# A Bestiary of Feynman Integral Calabi-Yau Geometries in Weighted Projective Space

Matthias Wilhelm, Niels Bohr Institute

Elliptics ‘19, AEI Golm

September 17th, 2019

Phys.Rev.Lett. 122 (2019) no.3, 031601 [1810.07689] with J. Bourjaily, A. McLeod, M. von Hippel

[1909.xxxxx] with J. Bourjaily, A. McLeod, C. Vergu, M. Volk, M. von Hippel

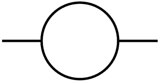
supported by VILLUM FONDEN

# Motivation

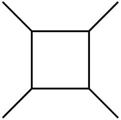
What numbers and functions occur in QFT?

What identities do they satisfy?

At one loop: Polylogarithms



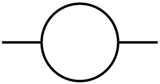
$$\overset{D=2}{\sim} \log\left(\frac{1+2\frac{m^2}{s}+\sqrt{1+4\frac{m^2}{s}}}{1+2\frac{m^2}{s}-\sqrt{1+4\frac{m^2}{s}}}\right),$$



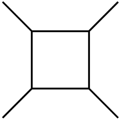
$$\overset{D=4}{\sim} \mathrm{Li}_2(\dots)+\dots$$

# One loop and Goncharov polylogarithms

At one loop: Polylogarithms



$$\overset{D=2}{\sim} \log\left(\frac{1+2\frac{m^2}{s}+\sqrt{1+4\frac{m^2}{s}}}{1+2\frac{m^2}{s}-\sqrt{1+4\frac{m^2}{s}}}\right),$$



$$\overset{D=4}{\sim} \mathrm{Li}_2(\dots)+\dots$$

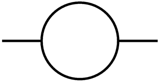
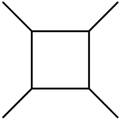
More generally: Goncharov polylogarithms [Chen (1977)], [Goncharov (1995)], ...

$$G(a_1,\dots,a_n;z)=\int_0^z \frac{dt}{t-a_1}G(a_2,\dots,a_n,t)$$

⇒ Well understood*

# One loop and Goncharov polylogarithms

At one loop: Polylogarithms

 $\overset{D=2}{\sim} \log\left(\frac{1+2\frac{m^2}{s}+\sqrt{1+4\frac{m^2}{s}}}{1+2\frac{m^2}{s}-\sqrt{1+4\frac{m^2}{s}}}\right)$,  $\overset{D=4}{\sim} \mathrm{Li}_2(\dots)+\dots$

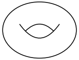
More generally: Goncharov polylogarithms [Chen (1977)], [Goncharov (1995)], ...

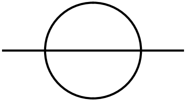
$$G(a_1,\dots,a_n;z)=\int_0^z \frac{dt}{t-a_1}G(a_2,\dots,a_n,t)$$

$\Rightarrow$ Well understood*

*For rational(!) arguments (see [Besier, van Straten, Weinzierl (2018)] for rationalizing roots)

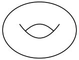
# Next stop: Elliptics

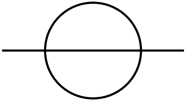
Beyond multiple polylogarithms: Integrals over elliptic curves 

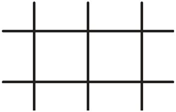
 


[Broadhurst, Fleischer, Tarasov (1993)], [. . . ,everybody in this room and their friends,. . . ]

# Next stop: Elliptics

Beyond multiple polylogarithms: Integrals over elliptic curves 

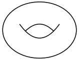


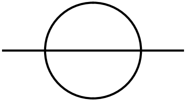



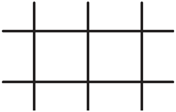
[Broadhurst, Fleischer, Tarasov (1993)], [. . . ,everybody in this room and their friends,. . . ]

[Paulos, Spradlin, Volovich (2012)], [Caron-Huot, Larsen (2012)], [Bourjaily, McLeod, Spradlin, von Hippel, MW (2017)]

# Next stop: Elliptics

Beyond multiple polylogarithms: Integrals over elliptic curves 



[Broadhurst, Fleischer, Tarasov (1993)], [. . . ,everybody in this room and their friends,. . . ]

[Paulos, Spradlin, Volovich (2012)], [Caron-Huot, Larsen (2012)], [Bourjaily, McLeod, Spradlin, von Hippel, MW (2017)]

$\Rightarrow$ Much recent progress! [. . . ,everybody in this room again,. . . ]

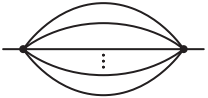
Elliptic multiple polylogarithms

$$\mathcal{E}\left(\begin{smallmatrix} n_1 \ldots n_k \\ c_1 \ldots c_k \end{smallmatrix}; x\right) = \int_0^x dt \Psi_{n_1}(c_1, t)\mathcal{E}\left(\begin{smallmatrix} n_2 \ldots n_k \\ c_2 \ldots c_k \end{smallmatrix}; t\right), \qquad \Psi_{-1}(c, x) = \frac{y_c}{y(x-c)} + \ldots$$

[Brown, Levin (2011)], [Brödel, Duhr, Dulat, Tancredi (2017)], [Brödel, Duhr, Dulat, Penante, Tancredi (2018)]

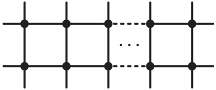
# This talk: Calabi-Yaus

## Beyond elliptics: Calabi-Yau manifolds



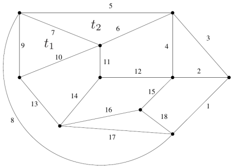
= picture of bananas, missing in the online version due to copyright

[Groote, Korner, Pivovarov (2005)], [Bloch, Kerr, Vanhove (2013,16)], . . . , [Brödel, Duhr, Dulat, Marzucca, Penante, Tancredi (2019)]



= picture of train tracks, missing in the online version due to copyright

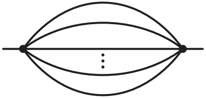
[Bourjaily, He, McLeod, von Hippel, MW (2017)]




[Brown, Schnetz (2010)]

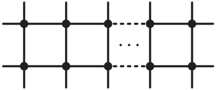
# This talk: Calabi-Yaus

Beyond elliptics: Calabi-Yau manifolds



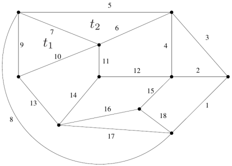
$=$ picture of bananas, missing in the online version due to copyright

[Groote, Korner, Pivovarov (2005)], [Bloch, Kerr, Vanhove (2013,16)], . . . , [Brödel, Duhr, Dulat, Marzucca, Penante, Tancredi (2019)]



$=$ picture of train tracks, missing in the online version due to copyright

[Bourjaily, He, McLeod, von Hippel, MW (2017)]




[Brown, Schnetz (2010)]

Questions:

- Further examples?
- What Calabi-Yau manifolds?
- How bad can it be (at given $L$, $D$)?

# Down the rabbit hole

picture of alice in wonderland looking down the rabbit hole, missing in the online version due to copyright

picture of tardigrade, missing in the online version due to copyright

# Table of contents

# Table of contents

# Numbers and functions beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?
$\rightarrow$ Different ways to see this (direct integration, leading singularities, differential equations)

# Numbers and functions beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?
→ Different ways to see this (direct integration, leading singularities, differential equations)

Direct integration: Consider a (Feynman) integral

$$I = \int_0^1 dx_1 dx_2 \dots dx_n \qquad \underbrace{\frac{1}{x_n^2 - P(x_1, \dots, x_{n-1})}}$$

# Numbers and functions beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?

→ Different ways to see this (direct integration, leading singularities, differential equations)

Direct integration: Consider a (Feynman) integral

$$I = \int_0^1 dx_1 dx_2 \dots dx_n \underbrace{\frac{1}{x_n^2 - P(x_1,\dots,x_{n-1})}}_{\frac{1}{2\sqrt{P(x_1,\dots,x_{n-1})}(x_n-\sqrt{P(x_1,\dots,x_{n-1})})}-\frac{1}{2\sqrt{P(x_1,\dots,x_{n-1})}(x_n+\sqrt{P(x_1,\dots,x_{n-1})})}}$$

# Numbers and functions beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?
→ Different ways to see this (direct integration, leading singularities, differential equations)

Direct integration: Consider a (Feynman) integral

$$I = \int_0^1 dx_1 dx_2 \dots dx_n \underbrace{\frac{1}{x_n^2 - P(x_1,\dots,x_{n-1})}}_{\frac{1}{2\sqrt{P(x_1,\dots,x_{n-1})}(x_n-\sqrt{P(x_1,\dots,x_{n-1})})} - \frac{1}{2\sqrt{P(x_1,\dots,x_{n-1})}(x_n+\sqrt{P(x_1,\dots,x_{n-1})})}}$$

$$= \int_0^1 dx_1 \dots dx_{n-1} \frac{\log(1-1/\sqrt{P(x_1,\dots,x_{n-1})})}{2\sqrt{P(x_1,\dots,x_{n-1})}} - \frac{\log(1+1/\sqrt{P(x_1,\dots,x_{n-1})})}{2\sqrt{P(x_1,\dots,x_{n-1})}}$$

# Numbers and functions beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?
$\rightarrow$ Different ways to see this (direct integration, leading singularities, differential equations)

Direct integration: Consider a (Feynman) integral

$$I = \int_0^1 dx_1 dx_2 \dots dx_n \underbrace{\frac{1}{x_n^2 - P(x_1, \dots, x_{n-1})}}_{\frac{1}{2\sqrt{P(x_1,\dots,x_{n-1})}(x_n - \sqrt{P(x_1,\dots,x_{n-1})})} - \frac{1}{2\sqrt{P(x_1,\dots,x_{n-1})}(x_n + \sqrt{P(x_1,\dots,x_{n-1})})}}$$

$$= \int_0^1 dx_1 \dots dx_{n-1} \frac{\log(1-1/\sqrt{P(x_1,\dots,x_{n-1})})}{2\sqrt{P(x_1,\dots,x_{n-1})}} - \frac{\log(1+1/\sqrt{P(x_1,\dots,x_{n-1})})}{2\sqrt{P(x_1,\dots,x_{n-1})}}$$

Case $n = 2$:
$\deg P(x_1) \leq 2 \Rightarrow$ Change of variables rationalizes $\sqrt{P(x_1)}$ $\rightarrow$MPL
$\deg P(x_1) \geq 3 \Rightarrow$ $\sqrt{P(x_1)}$ cannot be rationalized $\not\rightarrow$MPL

# Numbers and functions beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?
$\rightarrow$ Different ways to see this (direct integration, leading singularities, differential equations)

Direct integration: Consider a (Feynman) integral

$$I = \int_0^1 dx_1 dx_2 \dots dx_n \underbrace{\frac{1}{x_n^2 - P(x_1,\dots,x_{n-1})}}_{\frac{1}{2\sqrt{P(x_1,\dots,x_{n-1})}(x_n-\sqrt{P(x_1,\dots,x_{n-1})})} - \frac{1}{2\sqrt{P(x_1,\dots,x_{n-1})}(x_n+\sqrt{P(x_1,\dots,x_{n-1})})}}$$

$$= \int_0^1 dx_1 \dots dx_{n-1} \frac{\log(1-1/\sqrt{P(x_1,\dots,x_{n-1})})}{2\sqrt{P(x_1,\dots,x_{n-1})}} - \frac{\log(1+1/\sqrt{P(x_1,\dots,x_{n-1})})}{2\sqrt{P(x_1,\dots,x_{n-1})}}$$

Case $n=2$:
$\deg P(x_1) \leq 2 \Rightarrow$ Change of variables rationalizes $\sqrt{P(x_1)}$ $\rightarrow$MPL
$\deg P(x_1) \geq 3 \Rightarrow$ $\sqrt{P(x_1)}$ cannot be rationalized $\nrightarrow$MPL

General $n$:
$\deg_{x_i} P(x_1,\dots,x_{n-1}) \geq 3$ for all $i$ $\Rightarrow$ No integration possible within MPLs

# Geometry of Feynman integrals

Define $y = \sqrt{P(x_1, \ldots, x_{n-1})}$

$$I = \int_0^1 dx_1 \ldots dx_{n-1} \frac{\log(1-1/y)}{2y} - \frac{\log(1+1/y)}{2y}$$

What is the geometry of the (complex) algebraic variety defined by $y^2 = P(x_1, \ldots, x_{n-1})$?

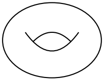
# Geometry of Feynman integrals

Define $y = \sqrt{P(x_1, \ldots, x_{n-1})}$

$$I = \int_0^1 dx_1 \ldots dx_{n-1} \frac{\log(1 - 1/y)}{2y} - \frac{\log(1 + 1/y)}{2y}$$

What is the geometry of the (complex) algebraic variety defined by $y^2 = P(x_1, \ldots, x_{n-1})$?

Case $n = 2$: Algebraic curve
$\deg P(x_1) = 3, 4 \Rightarrow$ Elliptic curve = Torus



# Geometry of Feynman integrals

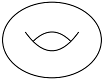
Define $y = \sqrt{P(x_1, \ldots, x_{n-1})}$

$$I = \int_0^1 dx_1 \ldots dx_{n-1} \frac{\log(1 - 1/y)}{2y} - \frac{\log(1 + 1/y)}{2y}$$

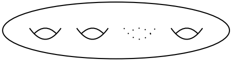
What is the geometry of the (complex) algebraic variety defined by $y^2 = P(x_1, \ldots, x_{n-1})$?

<u>Case $n = 2$:</u> Algebraic curve

$\deg P(x_1) = 3, 4 \Rightarrow$ Elliptic curve $=$ Torus



$\deg P(x_1) = 2g + 1, 2g + 2 \Rightarrow$ Riemann surface of genus $g$



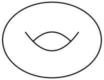
# Geometry of Feynman integrals

Define $y = \sqrt{P(x_1, \dots, x_{n-1})}$

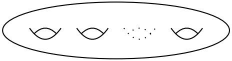
$$I = \int_0^1 dx_1 \dots dx_{n-1} \frac{\log(1 - 1/y)}{2y} - \frac{\log(1 + 1/y)}{2y}$$

What is the geometry of the (complex) algebraic variety defined by $y^2 = P(x_1, \dots, x_{n-1})$?

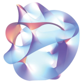
Case $n = 2$: Algebraic curve
$\deg P(x_1) = 3, 4 \Rightarrow$ Elliptic curve $=$ Torus



$\deg P(x_1) = 2g + 1, 2g + 2 \Rightarrow$ Riemann surface of genus $g$



General $n$: $n - 1$ (complex) dimensional variety
$P(x_1, \dots, x_{n-1}) = \sum_{\alpha_1, \dots, \alpha_{n-1}} c_{\alpha_1, \dots, \alpha_{n-1}} x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}}$ with $c_{\alpha_1, \dots, \alpha_{n-1}} = 0$ for $\alpha_1 + \dots + \alpha_{n-1} > 2n \Rightarrow$ Calabi-Yau (possibly singular)



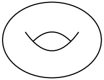
# Geometry of Feynman integrals

Define $y = \sqrt{P(x_1, \dots, x_{n-1})}$

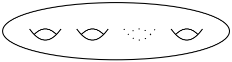
$$I = \int_0^1 dx_1 \dots dx_{n-1} \frac{\log(1 - 1/y)}{2y} - \frac{\log(1 + 1/y)}{2y}$$

What is the geometry of the (complex) algebraic variety defined by $y^2 = P(x_1, \dots, x_{n-1})$?

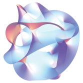
Case $n = 2$: Algebraic curve
$\deg P(x_1) = 3, 4 \Rightarrow$ Elliptic curve $=$ Torus



$\deg P(x_1) = 2g + 1, 2g + 2 \Rightarrow$ Riemann surface of genus $g$



General $n$: $n - 1$ (complex) dimensional variety
$P(x_1, \dots, x_{n-1}) = \sum_{\alpha_1,\dots,\alpha_{n-1}} c_{\alpha_1,\dots,\alpha_{n-1}} x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}}$ with $c_{\alpha_1,\dots,\alpha_{n-1}} = 0$ for $\alpha_1 + \dots + \alpha_{n-1} > 2n \Rightarrow$ Calabi-Yau (possibly singular)



Both cases studied in string theory! Which Calabi-Yaus?

# Reminder: Calabi-Yau manifolds

Calabi-Yau $(n-1)$-fold:

- Compact $(n-1)$-dimensional Kähler manifold
- Ricci flat
- vanishing first Chern class

picture of Eugenio Calabi, missing in the online version due to copyright

picture of Shing-Tung Yau, missing in the online version due to copyright

# Reminder: Calabi-Yau manifolds

Calabi-Yau $(n-1)$-fold:

- Compact $(n-1)$-dimensional Kähler manifold
- Ricci flat
- vanishing first Chern class

picture of Eugenio Calabi, missing in the online version due to copyright

picture of Shing-Tung Yau, missing in the online version due to copyright

Very abstract... hard to check ...

Classes of examples?

# Calabi-Yau manifolds in weighted projective space

*n*-dimensional projective space:

$$\mathbb{P}_n \quad = \mathbb{C}^{n+1} \setminus \{0\}/(x_1, \dots, x_{n+1}) \sim (\lambda \quad x_1, \dots, \lambda \quad x_{n+1})$$

# Calabi-Yau manifolds in weighted projective space

*n*-dimensional weighted projective space:

$$\mathbb{WP}^{\omega_1,\ldots,\omega_{n+1}} = \mathbb{C}^{n+1} \setminus \{0\} / (x_1, \ldots, x_{n+1}) \sim (\lambda^{\omega_1} x_1, \ldots, \lambda^{\omega_{n+1}} x_{n+1})$$

# Calabi-Yau manifolds in weighted projective space

$n$-dimensional weighted projective space:

$$\mathbb{WP}^{\omega_1,\ldots,\omega_{n+1}} = \mathbb{C}^{n+1} \setminus \{0\}/(x_1,\ldots,x_{n+1}) \sim (\lambda^{\omega_1}x_1,\ldots,\lambda^{\omega_{n+1}}x_{n+1})$$

$Q(x_1,\ldots x_{n+1})$ homogeneous polynomial in $\mathbb{WP}^{\omega_1,\ldots,\omega_{n+1}}$ with $\sum_i \omega_i = \deg Q$
$\Rightarrow$ $Q = 0$ defines a Calabi-Yau $(n-1)$-fold

Cover of the book "Calabi-Yau manifolds: A bestiary for physicists" by Tristan Hübsch, missing in the online version due to copyright

$n$-dimensional weighted projective space:

$$\mathbb{WP}^{\omega_1,\dots,\omega_{n+1}} = \mathbb{C}^{n+1} \setminus \{0\}/(x_1,\dots,x_{n+1}) \sim (\lambda^{\omega_1}x_1,\dots,\lambda^{\omega_{n+1}}x_{n+1})$$

$Q(x_1,\dots x_{n+1})$ homogeneous polynomial
in $\mathbb{WP}^{\omega_1,\dots,\omega_{n+1}}$ with $\sum_i \omega_i = \deg Q$
$\Rightarrow$ $Q = 0$ defines a Calabi-Yau $(n-1)$-fold

Cave: possibly singular, i.e. $\exists$ solutions to $\nabla Q = 0$

Cover of the book "Calabi-Yau manifolds: A bestiary for physicists" by Tristan Hübsch, missing in the online version due to copyright

# Calabi-Yau manifolds in weighted projective space

$n$-dimensional weighted projective space:

$$\mathbb{WP}^{\omega_1,\ldots,\omega_{n+1}} = \mathbb{C}^{n+1} \setminus \{0\}/(x_1,\ldots,x_{n+1}) \sim (\lambda^{\omega_1}x_1,\ldots,\lambda^{\omega_{n+1}}x_{n+1})$$

$Q(x_1,\ldots x_{n+1})$ homogeneous polynomial in $\mathbb{WP}^{\omega_1,\ldots,\omega_{n+1}}$ with $\sum_i \omega_i = \deg Q$
$\Rightarrow$ $Q = 0$ defines a Calabi-Yau $(n-1)$-fold

Cover of the book "Calabi-Yau manifolds: A bestiary for physicists" by Tristan Hübsch, missing in the online version due to copyright

Cave: possibly singular, i.e. $\exists$ solutions to $\nabla Q = 0$

Here: $Q(x_1,\ldots,x_n,y) = y^2 - P(x_1,\ldots,x_n)$ in $\mathbb{WP}^{1,\ldots,1,n}$

# Table of contents

# Symanzik representation of Feynman integrals

Feynman integral in $D$ dimensions with $L$ loops, $E$ internal (unit) propagators and trivial numerators:

$$I = \int_{x_i \geq 0} [d^{E-1} x_i] \frac{\mathfrak{U}^{E-(L+1)D/2}}{\mathfrak{F}^{E-LD/2}}$$

where

$$\mathfrak{U} = \sum_{\{T\} \in \mathfrak{T}_1} \prod_{e_i \notin T} x_i, \qquad \mathfrak{F} = \mathfrak{U} \sum_{e_i} x_i m_i^2 + \sum_{\{T_1, T_2\} \in \mathfrak{T}_2} s_{T_1, T_2} \prod_{e_i \notin T_1 \cup T_2} x_i$$

$\mathfrak{T}_k$: set of spanning $k$-forests of the graph
$m_i$: mass of propagator $i$
$s_{T_1, T_2}$: square of momenta flowing from $T_1$ into $T_2$

# Symanzik representation of Feynman integrals

$$I = \int_{x_i \geq 0} [d^{E-1} x_i] \frac{\mathfrak{U}^{E-(L+1)D/2}}{\mathfrak{F}^{E-LD/2}}$$

Two interesting subclasses for even $D$:

# Symanzik representation of Feynman integrals

$$I = \int_{x_i \geq 0} [d^{E-1} x_i] \frac{\mathfrak{U}^{E-(L+1)D/2}}{\mathfrak{F}^{E-LD/2}}$$

Two interesting subclasses for even $D$:

- $E = LD/2 \Rightarrow I = \int_{x_i \geq 0} [d^{E-1} x_i] \frac{1}{\mathfrak{U}^{D/2}}$ [Brown (2009)]
  - only $\mathfrak{U}$ polynomial $\Rightarrow$ no kinematics

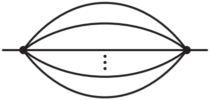
# Symanzik representation of Feynman integrals

$$I = \int_{x_i \geq 0} [d^{E-1} x_i] \frac{\mathfrak{U}^{E-(L+1)D/2}}{\mathfrak{F}^{E-LD/2}}$$

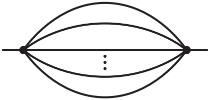
Two interesting subclasses for even $D$:

- $E = LD/2 \Rightarrow I = \int_{x_i \geq 0} [d^{E-1} x_i] \frac{1}{\mathfrak{U}^{D/2}}$ [Brown (2009)]
  - only $\mathfrak{U}$ polynomial $\Rightarrow$ no kinematics
- 'Marginal integrals': $E = (L+1)D/2 \Rightarrow I = \int_{x_i \geq 0} [d^{E-1} x_i] \frac{1}{\mathfrak{F}^{D/2}}$
  - only $\mathfrak{F}$ polynomial
  - compatible with $D$-gon power counting
  - compatible with maximal weight
  - avoid dimensional regularization if finite
  - $D = 2$: banana integrals (and their decorations with tadpoles)
  - $D = 4$: one-loop box and many more!

[Bourjaily, McLeod, von Hippel, MW (2018)]

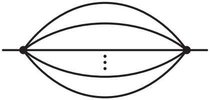
$I =$  $=$ picture of bananas, missing in the online version due to copyright $= \int_{x_i \geq 0} [d^L x_i] \frac{1}{\mathfrak{F}}$

# Marginal integrals in $D = 2$: Massive Banana Integrals

$I =$  $=$ picture of bananas, missing in the online version due to copyright $= \int_{x_i \geq 0} [d^L x_i] \frac{1}{\mathfrak{F}}$

$$\mathfrak{U} = \sum_{\{T\} \in \mathfrak{T}_1} \prod_{e_i \notin T} x_i = \sum_j \prod_{i \neq j} x_i = \Big( \prod_i x_i \Big) \sum_i \frac{1}{x_i}$$

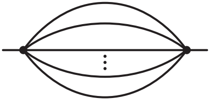
# Marginal integrals in $D = 2$: Massive Banana Integrals

$I =$  $=$ picture of bananas, missing in the online version due to copyright $= \int_{x_i \geq 0} [d^L x_i] \frac{1}{\mathfrak{F}}$

$$\mathfrak{U} = \sum_{\{T\} \in \mathfrak{T}_1} \prod_{e_i \notin T} x_i = \sum_j \prod_{i \neq j} x_i = \Big(\prod_i x_i\Big) \sum_i \frac{1}{x_i}$$

$$\mathfrak{F} = \mathfrak{U} \sum_{e_i} x_i m_i^2 + \sum_{\{T_1, T_2\} \in \mathfrak{T}_2} s_{T_1, T_2} \prod_{e_i \notin T_1 \cup T_2} x_i$$

$$= \Big(\prod_i x_i\Big) \Big[ \Big(\sum_i \frac{1}{x_i}\Big) \sum_i x_i m_i^2 + s \Big]$$

$I =$  $=$ picture of bananas, missing in the online version due to copyright $= \int_{x_i \geq 0} [d^L x_i] \frac{1}{\mathfrak{F}}$

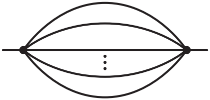
$$\mathfrak{U} = \sum_{\{T\} \in \mathfrak{T}_1} \prod_{e_i \notin T} x_i = \sum_j \prod_{i \neq j} x_i = \Big(\prod_i x_i\Big) \sum_i \frac{1}{x_i}$$

$$\mathfrak{F} = \mathfrak{U} \sum_{e_i} x_i m_i^2 + \sum_{\{T_1, T_2\} \in \mathfrak{T}_2} s_{T_1, T_2} \prod_{e_i \notin T_1 \cup T_2} x_i$$

$$= \Big(\prod_i x_i\Big) \Big[\Big(\sum_i \frac{1}{x_i}\Big) \sum_i x_i m_i^2 + s\Big]$$

$\mathfrak{F}$ quadratic in all $x_i$ ($m_i^2 \neq 0$ for all $i$)

# Marginal integrals in $D = 2$: Massive Banana Integrals

$I =$  $=$ picture of bananas, missing in the online version due to copyright $= \int_{x_i \geq 0} [d^L x_i] \frac{1}{\mathfrak{F}}$

$$\mathfrak{U} = \sum_{\{T\} \in \mathfrak{T}_1} \prod_{e_i \notin T} x_i = \sum_j \prod_{i \neq j} x_i = \Big(\prod_i x_i\Big) \sum_i \frac{1}{x_i}$$

$$\mathfrak{F} = \mathfrak{U} \sum_{e_i} x_i m_i^2 + \sum_{\{T_1, T_2\} \in \mathfrak{T}_2} s_{T_1, T_2} \prod_{e_i \notin T_1 \cup T_2} x_i$$

$$= \Big(\prod_i x_i\Big) \Big[ \Big( \sum_i \frac{1}{x_i} \Big) \sum_i x_i m_i^2 + s \Big]$$

$\mathfrak{F}$ quadratic in all $x_i$ ($m_i^2 \neq 0$ for all $i$)

Integrate in $x_{L+1}$ $\Rightarrow$ Logarithms and $y = \sqrt{\text{discriminant}_{x_{L+1}}(\mathfrak{F})}$

# Geometry of banana graphs

Projective polynomial of homogeneous degree $2L$ (quartic in each $x_i$):

$$
\begin{aligned}
P(x_1,\ldots,x_L) &= \text{discriminant}_{x_{L+1}}(\mathfrak{F}) \\
&= \Big(\prod_{i\neq L+1} x_i^2\Big)\Bigg[\Big(s+m_{L+1}^2+\Big(\sum_{i\neq L+1}\frac{1}{x_i}\Big)\sum_{i\neq L+1} x_i m_i^2\Big)^2 \\
&\qquad - 4m_{L+1}^2\Big(\sum_{i\neq L+1}\frac{1}{x_i}\Big)\sum_{i\neq L+1} x_i m_i^2\Bigg]
\end{aligned}
$$

Projective polynomial of homogeneous degree $2L$ (quartic in each $x_i$):

$$P(x_1,\ldots,x_L) = \text{discriminant}_{x_{L+1}}(\mathfrak{F})$$
$$= \Big(\prod_{i\neq L+1} x_i^2\Big)\Bigg[\Big(s+m_{L+1}^2+\Big(\sum_{i\neq L+1}\frac{1}{x_i}\Big)\sum_{i\neq L+1}x_i m_i^2\Big)^2 - 4m_{L+1}^2\Big(\sum_{i\neq L+1}\frac{1}{x_i}\Big)\sum_{i\neq L+1}x_i m_i^2\Bigg]$$

$Q(x_1,\ldots,x_L,y) = y^2 - P(x_1,\ldots,x_L)$ is a homogeneous polynomial of degree $2L$ in $L$-dimensional weighted projective space $\mathbb{WP}^{1,\ldots,1,L} = \mathbb{C}^{L+1}\setminus\{0\}/((x_1,\ldots,x_L,y)\sim(\lambda x_1,\ldots,\lambda x_L,\lambda^L y))$

# Geometry of banana graphs

Projective polynomial of homogeneous degree $2L$ (quartic in each $x_i$):

$$
\begin{aligned}
P(x_1,\ldots,x_L) &= \text{discriminant}_{x_{L+1}}(\mathfrak{F}) \\
&= \Big(\prod_{i\neq L+1} x_i^2\Big)\Bigg[\Big(s+m_{L+1}^2+\Big(\sum_{i\neq L+1}\frac{1}{x_i}\Big)\sum_{i\neq L+1} x_i m_i^2\Big)^2 \\
&\qquad - 4m_{L+1}^2\Big(\sum_{i\neq L+1}\frac{1}{x_i}\Big)\sum_{i\neq L+1} x_i m_i^2\Bigg]
\end{aligned}
$$

$Q(x_1,\ldots,x_L,y) = y^2 - P(x_1,\ldots,x_L)$ is a homogeneous polynomial of degree $2L$ in $L$-dimensional weighted projective space $\mathbb{WP}^{1,\ldots,1,L} = \mathbb{C}^{L+1}\setminus\{0\}/((x_1,\ldots,x_L,y)\sim(\lambda x_1,\ldots,\lambda x_L,\lambda^L y))$

Cover of the book "Calabi-Yau manifolds: A bestiary for physicists" by Tristan

$\sum$ weights = degree $\Rightarrow$ $Q=0$ defines a (singular)

# Marginal massless integrals in $D = 4$

Consider a finite massless integral in $D = 4$ with $E = 2L + 2$ propagators

$$I = \int_{x_i \geq 0} [d^{2L+1} x_i] \frac{1}{\mathfrak{F}^2}$$

# Marginal massless integrals in $D = 4$

Consider a finite massless integral in $D = 4$ with $E = 2L + 2$ propagators

$$I = \int_{x_i \geq 0} [d^{2L+1} x_i] \frac{1}{\mathfrak{F}^2}$$

Massless $\Rightarrow$ $\mathfrak{F}$ is linear in all $x_i$. Write $\mathfrak{F} = \mathfrak{F}_0^{(j)} + x_j \mathfrak{F}_1^{(j)}$

$$I = \int_{x_i \geq 0} [d^{2L} x_i] \frac{1}{\mathfrak{F}_0^{(j)} \mathfrak{F}_1^{(j)}}$$

# Marginal massless integrals in $D = 4$

Consider a finite massless integral in $D = 4$ with $E = 2L + 2$ propagators

$$I = \int_{x_i \geq 0} [d^{2L+1} x_i] \frac{1}{\mathfrak{F}^2}$$

Massless $\Rightarrow$ $\mathfrak{F}$ is linear in all $x_i$. Write $\mathfrak{F} = \mathfrak{F}_0^{(j)} + x_j \mathfrak{F}_1^{(j)}$

$$I = \int_{x_i \geq 0} [d^{2L} x_i] \frac{1}{\mathfrak{F}_0^{(j)} \mathfrak{F}_1^{(j)}}$$

$\mathfrak{F}$ is linear in all $x_i$ $\Rightarrow$ $\mathfrak{F}_0^{(j)}$ and $\mathfrak{F}_1^{(j)}$ linear in all $x_{i \neq j}$
Write $\mathfrak{F}_i^{(j)} = \mathfrak{F}_{i,0}^{(j,k)} + x_k \mathfrak{F}_{i,1}^{(j,k)}$

$$I = \int_{x_i \geq 0} [d^{2L-1} x_i] \frac{\log\left(\mathfrak{F}_{0,0}^{(j,k)} \mathfrak{F}_{1,1}^{(j,k)}\right) - \log\left(\mathfrak{F}_{0,1}^{(j,k)} \mathfrak{F}_{1,0}^{(j,k)}\right)}{\mathfrak{F}_{0,0}^{(j,k)} \mathfrak{F}_{1,1}^{(j,k)} - \mathfrak{F}_{0,1}^{(j,k)} \mathfrak{F}_{1,0}^{(j,k)}}$$

# Marginal massless integrals in $D = 4$

Consider a finite massless integral in $D = 4$ with $E = 2L + 2$ propagators

$$I = \int_{x_i \geq 0} [d^{2L+1} x_i] \frac{1}{\mathfrak{F}^2}$$

Massless $\Rightarrow$ $\mathfrak{F}$ is linear in all $x_i$. Write $\mathfrak{F} = \mathfrak{F}_0^{(j)} + x_j \mathfrak{F}_1^{(j)}$

$$I = \int_{x_i \geq 0} [d^{2L} x_i] \frac{1}{\mathfrak{F}_0^{(j)} \mathfrak{F}_1^{(j)}}$$

$\mathfrak{F}$ is linear in all $x_i$ $\Rightarrow$ $\mathfrak{F}_0^{(j)}$ and $\mathfrak{F}_1^{(j)}$ linear in all $x_{i \neq j}$
Write $\mathfrak{F}_i^{(j)} = \mathfrak{F}_{i,0}^{(j,k)} + x_k \mathfrak{F}_{i,1}^{(j,k)}$

$$I = \int_{x_i \geq 0} [d^{2L-1} x_i] \frac{\log\left(\mathfrak{F}_{0,0}^{(j,k)} \mathfrak{F}_{1,1}^{(j,k)}\right) - \log\left(\mathfrak{F}_{0,1}^{(j,k)} \mathfrak{F}_{1,0}^{(j,k)}\right)}{\mathfrak{F}_{0,0}^{(j,k)} \mathfrak{F}_{1,1}^{(j,k)} - \mathfrak{F}_{0,1}^{(j,k)} \mathfrak{F}_{1,0}^{(j,k)}}$$

If $\mathfrak{F}_{0,0}^{(j,k)} \mathfrak{F}_{1,1}^{(j,k)} - \mathfrak{F}_{0,1}^{(j,k)} \mathfrak{F}_{1,0}^{(j,k)}$ is quadratic in all $x_l$: Integrate in $x_l$
$\Rightarrow$ Dilogarithms and $y = \sqrt{P}$ with $P = \mathrm{discr}_{x_l}(\mathfrak{F}_{0,0}^{(j,k)} \mathfrak{F}_{1,1}^{(j,k)} - \mathfrak{F}_{0,1}^{(j,k)} \mathfrak{F}_{1,0}^{(j,k)})$

# Geometry

If $P$ is an irreducible cubic or higher for all $j, k, l$
$\Rightarrow$ No further polylogarithmic integration possible

# Geometry

If $P$ is an irreducible cubic or higher for all $j, k, l$
$\Rightarrow$ No further polylogarithmic integration possible

$Q = y^2 - P$ is a homogeneous polynomial of degree $4L - 2$ in $(2L - 1)$-dimensional weighted projective space
$\mathbb{WP}^{1,\dots,1,2L-1} = \mathbb{C}^{2L} \setminus \{0\}/((x_1, \dots, x_L, y) \sim (\lambda x_1, \dots, \lambda x_L, \lambda^{2L-1} y))$

If $P$ is an irreducible cubic or higher for all $j, k, l$
$\Rightarrow$ No further polylogarithmic integration possible

$Q = y^2 - P$ is a homogeneous polynomial of degree $4L - 2$ in $(2L-1)$-dimensional weighted projective space
$\mathbb{WP}^{1,\dots,1,2L-1} = \mathbb{C}^{2L} \setminus \{0\}/((x_1, \dots, x_L, y) \sim (\lambda x_1, \dots, \lambda x_L, \lambda^{2L-1} y))$

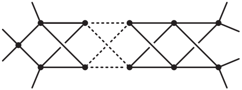
Cover of the book "Calabi-Yau manifolds: A bestiary for physicists" by Tristan Hübsch, missing in the online version due to copyright

$\sum$ weights = degree $\Rightarrow Q = 0$ defines a (possibly singular)

$(2L-2)$-dimensional Calabi-Yau manifold

# Tardigrades, amoebas and paramecia – a bestiary

Examples of finite, four-dimensional, massless, marginal Feynman integrals with $(2L-2)$-dimensional Calabi-Yau manifolds:

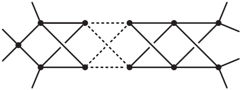
$L$ even: 



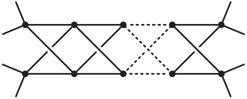
[Bourjaily, McLeod, von Hippel, MW (2018)]

# Tardigrades, amoebas and paramecia – a bestiary

Examples of finite, four-dimensional, massless, marginal Feynman integrals with $(2L-2)$-dimensional Calabi-Yau manifolds:

$L$ even: 



$L \geq 5$ odd: 

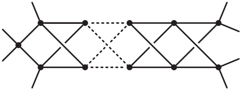
picture of an amoba, missing in the online version due to copyright

[Bourjaily, McLeod, von Hippel, MW (2018)]

# Tardigrades, amoebas and paramecia – a bestiary

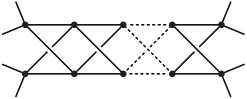
Examples of finite, four-dimensional, massless, marginal Feynman integrals with $(2L-2)$-dimensional Calabi-Yau manifolds:

$L$ even:



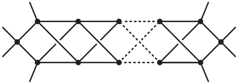


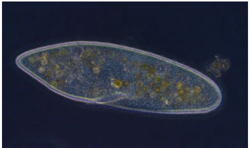
$L \geq 5$ odd:



picture of an amoba, missing in the online version due to copyright

$L \geq 1$ odd:







[Bourjaily, McLeod, von Hippel, MW (2018)]

# A bound on geometric rigidity

Geometric ‘rigidity’ $\simeq$ degree of ‘non-polylogarithmicity’

- Bounded in 4D massless marginal integrals by $2L - 2$
- Bound saturated by integrals above
- One-loop integrals are polylogarithmic
- (Polylogarithmic) $L$-loop integrals in 4D have maximum weight $2L$

$\Rightarrow$ Conjecture: same bound $2L - 2$ for all Feynman integrals in 4D

[Bourjaily, McLeod, von Hippel, MW (2018)]

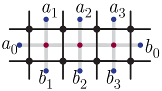
# Table of contents

# Planar three-loop examples

Loop-by-loop Feynman parametrization $\rightarrow$ rational 6-fold integral rep.
$+$ Residue analysis ($=$ leading singularity)

Triple box with K3 in $\mathbb{WP}^{1,1,1,3}$

$\mathfrak{T}^{(3)} =$

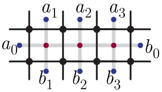



Component of three-loop 12-particle superamplitudes [Bourjaily, He, McLeod, von Hippel, MW (2018)]

# Planar three-loop examples

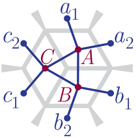
Loop-by-loop Feynman parametrization $\to$ rational 6-fold integral rep.
$+$ Residue analysis ($=$ leading singularity)

Triple box with K3 in $\mathbb{WP}^{1,1,1,3}$

$\mathfrak{T}^{(3)} =$ 


Component of three-loop 12-particle superamplitudes [Bourjaily, He, McLeod, von Hippel, MW (2018)]

Wheel with $CY_3$ in $\mathbb{WP}^{1,1,1,1,4}$

$\mathfrak{W}^{(3)} =$ 


[Bourjaily, McLeod, Vergu, von Hippel, Volk, MW (to apppear)]

# Table of contents

# Calabi-Yau hypersurfaces in weighted projective space

Goal: Analyze Calabi-Yau hypersurfaces $Q = 0$ embedded in $n$-dimensional weighted projective space $\mathbb{WP}^{1,\dots,1,n}$

Cave: $\exists$ Singularities $=$ solutions to $\nabla Q = 0$ $\Rightarrow$ Desingularize using a complex structure deformation $=$ Generic coefficients in $Q$

[Bourjaily, McLeod, Vergu, von Hippel, Volk, MW (to apppear)]

# Calabi-Yau hypersurfaces in weighted projective space

Goal: Analyze Calabi-Yau hypersurfaces $Q = 0$ embedded in $n$-dimensional weighted projective space $\mathbb{WP}^{1,\ldots,1,n}$

Cave: $\exists$ Singularities $=$ solutions to $\nabla Q = 0 \Rightarrow$ Desingularize using a complex structure deformation $=$ Generic coefficients in $Q$

Moduli space

$Q(x_1, \ldots, x_n, y)$ homogeneous of degree $2n$

Rescaling and shift in $y \Rightarrow Q(x_1, \ldots, x_n, y) = y^2 - P(x_1, \ldots, x_n)$

[Bourjaily, McLeod, Vergu, von Hippel, Volk, MW (to apppear)]

# Calabi-Yau hypersurfaces in weighted projective space

Goal: Analyze Calabi-Yau hypersurfaces $Q = 0$ embedded in $n$-dimensional weighted projective space $\mathbb{WP}^{1,\ldots,1,n}$

Cave: $\exists$ Singularities = solutions to $\nabla Q = 0 \Rightarrow$ Desingularize using a complex structure deformation = Generic coefficients in $Q$

Moduli space

$Q(x_1, \ldots, x_n, y)$ homogeneous of degree $2n$

Rescaling and shift in $y \Rightarrow Q(x_1, \ldots, x_n, y) = y^2 - P(x_1, \ldots, x_n)$

$P(x_1, \ldots, x_n)$ has $\binom{3n-1}{n-1}$ coefficients

$GL(n)$ eliminates $n^2 \Rightarrow \binom{3n-1}{n-1} - n^2$ (complex structure) moduli

[Bourjaily, McLeod, Vergu, von Hippel, Volk, MW (to apppear)]

# Calabi-Yau hypersurfaces in weighted projective space

Goal: Analyze Calabi-Yau hypersurfaces $Q = 0$ embedded in $n$-dimensional weighted projective space $\mathbb{WP}^{1,\dots,1,n}$

Cave: $\exists$ Singularities = solutions to $\nabla Q = 0 \Rightarrow$ Desingularize using a complex structure deformation = Generic coefficients in $Q$

Moduli space

$Q(x_1, \dots, x_n, y)$ homogeneous of degree $2n$
Rescaling and shift in $y \Rightarrow Q(x_1, \dots, x_n, y) = y^2 - P(x_1, \dots, x_n)$
$P(x_1, \dots, x_n)$ has $\binom{3n-1}{n-1}$ coefficients
$GL(n)$ eliminates $n^2 \Rightarrow \binom{3n-1}{n-1} - n^2$ (complex structure) moduli

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| #moduli | 1 | 19 | 149 | 976 | 6152 | 38711 | 245093 | 1562194 |

$\Rightarrow$ Concrete integrals sweep out a tiny part!

[Bourjaily, McLeod, Vergu, von Hippel, Volk, MW (to apppear)]

# Characterization: Cycles and differential forms

Reminder

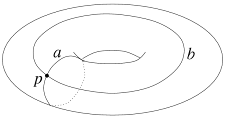
- Homology group: $H_k = \langle k\text{-cycles} \rangle$
- Betti numbers: $b_k = \operatorname{rank} H_k$

# Characterization: Cycles and differential forms

Reminder

- Homology group: $H_k = \langle k\text{-cycles}\rangle$
- Betti numbers: $b_k = \text{rank}\, H_k$

Example: Torus



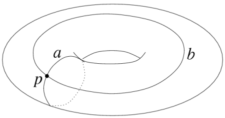

- e.g. $H_1 = \langle a, b\rangle$; $b_0 = 1$, $b_1 = 2$, $b_2 = 1$

# Characterization: Cycles and differential forms

Reminder

- Homology group: $H_k = \langle k\text{-cycles}\rangle$
- Betti numbers: $b_k = \operatorname{rank} H_k$
- De Rham cohomology: $H^k_{dR} = \dfrac{\ker d : k\text{-forms} \to (k+1)\text{-forms}}{d((k-1)\text{-forms})}$
- Poincaré duality + de Rham theorem $\Rightarrow$ $\dim H^k_{dR} = b_k$

Example: Torus



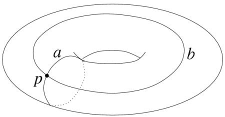

- e.g. $H_1 = \langle a, b\rangle$; $b_0 = 1$, $b_1 = 2$, $b_2 = 1$

# Characterization: Cycles and differential forms

Reminder

- Homology group: $H_k = \langle k\text{-cycles} \rangle$
- Betti numbers: $b_k = \operatorname{rank} H_k$
- De Rham cohomology: $H^k_{dR} = \dfrac{\ker d : k\text{-forms} \to (k+1)\text{-forms}}{d((k-1)\text{-forms})}$
- Poincaré duality $+$ de Rham theorem $\Rightarrow \dim H^k_{dR} = b_k$

Example: Torus



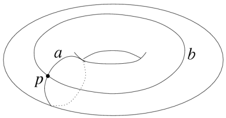

- e.g. $H_1 = \langle a, b \rangle$; $b_0 = 1$, $b_1 = 2$, $b_2 = 1$
- $H^1_{dR} = \langle [dz_1], [dz_2] \rangle$

# Characterization: Cycles and differential forms

Reminder

- Homology group: $H_k = \langle k\text{-cycles}\rangle$
- Betti numbers: $b_k = \text{rank}\, H_k$
- De Rham cohomology: $H^k_{dR} = \dfrac{\ker d : k\text{-forms} \to (k+1)\text{-forms}}{d((k-1)\text{-forms})}$
- Poincaré duality + de Rham theorem $\Rightarrow$ $\dim H^k_{dR} = b_k$
- Complex manifold: $d = \partial + \bar{\partial} \Rightarrow k\text{-forms} = \bigoplus_{p+q=k}(p,q)\text{-forms}$
- Dolbeault cohomology: $H^{p,q}_{\bar{\partial}} = \dfrac{\ker \partial \cap (p,q)\text{-forms}}{\bar{\partial}(p,q-1)\text{-forms}}$
- Hodge numbers: $h^{p,q} = \dim H^{p,q}_{\bar{\partial}}$, $\sum_{p+q=k} h^{p,q} = b_k$

Example: Torus



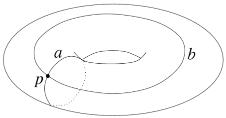

- e.g. $H_1 = \langle a, b\rangle$; $b_0 = 1,\ b_1 = 2,\ b_2 = 1$
- $H^1_{dR} = \langle [dz_1], [dz_2]\rangle$

# Characterization: Cycles and differential forms

Reminder

- Homology group: $H_k = \langle k\text{-cycles} \rangle$
- Betti numbers: $b_k = \operatorname{rank} H_k$
- De Rham cohomology: $H^k_{dR} = \dfrac{\ker\, d : k\text{-forms} \to (k+1)\text{-forms}}{d((k-1)\text{-forms})}$
- Poincaré duality $+$ de Rham theorem $\Rightarrow$ $\dim H^k_{dR} = b_k$
- Complex manifold: $d = \partial + \bar{\partial} \Rightarrow k\text{-forms} = \bigoplus_{p+q=k} (p,q)\text{-forms}$
- Dolbeault cohomology: $H^{p,q}_{\bar{\partial}} = \dfrac{\ker \partial \cap (p,q)\text{-forms}}{\bar{\partial}(p,q-1)\text{-forms}}$
- Hodge numbers: $h^{p,q} = \dim H^{p,q}_{\bar{\partial}}$, $\sum_{p+q=k} h^{p,q} = b_k$

Example: Torus




- e.g. $H_1 = \langle a, b \rangle$; $b_0 = 1$, $b_1 = 2$, $b_2 = 1$
- $H^1_{dR} = \langle [dz_1], [dz_2] \rangle$
- $H^{1,0}_{\bar{\partial}} = \langle [dz] \rangle$, $H^{0,1}_{\bar{\partial}} = \langle [d\bar{z}] \rangle$
- $h^{0,0} = h^{1,0} = h^{0,1} = h^{1,1} = 1$

# Hypersurfaces in weighted projective space $\mathbb{WP}^{1,\ldots,1,n}$

Hodge numbers via Batyrevs formalism and generalizations thereof, implemented in `PALP` $\subset$ `SageMath` [Kreuzer, Skarke (2002)]

[Bourjaily, McLeod, Vergu, von Hippel, Volk, MW (to apppear)]

# Hypersurfaces in weighted projective space $\mathbb{WP}^{1,\dots,1,n}$

Hodge numbers via Batyrevs formalism and generalizations thereof, implemented in `PALP` $\subset$ `SageMath` [Kreuzer, Skarke (2002)]

Hodge diamond for elliptic curve, $n = 2$:

$$\begin{matrix} & h^{0,0} & \\ h^{1,0} & & h^{0,1} \\ & h^{1,1} & \end{matrix} = \begin{matrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{matrix}$$

[Bourjaily, McLeod, Vergu, von Hippel, Volk, MW (to apppear)]

# Hypersurfaces in weighted projective space $\mathbb{WP}^{1,\ldots,1,n}$

Hodge numbers via Batyrevs formalism and generalizations thereof, implemented in PALP ⊂ SageMath [Kreuzer, Skarke (2002)]

Hodge diamond for K3 surface, $n = 3$:

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$$

[Bourjaily, McLeod, Vergu, von Hippel, Volk, MW (to apppear)]

Hodge numbers via Batyrevs formalism and generalizations thereof, implemented in PALP $\subset$ SageMath [Kreuzer, Skarke (2002)]

Hodge diamond for Calabi-Yau threefold, $n = 4$:

$$
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{3,0} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}
=
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 149 & & 149 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

[Bourjaily, McLeod, Vergu, von Hippel, Volk, MW (to apppear)]

Hodge numbers via Batyrevs formalism and generalizations thereof, implemented in `PALP` $\subset$ `SageMath` [Kreuzer, Skarke (2002)]

Hodge diamond for Calabi-Yau fourfold, $n = 5$:

$$
\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{3,0} & \\
h^{4,0} & & h^{3,1} & & h^{2,2} & & h^{1,3} & & h^{0,4} \\
 & h^{4,1} & & h^{3,2} & & h^{2,3} & & h^{1,4} & \\
 & & h^{4,2} & & h^{3,3} & & h^{2,4} & & \\
 & & & h^{4,3} & & h^{3,4} & & & \\
 & & & & h^{4,4} & & & &
\end{array}
=
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 976 & & 3952 & & 976 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

[Bourjaily, McLeod, Vergu, von Hippel, Volk, MW (to apppear)]

Hodge numbers via Batyrevs formalism and generalizations thereof, implemented in `PALP` $\subset$ `SageMath` [Kreuzer, Skarke (2002)]

Hodge diamond for Calabi-Yau fourfold, $n = 5$:

$$
\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{3,0} & \\
h^{4,0} & & h^{3,1} & & h^{2,2} & & h^{1,3} & & h^{0,4} \\
 & h^{4,1} & & h^{3,2} & & h^{2,3} & & h^{1,4} & \\
 & & h^{4,2} & & h^{3,3} & & h^{2,4} & & \\
 & & & h^{4,3} & & h^{3,4} & & & \\
 & & & & h^{4,4} & & & &
\end{array}
=
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 976 & & 3952 & & 976 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

(Calculated up to $n = 7$)

[Bourjaily, McLeod, Vergu, von Hippel, Volk, MW (to apppear)]

# Hypersurfaces in weighted projective space $\mathbb{WP}^{1,\ldots,1,n}$

Hodge numbers via Batyrevs formalism and generalizations thereof, implemented in `PALP` $\subset$ `SageMath` [Kreuzer, Skarke (2002)]

Hodge diamond for Calabi-Yau fourfold, $n = 5$:

$$
\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{3,0} & \\
h^{4,0} & & h^{3,1} & & h^{2,2} & & h^{1,3} & & h^{0,4} \\
 & h^{4,1} & & h^{3,2} & & h^{2,3} & & h^{1,4} & \\
 & & h^{4,2} & & h^{3,3} & & h^{2,4} & & \\
 & & & h^{4,3} & & h^{3,4} & & & \\
 & & & & h^{4,4} & & & &
\end{array}
=
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 976 & & 3952 & & 976 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

(Calculated up to $n = 7$)

Observation: Compatible with embedding in projective space

[Bourjaily, McLeod, Vergu, von Hippel, Volk, MW (to apppear)]

# Hypersurfaces in weighted projective space $\mathbb{WP}^{1,\ldots,1,n}$

Hodge numbers via Batyrevs formalism and generalizations thereof, implemented in `PALP` $\subset$ `SageMath` [Kreuzer, Skarke (2002)]

Hodge diamond for Calabi-Yau fourfold, $n = 5$:

$$
\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{3,0} & \\
h^{4,0} & & h^{3,1} & & h^{2,2} & & h^{1,3} & & h^{0,4} \\
 & h^{4,1} & & h^{3,2} & & h^{2,3} & & h^{1,4} & \\
 & & h^{4,2} & & h^{3,3} & & h^{2,4} & & \\
 & & & h^{4,3} & & h^{3,4} & & & \\
 & & & & h^{4,4} & & & &
\end{array}
=
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 976 & & 3952 & & 976 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

(Calculated up to $n = 7$)

Observation: Compatible with embedding in projective space

Closed formula for the Euler characteristic

$$\chi = \sum_k (-1)^k b_k = \frac{1 - (1-2n)^n + 2n^2}{2n}$$

[Bourjaily, McLeod, Vergu, von Hippel, Volk, MW (to apppear)]

# Table of contents

# Conclusions and Outlook

Conclusion

- Many quantities in QFT can be expressed in terms of MPLs
- Simplest functions beyond MPLs are eMPLs (much recent progress)
- Beyond: Infinite families of CY's $\subset$ $n$-dimensional $\mathbb{WP}^{1,\ldots,1,n}$
- Massive and massless examples, planar and non-planar, relevant for SM but occur even in planar $\mathcal{N}=4$ SYM theory
- Conjectured bound saturated for marginal integrals: $CY_{2L-2}$ at $L$ loops

# Conclusions and Outlook

Conclusion

- Many quantities in QFT can be expressed in terms of MPLs
- Simplest functions beyond MPLs are eMPLs (much recent progress)
- Beyond: Infinite families of CY's $\subset$ $n$-dimensional $\mathbb{WP}^{1,\dots,1,n}$
- Massive and massless examples, planar and non-planar, relevant for SM but occur even in planar $\mathcal{N}=4$ SYM theory
- Conjectured bound saturated for marginal integrals: $CY_{2L-2}$ at $L$ loops

Outlook

- More detailed analysis of geometries (using e.g. DEs, such as in [Brown, Schnetz (2010)], [Besier, Festi, Harrison, Naskrecki (2019)], [Verrill (1996)], [Vanhove, talk at StringMath 2019])
- Understanding of the class of functions beyond eMPLs and their identities

$\rightarrow$ Understanding of physical observables involving them

# Conclusions and Outlook

Conclusion

- Many quantities in QFT can be expressed in terms of MPLs
- Simplest functions beyond MPLs are eMPLs (much recent progress)
- Beyond: Infinite families of CY's $\subset$ $n$-dimensional $\mathbb{WP}^{1,\ldots,1,n}$
- Massive and massless examples, planar and non-planar, relevant for SM but occur even in planar $\mathcal{N}=4$ SYM theory
- Conjectured bound saturated for marginal integrals: $CY_{2L-2}$ at $L$ loops

Thank you!

Outlook

- More detailed analysis of geometries (using e.g. DEs, such as in [Brown, Schnetz (2010)], [Besier, Festi, Harrison, Naskrecki (2019)], [Verrill (1996)], [Vanhove, talk at StringMath 2019])
- Understanding of the class of functions beyond eMPLs and their identities

$\rightarrow$ Understanding of physical observables involving them