ and some **Φ_1, Φ_2, Φ_3**

Tricky!

- Hyper-angular momenta – so(6) algebra:

$$K_{\mu\nu} \equiv i(x_\mu \partial_\nu - x_\nu \partial_\mu) | \mu, \nu = 1, \ldots 6$$

$$T \ = \ \frac{m}{2}\dot{R}^2 + \frac{K^2_{\mu\nu}}{2mR^2}$$

Particle permutations

- Transformations are easily inferred since:

$$P_{12} : \boldsymbol{\rho} \to -\boldsymbol{\rho}, \boldsymbol{\lambda} \to \boldsymbol{\lambda} \Rightarrow \boldsymbol{X}^{\pm} \to \boldsymbol{X}^{\mp}$$

$$P_{13} : \boldsymbol{X}^{\pm} \to e^{\mp\frac{2\pi}{3}i}\boldsymbol{X}^{\mp}$$

$$P_{23} : \boldsymbol{X}^{\pm} \to e^{\pm\frac{2\pi}{3}i}\boldsymbol{X}^{\mp}$$

$$\Downarrow$$

$$P_{12} : Q \to -Q, \nu \to -\nu, K \to K$$

$$\Downarrow$$

$$P_{12} : \mathcal{Y}_{J,m}^{KQv}(X) \to \mathcal{Y}_{J,m}^{K,-Q,-v}(X)$$

…

Goal in 3-particle case

- Use c.m. system and split the problem into radial and angular parts
- Interaction is not radial-only, but in all realistic interaction potentials "radial" component is dominant – starting point for perturbation approach
- Solve angular part by decomposition to (hyper)spherical harmonics
- Account for some special dynamical symmetries (e.g. Y-string three-quark potential)
- Harmonics provide manifest permutation and rotation properties
- Applications: three quark systems, molecular physics, atomic physics (helium atom), positronium ion…

Hyper-spherical coordinates

- Triangle shape-space parameters:

$$\hat{\boldsymbol{n}} \;=\; (\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) = (\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2, 2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}, 2|\boldsymbol{\lambda}\times\boldsymbol{\rho}|)$$

$$R = \sqrt{\boldsymbol{n}_1^2 + \boldsymbol{n}_2^2 + \boldsymbol{n}_3^2} = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2} = \sqrt{\sum_\mu x_\mu^2}$$

Smith-Iwai Choice of angles

$$(\sin\alpha)^2 \;=\; \frac{\boldsymbol{n}_1^2 + \boldsymbol{n}_2^2}{R^2} = 1 - \left(\frac{2\boldsymbol{\rho}\times\boldsymbol{\lambda}}{R^2}\right)^2$$

$$\phi \;=\; -\tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}\left(\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2}\right)$$

- Plus angles that fix the position/orientation of the triangle plane (some $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$, $\boldsymbol{\Phi}_3$)

6 dim spherical harmonics = ???

- Let us recall a few facts about standard 3D s.h.
  - Functions on sphere: $\mathcal{Y}^{J \leftarrow \text{UIR of } SO(3)}_{m \leftarrow \text{UIR of } SO(2)\subset SO(3)}$

$$\boldsymbol{J}^2\mathcal{Y}^J_m = J(J+1)\mathcal{Y}^J_m \qquad J_3\mathcal{Y}^J_m = m\mathcal{Y}^J_m$$

  - Orthogonal:

$$\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\mathcal{Y}^J_m\mathcal{Y}^{J'*}_{m'}\,d\Omega = \delta_{JJ'}\delta mm'$$

  - E.g.:

$$Y_1^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta \quad = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot\frac{(x-iy)}{r}$$

$$Y_1^{0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\cos\theta \quad = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\frac{z}{r}$$

$$Y_1^{1}(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta \quad = \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\cdot\frac{(x+iy)}{r}$$

$$\boldsymbol{\nabla}^2\mathcal{P}^J_m = 0$$

$$Y_2^{-2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta \quad = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x-iy)^2}{r^2}$$

$$Y_2^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot\cos\theta \quad = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x-iy)z}{r^2}$$

$$Y_2^{0}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot(3\cos^2\theta-1) \quad = \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot\frac{(2z^2-x^2-y^2)}{r^2}$$

$$Y_2^{1}(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot\cos\theta \quad = \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x+iy)z}{r^2}$$

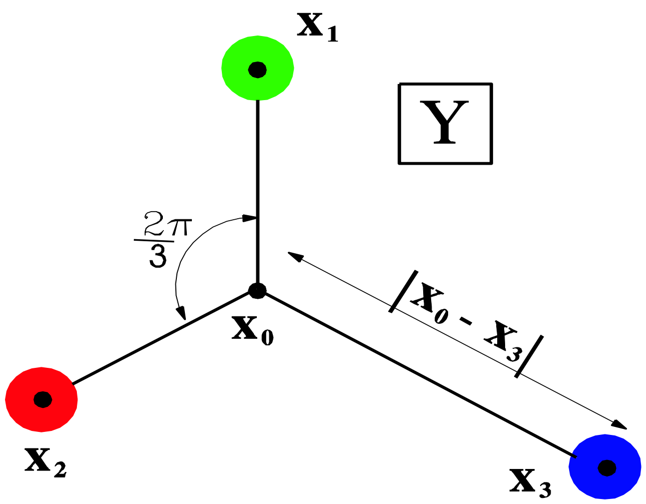
$$Y_2^{2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta \quad = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot\frac{(x+iy)^2}{r^2}$$

D-dim hyper-spherical harmonics

- Intuitively: natural basis for functions on D-dim sphere
- Functions on SO(D)/SO(D-1) – transform as traceless symmetric tensor representations (only a subset of all tensorial UIRs)
- UIR labeled by single integer *K,* highest weight (*K*, 0, 0,…) <=> *K* boxes in a single row <=> *K*(*K*+D-2) quadratic Casimir eigenvalue
- Homogenous harmonic polynomials (obeying Laplace eq. = traceless) of order *K* restricted to unit sphere
- Harmonics of order *K* are further labeled by appropriate quantum numbers, usually related to SO(D) subgroups

U $$\boxed{[Q, \boldsymbol{\rho} \times \boldsymbol{\lambda}] = 0}$$ $SO(6)$

- Co $Q \equiv$ … $= 0$ $1, 2, 3.$



Y-string potential = the shortest sum of string lengths ← function of triangle area

$J_{ij} \equiv$ … $),$

Q_i … $,$

SO(3) rotations

U(3)

SO(6)

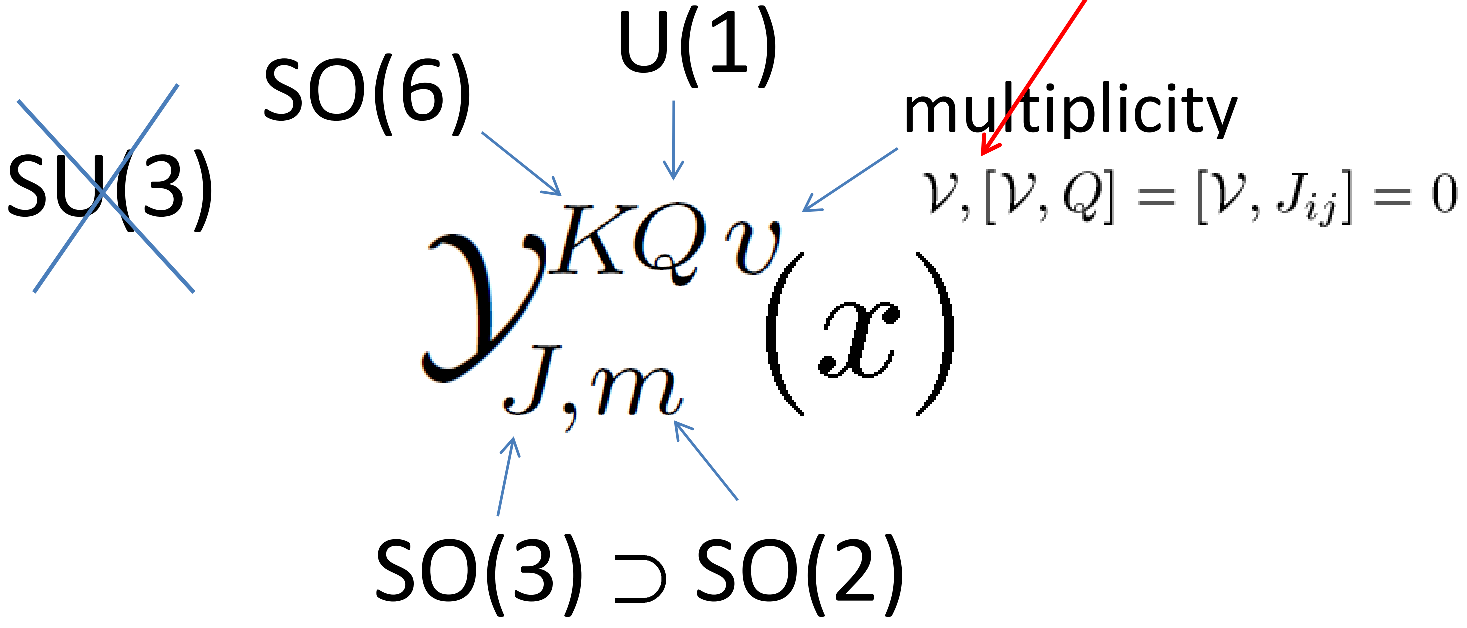
$$\Delta J_{ij} \equiv i(X_i^+ \frac{\partial}{\partial X_j^-} + X_i^- \frac{\partial}{\partial X_j^+} - X_j^+ \frac{\partial}{\partial X_i^-} - X_j^- \frac{\partial}{\partial X_i^+}),$$

$$W_{ij} \equiv X_i^+ \frac{\partial}{\partial X_j^-} - X_i^- \frac{\partial}{\partial X_j^+} - X_j^+ \frac{\partial}{\partial X_i^-} + X_j^- \frac{\partial}{\partial X_i^+}.$$

Quantum numbers

E.g. in SU(3) context often is used operator $\sum_{ij} J_i Q_{ij} J_j$

- Labels of SO(6) hyper-spherical harmonics



~~SU(3)~~

SO(6)

U(1)

multiplicity

$\mathcal{V}, [\mathcal{V}, Q] = [\mathcal{V}, J_{ij}] = 0$

$$\mathcal{Y}^{KQ\,v}_{J,m}(x)$$

SO(3) ⊃ SO(2)

$$U(1) \otimes SO(3)_{rot} \subset U(3) \subset SO(6)$$

Long way to the explicit expressions…

- Building blocks – two SO(3) vectors $\boldsymbol{X}^+$ and $\boldsymbol{X}^-$
- Start from polynomials sharp in Q:

$$\mathcal{P}^{d_+ d_-}_{J_+ m_+ J_- m_-}(X) = \left(\boldsymbol{X}^+ \cdot \boldsymbol{X}^+\right)^{\frac{d_+ - J_+}{2}} \tilde{\mathcal{Y}}^{J_+}_{3,m_+}(X^+) \left(\boldsymbol{X}^- \cdot \boldsymbol{X}^-\right)^{\frac{d_- - J_-}{2}} \tilde{\mathcal{Y}}^{J_-}_{3,m_-}(X^-)$$

- Define “core polynomials” sharp in J, m and Q:

$$\mathcal{P}^{\overline{K}Q}_{(J_+ J_-)J,m}(X) \equiv \sum_{m_+,m_-} C^{J_+ J_- J}_{m_+ m_- m} \mathcal{P}^{\frac{\overline{K}+Q}{2} \frac{\overline{K}-Q}{2}}_{J_+ m_+ J_- m_-}(X)$$

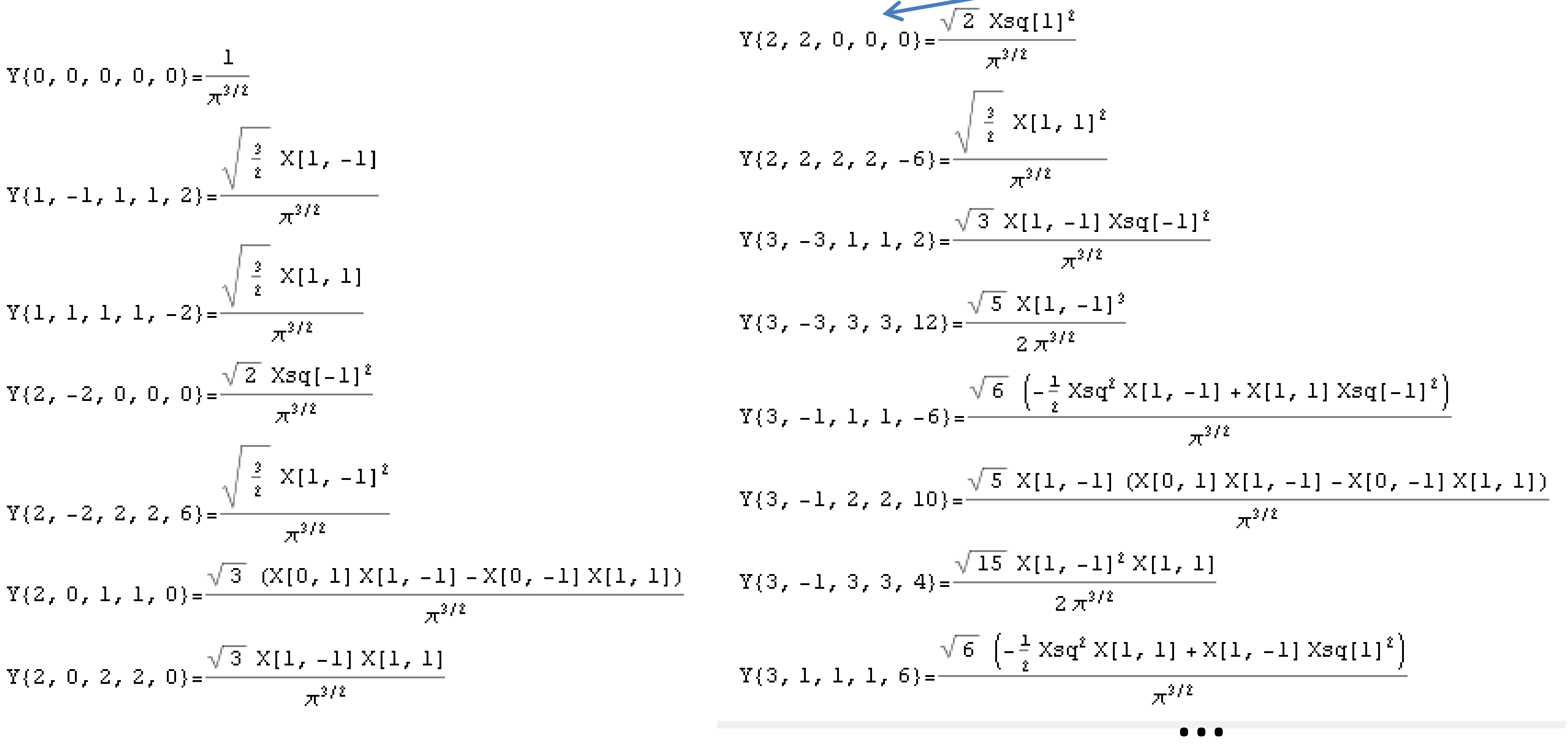
- Make them harmonic by finding ortho-complement w.r.t. polynomials with lesser *K, i.e.:*

$$\mathcal{P}\mathcal{H}^{\overline{K}Q}_{(J_+ J_-)J,m}(X) = \mathcal{P}^{\overline{K}Q}_{(J_+ J_-)J,m}(X) - \sum_a c_a R^{\overline{K}-K_a} \mathcal{P}_a(X),$$

- Finally, remove remaining degeneracy, i.e. introduce multiplicity label.

After all that we can…

…explicitly calculate the harmonics in Wolfram Mathematica…



$\mathcal{Y}\{K, Q, J, m, \nu\}$

$$\mathrm{Y}\{0,0,0,0,0\}=\frac{1}{\pi^{3/2}}$$

$$\mathrm{Y}\{1,-1,1,1,2\}=\frac{\sqrt{\frac{3}{2}}\ \mathrm{X}[1,-1]}{\pi^{3/2}}$$

$$\mathrm{Y}\{1,1,1,1,-2\}=\frac{\sqrt{\frac{3}{2}}\ \mathrm{X}[1,1]}{\pi^{3/2}}$$

$$\mathrm{Y}\{2,-2,0,0,0\}=\frac{\sqrt{2}\ \mathrm{Xsq}[-1]^2}{\pi^{3/2}}$$

$$\mathrm{Y}\{2,-2,2,2,6\}=\frac{\sqrt{\frac{3}{2}}\ \mathrm{X}[1,-1]^2}{\pi^{3/2}}$$

$$\mathrm{Y}\{2,0,1,1,0\}=\frac{\sqrt{3}\ (\mathrm{X}[0,1]\,\mathrm{X}[1,-1]-\mathrm{X}[0,-1]\,\mathrm{X}[1,1])}{\pi^{3/2}}$$

$$\mathrm{Y}\{2,0,2,2,0\}=\frac{\sqrt{3}\ \mathrm{X}[1,-1]\,\mathrm{X}[1,1]}{\pi^{3/2}}$$

$$\mathrm{Y}\{2,2,0,0,0\}=\frac{\sqrt{2}\ \mathrm{Xsq}[1]^2}{\pi^{3/2}}$$

$$\mathrm{Y}\{2,2,2,2,-6\}=\frac{\sqrt{\frac{3}{2}}\ \mathrm{X}[1,1]^2}{\pi^{3/2}}$$

$$\mathrm{Y}\{3,-3,1,1,2\}=\frac{\sqrt{3}\ \mathrm{X}[1,-1]\,\mathrm{Xsq}[-1]^2}{\pi^{3/2}}$$

$$\mathrm{Y}\{3,-3,3,3,12\}=\frac{\sqrt{5}\ \mathrm{X}[1,-1]^3}{2\,\pi^{3/2}}$$

$$\mathrm{Y}\{3,-1,1,1,-6\}=\frac{\sqrt{6}\ \left(-\frac{1}{2}\mathrm{Xsq}^2\,\mathrm{X}[1,-1]+\mathrm{X}[1,1]\,\mathrm{Xsq}[-1]^2\right)}{\pi^{3/2}}$$

$$\mathrm{Y}\{3,-1,2,2,10\}=\frac{\sqrt{5}\ \mathrm{X}[1,-1]\ (\mathrm{X}[0,1]\,\mathrm{X}[1,-1]-\mathrm{X}[0,-1]\,\mathrm{X}[1,1])}{\pi^{3/2}}$$

$$\mathrm{Y}\{3,-1,3,3,4\}=\frac{\sqrt{15}\ \mathrm{X}[1,-1]^2\,\mathrm{X}[1,1]}{2\,\pi^{3/2}}$$

$$\mathrm{Y}\{3,1,1,1,6\}=\frac{\sqrt{6}\ \left(-\frac{1}{2}\mathrm{Xsq}^2\,\mathrm{X}[1,1]+\mathrm{X}[1,-1]\,\mathrm{Xsq}[1]^2\right)}{\pi^{3/2}}$$

…

Particle permutations

- Transformations are easily inferred since:

$$\begin{aligned}
\mathcal{T}_{12}: &\quad \boldsymbol{\lambda} \to \boldsymbol{\lambda}, \quad \boldsymbol{\rho} \to -\boldsymbol{\rho}, \\
\mathcal{T}_{23}: &\quad \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} + \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} + \frac{\sqrt{3}}{2}\boldsymbol{\lambda}, \\
\mathcal{T}_{31}: &\quad \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} - \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} - \frac{\sqrt{3}}{2}\boldsymbol{\lambda}.
\end{aligned}$$

⇩

$$\mathcal{T}_{12}: X_i^{\pm} \to X_i^{\mp}, \quad \mathcal{T}_{23}: X_i^{\pm} \to e^{\pm\frac{2i\pi}{3}} X_i^{\mp}, \quad \mathcal{T}_{31}: X_i^{\pm} \to e^{\mp\frac{2i\pi}{3}} X_i^{\mp}.$$

⇩

$$\mathcal{T}_{ab}: Q \to -Q, K \to K, J_{ij} \to J_{ij}, \nu \to \pm\nu$$

“Core polynomials”

- Building blocks – two SO(3) vectors $\boldsymbol{X}^+$ and $\boldsymbol{X}^-$
- Start from polynomials sharp in Q:

$$\mathcal{P}^{d_+ d_-}_{J_+ m_+ J_- m_-}(X) = \left(\boldsymbol{X}^+ \cdot \boldsymbol{X}^+\right)^{\frac{d_+ - J_+}{2}} \tilde{\mathcal{Y}}^{J_+}_{3,m_+}(X^+) \;\Big|\; \left(\boldsymbol{X}^- \cdot \boldsymbol{X}^-\right)^{\frac{d_- - J_-}{2}} \tilde{\mathcal{Y}}^{J_-}_{3,m_-}(X^-)$$

- Define “core polynomials” sharp in J, m and Q:

$$\mathcal{P}^{\overline{K}Q}_{(J_+ J_-)J,m}(X) \equiv \sum_{m_+,m_-} C^{J_+ J_- J}_{m_+ m_- m} \mathcal{P}^{\frac{\overline{K}+Q}{2}\,\frac{\overline{K}-Q}{2}}_{J_+ m_+ J_- m_-}(X)$$

$$\boxed{J_+ + J_- = J \text{ or } J_+ + J_- = J + 1}$$

Core polynomial certainly contains component with $\overline{K} = K$ but also lower K components

“Harmonizing” polynomials

- Let $\mathcal{P}_a(X), a = 1, 2, 3...$ be shortened notation for all core polynomials with *K* values less than some given $\overline{K}$
- Harmonic polynomials are obtained as ortho-complement w.r.t. polynomials with lesser *K, i.e.:*

$$\mathcal{P}\mathcal{H}^{\overline{K}Q}_{(J_+J_-)J,m}(X) = \mathcal{P}^{\overline{K}Q}_{(J_+J_-)J,m}(X) - \sum_a c_a R^{\overline{K}-K_a}\mathcal{P}_a(X),$$

where c_a are deduced from requirement*:*

$$\left\langle \mathcal{P}_a \middle| \mathcal{P}\mathcal{H}^{\overline{K}Q}_{(J_+J_-)J,m} \right\rangle = 0$$

Scalar product of core polynomials

$$c_a = \sum_b (M^{-1})_{ab} A_b, \quad \text{with :} \quad M_{ab} \equiv \left\langle \mathcal{P}_a \middle| \mathcal{P}_b \right\rangle, \; A_a \equiv \left\langle \mathcal{P}_a \middle| \mathcal{P}\mathcal{H}^{\overline{K}Q}_{(J_+J_-)J,m} \right\rangle$$

Scalar product of polynomials on

$$\left\langle \mathcal{P}^{\overline{K}'Q'}_{(J'_+J'_-)J',m'} \middle| \mathcal{P}^{\overline{K}Q}_{(J_+J_-)J,m} \right\rangle = \delta_{mm'} \left\langle \mathcal{P}^{\overline{K}'Q'}_{(J'_+J'_-)J',J'} \middle| \mathcal{P}^{\overline{K}Q}_{(J_+J_-)J,J} \right\rangle$$

$$= \begin{cases} \frac{2\pi^3 \delta_{QQ'} \delta_{JJ'} \delta_{mm'}}{(2+\frac{\overline{K}+\overline{K}'}{2})!} \sum_{l=0}^{\frac{k^+}{2}} 2^{2l+J_+ + J'_-} \binom{\frac{k^+}{2}}{l} \cdot \\ \binom{\frac{k^+}{2}+J_+-J'_+}{\frac{k^+}{2}-l} (l+J_+)!(l+J'_-)!(k^+-2l)! & \text{if } \overline{K}-J \equiv \overline{K}'-J' \equiv 0 \pmod 2 \\ \\ \frac{2\pi^3 \delta_{QQ'} \delta_{JJ'} \delta_{mm'}}{(2+\frac{\overline{K}+\overline{K}'}{2})!} \frac{2\sqrt{J_+J_-\,J'_+J'_-}}{1+J} \sum_{l=0}^{\frac{k^+}{2}} 2^{2l+J_+ + J'_-} \binom{\frac{k^+}{2}}{l} \cdot \\ \Big(\binom{\frac{k^+}{2}+J_+-J'_+}{\frac{k^+}{2}-l} (l+J_+-1)!(l+J'_- -1)!(k^+-2l+1)! \frac{l+J+1}{2} & \text{if } \overline{K}-J \equiv \overline{K}'-J' \equiv 1 \pmod 2 \\ -\binom{\frac{k^+}{2}+J_+-J'_+}{\frac{k^+}{2}-l-1} (l+J_+)!(l+J'_-)!(k^+-2l)! \Big) \\ \\ 0 & \text{if } \overline{K}-J \not\equiv \overline{K}'-J' \pmod 2, \end{cases}$$

- that for core polynomials eventually leads to a closed-form expression…
- Integral of any number of polynomials can be evaluated (e.g. matrix elements)

E.g. this can be $(\boldsymbol{\rho} \times \boldsymbol{\lambda})^2$ or often used operator $\sum_{ij} J_i Q_{ij} J_j$

Multiplicity

- Exist nonorthogonal $\mathcal{P}_{\mathcal{H}^{KQ}_{(J_+ J_-)J,m}}(X)$ and $\mathcal{P}_{\mathcal{H}^{KQ}_{(J'_+ J'_-)J,m}}(X)$
- Degenerated subspace: $\{\mathcal{P}_{\mathcal{H}a} | a = 1, 2, \ldots \dim V^{K,Q}_{J,m}\}$
- We remove multiplicity by using physically appropriate operator $\mathcal{V}, [\mathcal{V}, Q] = [\mathcal{V}, J_{ij}] = 0$ and obtain orthonormalized spherical harmonic polynomials as:

$$Y_a(X) = \sum_b (M^{-\frac{1}{2}} U)_{ab} \mathcal{P}_{\mathcal{H}b}$$

- where $M_{ab} \equiv \left\langle \mathcal{P}_{\mathcal{H}a} \middle| \mathcal{P}_{\mathcal{H}b} \right\rangle$ and *U* is a matrix such that:

$$U^{-1}(M^{-\frac{1}{2}} \mathcal{V} M^{-\frac{1}{2}})U = diag(v_1, v_2, \ldots v_{dim})$$

Now we can solve problems by h.s.h. decomposition

- Schrodinger equation – coupled d.e. in $\psi^{\mathrm{K}}_{[\mathrm{m}]}(R)$:

$$-\frac{1}{2\mu}\left[\frac{d^2}{dR^2}+\frac{5}{R}\frac{d}{dR}-\frac{\mathrm{K}(\mathrm{K}+4)}{R^2}+2\mu E\right]\psi^{\mathrm{K}}_{[\mathrm{m}]}(R)+V_{\mathrm{eff.}}(R)\sum_{\mathrm{K}',[\mathrm{m}']}C^{\mathrm{K}\;\mathrm{K}'}_{[\mathrm{m}]\,[\mathrm{m}']}\psi^{\mathrm{K}'}_{[\mathrm{m}']}(R)=0$$

- where:

$$C^{\mathrm{K}''\;\mathrm{K}'}_{[\mathrm{m}'']\,[\mathrm{m}']}=\delta_{\mathrm{K}'',\mathrm{K}'}\delta_{[\mathrm{m}''],[\mathrm{m}']}+\pi\sqrt{\pi}\textstyle\sum_{\mathrm{K}>0,Q}^{\infty}\frac{v^{3-\mathrm{body}}_{\mathrm{K},Q}}{v^{3-\mathrm{body}}_{00}}\langle\mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5)|\mathcal{Y}^{\mathrm{K}Qv}_{00}(\alpha,\phi)|\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle$$

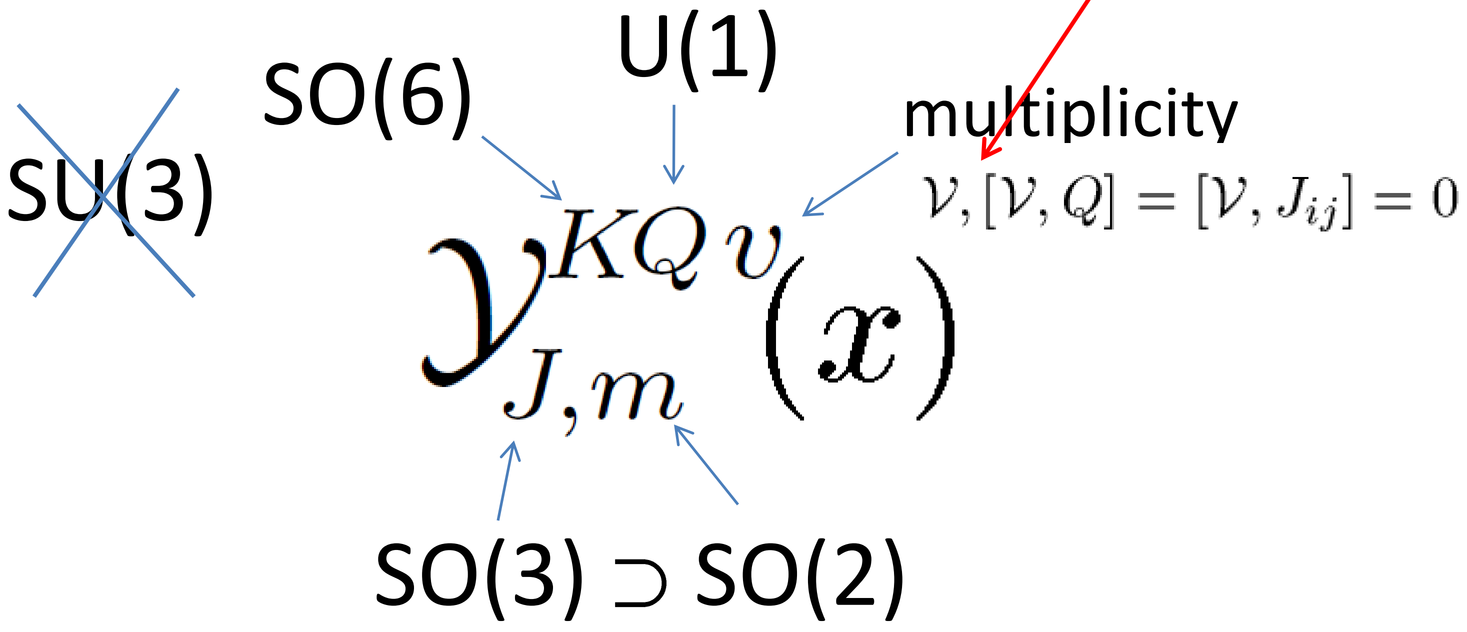
- In the first order p.t. this can be diagonalized into:

$$\left[\frac{d^2}{dR^2}+\frac{5}{R}\frac{d}{dR}-\frac{\mathrm{K}(\mathrm{K}+4)}{R^2}+2\mu(E-V^{\mathrm{K}}_{[\mathrm{m_d}]}(R))\right]\psi^{\mathrm{K}}_{[\mathrm{m_d}]}(R)=0$$

Quantum numbers

E.g. in SU(3) context often is used operator $\sum_{ij} J_i Q_{ij} J_j$

- Labels of SO(6) hyper-spherical harmonics



$$\mathcal{V}, [\mathcal{V}, Q] = [\mathcal{V}, J_{ij}] = 0$$

$$\mathcal{Y}^{KQ\,v}_{J,m}(x)$$

$$U(1) \otimes SO(3)_{rot} \subset U(3) \subset SO(6)$$