Conformal anomaly and fluid dynamics

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Outline

- Introduction and motivation
- Nonequilibrium deviation from the distribution function
- Equations of hydrodynamics with mean field effects
- Bulk viscosity in the relaxation time approximation
- Kubo formula for the bulk relaxation time
- Summary

Bulk viscosity

Hydrodynamics: a long-wavelength effective description of interacting systems

Conservation laws + equation of state
Transport coefficients: details of microscopic dynamics

Bulk viscosity – a measure of conformal anomaly

- weak coupling - perturbative QCD: $$\frac{\zeta}{\eta} \propto \left(\frac{1}{3} - c_s^2\right)^2$$
- strong (infinite) coupling – string theories: $$\frac{\zeta}{\eta} \propto \left(\frac{1}{3} - c_s^2\right)$$

Buchel bound:

$$\frac{\zeta}{\eta} \geq 2\left(\frac{1}{3} - c_s^2\right)$$

- Intermediate coupling – lattice QCD, ansatz for the bulk viscosity spectral function

Uncertaintities too large
Ansatz not reliable

- Phenomenology

Bulk viscosity strongly peaked near the critical temperature and model-dependent

Conformal anomaly

Trace of the energy-momentum tensor different from zero

Nonconformality parameters:

- Microscopic: m_0 - zero-temperature mass

  β_λ - fixes the coupling as a function of the energy scale
- Macroscopic: $\epsilon - 3P, \quad \frac{1}{3} - c_s^2$ $\qquad c_s^2 = \frac{dP/dT}{d\epsilon/dT}$

Motivation: - understand physics imposed by conformal anomaly
- work out details on parameters quantifying conformal anomaly: bulk viscosity and its relaxation time

$$\partial_t \Pi = -\frac{\Pi - \Pi_{\mathrm{NS}}}{\tau_\Pi}$$

Formulation of hydrodynamic equations with medium dependent effects

Nonequilibrium vs. equilibrium

The system under study is made of weakly interacting scalar particles, both classical and quantum statistics are considered

	Equilibrium (well defined state)	**Nonequilibrium** (small deviations, perturbative corrections to equilibrium quantities)
Quasiparticle thermal mass	$m_{\rm eq} \equiv m_{\rm eq}(x)$	$m_{\rm th} \equiv m_{\rm th}(x)$
Quasiparticle mass	$m_x = \sqrt{m_0^2 + m_{\rm eq}^2}$	$\tilde{m}_x = \sqrt{m_0^2 + m_{\rm th}^2}$
Quasiparticle energy	$E_k = \sqrt{\mathbf{k}^2 + m_x^2}$	$\mathcal{E}_k = \sqrt{\mathbf{k}^2 + \tilde{m}_x^2}$
Quasiparticle four-momentum	$k^\mu \equiv (k_0, \mathbf{k}) = (E_k, \mathbf{k})$	$\tilde{k}^\mu \equiv (\tilde{k}_0, \mathbf{k}) = (\mathcal{E}_k, \mathbf{k})$
Lorentz invariant measure	$dK = d^3\mathbf{k}/[(2\pi)^3 E_k]$	$d\mathcal{K} = d^3\mathbf{k}/[(2\pi)^3 \mathcal{E}_k]$
Distribution function	$f_0 = 1/[e^{\beta E_k} - 1]$	$f = f_0 + \Delta f$

Nonequilibrium deviation from the equilibrium distribution function

Boltzmann equation with the mean field contribution

$$\boxed{(\tilde{k}^\mu \partial_\mu - \mathcal{E}_k \nabla \mathcal{E}_k \cdot \nabla_k) f = C[f]}$$

All quantities entering the equation are x-dependent

$$f(x,k) = \underbrace{f_{\rm th}(x,k)}_{\text{retains equilibrium form}} + \delta f(x,k) = f_0(x,k) + \delta f_{\rm th}(x,k) + \delta f(x,k)$$

$$\Delta f(x,k) = \delta f_{\rm th}(x,k) + \delta f(x,k)$$

$$f_{\rm th}(x,k) \equiv f_0(x,k)|_{m_0^2 + m_{\rm eq}^2(x) \to m_0^2 + m_{\rm eq}^2(x) + \Delta m_{\rm th}^2(x)} = \left[\exp\left(\sqrt{\mathbf{k}^2 + m_0^2 + m_{\rm eq}^2(x) + \underbrace{\Delta m_{\rm th}^2(x)}_{\text{correction from the nonequilibrium thermal mass}}}\,\beta(x)\right) - 1\right]^{-1}$$

$$\Delta f = \delta f - \beta f_0(1+f_0) \frac{\Delta m_{\rm th}^2}{2E_k}$$

$$\boxed{\Delta f = \delta f - T^2 \frac{dm_{\rm eq}^2}{dT^2} \frac{f_0(1+f_0)}{E_k} \frac{\int dK \delta f}{\int dK E_k f_0(1+f_0)}}$$

$$T^2 \frac{dm_{\rm eq}^2}{dT^2} = m_{\rm eq}^2 + aT^2 \beta_\lambda$$

Equations of hydrodynamics

Local equilibrium hydrodynamics

Energy-momentum tensor:

$$T_0^{\mu\nu} = \int dK k^\mu k^\nu f_0 - g^{\mu\nu} U_0$$

mean-field contribution

- thermodynamic consistency of hydrodynamic equations
- conservation of energy and momentum

$$T_0^{\mu\nu} = \epsilon_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu}$$

$$dU_0 = \frac{q_0}{2} dm_{\mathrm{eq}}^2$$

Energy density and pressure:

$$\epsilon_0 = \bar{\epsilon}_0 - U_0, \qquad \bar{\epsilon}_0 = \int dK (u_\mu k^\mu)^2 f_0$$

$$P_0 = \bar{P}_0 + U_0 \qquad \bar{P}_0 = -\frac{1}{3} \int dK \Delta^{\mu\nu} k_\mu k_\nu f_0$$

- Enthalpy not changed: $\bar{\epsilon}_0 + \bar{P}_0 = \epsilon_0 + P_0$
- Thermodynamic relation satisfied: $T s_0 = T \frac{dP_0}{dT} = \epsilon_0 + P_0$

Nonequilibrium hydrodynamics

Energy-momentum tensor:

nonequilibrium mean-field contribution

$$T^{\mu\nu} = \int d\mathcal{K} \tilde{k}^\mu \tilde{k}^\nu f - g^{\mu\nu} U \qquad U = U_0 + \Delta U \qquad \Delta U = \frac{q_0}{2} \Delta m_{\rm th}^2$$

All quantities contain nonequilibrium thermal mass correction

$$T^{\mu\nu} = T_0^{\mu\nu} + \Delta T^{\mu\nu}$$

Particular components:

$$\begin{aligned}
\Delta T^{00} &= \int dK E_k^2 \Delta f \\
\Delta T^{0i} &= \int dK E_k k^i \Delta f \\
\Delta T^{ij} &= \int dK k^i k^j \Delta f - \frac{\Delta m_{\rm th}^2}{2} \int dK \frac{k^i k^j}{E_k^2} f_0 + \delta^{ij} \frac{\Delta m_{\rm th}^2}{2} \int dK f_0
\end{aligned}$$

Equations of hydrodynamics

Nonequilibrium hydrodynamics (local rest frame)

Landau matching is defined by the eigenvalue problem: $u_\mu T^{\mu\nu} = \epsilon u^\nu$

Local rest frame: $u^\mu = (1,0,0,0)$ $\Rightarrow$ $T^{00} = \epsilon$, $T^{0i} = 0$ $\Rightarrow$ $\Delta T^{00} = 0$, $\Delta T^{0i} = 0$

Landau matching conditions:

$$\int dK E_k k^i \delta f = 0$$

$$\int dK \left[E_k^2 - T^2 \frac{dm_{\rm eq}^2}{dT^2} \right] \delta f = 0$$

Contains the medium correction

Viscous corrections:

$$\Delta T^{ij} = \int dK k^i k^j \delta f \Rightarrow \pi^{ij} = \int dK k^{\langle i} k^{j \rangle} \delta f, \quad \Pi = \frac{1}{3} \int dK \mathbf{k}^2 \delta f$$

Known structures but x-dependent mass enters the equations

Bulk viscosity

Anderson-Witting model

Boltzmann equation in the Anderson-Witting model

$$\boxed{\left(\tilde{k}^\mu \partial_\mu - \mathcal{E}_k \nabla \mathcal{E}_k \cdot \nabla_k\right) f = -\frac{(u \cdot \tilde{k})}{\tau_R} \Delta f}$$

LHS of the Boltzmann equation dictates the form of RHS

$$\delta f(k) = f_0(k)(1 + f_0(k))\phi(k)$$

$$\Delta f(k) = f_0(k)(1 + f_0(k)) \left(\phi(k) - \frac{T^2}{E_k} \frac{dm_{\mathrm{eq}}^2}{dT^2} \frac{\int dK \phi(k) f_0(k)(1 + f_0(k))}{\int dK E_k f_0(k)(1 + f_0(k))} \right)$$

$\phi = \phi_{\mathrm{s}} + \phi_{\mathrm{b}}$ (shear part + bulk part)

Transport coefficients

Anderson-Witting model: bulk viscosity

Solution of the A-W model for the bulk part:

$$\phi_{\mathrm{b}}(k) = \beta\tau_R(\partial_i u^i)(c_s^2 - 1/3)\left(E_k - \frac{1}{E_k}\frac{J_{3,0} - T^2(dm_{\mathrm{eq}}^2/dT^2)J_{1,0}}{J_{1,0} - T^2(dm_{\mathrm{eq}}^2/dT^2)J_{-1,0}}\right) \qquad \delta f = f_0(1+f_0)\phi$$

Bulk viscosity can be computed using: $\Pi = M\int dK\delta f$ and $\Pi = -\zeta\partial_i u^i$

Nonconformality parameter:

Microscopic: $M = -\frac{1}{3}\left(m_0^2 - a\beta_\lambda T^2\right)$

the consequence of mean field corrections

Macroscopic: $c_s^2 = \frac{dP_0/dT}{d\epsilon_0/dT}$ $\Rightarrow$ $\frac{1}{3} - c_s^2 = -\frac{MJ_{1,0}}{J_{3,0} - T^2(dm_{\mathrm{eq}}^2/dT^2)J_{1,0}}$

Bulk viscosity

Anderson-Witting model

Shear viscosity is not influenced by the mean field in the leading order

$$\frac{\eta}{\tau_R} = \frac{\epsilon_0 + P_0}{5}$$

Bulk viscosity of the Boltzmann (classical) gas:

$$\frac{\zeta_{\mathrm{Boltz}}}{\tau_R} \propto T^4 \left(\frac{1}{3} - c_s^2\right)^2$$

Bulk viscosity of the Bose-Einstein (quantum) gas:

$$\frac{\zeta}{\tau_R} \propto T^4 \left(\frac{1}{3} - c_s^2\right)^2 \underbrace{\frac{T}{m_x}}$$

Effect of the cut-off of infrared divergencies

Relaxation time approximation can be too crude to obtain a reliable form of bulk viscosity

Kubo formula for bulk relaxation time

Quantum-theoretical approach

Hydrodynamic modes

no other currents coupled to the energy-momentum tensor → two hydrodynamic modes

$$\partial_t \pi^{ij} = -\frac{\pi^{ij} - \pi^{ij}_{\mathrm{NS}}}{\tau_\pi}$$

$$\partial_\mu T^{\mu\nu} = 0 \quad + \quad \partial_t \Pi = -\frac{\Pi - \Pi_{\mathrm{NS}}}{\tau_\Pi}$$

shear mode:

$$0 = -\omega^2 \tau_\pi - i\omega + D_T \mathbf{k}^2$$

sound mode:

$$\begin{aligned} 0 = & -\omega^2 + v_s^2 \mathbf{k}^2 + i\omega(\tau_\pi + \tau_\Pi) - i\left(\frac{4D_T}{3} + \gamma + v_s^2(\tau_\pi + \tau_\Pi)\right)\omega \mathbf{k}^2 \\ & + \tau_\pi \tau_\Pi \omega^4 - \left(\tau_\pi \tau_\Pi v_s^2 + \tau_\Pi \frac{4D_T}{3} + \tau_\pi \gamma\right)\omega^2 \mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4) \end{aligned}$$

Linear response theory

Viscous hydrodynamics is a perfect realization of the linear response theory

deviation of a given observable from equilibrium $\longrightarrow$ equilibrium retarded response function

Linear response to transverse fluctuations:

$$\delta\langle \hat{T}^{x0}(t,k_y)\rangle = \beta_x(k_y)\int dt'\bar{G}_R^{x0,x0}(t-t',k_y)\theta(-t')e^{\varepsilon t'}$$

direction of the fluid velocity — direction of the momentum diffusion

Linear response to longitudinal fluctuations:

$$\delta\langle \hat{T}^{00}(t,\mathbf{k})\rangle = \beta_0(\mathbf{k})\int dt'\bar{G}_R^{00,00}(t-t',\mathbf{k})\theta(-t')e^{\varepsilon t'}$$

Quantum-theoretical approach

Gravitational Ward identity

conservation of the energy-momentum current in terms of the correlation function

$$\boxed{k_\alpha\left(\bar{G}^{\alpha\beta,\mu\nu}(k) - g^{\beta\mu}\langle\hat{T}^{\alpha\nu}\rangle - g^{\beta\nu}\langle\hat{T}^{\alpha\mu}\rangle + g^{\alpha\beta}\langle\hat{T}^{\mu\nu}\rangle\right) = 0}$$

- Stress-energy tensor represents the conservation laws and the generators of the space-time evolution
- Ward identity introduces constraints on the stress-energy response functions

stress-energy retarded correlation function

$$\begin{aligned}\bar{G}_R^{ij,mn}(x,y) &= -\delta^{(4)}(x-y)\left(\delta^{jm}\langle\hat{T}^{in}(y)\rangle + \delta^{jn}\langle\hat{T}^{im}(y)\rangle - \delta^{ij}\langle\hat{T}^{mn}(y)\rangle\right) \\ &\quad -i\theta(x_0-y_0)\langle[\hat{T}^{ij}(x),\hat{T}^{mn}(y)]\rangle\end{aligned}$$

Parametrization of the longitudinal fluctuations response function

1. Consequences of gravitational Ward identity

$$\omega^4 \bar{G}_R^{00,00}(\omega, \mathbf{k}) = \omega^4 \epsilon - \omega^2 \mathbf{k}^2 (\epsilon + P) + \mathbf{k}^4 \bar{G}_L(\omega, \mathbf{k})$$

2. Hydrodynamic limit $\omega \to 0$

$$\bar{G}_L(\omega, \mathbf{k}) \approx \frac{\omega^2}{\mathbf{k}^2}(\epsilon + P) + \frac{\omega^4}{\mathbf{k}^4}\left(\bar{G}_R^{00,00}(0, \mathbf{k}) - \epsilon\right)$$

3. General properties of the retarded Green function

$$\mathrm{Re}\, G_R(\omega, \mathbf{k}) = \mathrm{Re}\, G_R(-\omega, \mathbf{k}) \qquad \mathrm{Im}\, G_R(\omega, \mathbf{k}) = -\mathrm{Im}\, G_R(-\omega, \mathbf{k})$$

Parametrization of the longitudinal fluctuation response function

Most general form of the function:

$$\boxed{\bar{G}_L(\omega,\mathbf{k}) = \frac{\omega^2(\epsilon + P + \omega^2 Q(\omega,\mathbf{k}))}{\mathbf{k}^2 - \frac{\omega^2}{Z(\omega,\mathbf{k})} + i\omega^3 R(\omega,\mathbf{k})}}$$

All functions *Q*, *Z*, and *R* have the forms: $Q(\omega,\mathbf{k}) = Q_R(\omega,\mathbf{k}) - i\omega Q_I(\omega,\mathbf{k})$

All components are real-valued even functions of ω and $\mathbf{k}$

Z_R and R_R have non-zero limits when $\omega \to 0, \ \mathbf{k} \to 0$

All other parts of *Q*, *R*, and *Z* have finite limits when $\omega \to 0, \ \mathbf{k} \to 0$

Only small frequency and wavevector limits of the correlation function are important

Constitutive relationships

pole structure of $\bar{G}_L$

$$
\begin{aligned}
0 \;=\; & \omega^4(Z_I(\omega,\mathbf{k})R_R(\omega,\mathbf{k}) + R_I(\omega,\mathbf{k})Z_{R1}(0,\mathbf{k})) - \omega^2\mathbf{k}^2 Z_{R2}(\omega,\mathbf{k}) \\
& -\omega^2 + \mathbf{k}^2 Z_{R1}(0,\mathbf{k}) - i\omega\mathbf{k}^2 Z_I(\omega,\mathbf{k}) + i\omega^3 R_R(\omega,\mathbf{k})Z_{R1}(0,\mathbf{k}) \\
& -i\omega^5(R_I(\omega,\mathbf{k})Z_I(\omega,\mathbf{k}) + R_R(\omega,\mathbf{k})Z_{R2}(\omega,\mathbf{k})) - \omega^6 R_I(\omega,\mathbf{k})Z_{R2}(\omega,\mathbf{k})
\end{aligned}
$$

dispersion relation of the sound mode

$$
\begin{aligned}
0 \;=\; & -\omega^2 + v_s^2\mathbf{k}^2 + i\omega(\tau_\pi + \tau_\Pi) - i\left(\frac{4D_T}{3} + \gamma + v_s^2(\tau_\pi + \tau_\Pi)\right)\omega\mathbf{k}^2 \\
& +\tau_\pi\tau_\Pi\omega^4 - \left(\tau_\pi\tau_\Pi v_s^2 + \tau_\Pi\frac{4D_T}{3} + \tau_\pi\gamma\right)\omega^2\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4)
\end{aligned}
$$

constitutive relationships

$$Z_{R1}(0,0) = v_s^2$$

$$Z_{R2}(0,0) = \tau_\pi\tau_\Pi v_s^2 + \tau_\Pi\frac{4D_T}{3} + \tau_\pi\gamma \qquad R_R(0,0) = \frac{\tau_\pi + \tau_\Pi}{v_s^2}$$

$$Z_I(0,0) = \frac{4D_T}{3} + \gamma + v_s^2(\tau_\pi + \tau_\Pi) \qquad R_I(0,0) = \frac{v_s^2\tau_\pi\tau_\Pi - v_s^2(\tau_\pi + \tau_\Pi)^2 - (4D_T/3 + \gamma)(\tau_\pi + \tau_\Pi)}{v_s^4}$$

Kubo formulas related to the bulk flow

$$\frac{4}{3}\eta + \zeta = \lim_{\omega,\mathbf{k}\to 0} \frac{1}{\omega} \mathrm{Im}\, \bar{G}_L(\omega, \mathbf{k})$$

$$\frac{4}{3}\eta\tau_\pi + \zeta\tau_\Pi + \underbrace{Q_R v_s^2}_{-2\kappa/3} = -\frac{1}{2} \lim_{\omega,\mathbf{k}\to 0} \partial_\omega^2 \mathrm{Re}\, \bar{G}_L(\omega, \mathbf{k})$$

from metric perturbation analysis

Combining these relations with the Kubo formulas related to the shear flow we get

Kubo formulas related to the bulk flow

$$\boxed{\begin{aligned} \zeta &= \lim_{\omega,\mathbf{k}\to 0} \frac{1}{\omega} \mathrm{Im}\, \bar{G}_R^{PP}(\omega, \mathbf{k}) \\ \zeta\tau_\Pi &= -\frac{1}{2} \lim_{\omega,\mathbf{k}\to 0} \partial_\omega^2\, \mathrm{Re}\bar{G}_R^{PP}(\omega, \mathbf{k}) \end{aligned}}$$

Summary and conclusions

- The form of the nonequilibrium correction to the distribution function found
- Fully consistent incorporation of thermal mean field in the hydrodynamical description of the dynamics of one-component systems
- The physics of bulk viscosity studied for the Boltzmann and Bose-Einstein gases
- Relaxation time approximation can be too crude to study bulk viscosity of the quantum gases with Bose-Einstein distribution
- Kubo formula for the bulk relaxation time found