Elliptics and integrability

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Goal: answer

- What is *integrability*?
- What is *elliptic integrability*?
- Where are all the elliptic functions?

Today's toy model

Classical M-dim many-body hamiltonian system:

$$H = \sum_{j=1}^{M} \frac{p_j^2}{2} + g \sum_{j<k}^{M} V(x_j - x_k), \qquad H : \mathcal{P} \to \mathbb{C}$$

with

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- g coupling constant

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The potentials

$$V(x) \sim \frac{1}{x^2}, \quad V(x) \sim \frac{1}{\sin^2 x}, \quad V(x) \sim \frac{1}{\sinh^2 x}, \quad V(x) \sim \wp(x)$$

are special and define the **Calogero-Sutherland-Moser** models.

Classical integrability

The classical CSM-models are **Liouville integrable**, i.e.

- there exist M **integrals of motion** $I_1 = H, I_2, \dots, I_M$ such that
  - the gradients dI_j are linearly independent
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This means

- the flows form a **regular foliation** of the phase space.
- the **Liouville-Arnold** theorem applies, implying there exist so-called **action-angle coordinates**:
  - *action*: $\tilde{p}_j$ such that $I_j = I_j\left(\{p_k\}\right)$
  - *angle*: $\tilde{x}_j \in \mathbb{T}$ or $\in \mathbb{R}$ that increase linearly under the flow of the I_k.

Example: harmonic oscillator

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The phase space $\mathbb{R}^2$ is foliated by circles of fixed radius ρ.

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Since the trace is *polynomial* in eigenvalues, it follows the **eigenvalues** are also conserved!

Elliptics appear

Consider again

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then there exists a Lax pair of $M \times M$-matrices

$$\begin{aligned} L_{jk} &= p_j \delta_{jk} + (1 - \delta_{jk}) f(x_j - x_k) \\ M_{jk} &= (1 - \delta_{jk}) h(x_j - x_k) - \delta_{jk} \sum_{n \neq j} V(x_j - x_n) \end{aligned}$$

if

$$f(x)h(y) - f(y)h(x) = f(x+y)(V(y) - V(x)).$$

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The most general **meromorphic** solution is $V(x) \sim \wp(x)$

Quantum case

Replacing $p_j \to \hat{p}_j = -\mathrm{i}\hbar \frac{\partial}{\partial x_j}$ yields the **Schrödinger operator** (or simply PDO)

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We could follow the Lax route, but now $I_k = \mathrm{tr} L^k$ is complicated and generically $[H, I_k] \neq 0$.

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Let us suppose we want meromorphic eigenfunctions

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General solution: $V(x) \sim \wp(x)$

Scattering: classical

x_1

x_2

$\vdots$

x_M

$t = -\infty$ $t = \infty$

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p_1^- x_1

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$\vdots$

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$$L = \begin{pmatrix} p_1^- & & \\ & \ddots & \\ & & p_M^- \end{pmatrix}$$

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Conclusion: scattering is *nondiffractive* and asymptotic momenta are *conserved*

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Remember: we have M integrals of motion I_j and the wavefunction Ψ has

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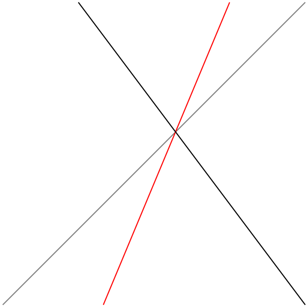
Bethe wave function: $\Psi(x) = \sum_{\tau \in S_M} A(\tau) e^{\mathrm{i}p \cdot x_\tau}$

Yang-Baxter equation



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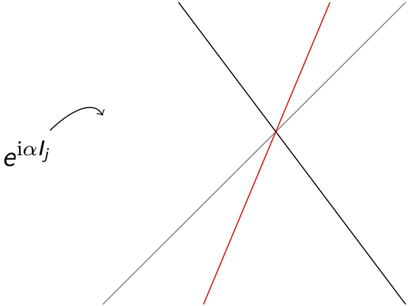

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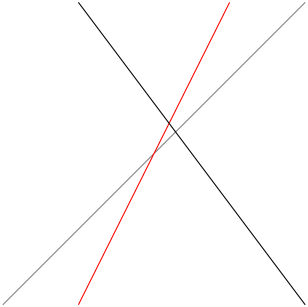

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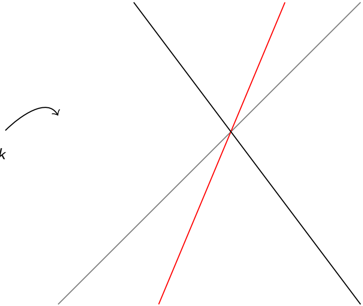
$e^{\mathrm{i}\alpha I_k}$



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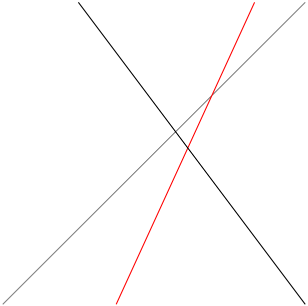

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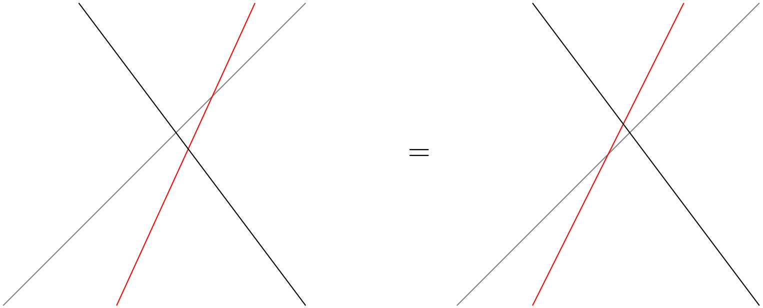

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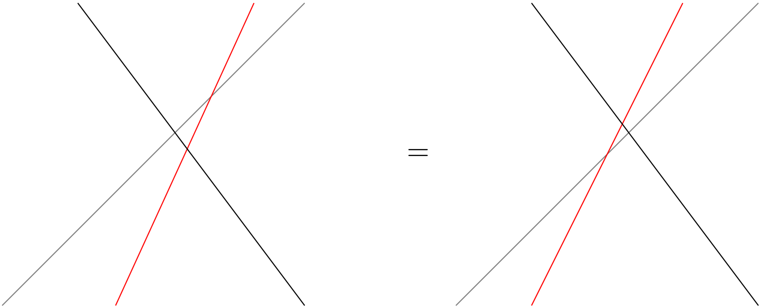

$$S_{23}S_{13}S_{12} = S_{12}S_{13}S_{23}$$

Spin

So far **spinless** particles, but **adding spin** is possible:

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with S_{jk} some spin function/operator.

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$$H = \sum_{j<k}^{M} V(x_j^* - x_k^*) S_{jk}.$$

A **spin chain**.

A web of chains

$$H = -J \sum_{j<k}^{L} V(j-k)\,\left(\sigma_j^{(x)}\sigma_k^{(x)} + \sigma_j^{(y)}\sigma_k^{(y)} + \sigma_j^{(z)}\sigma_k^{(z)}\right)$$

$$\begin{tikzcd}
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Heisenberg XXX

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Quantum CSM:

$$-\mathrm{i}\hbar\frac{\partial}{\partial x_j} \to \hat{p}_j := e^{-\mathrm{i}\hbar\alpha\frac{\partial}{\partial x_j}}$$

turns a PDO into a **finite-difference** operator. Modulo ordering these yield the quantum **Ruijsenaars-Schneider** models.

Deforming the operator

So far we have only talked about **potentials** V, but we can deform the **operators** too.

Quantum CSM:

$$-\mathrm{i}\hbar\frac{\partial}{\partial x_j} \to \hat{p}_j := e^{-\mathrm{i}\hbar\alpha\frac{\partial}{\partial x_j}}$$

turns a PDO into a **finite-difference** operator. Modulo ordering these yield the quantum **Ruijsenaars-Schneider** models. Very recently, sense has been made of a further deformation

$$\hat{p}_j \to \sum \# \hat{p}_\circ \dots \hat{p}_\circ$$

creating a **doubly-elliptic** model.

Spin chain operators

First *simple*: Heisenberg spin chain $V(x) = \delta_{|x|,1}$

Name	Operator	symmetry	R matrix
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- Forms a **building block** for the hamiltonian in a *fixed* recipe
- has characteristic *analytical* behaviour in an auxiliary parameter
- solves the quantum **Yang-Baxter equation**

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}, \qquad R_{jk} = R_{jk}(u_j - u_k)$$

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$$\varepsilon_M = \sum^{M} \varepsilon_1$$

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- that solving the spectral problem involves equations of rational/trigonometric/elliptic type, the **Bethe Ansatz Equations**

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Open question: How does this structure carry over beyond the nearest-neighbour case, i.e. other V?

Vertex Models

Present day topics

Directions I have not explored:

- Vertex models: dynamical R matrices $\leftrightarrow$ height models/RSOS models
- Algebra's: q-Virasoro, (double-affine) Hecke algebra
- Special functions: generalized β integrals, elliptic hypergeometrics, differential Galois theory
- quantum dilogarithms (from quantum Teichmüller space)

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Possibilities:

- XYZ correlation functions
- How 6D $\mathcal{N} = (2, 0)$ SYM relates to XYZ?

Part 2

Goal: Understand what happens to quantum integrability beyond the Heisenberg case

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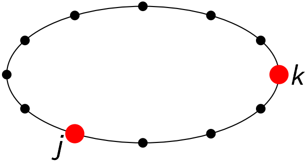
Goal: Understand what happens to quantum integrability beyond the Heisenberg case

Today:

- Study the elliptic deformation of Heisenberg XXX
- Improve its solution
- Show what is left of energy additivity and factorised scattering
- Is the spectral problem rational?

Inozemtsev's elliptic spin chain [Inozemtsev, 1989]

$$H \sim \sum_{j<k}^{L} \wp(j-k)\frac{P_{jk}-1}{2}$$



$H : \mathcal{H} \to \mathcal{H}$ with $\mathcal{H} \coloneqq (\mathbb{C}|\uparrow\rangle \oplus \mathbb{C}|\downarrow\rangle)^{\otimes L}$

XXX Spin operator: $\frac{P_{jk}-1}{2} = \sigma_j^{(x)}\sigma_k^{(x)} + \sigma_j^{(y)}\sigma_k^{(y)} + \sigma_j^{(z)}\sigma_k^{(z)}$

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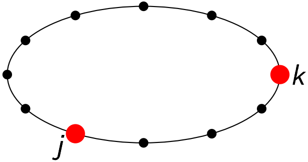

Properties

- $\mathfrak{sl}_2$-invariant, i.e. isotropic
- translation invariant
- pairwise long-range interactions
- periodic

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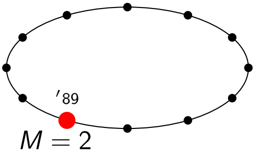

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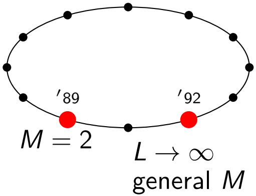

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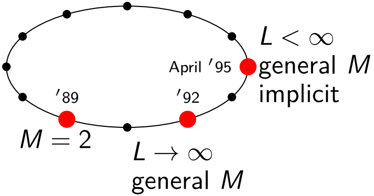

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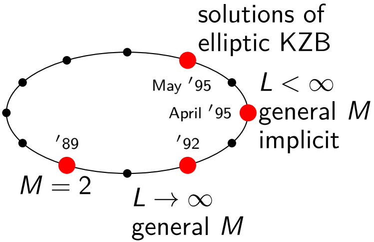

solutions of elliptic KZB

May '95

$L < \infty$ general M implicit

April '95

'89

'92

$M = 2$

$L \to \infty$ general M

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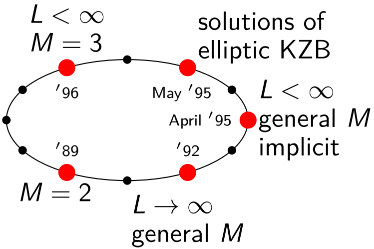

$L < \infty$
$M = 3$

solutions of
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$'96$

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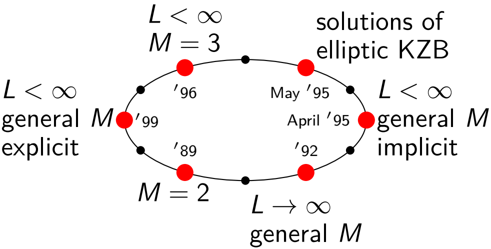

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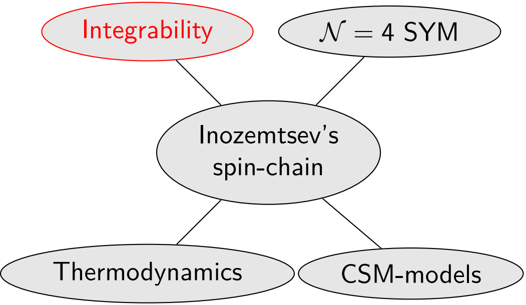


Integrability

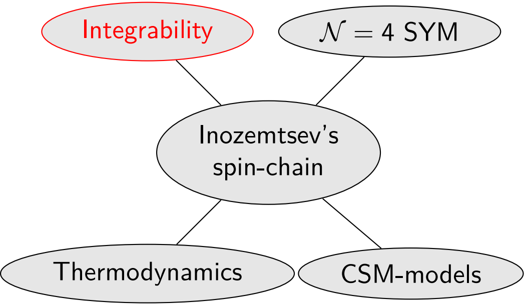
$\mathcal{N} = 4$ SYM

Inozemtsev's spin-chain

Thermodynamics

CSM-models

Literature



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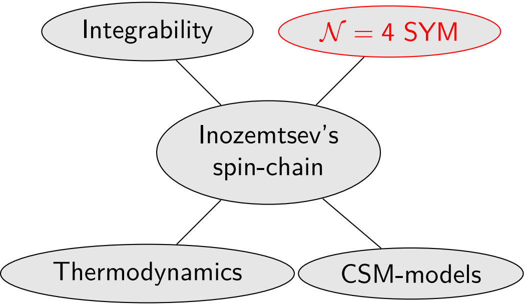
Status:

- No R matrix, no Yang-Baxter equation
- but a quantum Lax pair L, M [Inozemtsev, 1989]
- And a proposed set of commuting charges [Inozemtsev,1996]
- But so far only $[J_1, J_2] = 0$ has been proven [Dittrich, Inozemtsev, 2006]

Literature



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CSM-models

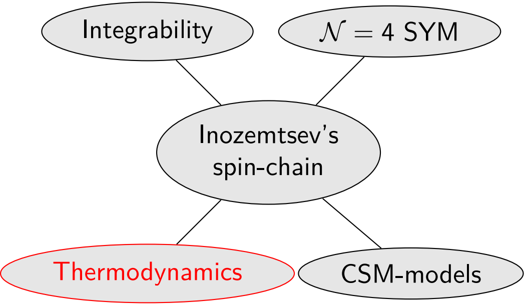
In the spectral problem of the **Dilatation operator** D

- $\text{spec(D)} = \text{spec(H)} + \mathcal{O}\left(\kappa^4\right)$

  [Serban, Staudacher, 2004]

- but discrepancies occur at fourth order

Literature



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- spin-spin correlation functions [Dittrich, Inozemtsev, 1997]
- central charge at critical point [Inozemtsev, Dörfel, 1993]
- Two-magnon bound states [Dittrich, Inozemtsev, 1997]
- Thermodynamic Bethe ansatz [Klabbers, 2016]

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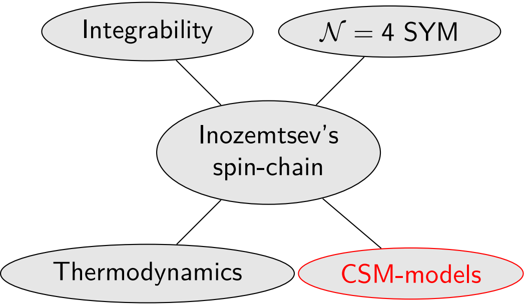

- integral representations for qKZB equation [Felder, Varchenko, 1995]

Inozemtsev's extended Bethe ansatz

Spectral problem:

$$H|\Psi\rangle = \varepsilon|\Psi\rangle.$$

Coordinate basis:

$$|{\downarrow\downarrow\uparrow}\ldots{\uparrow\downarrow}\ldots\uparrow\ldots\downarrow\rangle \text{ with } \uparrow \text{ at } \vec{n} = (n_1, n_2, \ldots, n_M)^T$$

Wavefunction component:

$$\langle{\downarrow\uparrow}\ldots{\downarrow\uparrow}\ldots\uparrow\ldots{\downarrow\downarrow}\,|\Psi\rangle = \Psi(\vec{n})$$

Heisenberg xxx

$$\text{ansatz}: \ \Psi(\vec{n}) = \sum_{\sigma\in S_M} A_\sigma(\vec{p}\,)e^{\mathrm{i}\vec{p}\cdot\vec{n}}$$

Imposing periodicity yields the Bethe equations

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$$\Psi(\vec{n}) := \sum_{\sigma\in S_M} \tilde{\Psi}_{\vec{p}}(\vec{n}_\sigma)\,\mathrm{e}^{-\mathrm{i}\,\vec{\delta p}\,\cdot\,\vec{n}_\sigma}$$

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$\tilde{\Psi}_{\vec{p}}$ solves the **elliptic CSM-model**: $H_{\text{CSM}}\tilde{\Psi}_{\vec{p}} = \tilde{\mathbf{E}}_M\tilde{\Psi}_{\vec{p}}$

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$$\tilde{\Psi}_{\vec{\tilde{p}}} = e^{\mathrm{i}\tilde{p}\cdot x} \sum_{\tau\in S_N} I(\tau) \prod_{\alpha}^{N} \chi_1$$

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- periodicity yields **transcendental equations** for $M(M-1)/2$ parameters

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How definitions change everything

Old ingredients:

- $V_{\mathrm{Ino}}(x) = \frac{\sinh^2 \kappa}{\kappa^2}\left(\wp(x) + \frac{\eta_2}{\omega}\right)$
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  with multiplicative quasiperiods
- $e^{\mathrm{i}\tilde{p}L}$, $e^{\mathrm{i}\tilde{p}\omega + 2\pi\mathrm{i}q}$, $\quad q = q(\vec{t}\,)$
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Now we use the **Legendre relation** to rewrite everything

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$$\boxed{\varepsilon_M = \sum_{m=1}^{M} \varepsilon_1(p_m) + \tilde{U}_2^{\mathrm{CSM}}}$$

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Periodicity of the energy:

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So ε_M is elliptic **on-shell**!

This turns the entire spectral problem into a **rational** one

Example: $M = 2$

$N = 1$ and we set $t_1 = -\gamma$ and $I = I_1 + I_2$. Define

$$\hat{x}_\gamma := \hat{\wp}(\gamma - \omega I/2), \quad \hat{y}_\gamma := \hat{\wp}'(\gamma - \omega I/2)$$

satisfying

$$\text{Weierstraß equation}: \quad \hat{y}_\gamma^2 = 4\hat{x}_\gamma^3 - g_2\hat{x}_\gamma - g_3$$

Constraint:

$$\frac{2}{L}\rho_2(\gamma) = \hat{\rho}_2(\gamma + I\omega) + \hat{\rho}_2(\gamma)$$

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of which the trivial solutions are precisely those with $\hat{y}_\gamma = 0$

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Energy:

$$\varepsilon_2 \sim \mathrm{cst}_I + (2 - 2/L)\hat{x}_\gamma + 2/L\hat{\rho}_2(\omega I)\frac{\hat{y}_0}{\hat{x}_\gamma - \hat{x}_0}$$
$$+ \frac{1}{2L^2}\sum_{n=0}^{L-1}\left(\frac{\hat{y}_\gamma + \hat{y}_{\omega n}}{\hat{x}_\gamma - \hat{x}_{\omega n}}\right)^2 - 1/2\left(\frac{\hat{y}_\gamma}{\hat{x}_\gamma - \hat{x}_0}\right)^2$$

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- **match** with the numerical spectrum ($L \leq 12$)
- Completeness

Summary:

We found a **new parametrisation** of the spectral problem, such that

- $\varepsilon_M = \sum_m \epsilon(r_m) + \tilde{U}^{\mathsf{CSM}}$, i.e. almost additive energies
- the spectral problem becomes fully **rational**
- All κ limits are much better behaved

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We found a **new parametrisation** of the spectral problem, such that

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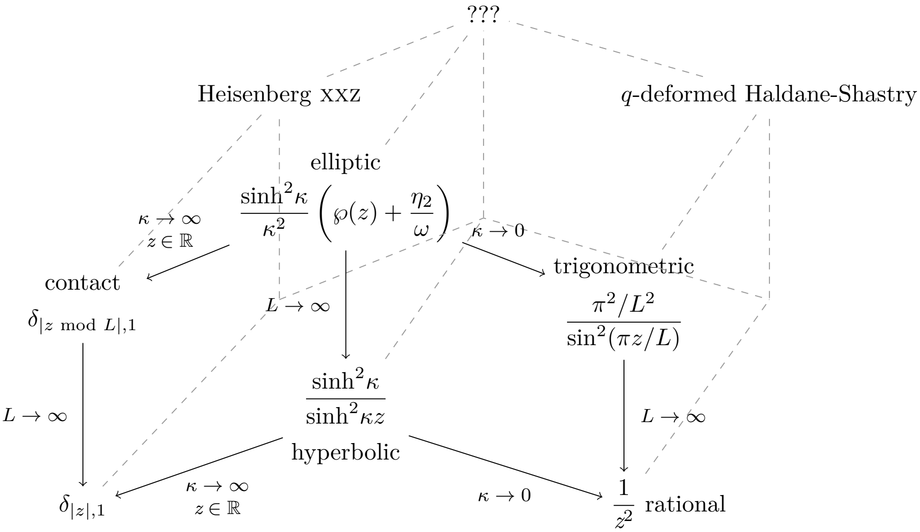
- the spectral problem becomes fully **rational**
- All κ limits are much better behaved

Future directions:

- True additivity
- Study completeness for $M > 2$, at least numerically
- Higher spin: does the spin chain $\leftrightarrow$ CSM relation hold beyond $s = 1/2$?
- XXZ

Limits



$$\begin{tikzcd}
\begin{array}{c}\text{contact}\\ \delta_{|z \bmod L|,1}\end{array} \arrow[d, "L \to \infty"'] & \begin{array}{c}\text{elliptic}\\ \dfrac{\sinh^2\kappa}{\kappa^2}\left(\wp(z)+\dfrac{\eta_2}{\omega}\right)\end{array} \arrow[l, "\begin{array}{c}\kappa\to\infty\\ z\in\mathbb{R}\end{array}"'] \arrow[r, "\kappa \to 0"] \arrow[d, "L \to \infty"'] & \begin{array}{c}\text{trigonometric}\\ \dfrac{\pi^2/L^2}{\sin^2(\pi z/L)}\end{array} \arrow[d, "L \to \infty"] \\
\delta_{|z|,1} & \begin{array}{c}\dfrac{\sinh^2\kappa}{\sinh^2\kappa z}\\ \text{hyperbolic}\end{array} \arrow[l, "\begin{array}{c}\kappa\to\infty\\ z\in\mathbb{R}\end{array}"] \arrow[r, "\kappa \to 0"'] & \dfrac{1}{z^2}\ \text{rational}
\end{tikzcd}$$

Constraints and energy

Extended Bethe ansatz equations

(I) $$\bar{\rho}_1(q_m + I_m\,\omega/L) = \mathrm{i}\,(\tilde{p}_m - 2\pi\, I_m/L)$$

(II) $$\sum_{\beta\,\in\, c^{-1}\{c_\alpha - 1,\, c_\alpha + 1\}} \rho_1(t_\alpha - t_\beta) \; - \; 2 \sum_{\beta\,\in\,(c^{-1}\{c_\alpha\})\setminus\{\alpha\}} \rho_1(t_\alpha - t_\beta) = \mathrm{i}\,(\tilde{p}_{c_\alpha} - \tilde{p}_{c_\alpha + 1})$$

(III) $$L\, q_m = \sum_{\alpha\,\in\, c^{-1}\{m\}} t_\alpha - \sum_{\alpha\,\in\, c^{-1}\{m-1\}} t_\alpha$$

Definitions

$$\rho_j(z) = \zeta(z) - \frac{\eta_j}{\omega_j} z$$

$$F_j(t) = \rho_j'(t) + \rho_j(t)^2 + 3\eta_j/\omega_j$$

$$\overline{\mathbb{L}} = (1, \omega)$$

Dispersion

$$\epsilon(p) \sim -\overline{\wp}\left(\frac{\omega p}{2\pi}\right) + \overline{\rho}_2\left(\frac{\omega p}{2\pi}\right)^2 + 2\bar{\eta}_2/\omega$$

Energy

$$\varepsilon_M \sim \mathcal{E}_M + \frac{M\,(M-1)\,\eta_2 - M\,\bar{\eta}_2}{\omega}$$

$$\mathcal{E}_M = \tilde{\mathbf{E}}_M + \frac{1}{2}\sum_{m=1}^{M} \bar{\wp}\left(q_m + \tfrac{I_m}{L}\omega\right)$$

$$\tilde{\mathbf{E}}_M = -\frac{M\,(M-1)}{L}\,\eta_1 + \sum_{m=1}^{M} \frac{\tilde{p}_m^2}{2} + \tilde{U}_1$$

$$\tilde{U}_j := \frac{1}{2}\sum_{\alpha=1}^{N}\left(\sum_{\beta\,\in\,(c^{-1}\{c_\alpha - 1\})} F_j(t_\alpha - t_\beta) - \sum_{\beta\,\in\, c^{-1}\{c_\alpha\}} F_j(t_\alpha - t_\beta)\right)$$