# Rademacher sums from Weierstrass to QED

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I give examples of situations in which **Rademacher sums** resolve the question of how to separate **quasi-periods** from periods.

At modular **weight 2**, I show how **Rademacher sums** restore to an **elliptic curve** a rational number **lost by Weierstrass**.

At higher genus, **Rademacher sums** emerge from **hyper-elliptic** curves.

Francis Brown encountered a **Rademacher sum** when studying quasi-periods at weight 12 and **level 1**. I show how to extend this work up to **weight 120**, by taking **Hadamard finite parts**.

In **QED**, 4-loop radiative corrections to the **magnetic moment** of the electron provide an example of quasi-periods, at **weight 4** and **level 6**, resolved by a **Rademacher sum**.

A combination of two **Rademacher sums** appears in QED at **6 loops**.

## Plan:

1. **Elliptic** curves up to level 50:
   an integer that **Weierstrass** forgot.

2. **Hyper-elliptic** curves up to level 71:
   quasi-periods for the **moonshine** primes.

3. **Quasi-periods** at level 1:
   results up to **weight 120**.

4. Laporta's 4-loop **magnetic moment** integrals:
   **quasi-periods** and **Rademacher sums**.

5. **6-loop** QED.

# 1 Elliptic curves up to level 50

At even weight $k$ and level $N$, I define **Rademacher sums**

$$R(k,N,m,n) = \sum_{c>0,\ \gcd(c,N)=1} \frac{2\pi I_{k-1}(4\pi\sqrt{mn/N}/c)}{\sqrt{nN/mc}} K(c,N,m,n)$$

which are sums of **Bessel** functions multiplied by **Kloosterman** sums

$$K(c,N,m,n) = \sum_{r\in[1,c],\ \gcd(r,c)=1} \exp\left(\frac{2\pi\mathrm{i}(mr-ns)}{c}\right)\Big|_{Nrs\ =\ 1\ \mathrm{mod}\ c}$$

with $R(2,1,1,1) = 196884$ famously exceeding by unity the dimension of the smallest non-trivial irreducible representation of the **monster group**.

My focus here is on Rademacher sums that are (almost certainly) **irrational** and (very probably) **transcendental** numbers.

Having discovered that 4-loop Feynman integrals determine $R(4,6,1,\pm1)$ at weight $k=4$ and level $N=6$, I turned my attention to weight $k=2$, which relates to **elliptic curves**.

Karl Weierstrass (1815–1897) gave a beautifully concise definition of the periods and quasi-periods of an elliptic curve $y^2 = 4x^3 - g_2 x - g_3$.
The **periods**, $(\omega_+, \omega_-)$, come from integrals of $\mathrm{d}x/y$.
Weierstrass **quasi-periods**, $(\eta_+^{\mathrm{W}}, \eta_-^{\mathrm{W}})$, from integrals of $x\mathrm{d}x/y$, satisfy

$$\omega_+ \eta_-^{\mathrm{W}} - \omega_- \eta_+^{\mathrm{W}} = 2\pi\mathrm{i}, \quad \eta_+^{\mathrm{W}} \omega_+ = \frac{\pi^2}{3} G_2\left(-\frac{\omega_-}{\omega_+}\right),$$

$$G_2(z) = 1 - 24 \sum_{n>0} \frac{n q^n}{1 - q^n}, \quad q = \exp(2\pi\mathrm{i}z).$$

Now I restore an integer that **Weierstrass forgot**. Consider the **quartic**

$$y^2 = Q(x) = x(x+4)(x+5)(x+9).$$

Its periods are delivered by the **arithmetic-geometric mean** of Gauss:

$$[\mathtt{agm}(4,5),\ \mathtt{agm}(3,5)] = \left[\frac{2\pi}{\omega_+}, \frac{-2\pi\mathrm{i}}{\omega_-}\right]$$

determining the weight 2 level 15 Rademacher sum

$$\sigma = R(2, 15, 1, -1) = \frac{\mathtt{agm}(4,5)\,\mathtt{agm}(3,5)}{4\pi}.$$

With a **Rademacher sum** $\rho = R(2,15,1,1)$, I define **quasi-periods**

$$\eta_+ = (\rho + \sigma)\omega_+, \quad \eta_- = (\rho - \sigma)\omega_-.$$

`Pari/GP` does **not** deliver these, since it is attuned to Weierstrass. Given the Cremona curve **15a1**, it delivers quasi-periods $\eta_\pm^{\mathrm{W}} = \eta_\pm - \frac{\mathbf{101}}{12}\omega_\pm$, where **101** is the integer that Weierstrass forgot. This is how it came about:

$$h^4 Q(1/h) = P(h) = (1+4h)(1+5h)(1+9h) = 180h^3 + \mathbf{101}h^2 + 18h + 1$$

and then **101** is lost if we force the roots of the cubic to sum to zero.

In terms of **Dedekind eta** quotients, the cubic $P(h)$ has a **modular parametrization** deriving from the level 15 **cusp form** $f = \eta_1\eta_3\eta_5\eta_{15}$:

$$h = \frac{1}{5}\left(\frac{\eta_3^5\eta_5}{\eta_1^5\eta_{15}} - 1\right) = q + 4q^2 + 12q^3 + 33q^4 + O(q^5)$$

$$d = \frac{q}{f}\frac{\mathrm{d}h}{\mathrm{d}q} = 1 + 9q + 46q^2 + 188q^3 + 647q^4 + O(q^5)$$

$$d^2 = P(h) = (1+4h)(1+5h)(1+9h), \quad \eta_n = q^{n/24}\prod_{k>0}(1-q^{nk}).$$

I have restored missing integers to all elliptic curves up to $N = 50$.

| $N$ | class | shift | coeffs | $N$ | class | shift | coeffs |
|---|---|---|---|---|---|---|---|
| 11 | $a$ | 188 | $[1]$ | 36 | $a$ | 24 | $[1]$ |
| 14 | $a$ | 109 | $[1]$ | 37 | $a$ | 60 | $[0,1]$ |
| **15** | $a$ | **101** | $[1]$ | 37 | $b$ | 100 | $[2,1]$ |
| 17 | $a$ | 81 | $[1]$ | 38 | $a$ | 457 | $[3,0,3]$ |
| 19 | $a$ | 64 | $[1]$ | 38 | $b$ | $-115$ | $[1,0,-1]$ |
| 20 | $a$ | 52 | $[1]$ | 39 | $a$ | $-127$ | $[1,0,-1]$ |
| 21 | $a$ | 61 | $[1]$ | 40 | $a$ | 36 | $[2]$ |
| 24 | $a$ | 44 | $[1]$ | 42 | $a$ | $-79$ | $[7/2,-1/2,-1]$ |
| 26 | $a$ | $-107$ | $[1,-1]$ | 43 | $a$ | 82 | $[0,1/2,1/2]$ |
| 26 | $b$ | 177 | $[1,1]$ | 44 | $a$ | $-20$ | $[2,-1]$ |
| 27 | $a$ | 36 | $[1]$ | 45 | $a$ | 57 | $[1,1]$ |
| 30 | $a$ | 13 | $[5/3,-1/3]$ | 46 | $a$ | $-195$ | $[7,2,-5]$ |
| 32 | $a$ | 24 | $[1]$ | 48 | $a$ | 28 | $[2]$ |
| **33** | $a$ | **137** | $[2,1]$ | 49 | $a$ | 21 | $[1]$ |
| 34 | $a$ | 37 | $[5/2,-1/2]$ | 50 | $a$ | 13 | $[1]$ |
| 35 | $a$ | 268 | $[1,1,1]$ | 50 | $b$ | 17 | $[1]$ |

Table 1: Rademacher–Weierstrass shifts for elliptic curves with conductors up to $N = 50$.

**Example:** The **Weierstrass** quasi-periods of Cremona curve **33a1** are $\eta_\pm - \frac{\mathbf{137}}{12}\omega_\pm$, where $\eta_\pm$ are the quasi-periods determined by a combination $2R(2,33,1,1) + R(2,33,2,1)$ of **Rademacher** sums, associated with

$$y^2 = (x+11)(x+15)(4x+33) = 4x^3 + \mathbf{137}x^2 + 1518x + 544.$$

## 2 Hyper-elliptic curves up to level 71

The majority of cusp forms of weight 2 are **not** of elliptic type. Consider for example level $N = 71$, which is the largest prime that divides the order of the monster group. Here we have a **hyper-elliptic** curve of **genus 6** and **degree 14**. A modular parametrization is achieved by

$$\begin{aligned} h &= \frac{f_6}{f_5 - 2f_6} = q - q^3 - q^4 + q^6 + O(q^8) \\ d &= \frac{h^5 q}{f_6} \frac{\mathrm{d}h}{\mathrm{d}q} = 1 + 2q - 3q^2 - 15q^3 - 13q^4 + 27q^5 + 62q^6 - 6q^7 + O(q^8) \\ d^2 &= P_1(h) P_2(h) \\ P_1(h) &= 1 - 7h^2 - 11h^3 + 5h^4 + 18h^5 + 4h^6 - 11h^7 \\ P_2(h) &= 1 + 4h + 5h^2 + h^3 - 3h^4 - 2h^5 + h^7 \end{aligned}$$

where $f_n = q^n + O(q^7)$ is a basis for the 6 cusp forms.

There are 6 pairs of periods and 6 pairs of quasi-periods to consider. The 6 pairs of periods come from Eichler integrals of **Hecke eigenforms** $F_n = \sum_{m=1}^{6} T_{n,m} f_m$ where the entries of matrix $\mathbf{T}$ are constructed from roots of the cubic $x^3 - x^2 - 4x + 3$, with **discriminant 257**.

For each eigenform $F_n$, I constructed a **weakly holomorphic** modular form $G_n$ such that periods of $F_n$ and the quasi-periods of $G_n$ determine a pair of numbers $(\rho_n, \sigma_n)$ in analogy with the elliptic case. These depend on an **embedding** of a number field. To transform them to **Rademacher sums**, I use the **transpose** of the matrix $\mathbf{T}$.

**Example:** At $N = 35$, with genus 3, I use

$$f = \eta_5^2\eta_7^2 = q + O(q^6), \quad h = \frac{\eta_1\eta_{35}}{\eta_5\eta_7} = q + O(q^2), \quad d = \frac{q}{f}\frac{\mathrm{d}h}{\mathrm{d}q} = 1 + O(q),$$

to parametrize a **hyper-elliptic curve** of degree 8

$$d^2 = (1 + h - h^2)(1 - 5h - 9h^3 - 5h^5 - h^6),$$

with a square root resolved by

$$d(z) = (1 - 2h - h^2)\left(\frac{\eta_1^4\eta_5^4 - 7^2\eta_7^4\eta_{35}^4}{2f^2h}\right) - \left(\frac{\eta_1^6\eta_7^6 - 5^3\eta_5^6\eta_{35}^6}{2f^3h}\right) = -d\left(\frac{-1}{35z}\right).$$

The space of **cusp forms** is spanned by $[f_1,\ f_2,\ f_3] = [1,\ h+h^2,\ h^2]f$, with $f_n = q^n + O(q^4)$. The space of **weakly** holomorphic forms is spanned by

$$\begin{aligned}
g_1 &= \frac{1-2h-5h^2-12h^3-41h^4-d}{2h^2}f = 35(2q^4+2q^5+q^6+O(q^7)),\\
g_2 &= \frac{1-h}{h}g_1 - 70h^2f = 35(6q^4+7q^5+9q^6+O(q^7)),\\
g_3 &= \frac{1-2h-h^2}{h^2}g_1 - 70h(1+2h)f = 35(15q^4+23q^5+38q^6+O(q^7)).
\end{aligned}$$

By construction $g_n(z)$ vanishes at the cusps at $z = \frac{1}{35}, \frac{1}{7}, \frac{1}{5}, \mathrm{i}\infty$.
It has an **exponential singularity** at small imaginary $z$, with

$$\varepsilon^2 g_n\left(\frac{\mathrm{i}\varepsilon}{\sqrt{N}}\right) = \frac{1}{Q^n} + O(Q), \quad Q = \exp\left(\frac{-2\pi}{\sqrt{N}\varepsilon}\right).$$

It solves a recurrence relation for Rademacher sums:

$$\sum_{m=1}^{\infty} R(2,N,m,n)q^m = g_n(z) + \sum_{m=1}^{3} R(2,N,m,n)f_m(z), \quad \text{for } n = 1,2,3.$$

The Hecke **eigenforms** are given by the **matrix** equation

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & \lambda_+ & \lambda_- \\ 1 & \lambda_- & \lambda_+ \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_2 \end{bmatrix}, \quad \lambda_\pm = \frac{\pm\sqrt{17}-1}{2}.$$

Let $\mathbf{T}$ denote the matrix above. I seek matrices $\mathbf{U}$ and $\mathbf{V}$ such that

$$[G_1,\, G_2,\, G_3] = [g_1,\, g_2,\, g_3]\mathbf{U}\mathbf{T}^{-1} + N[f_1,\, f_2,\, f_3]\mathbf{V}\mathbf{T}^{-1}$$

are the weakly holomorphic **eigenpartners** of the eigenforms $[F_1,\, F_2,\, F_3]$. The eigenpartnership is as follows. Consider **Eichler integrals**

$$P_n = \int_{z_1}^{z_2} F_n(z)\mathrm{d}z, \quad Q_n = \int_{z_1}^{z_2} G_n(z)\mathrm{d}z, \quad [z_1, z_2] = \left[\frac{\mathrm{i}-A}{N}, \frac{\mathrm{i}+B}{N}\right]$$

along the **horizontal** path with $\Im z = 1/N$, with integers $A$ and $B$ such that $AB \equiv 1 \bmod N$. I require constants, $\rho_n$ and $\sigma_n$, such that

$$\Im((Q_n + \rho_n P_n)P_n) = 0, \quad \Im((\overline{Q}_n - \sigma_n P_n)P_n) = 0,$$
$$R(2, N, +1, n) = \sum_{m=1}^{3} \rho_m T_{m,n}, \quad R(2, N, -1, n) = \sum_{m=1}^{3} \sigma_m T_{m,n}.$$

With $N = 35$, I take $A = 3$, $B = 12$ and determine the required matrices

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 9 \end{bmatrix}.$$

This pattern continues in the hyper-elliptic cases $N = 39, 41, 47, 59, 71$, with genera up to $g = 6$. The matrix $\mathbf{U}$ is **diagonal** with elements $U_{n,n} = n^{k-1}$, at weight $k$. The matrix $\mathbf{V}$ is **symmetric**, with $V_{n,1} = 0$. In the present case, with weight $k = 2$, I obtained

$$\mathbf{V}_{39} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}, \quad \mathbf{V}_{41} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix}, \quad \mathbf{V}_{47} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 7 \\ 0 & 1 & 7 & 16 \end{bmatrix},$$

$$\mathbf{V}_{59} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 2 & 0 & 2 & 4 \\ 0 & -1 & 2 & 6 & 14 \\ 0 & 0 & 4 & 14 & 35 \end{bmatrix}, \quad \mathbf{V}_{71} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 2 & 3 & 11 \\ 0 & -1 & 4 & 3 & 14 & 28 \\ 0 & 2 & 3 & 11 & 28 & 59 \end{bmatrix}.$$

## 3 Quasi-periods at level 1 up to weight 120

**Francis Brown** posted ideas [arXiv:1710.07912] on quasi-periods associated to modular forms. A definition of these has been strangely elusive at weights greater than 2. For the weight 12 level 1 cusp form

$$\Delta(z) = \eta_1^{24} = q\prod_{n>0}(1-q^n)^{24} = \frac{\Delta(-1/z)}{z^{12}}$$

with $q = \exp(2\pi \mathrm{i} z)$, **periods** are defined via $L(\Delta, s)$ which has 11 critical values at integers $s \in [1, 11]$. At odd integers these are given, up to rational multiples of powers of $\pi$, by $\omega_+$, while at even integers we use $\omega_-$. Specifically, in terms of $L(\Delta, 5)$ and $L(\Delta, 6)$, the **periods** are

$$\begin{aligned}
\omega_+ &= -70(2\pi)^{11}\int_0^\infty \Delta(\mathrm{i}y)y^4\mathrm{d}y\\
&= -68916772.809595194754310124655331030439069969\,1\ldots\\
\omega_- &= -6(2\pi)^{11}\int_0^\infty \Delta(\mathrm{i}y)y^5\mathrm{d}y\\
&= -5585015.3793104018668771392637962751296350334\,3\ldots
\end{aligned}$$

Brown associates **quasi-periods** with the **weakly** holomorphic modular form $\Delta'(z)$, defined in terms of Klein's $j$-invariant by

$$\begin{aligned}\Delta'(z) &= (j^2 - 1464j + 142236)\Delta(z) = 1/q + O(q^2),\\ j &= \frac{1}{\Delta(z)}\left(1 + 240\sum_{n>0}\frac{n^3q^n}{1-q^n}\right)^3 = \frac{1}{q} + 744 + 196884q + O(q^2)\end{aligned}$$

Numerical values of

$$\begin{aligned}\eta_+ &= 127202100647.17709477317161298610877494045988\ldots\\ \eta_- &= 10276732343.649132750817193072400920908893990\ldots\end{aligned}$$

are obtainable from a **determinant** and **permanent**,

$$\begin{aligned}\omega_+\eta_- - \omega_-\eta_+ &= (2\pi)^{11}10!\\ \frac{\omega_+\eta_- + \omega_-\eta_+}{4\pi\omega_+\omega_-} &= -\sum_{c>0}\frac{I_{11}(4\pi/c)}{c}\sum_{r\in(\mathbf{Z}/\mathbf{Z}c)^*}\exp\left(\frac{2\pi\mathrm{i}(r-s)}{c}\right)\Bigg|_{rs=1\,\mathrm{mod}\,c}\end{aligned}$$

with a **Rademacher** sum over Bessel functions and **Kloosterman** sums.

Integration of $\Delta'(z)z^{s-1}\mathrm{d}z$ along the imaginary axis $z = \mathrm{i}y$ requires one to handle the **singularity** $1/q$ in $\Delta'$ at large $y$, where $q$ is small.

The indefinite integral of $(\log(q))^{s-1}/q^2$ with respect to $q$ is easily performed for integers $s > 0$. Then one simply drops singular terms at the lower limit $q = 0$ while retaining them at the involution point $y = 1$, corresponding to $q = \exp(-2\pi)$. For $y < 1$, the involution $y \to 1/y$ leads to a similar prescription for the **Hadamard finite part**.

The **quasi-period polynomial** from integration of $(X - zY)^{10}\Delta'(z)\mathrm{d}z$ has a term $X^{10} - Y^{10}$, absent from the period polynomial.

From experience with Hecke eigenforms at level 71 and weight 2, I was able to handle all level 1 cases up to **weight 120**, with **10 eigenforms**.

The construction of 10 weakly holomorphic **eigenpartners** at weight $k = 120$ involves matrices with **large** integers. For example $U_{10,10} = 10^{119}$, while $V_{10,10}$ has 150 digits:

```
71743551479043323106025847609165970529550636954817\
15811526255378756621657659867939924492739540179038\
69536592195311460547291447667045944131309536870400.
```

# 4 The electron's magnetic moment

The **magnetic moment** of the electron, in Bohr magnetons, has QED contributions $\sum_{L\geq 0} a_L(\alpha/\pi)^L$ given up to $L = 4$ loops by

$$\begin{aligned}
a_0 &= 1 \quad [\textbf{Dirac}, 1928] \\
a_1 &= 0.5 \quad [\textbf{Schwinger}, 1947] \\
a_2 &= -0.32847896557919378458217281696489239241111929867962\ldots \\
a_3 &= 1.18124145658720000627475398221287785336878939093213\ldots \\
a_4 &= -1.91224576492644557415264716743983005406087339065872\ldots
\end{aligned}$$

**Petermann** and **Sommerfield** [1957] obtained

$$a_2 = \frac{197}{144} + \frac{\zeta(2)}{2} + \frac{3\zeta(3) - 2\pi^2 \log 2}{4}\,.$$

**Laporta and Remiddi** [1996] encountered $U_{3,1} = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n}$ in

$$\begin{aligned}
a_3 \;=\; & \frac{28259}{5184} + \frac{17101\zeta(2)}{135} + \frac{139\zeta(3) - 596\pi^2 \log 2}{18} \\
& - \frac{39\zeta(4) + 400U_{3,1}}{24} - \frac{215\zeta(5) - 166\zeta(3)\zeta(2)}{24}\,.
\end{aligned}$$

## 4.1 The first non-polylog

A **Bessel moment**

$$\begin{aligned} B &= -\int_0^\infty \frac{27550138x + 35725423x^3}{48600} I_0(x)K_0^5(x)\mathrm{d}x \\ &= -1483.68505914882529459059985184510836700500152630607810\ldots \end{aligned}$$

occurs at weight 4 in the breath-taking evaluation by **Stefano Laporta** [arXiv:1704.06996] of **4800 digits** of

$$a_4 = P + B + E + U \approx 2650.565 - 1483.685 - 1036.765 - 132.027 \approx -1.912$$

where $P$ comprises polylogs and $E$ comprises integrals, with weights 5, 6 and 7, whose integrands contain logs and products of elliptic integrals. $U$ comes from 6 light-by-light master integrals, still under investigation.

The weight-4 non-polylog term $B$ has $N = 6$ Bessel functions, with 5 instances of $K_0(x)$, from 5-fermion intermediate states. The sibling of $K_0(x)$ is $I_0(x) = \sum_{k\geq0}((x/2)^k/k!)^2$, from Fourier transformation.

Both master integrals in $B$ occur in $D = 2$ spacetime dimensions.

### 4.2 A simple determinant of Bessel moments

Consider **Bessel moments** of the form

$$M(a,b,c) = \int_0^\infty I_0^a(x) K_0^b(x) x^c \mathrm{d}x.$$

$2^L M(1, L+1, 1)$ is an $L$-loop **sunrise integral** at $D = 2$, on shell:

$$S_L(t) = \int_0^\infty \frac{\mathrm{d}x_1}{x_1} \cdots \int_0^\infty \frac{\mathrm{d}x_L}{x_L} \frac{1}{(1 + \sum_{j=1}^L x_j)(1 + \sum_{k=1}^L 1/x_k) - t}$$

$$S_4(1) = 2^4 M(1,5,1) = 2^4 \int_0^\infty I_0(x) K_0^5(x) x \mathrm{d}x.$$

Laporta encountered $M(1,5,1)$ as a master integral at $D = 4$. He also encountered $M(1,5,3)$, which is obtained by differentiation of $S_4(t)$ before setting $t = 1$. Now look at the simple **determinant**

$$\det \begin{bmatrix} M(1,5,1) & M(1,5,3) \\ M(2,4,1) & M(2,4,3) \end{bmatrix} = \frac{\pi^4}{24^2}$$

$M(2,4,1)$ comes from cutting an internal line. It occurred at stages of Laporta's $\varepsilon$-expansions. $M(2,4,3)$ comes from a cut and differentiation.

With $q=\exp(2\pi\mathrm{i}z)$ and $\Im(z)>0$, the **Dedekind eta** function satisfies

$$\eta(z)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)=\sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24}=\frac{\eta(-1/z)}{\sqrt{-\mathrm{i}z}}.$$

With $\eta_n=\eta(nz)$ I define the weight-4 level-6 **cusp form**

$$f_{4,6}(z)=(\eta_1\eta_2\eta_3\eta_6)^2=\sum_{n>0}A_6(n)q^n=\frac{f_{4,6}(-1/(6z))}{6^2z^4}.$$

For $\Re s>5/2$, there is a convergent **L-series**

$$L_6(s)=\sum_{n>0}\frac{A_6(n)}{n^s}=\frac{1}{1+2^{1-s}}\frac{1}{1+3^{1-s}}\prod_{p>3}\frac{1}{1-A_6(p)p^{-s}+p^{3-2s}}.$$

Its analytic continuation is provided by the **Eichler integral**

$$L_6(s)=\frac{(2\pi)^s}{\Gamma(s)}\int_0^\infty f_{4,6}(\mathrm{i}y)y^{s-1}\mathrm{d}y$$

with **critical** values related to **Bessel moments** as follows

$$L_6(\mathbf{2})=\frac{2}{\pi^2}M(1,5,1)=\frac{2}{3}M(3,3,1),\ \ L_6(\mathbf{1})=\frac{2}{\pi^2}M(2,4,1)=\frac{3}{\pi^2}L_6(\mathbf{3}).$$

### 4.3 Quasi-periods in QED

At weight 12 and level 1, the Eichler integrals for the quasi-periods of $\Delta'(z)$ blow up **exponentially** as $z \to \mathrm{i}\infty$ and as $z \to 0$. I resolved that issue by taking **Hadamard finite parts**. At **weight 4** and **level 6**, in QED, the quasi-periods come from **convergent** Eichler integrals

$$\frac{D_2}{2} = \frac{M(1,5,1)}{\pi^4} = \frac{4M(1,5,3)}{\pi^4} + \frac{5E_2}{18}$$

$$\frac{3D_1}{5} = \frac{M(2,4,1)}{\pi^3} = \frac{4M(2,4,3)}{\pi^3} + \frac{E_1}{3}$$

$$\begin{bmatrix} D_s \\ E_s \end{bmatrix} = -\int_{1/\sqrt{3}}^{\infty} \begin{bmatrix} f_{4,6}\left(\frac{1+\mathrm{i}y}{2}\right) \\ g_{4,6}\left(\frac{1+\mathrm{i}y}{2}\right) \end{bmatrix} y^{s-1}\mathrm{d}y,$$

$$g_{4,6} = \frac{(w^2-3)^2(w^4+9)f_{4,6}}{8w^4} = 5q + 102q^2 + 945q^3 + O(q^4),$$

$$w = 3\,\frac{\eta_2^2\eta_3^4}{\eta_1^4\eta_6^2}, \quad f_{4,6} = q - 2q^2 - 3q^3 + O(q^4),$$

$$D_1E_2 - D_2E_1 = \frac{1}{24\pi^3},$$

satisfying the determinant criterion. What about the permanent?

### 4.4 Rademacher sums in QED

From ratios of Feynman integrals, I form

$$\frac{M(1,5,1)+M(1,5,3)}{5M(1,5,1)} = \frac{\rho-\sigma}{144}, \qquad \frac{M(2,4,1)+M(2,4,3)}{5M(2,4,1)} = \frac{\rho+\sigma}{144},$$

which I here relate to the **Rademacher sums**

$$\begin{aligned}
\rho &= \sum_{c=1,5 \bmod 6} \frac{2\pi \mathbf{I}_3(4\pi/(\sqrt{6}c))}{\sqrt{6}c} \sum_{r\in(\mathbf{Z}/\mathbf{Z}c)^*} \exp\left(\frac{2\pi\mathrm{i}(r\ -\ s)}{c}\right)\Bigg|_{6rs=1 \bmod c} \\
\sigma &= \sum_{c=1,5 \bmod 6} \frac{2\pi \mathbf{J}_3(4\pi/(\sqrt{6}c))}{\sqrt{6}c} \sum_{r\in(\mathbf{Z}/\mathbf{Z}c)^*} \exp\left(\frac{2\pi\mathrm{i}(r\ +\ s)}{c}\right)\Bigg|_{6rs=1 \bmod c} \\
&= \frac{\pi^4}{40M(1,5,1)M(2,4,1)} = \frac{1}{6(4\pi)^3(f_{4,6},f_{4,6})} = 0.8852376196\ldots
\end{aligned}$$

with an oscillating Bessel function $\mathbf{J}_3(x) = \mathrm{i}\mathbf{I}_3(\mathrm{i}x)$ in $\sigma$, which is proportional to the inverse of the **Petersson norm** of the weight 4 level 6 modular form $f_{4,6}$. My evaluations of Feynman integrals determine 10,000 decimal digits of $\rho = 30.58572642\ldots$. I conclude that the **quasi-periods** are proportional to $M(1,5,1)+M(1,5,3)$ and $M(2,4,1)+M(2,4,3)$.

## 5 Quasi-periods in six-loop QED

Consider the **Fourier** expansion of the weight-6 level-6 **cusp form**

$$f_{6,6}(z) = \frac{\eta_2^9\eta_3^9}{\eta_1^3\eta_6^3} + \frac{\eta_1^9\eta_6^9}{\eta_2^3\eta_3^3} = \sum_{n>0} A_8(n)q^n = -\frac{f_{6,6}(-1/(6z))}{6^3 z^6}.$$

For $\Re s > 7/2$, there is a convergent **L-series**

$$L_8(s) = \sum_{n>0}\frac{A_8(n)}{n^s} = \frac{1}{1-2^{2-s}}\frac{1}{1+3^{2-s}}\prod_{p>3}\frac{1}{1-A_8(p)p^{-s}+p^{5-2s}}.$$

Its analytic continuation is provided by the **Eichler integral**

$$L_8(s) = \frac{(2\pi)^s}{\Gamma(s)}\int_0^\infty f_{6,6}(\mathrm{i}y)y^{s-1}\mathrm{d}y$$

with **critical** values related to **Bessel moments** as follows

$$L_8(\mathbf{4}) = \frac{4}{9\pi^2}M(1,7,1) = \frac{4}{9}M(3,5,1) = \frac{\pi^2}{9}L_8(\mathbf{2}),$$

$$L_8(\mathbf{5}) = \frac{4}{27}M(2,6,1) = \frac{2\pi^2}{21}M(4,4,1) = \frac{2\pi^2}{21}L_8(\mathbf{3}) = \frac{\pi^4}{54}L_8(\mathbf{1}).$$

### 5.1 Eichler integrals at 6 loops

Eichler integrals appear in the weight 6 level 6 determinant, as follows:

$$
\begin{aligned}
\det\left[\begin{array}{ll} M(1,7,1) & 32M(1,7,3)-64M(1,7,5) \\ M(2,6,1) & 32M(2,6,3)-64M(2,6,5)\end{array}\right] &= \frac{5\pi^6}{192},\\
\frac{F_2}{4} = \frac{M(1,7,1)}{\pi^6} &= \frac{32M(1,7,3)-64M(1,7,5)}{\pi^6}+\frac{35G_2}{108},\\
\frac{9F_1}{28} = \frac{M(2,6,1)}{\pi^5} &= \frac{32M(2,6,3)-64M(2,6,5)}{\pi^5}+\frac{5G_1}{12},\\
\left[\begin{array}{c} F_s \\ G_s \end{array}\right] &= -\int_{1/\sqrt{3}}^{\infty}\left[\begin{array}{c} f_{6,6}\left(\frac{1+\mathrm{i}y}{2}\right) \\ g_{6,6}\left(\frac{1+\mathrm{i}y}{2}\right)\end{array}\right](3y^2-1)y^{s-1}\mathrm{d}y,\\
g_{6,6} = \frac{(w^2-3)^4 f_{6,6}}{16w^4} &= q+36q^2+567q^3+5264q^4+O(q^5),\\
w = 3\,\frac{\eta_2^2\eta_3^4}{\eta_1^4\eta_6^2}, \quad f_{6,6} &= q+4q^2-9q^3+16q^4+O(q^5),\\
F_1G_2-F_2G_1 &= \frac{1}{4\pi^5}.
\end{aligned}
$$

### 5.2 Rademacher sums at 6 loops

At weight 6 and level 6, there are **two** independent irrational Rademacher sums in play. I found that

$$\frac{G_1 + 9F_1}{F_1} + \frac{G_2 + 9F_2}{F_2} = \frac{2R(6,6,2,1) - 5R(6,6,1,1)}{12}.$$

It does **not** appear to be possible to determine any other combination of $R(6,6,2,1)$ and $R(6,6,1,1)$ from **8-Bessel moments**, since the relations

$$M(1,7,1) = 72M(1,7,3) + 7\pi^4/48, \quad M(2,6,1) = 72M(2,6,3)$$

lead to determinations of $M(a,b,c)$ by periods and quasi-periods in the the cases $(a,b) = (1,7)$ and $(a,b) = (2,6)$. Nor does the case $(a,b) = (3,5)$ yield essentially new numbers. **Instead**, the combination

$$3R(6,6,2,1) - 3R(6,6,1,1) = R(6,3,1,1) + 3^5$$

indicates a connection to the **old space** at weight 6, coming from level $N = 3$ and spanned by $f_{6,3}(z) = (\eta_1\eta_3)^6$ and $f_{6,3}(2z) = (\eta_2\eta_6)^6$.

## Conclusions

1. Rademacher sums define quasi-periods for elliptic curves, restoring an integer that Weierstrass forgot.
2. In the hyper-elliptic cases at levels $N = 35, 39, 41, 47, 59, 71$, matrix transformations relate Rademacher sums to Eichler integrals of weakly holomorphic forms.
3. The most difficult case is level $N = 1$, since there are only two cusps and the weakly holomorphic forms blow up, exponentially, at each. This problem was overcome by taking Hadamard finite parts, leading to explicit relations between Rademacher sums and quasi-periods for all weights up to 120.
4. QED benefits neatly from level 6. At 4 loops, a Rademacher sum resolves the separation of quasi-periods from periods.
5. A combination of two Rademacher sums appears in QED at 6 loops.