Differential equations for elliptic Feynman integrals

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Part I: **Differential equations**

Part II: **One elliptic curve, one variable**

Part III: **One elliptic curve, several variables**

Standard tools

- Integration-by-parts identities

  Tkachov ’81, Chetyrkin ’81

- the method of differential equations

  Kotikov ’90, Remiddi ’97, Gehrmann and Remiddi ’99

- Laporta algorithm and computer implementations

  Laporta ’01,

  `REDUZE` von Manteuffel, Studerus ’12,

  `FIRE` Smirnov ’15,

  `KIRA` Maierhöfer, Usovitsch, Uwer ’ 17

Notation

$N_F = N_{\mathrm{Fibre}}$: Number of master integrals,
master integrals denoted by $I = (I_1, \dots, I_{N_F})$.

$N_B = N_{\mathrm{Base}}$: Number of kinematic variables,
kinematic variables denoted by $x = (x_1, \dots, x_B)$.

$N_L = N_{\mathrm{Letters}}$: Number of letters,
differential one-forms denoted by $\omega = (\omega_1, \dots, \omega_L)$.

Differential equations

System of differential equations

$$dI + AI = 0,$$

where $A(\varepsilon, x)$ is a matrix-valued one-form

$$A = \sum_{i=1}^{N_B} A_i dx_i.$$

The matrix-valued one-form A satisfies the integrability condition

$$dA + A \wedge A = 0 \qquad \text{(flat Gauß-Manin connection)}.$$

Computation of Feynman integrals reduced to solving differential equations!

Simple differential equations

The system of differential equations is particular simple, if A is of the form

$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k,$$

where

- C_k is a $N_F \times N_F$-matrix, whose entries are (rational or integer) numbers,
- the only dependence on ε is given by the explicit prefactor,
- the differential one-forms ω_k have only simple poles.

Henn '13

Iterated integrals

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \to M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \quad = \quad \gamma^* \omega_j.$$

The iterated integral is defined by (Chen '77)

$$I_\gamma(\omega_1, \ldots, \omega_k; \lambda) \quad = \quad \int_0^{\lambda} d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \ldots \int_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Computation of Feynman integrals reduced to transforming the system of differential equations to a simple form!

Multiple polylogarithms

If all ω_k's are of the form

$$\omega_k = d \ln p_k(x),$$

where the p_k's are polynomials in the variables x, then (after factorisation of univariate polynomials)

$$f_j = \frac{d\lambda}{\lambda - z_j}$$

and all iterated integrals are multiple polylogarithms:

$$G(z_1, \ldots, z_k; \lambda) = \int_0^{\lambda} \frac{d\lambda_1}{\lambda_1 - z_1} \int_0^{\lambda_1} \frac{d\lambda_2}{\lambda_2 - z_2} \ldots \int_0^{\lambda_{k-1}} \frac{d\lambda_k}{\lambda_k - z_k}$$

Transformations

- Change the basis of the master integrals

$$I' = U I,$$

where $U(\varepsilon, x)$ is a $N_F \times N_F$-matrix. The new connection matrix is

$$A' = U A U^{-1} + U d U^{-1}.$$

- Perform a coordinate transformation on the base manifold:

$$x'_i = f_i(x), \qquad 1 \leq i \leq N_B.$$

The connection transforms as

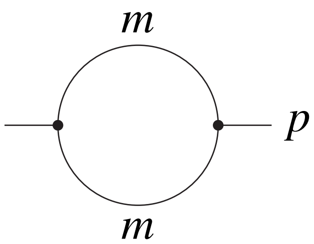
$$A = \sum_{i=1}^{N_B} A_i dx_i \qquad \Rightarrow \qquad A' = \sum_{i,j=1}^{N_B} A_i \frac{\partial x_i}{\partial x'_j} dx'_j.$$

Change of coordinates

A change of variables is already required for the one-loop two-loop function, where one encounters ($x = p^2/m^2$)

$$\frac{dx}{\sqrt{-x(4-x)}}.$$

Here, a change of variables in the base manifold



$$x = -\frac{(1-x')^2}{x'}$$

will rationalise the square root and transform

$$\frac{dx}{\sqrt{-x(4-x)}} = \frac{dx'}{x'}$$

Transformations in the case of multiple polylogarithm

- Change the basis of the master integrals

$$I' = UI$$

Systematic algorithms if U is rational in the kinematic variables:

Henn '13; Gehrmann, von Manteuffel, Tancredi, Weihs '14; Argeri et al. '14; Lee '14; Meyer '16; Prausa '17; Gituliar, Magerya '17; Lee, Pomeransky '17;

- Perform a coordinate transformation on the base manifold:

$$x_i' = f_i(x)$$

Algorithms to rationalise square roots:

Becchetti, Bonciani, '17, Besier, van Straten, S.W., '18

Simple differential equations

$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k, \qquad \text{with } \omega_k \text{ only simple poles.}$$

This form can be reached for many Feynman integrals evaluating to multiple polylogarithms.

Remark: Two-loop electroweak-QCD corrections to Drell-Yan

Heller, von Manteuffel, Schabinger, '19

Simple differential equations beyond multiple polylogarithms

Can the system of differential equations be brought into the form

$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k, \qquad \text{with } \omega_k \text{ only simple poles}$$

for Feynman integrals not evaluating to multiple polylogarithms?

Some explicit examples:

Integral	ε-form	simple poles	comments
equal mass sunrise	yes	yes	$N_B = 1$, 1 elliptic curve
unequal mass sunrise	yes	yes	$N_B = 3$, 1 elliptic curve
topbox	yes	?	$N_B = 2$, 3 elliptic curves

Part II

One elliptic curve, one variable

(The equal mass sunrise integral)

$$x = \frac{p^2}{m^2}$$

The elliptic curve

How to get the elliptic curve?

- From the Feynman graph polynomial:

$$-x_1x_2x_3x + (x_1 + x_2 + x_3)(x_1x_2 + x_2x_3 + x_3x_1) = 0$$

- From the maximal cut:

$$v^2 - (u - x)(u - x + 4)\left(u^2 + 2u + 1 - 4x\right) = 0$$

Baikov '96; Lee '10; Kosower, Larsen, '11; Caron-Huot, Larsen, '12; Frellesvig, Papadopoulos, '17; Bosma, Sogaard, Zhang, '17; Harley, Moriello, Schabinger, '17

The periods ψ_1, ψ_2 of the elliptic curve are solutions of the homogeneous differential equation.

Adams, Bogner, S.W., '13; Primo, Tancredi, '16

Variables

Recall

$$x = \frac{p^2}{m^2}.$$

Set

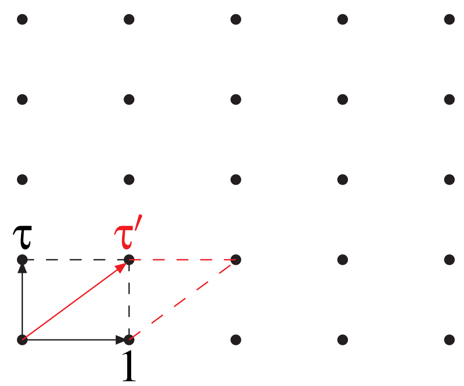
$$\tau = \frac{\psi_2}{\psi_1}, \qquad q = e^{2i\pi\tau}.$$

Change variable from x to τ (or q).

Bloch, Vanhove, '13

Bases of lattices

The periods ψ_1 and ψ_2 generate a lattice. Any other basis as good as (ψ_2, ψ_1).
Convention: Normalise $(\psi_2, \psi_1) \to (\tau, 1)$ where $\tau = \psi_2/\psi_1$.



Change of basis: $$\begin{pmatrix} \psi_2' \\ \psi_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix},$$

Transformation should be invertible: $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}),$$

In terms of τ and τ': $$\tau' = \frac{a\tau + b}{c\tau + d}$$

The ε-form of the differential equation for the sunrise

It is not possible to obtain an ε-form by a rational/algebraic change of variables and/or a rational/algebraic transformation of the basis of master integrals.

However by factoring off the (non-algebraic) expression ψ_1/π from the master integrals in the sunrise sector one obtains an ε-form:

$$I_1 = 4\varepsilon^2 S_{110}\left(2-2\varepsilon,x\right), \quad I_2 = -\varepsilon^2 \frac{\pi}{\psi_1} S_{111}\left(2-2\varepsilon,x\right), \quad I_3 = \frac{1}{\varepsilon}\frac{1}{2\pi i}\frac{d}{d\tau} I_2 + \frac{1}{24}\left(3x^2-10x-9\right)\frac{\psi_1^2}{\pi^2} I_2.$$

If in addition one makes a (non-algebraic) change of variables from x to τ, one obtains

$$\frac{d}{d\tau}\vec{I} \;=\; \varepsilon\, A(\tau)\, \vec{I},$$

where $A(\tau)$ is an ε-independent 3×3-matrix whose entries are modular forms.

The ε-form of the differential equation for the sunrise

The matrix $A(\tau)$ is given by

$$A(\tau) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -f_2(\tau) & 1 \\ \frac{1}{4} f_3(\tau) & f_4(\tau) & -f_2(\tau) \end{pmatrix},$$

where f_2, f_3 and f_4 are modular forms of $\Gamma_1(6)$ of modular weight 2, 3 and 4, respectively.

I_1, I_2 and I_3 are expressed as iterated integrals of modular forms to all orders in ε.

Adams, S.W., ’17, ’18

Simple poles at $\tau = i\infty$

A modular form $f_k(\tau)$ is by definition holomorphic at the cusp and has a q-expansion

$$f_k(\tau) = a_0 + a_1 q + a_2 q^2 + \dots, \qquad q = \exp(2\pi i \tau)$$

The transformation $q = \exp(2\pi i \tau)$ transforms the point $\tau = i\infty$ to $q = 0$ and we have

$$2\pi i f_k(\tau) d\tau = \frac{dq}{q}\left(a_0 + a_1 q + a_2 q^2 + \dots\right).$$

Thus a modular form non-vanishing at the cusp $\tau = i\infty$ has a simple pole at $q = 0$.

Part III

One elliptic curve, several variables

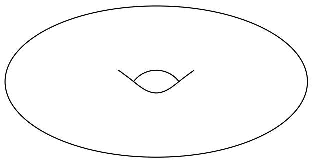
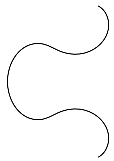
(The unequal mass sunrise integral)

$$x = \frac{p^2}{m_3^2}, \qquad y_1 = \frac{m_1^2}{m_3^2}, \qquad y_2 = \frac{m_2^2}{m_3^2}$$

Moduli spaces

$\mathcal{M}_{g,n}$: Space of isomorphism classes of smooth (complex, algebraic) curves of genus g with n marked points.

Recall:

real surface  $\Leftrightarrow$  complex curve

$$\dim \mathcal{M}_{g,n} \quad = \quad 3g + n - 3.$$

Coordinates

Genus 0: $\dim \mathcal{M}_{0,n} = n-3$.
Sphere has a unique shape
Use Möbius transformation to fix $z_{n-2} = 1, \quad z_{n-1} = \infty, \quad z_n = 0$
Coordinates are $(z_1, \ldots, z_{n-3})$

Genus 1: $\dim \mathcal{M}_{1,n} = n$.
One coordinate describes the shape of the torus
Use translation to fix $z_n = 0$
Coordinates are $(\tau, z_1, \ldots, z_{n-1})$

In particular:

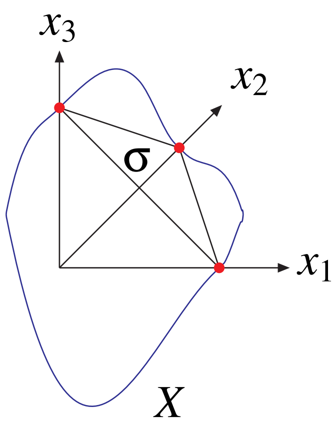
$\dim \mathcal{M}_{1,1} = 1$	with coordinate τ,	(equal mass sunrise)
$\dim \mathcal{M}_{1,3} = 3$	with coordinates τ, z_1, z_2,	(unequal mass sunrise).

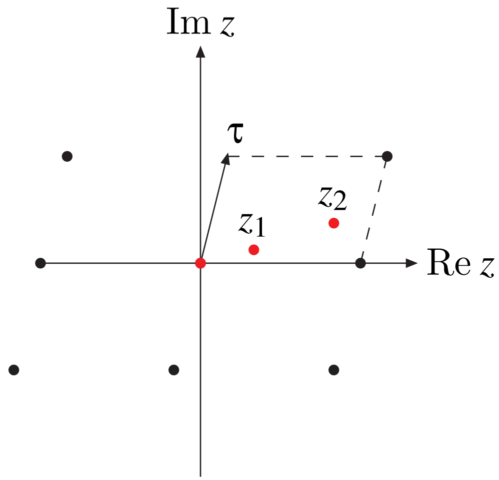
How to find z_1 and z_2?

In the Feynman parameter representation there are two objects of interest:

- the domain of integration σ,
- the zero set X of the second graph polynomial.

X and σ intersect at three points:





Master integrals

$$
\begin{aligned}
J_4 &= \varepsilon^2 \frac{\pi}{\psi_1} S_{111}, \\
J_5 &= \varepsilon \left[\frac{\left(m_1^2+m_2^2-2m_3^2\right)}{\mu^2} S_{111} + \frac{\left(t-m_1^2-3m_2^2+3m_3^2\right) m_1^2}{\mu^4} S_{211} + \frac{\left(t-3m_1^2-m_2^2+3m_3^2\right) m_2^2}{\mu^4} S_{121} - \frac{2\left(t-m_3^2\right) m_3^2}{\mu^4} S_{112} \right] \\
&\quad + \frac{2\varepsilon^2}{\left(3t^2-2M_{100}t+\Delta\right)\mu^2} \times \left[7\left(m_1^2+m_2^2-2m_3^2\right) t^2 - 2\left(3m_1^4+3m_2^4-6m_3^4+m_1^2m_3^2+m_2^2m_3^2-2m_1^2m_2^2\right) t \right. \\
&\quad \left. + \left(m_1^2+m_2^2-2m_3^2\right)\Delta \right] S_{111} + F_{54} J_4, \\
J_6 &= \varepsilon \left[\frac{\left(m_1^2-m_2^2\right)}{\mu^2} S_{111} + \frac{\left(t-m_1^2+m_2^2-m_3^2\right) m_1^2}{\mu^4} S_{211} - \frac{\left(t+m_1^2-m_2^2-m_3^2\right) m_2^2}{\mu^4} S_{121} - \frac{2\left(m_1^2-m_2^2\right) m_3^2}{\mu^4} S_{112} \right] \\
&\quad + \frac{2\varepsilon^2\left(m_1^2-m_2^2\right)}{\left(3t^2-2M_{100}t+\Delta\right)\mu^2} \left[7t^2 - 2\left(3m_1^2+3m_2^2-m_3^2\right) t + \Delta \right] S_{111} + F_{64} J_4, \\
J_7 &= \frac{1}{\varepsilon} \frac{\psi_1^2}{2\pi i W_t} \frac{d}{dt} J_4 + \frac{\varepsilon^2}{8} \frac{1}{\left(3t^2-2M_{100}t+\Delta\right)^2 \mu^4} \left[9t^6 - 22M_{100}t^5 + \left(50M_{110}-M_{200}\right) t^4 + \left(44M_{300}-76M_{210}+216M_{111}\right) t^3 \right. \\
&\quad \left. + \left(-41M_{400}+84M_{310}-86M_{220}-52M_{211}\right) t^2 + 2\Delta\left(-5M_{300}+5M_{210}-2M_{111}\right) t - \Delta^3 \right] \frac{\psi_1}{\pi} S_{111} \\
&\quad - \frac{1}{8} F_{64} J_6 - \frac{1}{24} F_{54} J_5 + F_{74} J_4.
\end{aligned}
$$

Technical details

The three functions F_{54}, F_{64}, F_{74}, appearing in the definition of J_5, J_6 and J_7 are given by

$$F_{54} = \frac{6i\mu^2}{(3t^2-2M_{100}t+\Delta)\,\psi_1}\left[\left(m_1^2-m_2^2+m_3^2-t\right)\frac{1}{y_1}\frac{dy_1}{d\tau}+\left(-m_1^2+m_2^2+m_3^2-t\right)\frac{1}{y_2}\frac{dy_2}{d\tau}\right],$$

$$F_{64} = \frac{2i\mu^2}{(3t^2-2M_{100}t+\Delta)\,\psi_1}\left[\left(3m_1^2+m_2^2-m_3^2-3t\right)\frac{1}{y_1}\frac{dy_1}{d\tau}-\left(m_1^2+3m_2^2-m_3^2-3t\right)\frac{1}{y_2}\frac{dy_2}{d\tau}\right],$$

$$\begin{aligned} F_{74} = & -\frac{\mu^4}{(3t^2-2M_{100}t+\Delta)^2\,\psi_1^2}\left[\left(3m_1^4+m_2^4+m_3^4-2m_2^2m_3^2-6m_1^2t+3t^2\right)\left(\frac{1}{y_1}\frac{dy_1}{d\tau}\right)^2\right. \\ & -\left(3m_1^4+3m_2^4-m_3^4+2m_1^2m_2^2-2m_1^2m_3^2-2m_2^2m_3^2-6\left(m_1^2+m_2^2-m_3^2\right)t+3t^2\right)\left(\frac{1}{y_1}\frac{dy_1}{d\tau}\right)\left(\frac{1}{y_2}\frac{dy_2}{d\tau}\right) \\ & \left.+\left(m_1^4+3m_2^4+m_3^4-2m_1^2m_3^2-6m_2^2t+3t^2\right)\left(\frac{1}{y_2}\frac{dy_2}{d\tau}\right)^2\right]. \end{aligned}$$

The differential equation for the unequal mass sunrise integral

There are 7 master integrals. After a redefinition of the basis of master integrals and a change of coordiantes from $(x, y_1, y_2) = (p^2/m_3^2, m_1^2/m_3^2, m_2^2/m_3^2)$ to (τ, z_1, z_2) one finds

$$A \quad = \quad \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k, \qquad \text{with } \omega_k \text{ only simple poles},$$

where ω_k is either

$$(2\pi)^{2-k} f_k(\tau) \frac{d\tau}{2\pi i},$$

where $f_k(\tau)$ is a modular form, or of the form

$$\omega_k(z_i, \tau) \quad = \quad (2\pi)^{2-k} \left[g^{(k-1)}(z_i, \tau) \, dz_i + (k-1) \, g^{(k)}(z_i, \tau) \frac{d\tau}{2\pi i} \right],$$

where $g^{(k)}(z, \tau)$ are functions appearing in the expansion of the Kronecker function.

Bogner, Müller-Stach, S.W., ’19

The Kronecker function

$$F(z,\alpha,\tau) \;=\; \pi\theta_1'(0,q)\frac{\theta_1(\pi(z+\alpha),q)}{\theta_1(\pi z,q)\,\theta_1(\pi\alpha,q)} = \frac{1}{\alpha}\sum_{k=0}^{\infty} g^{(k)}(z,\tau)\alpha^k, \qquad q = e^{i\pi\tau}$$

Properties of $g^{(k)}(z,\tau)$:

- only simple poles as a function of z

- quasi-periodic as a function of z: Periodic by 1, quasi-periodic by τ.

- almost modular: Nice modular transformation properties only spoiled by divergent Eisenstein series $E_1(z,\tau)$.

Brown, Levin, '11,

Broedel, Duhr, Dulat, Penante, Tancredi, '18

The differential one-forms

Occurring differential forms ($z_3 = 1 - z_1 - z_2$):

$$\omega_k(z_j, N\tau) = (2\pi)^{2-k}\left[g^{(k-1)}(z_j, N\tau)\, dz_j + N(k-1)\, g^{(k)}(z_j, N\tau)\frac{d\tau}{2\pi i}\right]$$

$$0 \le k \le 4, \quad 1 \le j \le 3, \quad 1 \le N \le 2$$

and (with Eisenstein series $e_2(\tau)$ and $e_4(\tau)$)

$$\eta_2(\tau) = [e_2(\tau) - 2e_2(2\tau)]\frac{d\tau}{2\pi i} \in \mathcal{M}_2(\Gamma_0(2))$$

$$\eta_4(\tau) = \frac{1}{(2\pi)^2} e_4(\tau)\frac{d\tau}{2\pi i} \in \mathcal{M}_4(\mathrm{SL}_2(\mathbb{Z}))$$

Remark : $0 \le k \le 4$

Integration along $\tau = \mathrm{const}$

Elliptic polylogarithms (holomorphic version, not double periodic)

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z; \tau\right) = \int\limits_0^z dz' \, g^{(n_1)}(z' - z_1, \tau) \, \widetilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; z'; \tau\right)$$

Broedel, Duhr, Dulat, Tancredi, ’17

We have

$$\omega_k(z_j, N\tau) \xrightarrow{\tau=\mathrm{const}} (2\pi)^{2-k} g^{(k-1)}(z_j, N\tau) \, dz_j$$

and

$$g^{(k)}(z, 2\tau) = \frac{1}{2}\left[g^{(k)}\left(\frac{z}{2}, \tau\right) + g^{(k)}\left(\frac{z}{2} + \frac{1}{2}, \tau\right)\right]$$

Integration along $z_1 = \text{const}$ and $z_2 = \text{const}$

Integration along τ.

In the equal mass case we have

$$z_1 = z_2 = z_3 = \frac{1}{3}$$

and the integration kernels reduce to modular forms of $\Gamma_1(6)$.

We recover the equal mass result in terms of iterated integrals of modular forms.

Conclusions

Computation of Feynman integrals is trivial, as soon as the system of differential equations is transformed to

$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k, \qquad \text{with } \omega_k \text{ only simple poles.}$$

This form can be reached for
- many Feynman integrals evaluating to multiple polylogarithms
- a few non-trivial elliptic examples

Open question: Any Feynman integral can be obtained from a system of differential equations of this form.

A constructive proof would gives us an algorithm to compute any Feynman integral.