Renormalization group and scattering-equivalent Hamiltonians on a coarse momentum grid

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Excited QCD 2020, Krynica Zdrój

Outline

1. Introduction
2. Phase shifts and the spectral-shift method
3. Similarity renormalization group and scattering

Introduction

What are natural scales in a problem?

Fundamental theories

Degrees of freedom are given and they are infinite

$$H_{QCD}|\psi\rangle = E|\psi\rangle$$

$$\begin{aligned} H_{QCD} &= H_q + H_{\bar{q}} + H_g + H_{Qqg} + H_{q\bar{q}} + \dots \ , \\ |\psi\rangle &= |0\rangle + |q\rangle + |\bar{q}\rangle + |g\rangle + |qg\rangle + |q\bar{q}\rangle + \dots \end{aligned}$$

Local interactions produce divergences $\int^{\infty}$

Effective theories

$\longrightarrow$ **Tool:** Głazek-Wilson Similarity Renormalization Group (SRG):

$$H_\lambda = U_\lambda H_0 U_\lambda^\dagger \ , \qquad U_\lambda|\psi_0\rangle = |\psi_\lambda\rangle \ , \quad H_\lambda|\psi_\lambda\rangle = H_0|\psi_0\rangle$$

Then, truncation of the Fock space makes sense for low energies

Cutoff $\int^{\Lambda} \longrightarrow \dfrac{\partial[\text{Observables}]}{\partial\Lambda} = 0, \qquad \Lambda \sim \Lambda_{\text{natural}}.$

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Recent QCD papers:

[EPJ. C78 (2018) no.11, 964] [PLB 773 (2017) 172-178]

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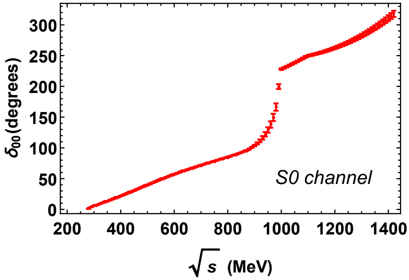
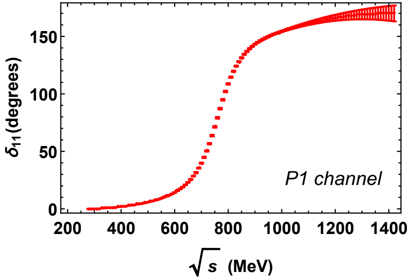
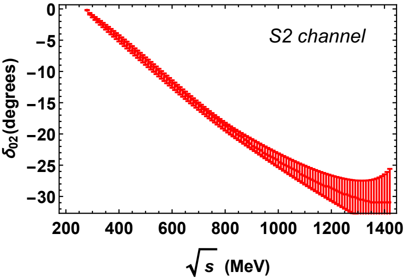
In this work we apply the SRG to the potential that describes $\pi\pi$ scattering.

$\pi\pi$ scattering phase shifts

Phase shifts δ_{LI}
data analysis taken from:
[García-Martín, Kamiński, Peláez, Ruiz de Elvira, Ynduráin, PRD83(2011)]







Kadyshevsky equation

- It is a 3D-reduction of the BSE
- It enables a relativistic Hamiltonian interpretation for the scattering problem
- Amenable for numerical analysis
- Useful in view of its application to the three-body interaction (i.e. omega decays into 3 pions..., and so on)

$$t(\vec{p}',\vec{p},\sqrt{s}) = v(\vec{p}',\vec{p}) + \int \frac{d^3q}{(2\pi)^3}\frac{v(\vec{p}',\vec{q})}{4E_q^2}\frac{t(\vec{q},\vec{p},\sqrt{s})}{\sqrt{s}-2E_q+i\varepsilon}$$

Kadyshevsky, Nucl. Phys. B6 125(1968)

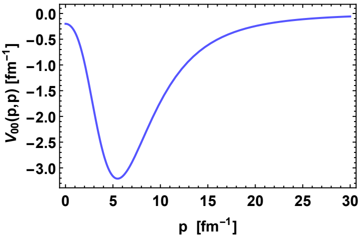
Phase shifts:

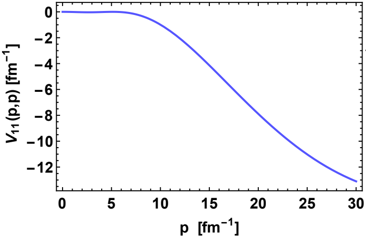
$$\tan\delta_l(p) = -\frac{\pi}{8E_p}\,p\,r_l(p,p,\sqrt{s})$$

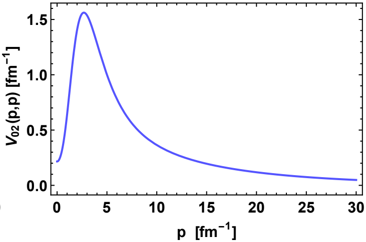
$$r_l(p',p,\sqrt{s}) = v_l(p',p) + \int_0^\infty dq\,\frac{q^2}{4E_q^2}v_l(p',q)\frac{1}{\sqrt{s}-2E_q}r_l(q,p,\sqrt{s})$$

Kadyshevsky equation

Model potential
Separable potential from
Mathelitsch and Garcilazo
PRC32 (1985)







Kadyshevsky equation

Its corresponding Hamiltonian in the center-of-mass system:

$$H\Psi_l(p) \equiv 2E_p\Psi_l(p) + \int dq \frac{q^2}{4E_q^2} v_l(p,q)\Psi_l(q)$$

The homogeneus Kadyshevsky equation reads

$$H\Psi_l(p) \quad = \quad \sqrt{s}\Psi_l(p)$$

Spectral-shift prescription:

[**Arriola, Szpigel, Timoteo, Ann.Phys.326 (2011) 398**]

For a given momentum grid, e.g. Gauss-Chebyshev grid

$$\int_0^\Lambda dp\, f(p) \to \sum_{n=1}^{N} w_n f(p_n)$$

$$p_n = \frac{\Lambda_{\rm num}}{2}[1-\cos(\pi/N(n-1/2))]$$

$$w_n = \frac{\Lambda_{\rm num}}{2}\sin(\pi/N(n-1/2))$$

Phase shifts can be obtained as

$$\delta_n = -\pi\frac{P_n - p_n}{w_n}\ , \qquad \text{where} \qquad \sqrt{s_n} = 2E_n = 2\sqrt{P_n^2 + m_\pi^2}$$

Spectral-shift prescriptions

Momentum-shift prescription:

Based on an **equidistant momentum** grid
[Fukuda & Newton, Phys. Rev. 103 (1956) 1558]

$$\delta_n = -\pi \frac{P_n - p_n}{\Delta p} \;, \qquad \Delta p = \text{const.}$$

Energy-shift prescription:

Based on an **equidistant energy** grid
[B.S. DeWitt, Phys. Rev. 103 (1956) 1565.]

$$\delta_n = -\pi \frac{\Delta E_n}{\Delta e} \;, \qquad \Delta E_n = \sqrt{P_n^2 + m^2} - \sqrt{p_n^2 + m^2} \;, \qquad \Delta e = \text{const.}$$

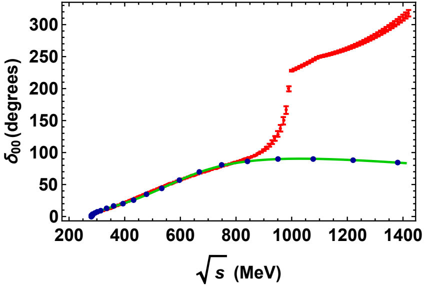
Chebyshev-angle shift prescription (ϕ-shift):

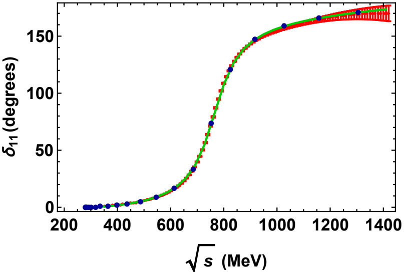
Based on **equidistant Chebyshev angle** grid (Gauss-Chebyshev grid)
[Gomez-Rocha, Arriola Phys.Lett. B800 (2020) 135107]

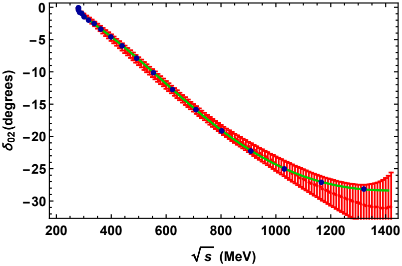
$$\delta_n = -\pi \frac{\Phi_n - \phi_n}{d\phi_n} \;, \quad p_n = \frac{\Delta}{2}(1-\cos\phi_n) \;, \quad \phi = \frac{\pi}{N}(1-\frac{1}{2}) \;, \quad P_n = \frac{\Delta}{2}(1-\cos\Phi_n)$$

Calculation of phase shifts δ_{LI}

Blue: ϕ-shift calculation
Green: Fit, standard prescription
Red: Experiment







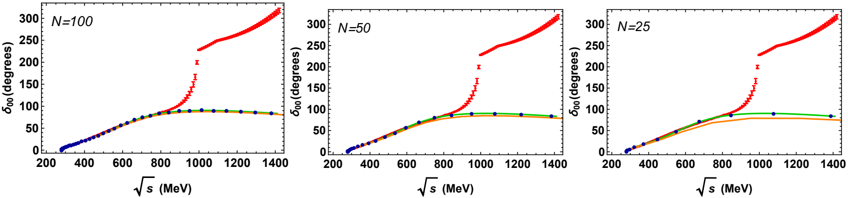
Advantages of the ϕ-shift method

Compare the ϕ-shift results • with other methods (Lippmann-Schwinger •):

$\Rightarrow$ *Simple formula, Fast calcuation*

and...

Results are less sensitive to grid effects!!



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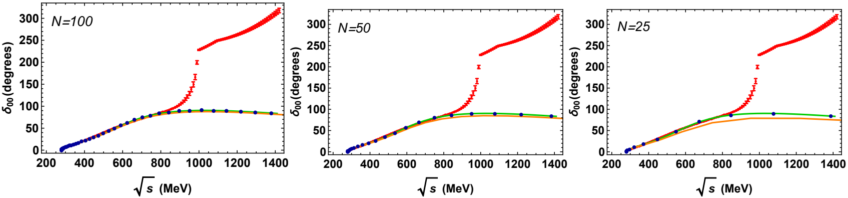

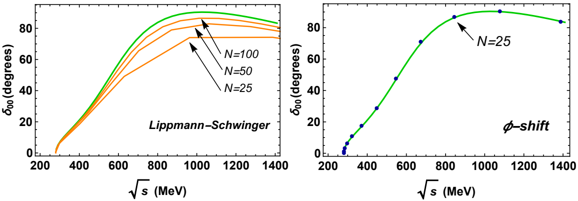

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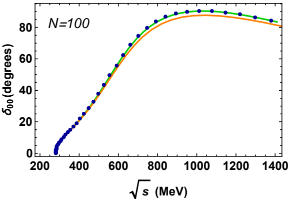

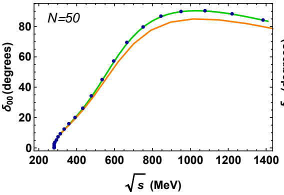

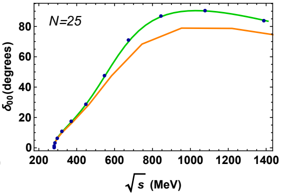

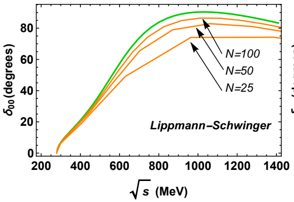

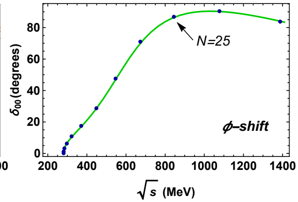

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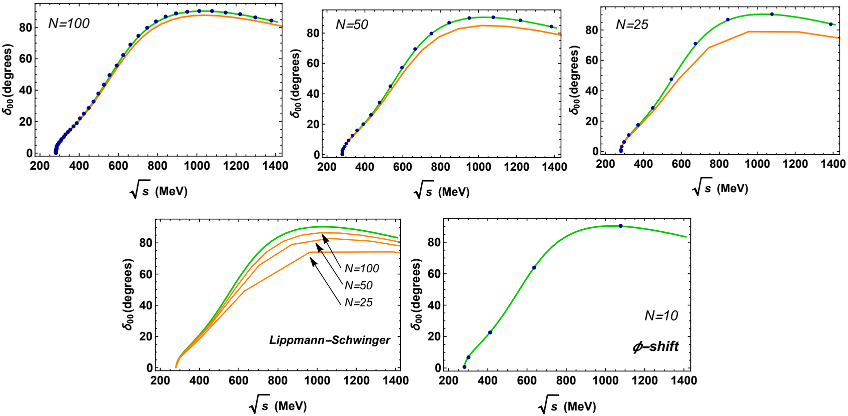

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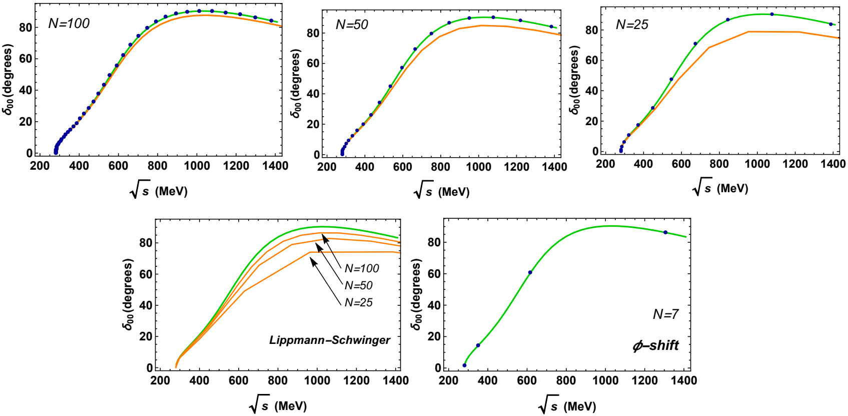

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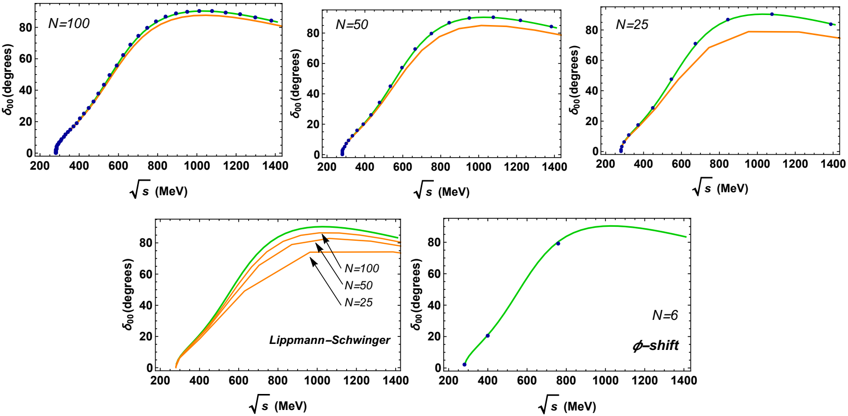

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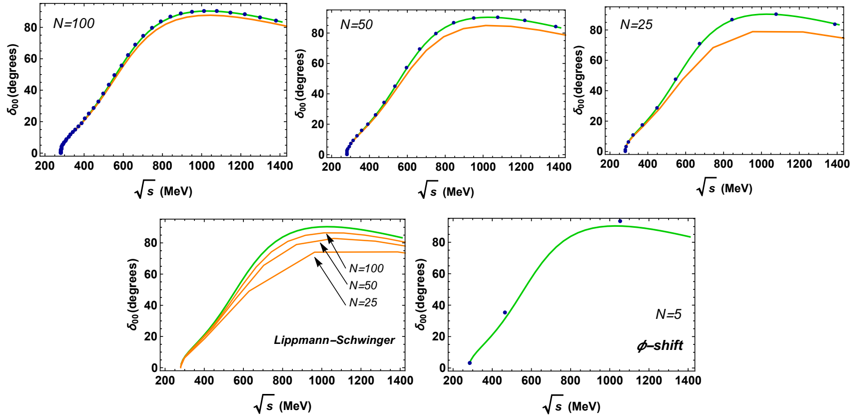

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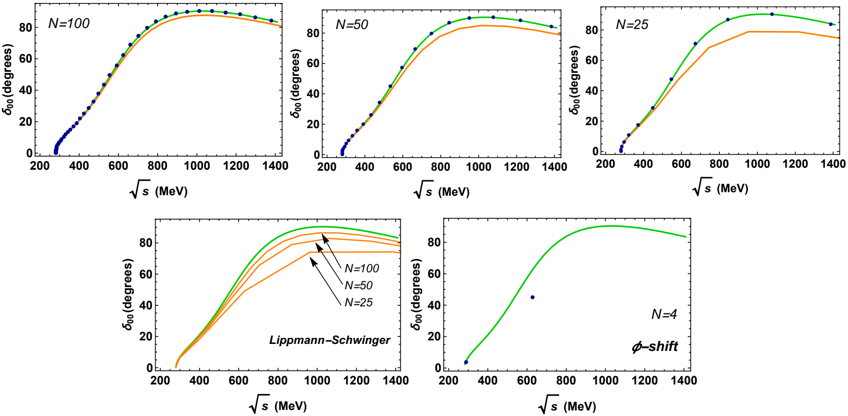
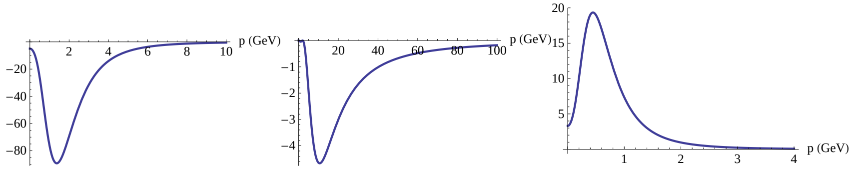


Phase shifts and the SRG

- In order to fit $\pi\pi$ phase shifts up to energies $\sqrt{s} \lesssim 1$GeV, a very high-energy momentum tail up to $\sqrt{s} \lesssim 10$GeV needs to be considered

Model potential taken from [Mathelitsch and Garzilazo PRC 32 (1985)] for $S0$, $P1$ and $S2$, respectively



- Disparity in energy scales: annoying
- Details of the interaction at short distances are so relevant??
- Is it possible to construct an effective theory and focus in a short energy range?

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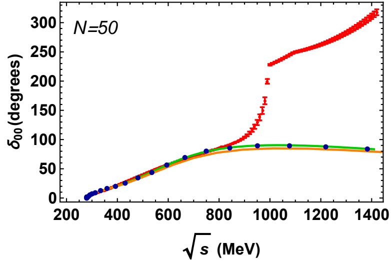

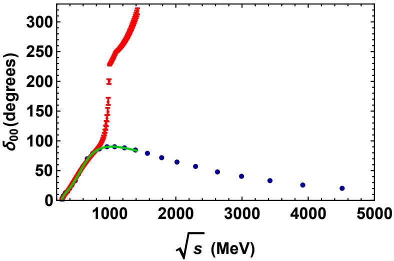

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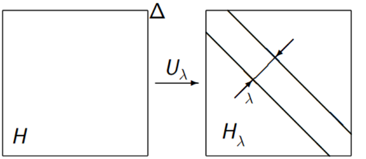
Similarity renormalization group

SRG employs a transformation that changes the cutoff to isolate Hamiltonians that produce cutoff-independent eigenvalues.

SRG allows to select the relevant energy scale

Initial theory   Effective theory



The transformation

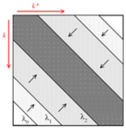
$$H_\lambda = U_\lambda H_0 U_\lambda^\dagger$$

does not change the spectrum

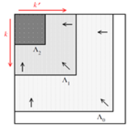
$$H|\psi\rangle = H_\lambda|\psi_\lambda\rangle = E|\psi\rangle$$

$$H_t' = [G_t, H_t] \qquad t = 1/\lambda^2$$

Different generators G_λ can be chosen, so that H_λ becomes



Diagonal



Block-diagonal

Evolution using the Crank-Nicolson algorithm

Consider the Kadyshevsky equation

$$H\Psi_l(p) = \sqrt{s}\Psi_l(p)$$

SRG evolution obeys

$$H_t' = [G_t, H_t]$$

The effective Hamiltonian is related to the initial one by a unitary transformation:

$$H_t = U_t H_0 U_t^\dagger$$

The unitary transformation must be such that

$$\frac{dU_t}{dt} = G_t U_t \equiv -i\mathcal{H} U_t$$

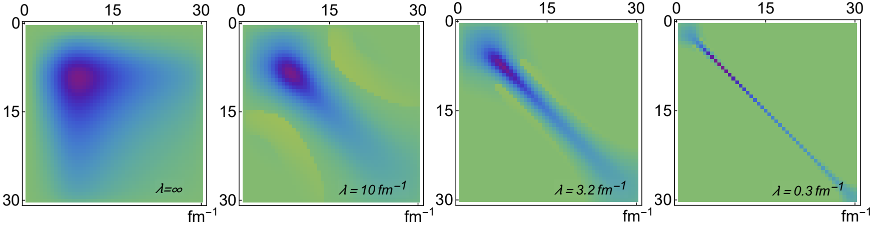
One can apply the Crank-Nicolson algorithm,

$$U_{n+1} \quad = \quad \left(1 - i\frac{dt}{2}\mathcal{H}_n\right)\left(1 + i\frac{dt}{2}\mathcal{H}_n\right)^{-1} U_n$$

with $-i\mathcal{H} = G_t$.

Effective Hamiltonian matrix for $S0$-wave

Evolution of $V(p, p')$ with Wilson-diagonal generator:



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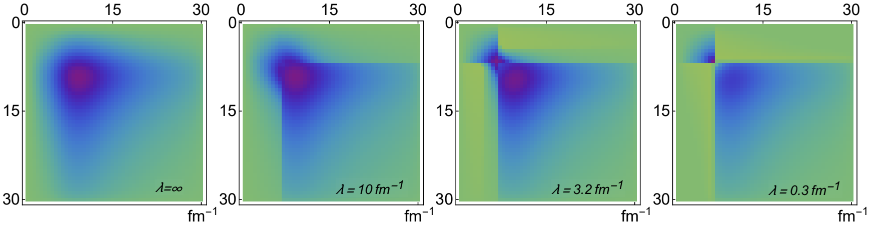
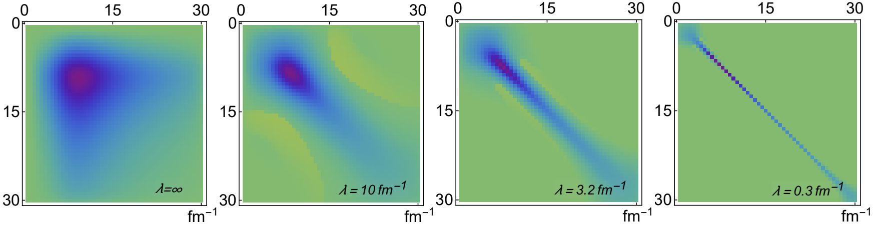


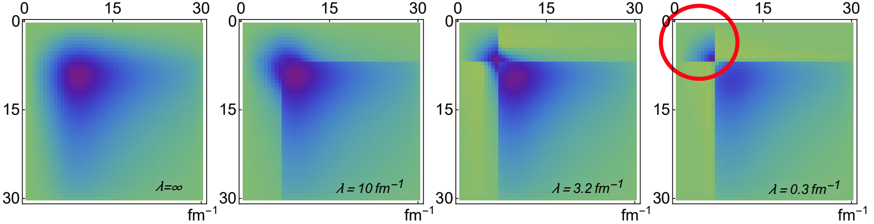
$\Rightarrow$ hope for finding an effective potential "contained" in the small matrix.

Effective Hamiltonian matrix for $S0$-wave

Evolution of $V(p,p')$ with Wilson-diagonal generator:



Evolution of $V(p,p')$ with Block-diagonal generator



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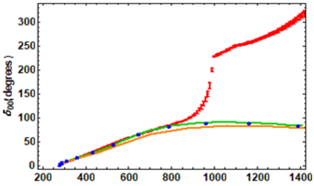
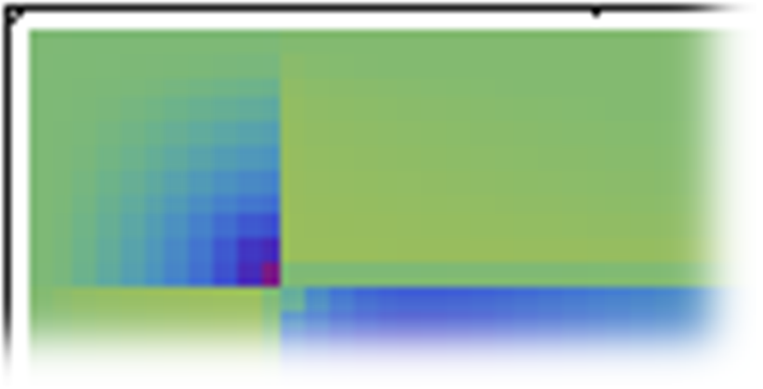
Effective Hamiltonian matrix

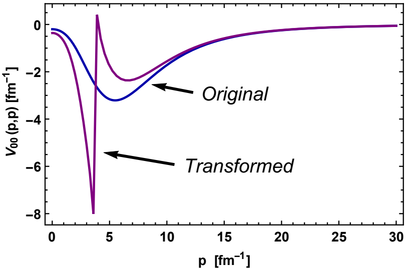


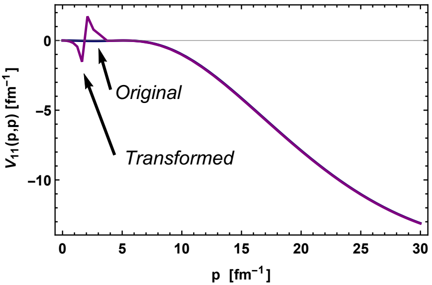
Physics of interest happens
in a small energy region
$(< \Lambda \simeq 1400$ *MeV*$)$

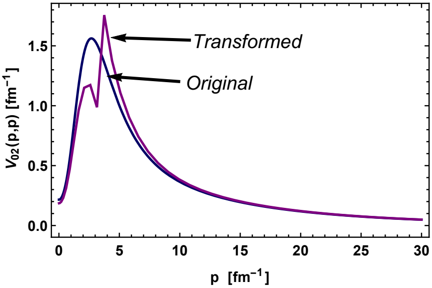


Effective Hamiltonian matrix

Comparison of potentials before and after the evolution



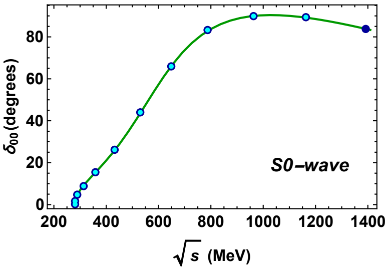


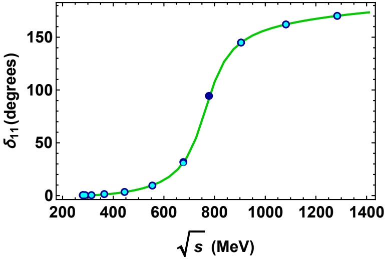


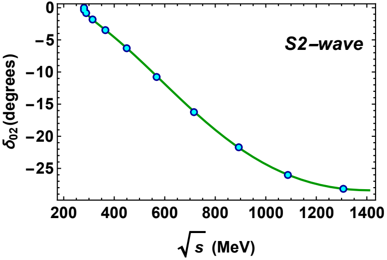
Effective Hamiltonian matrix

Phase shifts δ_{LI} calculated using
- original matrix (N=50)
- small block (N=11)

⇒Phase shifts are reproduced with a much smaller number of points !!!







Conclusions

- **ϕ-shift method: a reliable tool for numerical calculations of scattering phase shifts**
  - Very accurate results, even for small N
  - Computationally cheaper that other conventional methods

- **SRG helps to treat the problem of long-tail potentials in hadronic physics**
  - Block-diagonal generator decouples low and high energy
  - A smaller matrix can be selected

⇒ **A smaller grid makes further calculations simpler:**
$\omega \to \pi\pi\pi$, A_1, molecules, etc.

Note: Same study has been done for NN and πN scattering (cf. M.G.R., Arriola, to appear in PRD, arXiv:1911.08990).

Acknowledgment

Thank you for your attention