IBPs without IBPs - Intersection theory and the vector space of Feynman integrals

Hjalte Frellesvig

Dipartimento di Fisica e Astronomia "Galileo Galilei", University of Padova.

September 20, 2019

JHEP

PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: *February 13, 2019*
REVISED: *April 10, 2019*
ACCEPTED: *May 5, 2019*
PUBLISHED: *May 24, 2019*

Decomposition of Feynman integrals on the maximal cut by intersection numbers

Hjalte Frellesvig,[a,b] Federico Gasparotto,[a,b] Stefano Laporta,[a,b] Manoj K. Mandal,[a,b] Pierpaolo Mastrolia,[a,b] Luca Mattiazzi[b,a] and Sebastian Mizera[c,d]

[a] *Dipartimento di Fisica e Astronomia, Università di Padova, Via Marzolo 8, 35131 Padova, Italy*
[b] *INFN, Sezione di Padova, Via Marzolo 8, 35131 Padova, Italy*
[c] *Perimeter Institute for Theoretical Physics, 31 Caroline St N, Waterloo, ON N2L 2Y5, Canada*
[d] *Department of Physics & Astronomy, University of Waterloo, 200 University Ave W, Waterloo, ON N2L 3G1, Canada*

E-mail: hjalte.frellesvig@pd.infn.it, federico.gasparotto@pd.infn.it, stefano.laporta@pd.infn.it, manojkumar.mandal@pd.infn.it, pierpaolo.mastrolia@pd.infn.it, luca.mattiazzi@pd.infn.it, smizera@pitp.ca

ABSTRACT: We elaborate on the recent idea of a direct decomposition of Feynman integrals onto a basis of master integrals on maximal cuts using intersection numbers. We begin by showing an application of the method to the derivation of contiguity relations for special functions, such as the Euler beta function, the Gauss ${}_2F_1$ hypergeometric function, and the Appell F_1 function. Then, we apply the new method to decompose Feynman integrals whose maximal cuts admit 1-form integral representations, including examples that have from two to an arbitrary number of loops, and/or from zero to an arbitrary number of legs. Direct constructions of differential equations and dimensional recurrence relations for Feynman integrals are also discussed. We present two novel approaches to decomposition-by-intersections in cases where the maximal cuts admit a 2-form integral representation, with a view towards the extension of the formalism to n-form representations. The decomposition formulae computed through the use of intersection numbers are directly verified to agree with the ones obtained using integration-by-parts identities.

KEYWORDS: Scattering Amplitudes, Differential and Algebraic Geometry

ARXIV EPRINT: 1901.11510

https://doi.org/10.1007/JHEP05(2019)153

JHEP05(2019)153

arXiv:1907.02000v1 [hep-th] 3 Jul 2019

Vector Space of Feynman Integrals and Multivariate Intersection Numbers

Hjalte Frellesvig,[1,2,*] Federico Gasparotto,[1,2,†] Manoj K. Mandal,[1,2,‡] Pierpaolo Mastrolia,[1,2,§] Luca Mattiazzi,[2,1,¶] and Sebastian Mizera[3,4,**]

[1] *Dipartimento di Fisica e Astronomia, Università di Padova, Via Marzolo 8, 35131 Padova, Italy*
[2] *INFN, Sezione di Padova, Via Marzolo 8, 35131 Padova, Italy*
[3] *Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada*
[4] *Department of Physics & Astronomy, University of Waterloo, Waterloo, ON N2L 3G1, Canada*

(Dated: July 4, 2019)

Feynman integrals obey linear relations governed by intersection numbers, which act as scalar products between vector spaces. We present a general algorithm for constructing multivariate intersection numbers relevant to Feynman integrals, and show for the first time how they can be used to solve the problem of integral reduction to a basis of master integrals by projections, and to directly derive functional equations fulfilled by the latter. We apply it to the derivation of contiguity relations for special functions admitting multi-fold integral representations, and to the decomposition of a few Feynman integrals at one- and two-loops, as first steps towards potential applications to generic multi-loop integrals.

INTRODUCTION

Scattering amplitudes encode crucial information about collision phenomena in our universe, from the smallest to the largest scales. Within the perturbative field-theoretical approach, the evaluation of multi-loop Feynman integrals is a mandatory operation for the determination of scattering amplitudes and related quantities.

Linear relations among Feynman integrals can be exploited to simplify the evaluation of scattering amplitudes: they can be used both for decomposing scattering amplitudes in terms of a basis of functions, referred to as *master integrals* (MIs), and for the evaluation of the latter. The standard procedure used to derive relations among Feynman integrals in dimensional regularization makes use of *integration-by-parts identities* (IBPs) [1], which are found by solving linear systems of equations [2] (see [3, 4] and references therein for reviews). Algebraic manipulations in these cases are very demanding, and efficient algorithms for solving large-size systems of linear equations have been recently devised, by making use of finite field arithmetic and rational functions reconstruction [5–7].

In [8], it was shown that *intersection numbers* [9] of differential forms can be employed to define (what amounts to) a *scalar product* on a *vector space* of Feynman integrals in a given family. Using this approach, projecting any multi-loop integral onto a basis of MIs is conceptually no different from decomposing a generic vector into a basis of a vector space. Within this new approach, relations among Feynman integrals can be derived avoiding the generation of intermediate, auxiliary expressions which are needed when applying Gauss elimination, as in the standard IBP-based approaches.

In the initial studies, [8, 10], this novel decomposition method was applied to the realm of special mathematical functions falling in the class of Lauricella functions, as well as to Feynman integrals on maximal cuts, *i.e.* with on-shell internal lines, mostly admitting a one-fold integral representation. Those results concerned a partial construction of Feynman integral relations, mainly limited to the determination of the coefficients of the MIs with the same number of denominators as the integral to decompose, which was achieved by means of intersection numbers for *univariate* forms.

In this paper, we make an important step further, and address the *complete* integral reduction, for the determination of *all* coefficients, including those associated to MIs corresponding to sub-graphs. In the current work, we discuss the one-loop massless four-point integral as a paradigmatic case, although the algorithm has been successfully applied to several other cases at one- and two-loop.

Generic Feynman integrals admit multi-fold integral representations. Their complete decomposition requires the evaluation of intersection numbers for multivariate rational differential forms. Intersection numbers of multivariate forms have been previously studied in [11–19]. Recently, a new recursive algorithm was introduced in [20]. In this letter, we present its refined implementation and application to Feynman integrals, which provide a major step towards large-scale applicability of our strategy for the reduction to MIs. The results of this work show potential for further applications ranging from particle physics, through condensed matter and statistical mechanics, to gravitational-wave physics, while making new connections to mathematics.

INTEGRALS AND DIFFERENTIAL FORMS

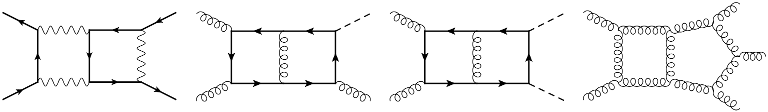
In this work, we focus on integrals of the hypergeometric type,

$$I = \int_{\mathcal{C}} u(\mathbf{z})\, \varphi(\mathbf{z}) \,, \qquad (1)$$

Università degli Studi di Padova

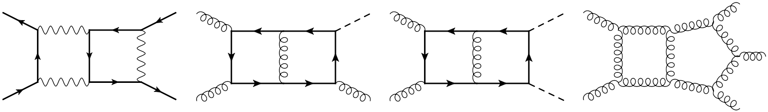
MARIE CURIE ACTIONS

Introduction



For state-of-the art two-loop scattering amplitude calculations
Feynman diagrams $\rightarrow \mathcal{O}(10000)$ Feynman integrals

Linear relations bring this down to $\mathcal{O}(100)$ *master integrals*



For state-of-the art two-loop scattering amplitude calculations
Feynman diagrams $\to \mathcal{O}(10000)$ Feynman integrals

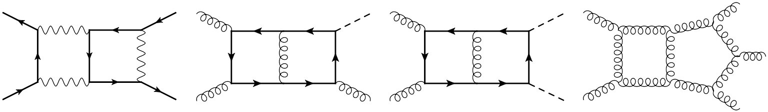
Linear relations bring this down to $\mathcal{O}(100)$ *master integrals*

Linear relations may be derived using IBP (integration by part) identities

$$\int \frac{\mathrm{d}^d k}{\pi^{d/2}} \frac{\partial}{\partial k^\mu} \frac{q^\mu N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} = 0$$

Systematic by Laporta's algorithm $\Rightarrow$ Solve a huge linear system.

Introduction



For state-of-the art two-loop scattering amplitude calculations
Feynman diagrams $\to \mathcal{O}(10000)$ Feynman integrals

Linear relations bring this down to $\mathcal{O}(100)$ *master integrals*

Linear relations may be derived using IBP (integration by part) identities

$$\int \frac{\mathrm{d}^d k}{\pi^{d/2}} \frac{\partial}{\partial k^\mu} \frac{q^\mu N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} = 0$$

Systematic by Laporta's algorithm $\Rightarrow$ Solve a huge linear system.

The linear relations are often informally referred to as IBPs as well.

The linear relations form a vector space

$$I = \sum_{i \in \text{masters}} c_i I_i$$

Subsectors are sub-spaces

The linear relations form a vector space

$$I = \sum_{i \in \text{masters}} c_i I_i$$

Subsectors are sub-spaces

Not all vector spaces are *inner product spaces*

$$\langle v| = \sum_i \langle v w_j \rangle \left(C^{-1}\right)_{ji} \langle v_i| \qquad \text{with} \qquad C_{ij} = \langle v_i w_j \rangle$$
$$= \sum_i c_i \, \langle v_i|$$

The linear relations form a vector space

$$I = \sum_{i \in \text{masters}} c_i I_i$$

Subsectors are sub-spaces

Not all vector spaces are *inner product spaces*

$$\langle v| = \sum_i \langle v w_j \rangle \left(C^{-1}\right)_{ji} \langle v_i| \qquad \text{with} \qquad C_{ij} = \langle v_i w_j \rangle$$
$$= \sum_i c_i \, \langle v_i|$$

If only there were a way to define an inner product for Feynman integrals...

Baikov representation

$$I = \int \frac{\mathrm{d}^d k_1}{\pi^{d/2}} \cdots \int \frac{\mathrm{d}^d k_L}{\pi^{d/2}} \frac{N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} \; = \; K \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^\gamma(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}}$$

The x_i are Baikov variables, $\mathcal{B}$ is the Baikov Polynomial, $\mathcal{C} = \{\mathcal{B} > 0\}$.

$$n = L(L+1)/2 + EL \qquad \gamma = (d - E - L - 1)/2$$

P. Baikov: *Nucl. Instrum. Meth.A* **389** (1997) 347–349, [hep-ph/9611449]

Theory

Baikov representation

$$I = \int \frac{\mathrm{d}^d k_1}{\pi^{d/2}} \cdots \int \frac{\mathrm{d}^d k_L}{\pi^{d/2}} \frac{N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} \; = \; K \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^\gamma(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}}$$

The x_i are Baikov variables, $\mathcal{B}$ is the Baikov Polynomial, $\mathcal{C} = \{\mathcal{B} > 0\}$.

$$n = L(L{+}1)/2{+}EL \qquad\qquad \gamma = (d{-}E{-}L{-}1)/2$$

P. Baikov: *Nucl. Instrum. Meth.A* **389** (1997) 347–349, [hep-ph/9611449]

The loop-by-loop version of Baikov representation can often decrease n

$$I = \tilde{K} \int_{\mathcal{C}} \mathrm{d}^{\tilde{n}} x \, \frac{\left(\prod_{j=1}^{2L-1} \mathcal{B}_j^{\gamma_j}(x) \right) N(x)}{x_1^{a_1} \cdots x_P^{a_P}}$$

HF and C. Papadopoulos, *JHEP* **04** (2017) 083, [arXiv:1701.07356]

Baikov representation

$$I = \int \frac{\mathrm{d}^d k_1}{\pi^{d/2}} \cdots \int \frac{\mathrm{d}^d k_L}{\pi^{d/2}} \frac{N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} \; = \; K \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^\gamma(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}}$$

The x_i are Baikov variables, $\mathcal{B}$ is the Baikov Polynomial, $\mathcal{C} = \{\mathcal{B} > 0\}$.

$$n = L(L{+}1)/2{+}EL \qquad \gamma = (d{-}E{-}L{-}1)/2$$

P. Baikov: *Nucl. Instrum. Meth.A* **389** (1997) 347–349, [hep-ph/9611449]

The loop-by-loop version of Baikov representation can often decrease n

$$I = \tilde{K} \int_{\mathcal{C}} \mathrm{d}^{\tilde{n}} x \, \frac{\left(\prod_{j=1}^{2L-1} \mathcal{B}_j^{\gamma_j}(x)\right) N(x)}{x_1^{a_1} \cdots x_P^{a_P}}$$

HF and C. Papadopoulos, *JHEP* **04** (2017) 083, [arXiv:1701.07356]

Baikov representation is suitable for *generalized unitarity cuts*

$\int \mathrm{d}x \to \oint \mathrm{d}x$. Preserve linear relations.

J. Bosma, M. Søgaard, Y. Zhang, *JHEP* **08** (2017) 051, [arXiv:1704.04255]

UNIVERSITÀ DEGLI STUDI DI PADOVA

MARIE CURIE ACTIONS

$$I = \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^\gamma(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}} = \int_{\mathcal{C}} u \phi$$

$u = \mathcal{B}^\gamma$ is a multivalued function in $\{x\}$

$\phi = \frac{N(x)}{x_1^{a_1} \cdots x_P^{a_P}} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a form

$$I = \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^{\gamma}(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}} = \int_{\mathcal{C}} u\phi = \langle \phi | \mathcal{C}]_{\omega}$$

$u = \mathcal{B}^{\gamma}$ is a multivalued function in $\{x\}$

$\phi = \frac{N(x)}{x_1^{a_1} \cdots x_P^{a_P}} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a form

$\omega = \mathrm{d}\log(u)$ is *the twist*

$\langle \phi | \mathcal{C}]_{\omega}$ is a pairing of a *twisted cycle* ($\mathcal{C}$) and a *twisted cocycle* (ϕ) (equivalence classes of contours and integrands respectively)

P. Mastrolia and S. Mizera, *Feynman Integrals and Intersection Theory*, JHEP **1902** (2019) 139

dim of the set of ϕs, is the number of master integrals.

$$I = \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^\gamma(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}} = \int_{\mathcal{C}} u\phi = \langle \phi | \mathcal{C}]_\omega$$

$u = \mathcal{B}^\gamma$ is a multivalued function in $\{x\}$

$\phi = \frac{N(x)}{x_1^{a_1} \cdots x_P^{a_P}} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a form

$\omega = \mathrm{d}\log(u)$ is *the twist*

$\langle \phi | \mathcal{C}]_\omega$ is a pairing of a *twisted cycle* ($\mathcal{C}$) and a *twisted cocycle* (ϕ)
(equivalence classes of contours and integrands respectively)

P. Mastrolia and S. Mizera, *Feynman Integrals and Intersection Theory*, JHEP **1902** (2019) 139

dim of the set of ϕs, is the number of master integrals.

Lee Pomeransky criterion:
nr. of master integrals $=$ nr. of solutions to $\omega = 0$

R. Lee and A. Pomeransky, *JHEP* **11** (2013) 165, [arXiv:1308.6676].

Università degli Studi di Padova

MARIE CURIE ACTIONS

The *intersection number* $\langle\phi|\xi\rangle$ is a pairing of a twisted cocycle ϕ with a *dual* twisted cocycle ξ

Lives up to all criteria for being a scalar product.

The *intersection number* $\langle\phi|\xi\rangle$ is a pairing of a twisted cocycle ϕ with a *dual* twisted cocycle ξ

Lives up to all criteria for being a scalar product.

When there is one integration variable z (ϕ and ξ are one-forms)

$$\langle\phi|\xi\rangle_\omega = \sum_{p\in\mathcal{P}} \mathsf{Res}_{z=p}(\psi_p\xi) \qquad\qquad (\mathrm{d}+\omega)\psi_p = \phi$$

$\mathcal{P}$ is the set of poles of ω.

$(\mathrm{d}+\omega)\psi_p = \phi$ can be solved with a series ansatz around $z=p$
$\psi_p = \sum_i \psi_p^{(i)}(z-p)^i$. The exact expressions are not needed for the Res.

The *intersection number* $\langle\phi|\xi\rangle$ is a pairing of a twisted cocycle ϕ with a *dual* twisted cocycle ξ

Lives up to all criteria for being a scalar product.

When there is one integration variable z (ϕ and ξ are one-forms)

$$\langle\phi|\xi\rangle_\omega = \sum_{p\in\mathcal{P}} \text{Res}_{z=p}(\psi_p\xi) \qquad (\mathrm{d}+\omega)\psi_p = \phi$$

$\mathcal{P}$ is the set of poles of ω.

$(\mathrm{d}+\omega)\psi_p = \phi$ can be solved with a series ansatz around $z = p$
$\psi_p = \sum_i \psi_p^{(i)}(z-p)^i$. The exact expressions are not needed for the Res.

References:

K. Cho and K. Matsumoto, *Intersection theory for twisted cohomologies and twisted Riemann's period relations*, Nagoya Math. J. **139** (1995) 67-86

K. Matsumoto, *Intersection numbers for logarithmic k-forms*, Osaka J. Math. **35** (1998) no. 4 873-893

S. Mizera, *Scattering Amplitudes from Intersection Theory*, Phys. Rev. Lett. **120** (2018) no. 14 141602

Summary:

$$I = \sum_{i \in \text{masters}} c_i I_i \quad \Leftrightarrow \quad \langle \phi | \mathcal{C}] = \sum_i c_i \langle \phi_i | \mathcal{C}]$$

with $I = \int_{\mathcal{C}} u\phi$. u is multivalued function,
ϕ is a form (rational pre-factor), $\omega = \mathrm{d}\log(u)$.

Summary:

$$I = \sum_{i \in \text{masters}} c_i I_i \quad \Leftrightarrow \quad \langle \phi | \mathcal{C}] = \sum_i c_i \langle \phi_i | \mathcal{C}]$$

with $I = \int_{\mathcal{C}} u\phi$. u is multivalued function,
ϕ is a form (rational pre-factor), $\omega = \mathrm{d}\log(u)$.

$$c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \qquad \text{with} \qquad C_{ij} = \langle \phi_i | \xi_j \rangle$$

Summary:

$$I = \sum_{i \in \text{masters}} c_i I_i \quad \Leftrightarrow \quad \langle \phi | \mathcal{C}] = \sum_i c_i \langle \phi_i | \mathcal{C}]$$

with $I = \int_{\mathcal{C}} u\phi$. u is multivalued function,
ϕ is a form (rational pre-factor), $\omega = \mathrm{d}\log(u)$.

$$c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \qquad \text{with} \qquad C_{ij} = \langle \phi_i | \xi_j \rangle$$

For one-forms:

$$\langle \phi | \xi \rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p}(\psi_p \xi) \qquad (\mathrm{d} + \omega)\psi_p = \phi$$

solve with series ansatz

Summary:

$$I = \sum_{i \in \text{masters}} c_i I_i \quad \Leftrightarrow \quad \langle\phi|\mathcal{C}] = \sum_i c_i \langle\phi_i|\mathcal{C}]$$

with $I = \int_{\mathcal{C}} u\phi$. u is multivalued function,
ϕ is a form (rational pre-factor), $\omega = \mathrm{d}\log(u)$.

$$c_i = \langle\phi|\xi_j\rangle (C^{-1})_{ji} \qquad \text{with} \qquad C_{ij} = \langle\phi_i|\xi_j\rangle$$

For one-forms:

$$\langle\phi|\xi\rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p}(\psi_p \xi) \qquad (\mathrm{d} + \omega)\psi_p = \phi$$

solve with series ansatz

References:

P. Mastrolia and S. Mizera, *Feynman Integrals and Intersection Theory*, JHEP **1902** (2019) 139

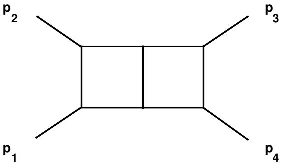
HF, F. Gasparotto, S. Laporta, M. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera, *Decomposition of Feynman integrals on the maximal cut by intersection numbers*, JHEP **1905** (2019) 153

HF, F. Gasparotto, M. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera, *Vector Space of Feynman Integrals and Multivariate Intersection Numbers*.

Università degli Studi di Padova

MARIE CURIE ACTIONS

Massless double box:

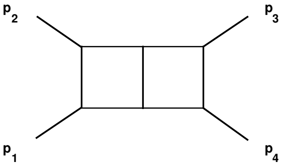


$$D_1 = k_1^2,\ \ D_2 = (k_1 - p_1)^2,\ \ D_3 = (k_1 - p_1 - p_2)^2,\ \ D_4 = (k_1 - k_2)^2,$$
$$D_5 = (k_2 - p_1 - p_2)^2,\ \ D_6 = (k_1 - p_1 - p_2 - p_3)^2,\ \ D_7 = k_2^2$$

$$n_{\mathsf{std}} = L(L+1)/2 + LE = 9$$

Example (double box)

Massless double box:



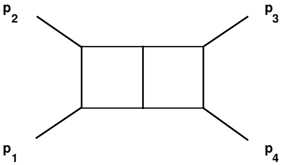
$$D_1 = k_1^2,\ \ D_2 = (k_1 - p_1)^2,\ \ D_3 = (k_1 - p_1 - p_2)^2,\ \ D_4 = (k_1 - k_2)^2,$$
$$D_5 = (k_2 - p_1 - p_2)^2,\ \ D_6 = (k_1 - p_1 - p_2 - p_3)^2,\ \ D_7 = k_2^2,\ \ D_8 = (k_2 - p_1)^2 = z.$$

$$n_{\mathsf{std}} = L(L+1)/2 + LE = 9 \quad \text{but} \quad n_{\mathsf{LBL}} = 8$$

$$I = \int \mathrm{d}^8 x \, \frac{u N(x)}{x_1^{a_1} \cdots x_7^{a_1}} \quad \to \quad I_{7\times\mathsf{cut}} = \int u_{7\times\mathsf{cut}}\, \phi \qquad u_{7\times\mathsf{cut}} = z^{d/2-3}(z+s)^{2-d/2}(z-t)^{d-5}$$

Example (double box)

Massless double box:



$$D_1 = k_1^2,\ \ D_2 = (k_1{-}p_1)^2,\ \ D_3 = (k_1{-}p_1{-}p_2)^2,\ \ D_4 = (k_1{-}k_2)^2,$$
$$D_5 = (k_2{-}p_1{-}p_2)^2,\ \ D_6 = (k_1{-}p_1{-}p_2{-}p_3)^2,\ \ D_7 = k_2^2,\ \ D_8 = (k_2{-}p_1)^2 = z.$$

$$n_{\text{std}} = L(L+1)/2 + LE = 9 \quad \text{but} \quad n_{\text{LBL}} = 8$$

$$I = \int \mathrm{d}^8 x \, \frac{u N(x)}{x_1^{a_1} \cdots x_7^{a_1}} \quad \rightarrow \quad I_{7\times\text{cut}} = \int u_{7\times\text{cut}}\, \phi \qquad u_{7\times\text{cut}} = z^{d/2-3}(z{+}s)^{2-d/2}(z{-}t)^{d-5}$$

$$\omega = \left(\frac{d-6}{2z} + \frac{4-d}{2(z{+}s)} + \frac{d-5}{z{-}t} \right) \mathrm{d}z \qquad \Rightarrow \qquad \nu = 2$$

Example (double box)

We want to reduce

$$I_{1111111;-2} = c_0 I_{1111111;0} + c_1 I_{1111111;-1} + \text{lower}$$

$$c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \qquad \text{with} \qquad C_{ij} = \langle \phi_i | \xi_j \rangle$$

$\phi = z^2 \, \mathrm{d}z \,, \quad \phi_1 = 1 \, \mathrm{d}z \,, \quad \phi_2 = z \, \mathrm{d}z$

Example (double box)

We want to reduce

$$I_{1111111;-2} = c_0 I_{1111111;0} + c_1 I_{1111111;-1} + \text{lower}$$

$$c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \qquad \text{with} \qquad C_{ij} = \langle \phi_i | \xi_j \rangle$$

$$\phi = z^2 \, \mathrm{d}z \,, \quad \phi_1 = 1 \, \mathrm{d}z \,, \quad \phi_2 = z \, \mathrm{d}z \,, \quad \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s} \right) \mathrm{d}z \,, \quad \xi_2 = \left(\tfrac{1}{z+s} - \tfrac{1}{z-t} \right) \mathrm{d}z \,,$$

The ξ basis is (almost) arbitrary. A dlog form is convenient

UNIVERSITÀ DEGLI STUDI DI PADOVA

MARIE CURIE ACTIONS

Example (double box)

We want to reduce

$$I_{1111111;-2} = c_0 I_{1111111;0} + c_1 I_{1111111;-1} + \text{lower}$$

$$c_i = \langle\phi|\xi_j\rangle(C^{-1})_{ji} \qquad \text{with} \qquad C_{ij} = \langle\phi_i|\xi_j\rangle$$

$$\phi = z^2\,\mathrm{d}z\,, \quad \phi_1 = 1\,\mathrm{d}z\,, \quad \phi_2 = z\,\mathrm{d}z\,, \quad \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\mathrm{d}z\,, \quad \xi_2 = \left(\tfrac{1}{z+s} - \tfrac{1}{z-t}\right)\mathrm{d}z\,,$$

The ξ basis is (almost) arbitrary. A dlog form is convenient

We need the intersection numbers

$$\{\langle\phi|\xi_1\rangle, \langle\phi|\xi_2\rangle, \langle\phi_1|\xi_1\rangle, \langle\phi_1|\xi_2\rangle, \langle\phi_2|\xi_1\rangle, \langle\phi_2|\xi_2\rangle\}$$

We want to reduce

$$I_{1111111;-2} = c_0 I_{1111111;0} + c_1 I_{1111111;-1} + \text{lower}$$

$$c_i = \langle\phi|\xi_j\rangle (C^{-1})_{ji} \qquad \text{with} \qquad C_{ij} = \langle\phi_i|\xi_j\rangle$$

$$\phi = z^2\,\mathrm{d}z\,, \quad \phi_1 = 1\,\mathrm{d}z\,, \quad \phi_2 = z\,\mathrm{d}z\,, \quad \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\mathrm{d}z\,, \quad \xi_2 = \left(\tfrac{1}{z+s} - \tfrac{1}{z-t}\right)\mathrm{d}z\,,$$

The ξ basis is (almost) arbitrary. A dlog form is convenient

We need the intersection numbers

$$\{\langle\phi|\xi_1\rangle, \langle\phi|\xi_2\rangle, \langle\phi_1|\xi_1\rangle, \langle\phi_1|\xi_2\rangle, \langle\phi_2|\xi_1\rangle, \langle\phi_2|\xi_2\rangle\}$$

Let us calculate

$$\langle\phi_1|\xi_1\rangle = \sum_{p\in\mathcal{P}} \text{Res}_{z=p}(\psi\xi_1) \qquad\qquad (\mathrm{d}+\omega)\psi = \phi_1$$

$$\omega = \left(\frac{d-6}{2z} + \frac{4-d}{2(z+s)} + \frac{d-5}{z-t}\right)\mathrm{d}z \quad \to \quad \text{Poles of } \omega\text{: } \mathcal{P} = \{0, -s, t, \infty\}$$

Let us start with $z = 0$

Example (double box)

$$\phi_1 = 1\,\mathrm{d}z\,,\;\; \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\mathrm{d}z\,,\;\; \mathcal{P} = \{0, -s, t, \infty\}\,,\;\; \langle \phi_1 | \xi_1 \rangle = \sum_{p \in \mathcal{P}} \underset{z=p}{\mathsf{Res}}(\psi \xi_1)\,,\;\; (\mathrm{d}+\omega)\psi = \phi_1$$

Example (double box)

$$\phi_1 = 1\,\mathrm{d}z\,,\ \ \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\mathrm{d}z\,,\ \ \mathcal{P} = \{0, -s, t, \infty\}\,,\ \ \langle \phi_1 | \xi_1 \rangle = \sum_{p\in\mathcal{P}} \underset{z=p}{\mathsf{Res}}(\psi \xi_1)\,,\ \ (\mathrm{d}+\omega)\psi = \phi_1$$

$$(\mathrm{d} + \omega)\psi_0 = \phi \qquad \Rightarrow \qquad \left(\partial_z + \frac{d-6}{2z} + \frac{4-d}{2(z+s)} + \frac{d-5}{z-t}\right)\psi_0 = 1$$

$$\psi_0 = \sum_i c_i z^i$$

Example (double box)

$$\phi_1 = 1\,\mathrm{d}z\,,\;\; \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\mathrm{d}z\,,\;\; \mathcal{P} = \{0, -s, t, \infty\}\,,\;\; \langle \phi_1 | \xi_1 \rangle = \sum_{p\in\mathcal{P}} \mathsf{Res}_{z=p}(\psi \xi_1)\,,\;\; (\mathrm{d}{+}\omega)\psi = \phi_1$$

$$(\mathrm{d} + \omega)\psi_0 = \phi \qquad \Rightarrow \qquad \left(\partial_z + \frac{d-6}{2z} + \frac{4-d}{2(z+s)} + \frac{d-5}{z-t}\right)\psi_0 = 1$$

$$\psi_0 = \sum_i c_i z^i \qquad c_{\leq 0} = 0\,, \quad c_1 = \frac{2}{d-4}\,, \quad c_2 = \frac{4(d-5)s + 2(d-4)t}{(d-4)(d-2)st}\,, \ldots$$

$$\mathsf{Res}_{z=0}\left(\left(\tfrac{2}{d-4} z + \mathcal{O}(z^2)\right)\left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\right) = 0$$

Example (double box)

$$\phi_1 = 1\,\mathrm{d}z\,, \quad \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\mathrm{d}z\,, \quad \mathcal{P} = \{0, -s, t, \infty\}\,, \quad \langle \phi_1 | \xi_1 \rangle = \sum_{p\in\mathcal{P}} \underset{z=p}{\mathsf{Res}}(\psi \xi_1)\,, \quad (\mathrm{d}+\omega)\psi = \phi_1$$

$$(\mathrm{d}+\omega)\psi_0 = \phi \qquad \Rightarrow \qquad \left(\partial_z + \frac{d-6}{2z} + \frac{4-d}{2(z+s)} + \frac{d-5}{z-t}\right)\psi_0 = 1$$

$$\psi_0 = \sum_i c_i z^i \qquad c_{\leq 0} = 0\,, \quad c_1 = \frac{2}{d-4}\,, \quad c_2 = \frac{4(d-5)s + 2(d-4)t}{(d-4)(d-2)st}\,, \ldots$$

$$\mathsf{Res}_{z=0}\left(\left(\tfrac{2}{d-4}z + \mathcal{O}(z^2)\right)\left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\right) = 0$$

Likewise in $-s$ and t: $\mathsf{Res}_{z\in\{-s,t\}}(\psi\xi_1) = 0$

Example (double box)

$$\phi_1 = 1\,\mathrm{d}z\,, \;\; \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\mathrm{d}z\,, \;\; \mathcal{P} = \{0, -s, t, \infty\}\,, \;\; \langle \phi_1 | \xi_1 \rangle = \sum_{p\in\mathcal{P}} \mathrm{Res}_{z=p}(\psi \xi_1)\,, \;\; (\mathrm{d}+\omega)\psi = \phi_1$$

$$(\mathrm{d}+\omega)\psi_0 = \phi \qquad \Rightarrow \qquad \left(\partial_z + \frac{d-6}{2z} + \frac{4-d}{2(z+s)} + \frac{d-5}{z-t}\right)\psi_0 = 1$$

$$\psi_0 = \sum_i c_i z^i \qquad c_{\leq 0} = 0\,, \quad c_1 = \frac{2}{d-4}\,, \quad c_2 = \frac{4(d-5)s + 2(d-4)t}{(d-4)(d-2)st}\,, \ldots$$

$$\mathrm{Res}_{z=0}\left(\left(\tfrac{2}{d-4}z + \mathcal{O}(z^2)\right)\left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\right) = 0$$

Likewise in $-s$ and t: $\mathrm{Res}_{z\in\{-s,t\}}(\psi\xi_1) = 0$

For $z = \infty$ we use $y = z^{-1}$ giving $\xi_1 = \frac{-s}{1+sy}dy$

$$(\mathrm{d}+\omega)\psi_\infty = \phi \qquad \Rightarrow \qquad \left(\partial_y + \frac{-1}{2y}\left((d-6) - \frac{d-4}{1+sy} + \frac{2(d-5)}{1-ty}\right)\right)\psi_\infty = \frac{-1}{y^2}$$

$$\psi_\infty = \sum_i c_i y^i \qquad c_{\leq -2} = 0\,, \quad c_{-1} = \frac{1}{d-5}\,, \quad c_0 = \frac{-((d-4)s + 2(d-5)t)}{2(d-6)(d-5)}\,, \ldots$$

$$\mathrm{Res}_{y=0}\left(\left(\tfrac{1}{d-5}\tfrac{1}{y} + \mathcal{O}(y^0)\right)\left(\tfrac{-s}{1+sy}\right)\right) = \frac{-s}{d-5}$$

Example (double box)

$$\phi_1 = 1\,\mathrm{d}z\,,\;\; \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\mathrm{d}z\,,\;\; \mathcal{P} = \{0, -s, t, \infty\}\,,\;\; \langle \phi_1 | \xi_1 \rangle = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p}(\psi \xi_1)\,,\;\; (\mathrm{d}+\omega)\psi = \phi_1$$

$$(\mathrm{d}+\omega)\psi_0 = \phi \qquad \Rightarrow \qquad \left(\partial_z + \frac{d-6}{2z} + \frac{4-d}{2(z+s)} + \frac{d-5}{z-t}\right)\psi_0 = 1$$

$$\psi_0 = \sum_i c_i z^i \qquad c_{\leq 0} = 0\,, \quad c_1 = \frac{2}{d-4}\,, \quad c_2 = \frac{4(d-5)s + 2(d-4)t}{(d-4)(d-2)st}\,, \ldots$$

$$\mathrm{Res}_{z=0}\left(\left(\tfrac{2}{d-4}z + \mathcal{O}(z^2)\right)\left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\right) = 0$$

Likewise in $-s$ and t: $\mathrm{Res}_{z \in \{-s,t\}}(\psi \xi_1) = 0$

For $z = \infty$ we use $y = z^{-1}$ giving $\xi_1 = \frac{-s}{1+sy}dy$

$$(\mathrm{d}+\omega)\psi_\infty = \phi \qquad \Rightarrow \qquad \left(\partial_y + \frac{-1}{2y}\left((d-6) - \frac{d-4}{1+sy} + \frac{2(d-5)}{1-ty}\right)\right)\psi_\infty = \frac{-1}{y^2}$$

$$\psi_\infty = \sum_i c_i y^i \qquad c_{\leq -2} = 0\,, \quad c_{-1} = \frac{1}{d-5}\,, \quad c_0 = \frac{-((d-4)s + 2(d-5)t)}{2(d-6)(d-5)}\,, \ldots$$

$$\mathrm{Res}_{y=0}\left(\left(\tfrac{1}{d-5}\tfrac{1}{y} + \mathcal{O}(y^0)\right)\left(\tfrac{-s}{1+sy}\right)\right) = \frac{-s}{d-5}$$

$$\langle \phi_1 | \xi_1 \rangle = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p}(\psi \xi_1) = \frac{-s}{d-5}$$

Università degli Studi di Padova

Marie Curie Actions

Example (double box)

$$c_i = \langle \phi | \xi_j \rangle (\mathbf{C}^{-1})_{ji} \qquad \text{with} \qquad \mathbf{C}_{ij} = \langle \phi_i | \xi_j \rangle$$

$$\phi = z^2 \, \mathrm{d}z \,, \quad \phi_1 = 1 \, \mathrm{d}z \,, \quad \phi_2 = z \, \mathrm{d}z \,, \quad \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s} \right) \mathrm{d}z \,, \quad \xi_2 = \left(\tfrac{1}{z+s} - \tfrac{1}{z-t} \right) \mathrm{d}z \,,$$

We needed the intersection numbers: $\Big\{ \langle \phi | \xi_1 \rangle, \, \langle \phi | \xi_2 \rangle, \, \langle \phi_1 | \xi_1 \rangle, \, \langle \phi_1 | \xi_2 \rangle, \, \langle \phi_2 | \xi_1 \rangle, \, \langle \phi_2 | \xi_2 \rangle \Big\}$

Using $\langle \phi | \xi \rangle = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p} (\psi_p \xi)$ with $(\mathrm{d} + \omega) \psi_p = \phi$, we get

$$\langle \phi | \xi_1 \rangle = \frac{s(4(d-5)t^2 - 3(d-4)(3d-14)s^2 - 4(d-5)(2d-9)st)}{4(d-5)(d-4)(d-3)} \,,$$

$$\langle \phi | \xi_2 \rangle = \frac{s(s+t)(3(d-4)(3d-14)s + 2(d-6)(d-5)t)}{4(d-5)(d-4)(d-3)} \,,$$

$$\langle \phi_1 | \xi_1 \rangle = \frac{-s}{d-5} \,, \qquad \langle \phi_1 | \xi_2 \rangle = \frac{s+t}{d-5} \,,$$

$$\langle \phi_2 | \xi_1 \rangle = \frac{s((3d-14)s + 2(d-5)t)}{2(d-5)(d-4)} \,, \qquad \langle \phi_2 | \xi_2 \rangle = \frac{-(3d-14)s(s+t)}{2(d-5)(d-4)} \,.$$

UNIVERSITÀ DEGLI STUDI DI PADOVA

MARIE CURIE ACTIONS

Example (double box)

$$c_i = \langle \phi | \xi_j \rangle (\mathbf{C}^{-1})_{ji} \qquad \text{with} \qquad \mathbf{C}_{ij} = \langle \phi_i | \xi_j \rangle$$

$$\phi = z^2 \, \mathrm{d}z \,, \quad \phi_1 = 1 \, \mathrm{d}z \,, \quad \phi_2 = z \, \mathrm{d}z \,, \quad \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s} \right) \mathrm{d}z \,, \quad \xi_2 = \left(\tfrac{1}{z+s} - \tfrac{1}{z-t} \right) \mathrm{d}z \,,$$

We needed the intersection numbers: $\Big\{ \langle \phi|\xi_1 \rangle, \langle \phi|\xi_2 \rangle, \langle \phi_1|\xi_1 \rangle, \langle \phi_1|\xi_2 \rangle, \langle \phi_2|\xi_1 \rangle, \langle \phi_2|\xi_2 \rangle \Big\}$

Using $\langle \phi | \xi \rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p}(\psi_p \xi)$ with $(\mathrm{d} + \omega)\psi_p = \phi$, we get

$$\langle \phi|\xi_1 \rangle = \frac{s(4(d-5)t^2 - 3(d-4)(3d-14)s^2 - 4(d-5)(2d-9)st)}{4(d-5)(d-4)(d-3)} \,,$$

$$\langle \phi|\xi_2 \rangle = \frac{s(s+t)(3(d-4)(3d-14)s + 2(d-6)(d-5)t)}{4(d-5)(d-4)(d-3)} \,,$$

$$\langle \phi_1|\xi_1 \rangle = \frac{-s}{d-5} \,, \qquad \langle \phi_1|\xi_2 \rangle = \frac{s+t}{d-5} \,,$$

$$\langle \phi_2|\xi_1 \rangle = \frac{s((3d-14)s + 2(d-5)t)}{2(d-5)(d-4)} \,, \qquad \langle \phi_2|\xi_2 \rangle = \frac{-(3d-14)s(s+t)}{2(d-5)(d-4)} \,.$$

$$I_{1111111;-2} = c_0 I_{1111111;0} + c_1 I_{1111111;-1} + \text{lower} \qquad c_0 = \frac{(d-4)st}{2(d-3)} \,, \quad c_1 = \frac{2t - 3(d-4)s}{2(d-3)} \,,$$

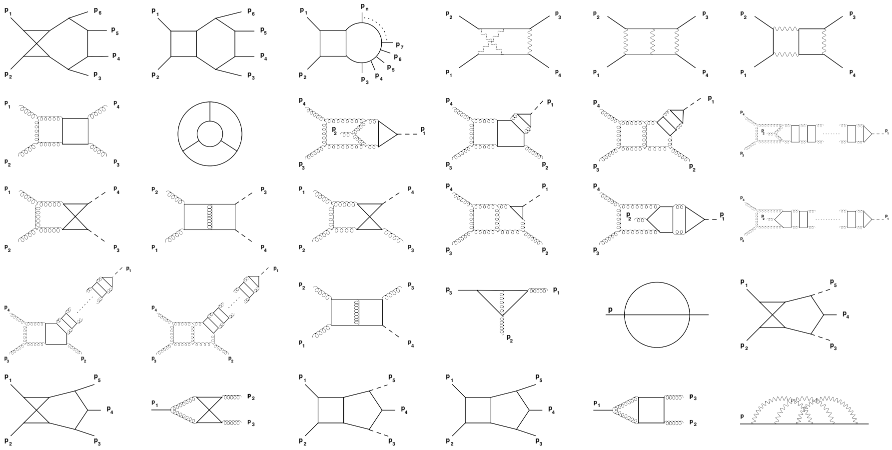
in agreement with FIRE

Further cases

We did $\mathcal{O}(30)$ examples in the paper

Further cases

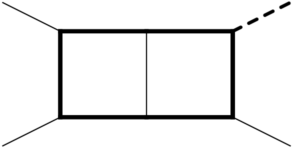
We did $\mathcal{O}(30)$ examples in the paper



Further cases

A planar integral contributing to NLO Higgs+jet production

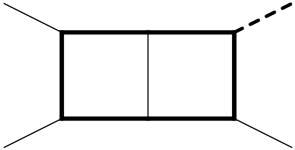
$H + j$ "Family A":



$$D_1 = k_2^2 - m_t^2 \,, \quad D_2 = (k_2 + p_1)^2 - m_t^2 \,, \quad D_3 = (k_2 + p_1 + p_2)^2 - m_t^2 \,, \quad D_4 = (k_1 + p_1 + p_2)^2 - m_t^2 \,,$$
$$D_5 = (k_1 + p_1 + p_2 + p_3)^2 - m_t^2 \,, \quad D_6 = k_1^2 - m_t^2 \,, \quad D_7 = (k_1 - k_2)^2$$

A planar integral contributing to NLO Higgs+jet production

$H + j$ "Family A":



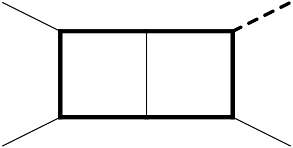
$$D_1 = k_2^2 - m_t^2\,, \quad D_2 = (k_2+p_1)^2 - m_t^2\,, \quad D_3 = (k_2+p_1+p_2)^2 - m_t^2\,, \quad D_4 = (k_1+p_1+p_2)^2 - m_t^2\,,$$
$$D_5 = (k_1+p_1+p_2+p_3)^2 - m_t^2\,, \quad D_6 = k_1^2 - m_t^2\,, \quad D_7 = (k_1-k_2)^2\,; \quad D_8 = (k_1+p_1)^2 - m_t^2 = z\,.$$

$$u = z^{d-5}(z^2+sz+m_t^2 s)^{\frac{4-d}{2}} \Big((m_H^2-s)^2 z^2 + 2(m_H^2-s)stz + st\big(4m_t^2(m_H^2-s-t)+st\big)\Big)^{\frac{d-5}{2}}$$

There are four master integrals.

A planar integral contributing to NLO Higgs+jet production

$H + j$ "Family A":



$$D_1 = k_2^2 - m_t^2\,, \quad D_2 = (k_2+p_1)^2 - m_t^2\,, \quad D_3 = (k_2+p_1+p_2)^2 - m_t^2\,, \quad D_4 = (k_1+p_1+p_2)^2 - m_t^2\,,$$
$$D_5 = (k_1+p_1+p_2+p_3)^2 - m_t^2\,, \quad D_6 = k_1^2 - m_t^2\,, \quad D_7 = (k_1-k_2)^2\,; \quad D_8 = (k_1+p_1)^2 - m_t^2 = z\,.$$

$$u = z^{d-5}(z^2+sz+m_t^2 s)^{\frac{4-d}{2}}\Big((m_H^2-s)^2 z^2 + 2(m_H^2-s)stz + st\big(4m_t^2(m_H^2-s-t)+st\big)\Big)^{\frac{d-5}{2}}$$

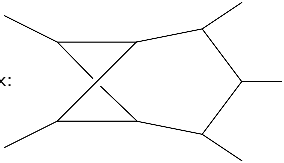
There are four master integrals.

$$I_{1111111;-1} = c_1 I_{1111111;0} + c_2 I_{1211111;0} + c_3 I_{1111211;0} + c_4 I_{1111112;0} + \text{lower}$$

The intersection procedure gives cs in agreement with Kira.

A non-planar integral contributing to NNLO 3-jet production

First non-planar pentabox:



$$D_1 = k_1^2\,, \quad D_2 = (k_1{+}p_1)^2\,, \quad D_3 = (k_1{-}k_2{-}p_2)^2\,, \quad D_4 = (k_1{-}k_2)^2\,, \quad D_5 = (k_2{+}p_1{+}p_2)^2\,,$$
$$D_6 = (k_2{+}p_1{+}p_2{+}p_3)^2\,, \quad D_7 = (k_2{+}p_1{+}p_2{+}p_3{+}p_4)^2\,, \quad D_8 = (k_2)^2\,; \quad D_9 = (k_2+p_1)^2 = z\,.$$

$$u = \Big(z\,(z+s_{12})\,\big(s_{35}\,z^2 + (s_{51}s_{12}{+}s_{12}s_{23}{-}s_{23}s_{34}{+}s_{34}s_{45}{-}s_{45}s_{51})z - s_{51}s_{12}s_{23}\big)\Big)^{\frac{d-6}{2}}$$

The Lee-Pomeransky criterion gives three master integrals in agreement with the literature.

$$I_{11111111;-3} = c_0 I_{11111111;0} + c_1 I_{11111111;-1} + c_2 I_{11111111;-2} + \text{lower}$$

Again the intersection procedure gives cs in agreement with the codes.

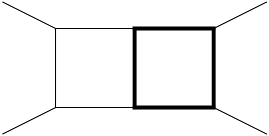
Further cases

An example of apparent discrepancy:

Further cases

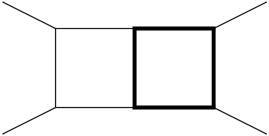
An example of apparent discrepancy:

Internally massive double-box:



$$D_1 = k_1^2\,, \quad D_2 = (k_1+p_1)^2\,, \quad D_3 = (k_1+p_1+p_2)^2\,, \quad D_4 = (k_2+p_1+p_2)^2 - m^2\,,$$
$$D_5 = (k_2-p_4)^2 - m^2\,, \quad D_6 = k_2^2 - m^2\,, \quad D_7 = (k_1-k_2)^2 - m^2\,; \quad D_8 = (k_1-p_4)^2 = z\,.$$

An example of apparent discrepancy:

Internally massive double-box: 

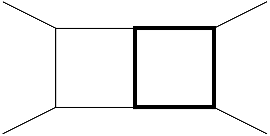
$$D_1 = k_1^2\,, \quad D_2 = (k_1{+}p_1)^2\,, \quad D_3 = (k_1{+}p_1{+}p_2)^2\,, \quad D_4 = (k_2{+}p_1{+}p_2)^2 - m^2\,,$$
$$D_5 = (k_2{-}p_4)^2 - m^2\,, \quad D_6 = k_2^2 - m^2\,, \quad D_7 = (k_1{-}k_2)^2 - m^2\,; \quad D_8 = (k_1{-}p_4)^2 = z\,.$$

The Lee-Pomeransky criterion gives three master integrals, but the literature mentions four!

$$I_{1111111;0}\,, \quad I_{1211111;0}\,, \quad I_{1111211;0}\,, \quad I_{1111112;0}\,.$$

An example of apparent discrepancy:

Internally massive double-box:



$$D_1 = k_1^2\,,\quad D_2 = (k_1+p_1)^2\,,\quad D_3 = (k_1+p_1+p_2)^2\,,\quad D_4 = (k_2+p_1+p_2)^2 - m^2\,,$$
$$D_5 = (k_2-p_4)^2 - m^2\,,\quad D_6 = k_2^2 - m^2\,,\quad D_7 = (k_1-k_2)^2 - m^2\,;\quad D_8 = (k_1-p_4)^2 = z\,.$$

The Lee-Pomeransky criterion gives three master integrals, but the literature mentions four!

$$I_{1111111;0}\,,\quad I_{1211111;0}\,,\quad I_{1111211;0}\,,\quad I_{1111112;0}\,.$$

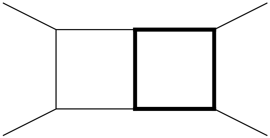
There is an extra relation relating 7-propagator sectors:

$$I_{01111111;1} = \tfrac{1}{2} I_{1111112;0} - \tfrac{1}{2} I_{1111211;0} - \tfrac{d-4}{4m^2} I_{1111011;1} + \tfrac{d-4}{2m^2} I_{1111110;1} + \text{lower}$$

The intersection theory knows this relation!
On the 7×cut there are three (checked numerically)

An example of apparent discrepancy:

Internally massive double-box:



$$D_1 = k_1^2\,, \quad D_2 = (k_1{+}p_1)^2\,, \quad D_3 = (k_1{+}p_1{+}p_2)^2\,, \quad D_4 = (k_2{+}p_1{+}p_2)^2 - m^2\,,$$
$$D_5 = (k_2{-}p_4)^2 - m^2\,, \quad D_6 = k_2^2 - m^2\,, \quad D_7 = (k_1{-}k_2)^2 - m^2\,; \quad D_8 = (k_1{-}p_4)^2 = z\,.$$

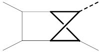
The Lee-Pomeransky criterion gives three master integrals, but the literature mentions four!

$$I_{1111111;0}\,, \quad I_{1211111;0}\,, \quad I_{1111211;0}\,, \quad I_{1111112;0}\,.$$

There is an extra relation relating 7-propagator sectors:

$$I_{01111111;1} = \tfrac{1}{2} I_{1111112;0} - \tfrac{1}{2} I_{1111211;0} - \tfrac{d-4}{4m^2} I_{1111011;1} + \tfrac{d-4}{2m^2} I_{1111110;1} + \text{lower}$$

The intersection theory knows this relation!
On the 7×cut there are three (checked numerically)

This also holds for $H{+}j$ fam. F  $(6 \to 4)$

(see arXiv:1907.13156 for fam. F.)

Does it only work for maximal cuts?

Does it only work for maximal cuts? NO!

Does it only work for maximal cuts? NO!

$$I = \int_{\mathcal{C}} u\hat{\phi}\,\mathrm{d}^n z = \sum_i c_i I_i \qquad \text{with} \qquad c_i = \langle\phi|\xi_j\rangle(\mathbf{C}^{-1})_{ji} \quad \mathbf{C}_{ij} = \langle\phi_i|\xi_j\rangle$$

but now $\langle\phi|\xi\rangle$ is a *multivariate intersection number*

K. Matsumoto, *Intersection numbers for logarithmic k-forms*, Osaka J. Math. **35** (1998) no. 4 873-893
S. Mizera, *Aspects of Scattering Amplitudes and Moduli Space Localization*, [arXiv:1906.02099]
HF, F. Gasparotto, M. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera,
Vector Space of Feynman Integrals and Multivariate Intersection Numbers, [arXiv:1907.02000]

Does it only work for maximal cuts? NO!

$$I = \int_{\mathcal{C}} u \hat{\phi}\, \mathrm{d}^n z = \sum_i c_i I_i \qquad \text{with} \qquad c_i = \langle \phi | \xi_j \rangle (\mathbf{C}^{-1})_{ji} \qquad \mathbf{C}_{ij} = \langle \phi_i | \xi_j \rangle$$

but now $\langle \phi | \xi \rangle$ is a *multivariate intersection number*

K. Matsumoto, *Intersection numbers for logarithmic k-forms*, Osaka J. Math. **35** (1998) no. 4 873-893
S. Mizera, *Aspects of Scattering Amplitudes and Moduli Space Localization*, [arXiv:1906.02099]
HF, F. Gasparotto, M. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera,
Vector Space of Feynman Integrals and Multivariate Intersection Numbers, [arXiv:1907.02000]

$${}_{\mathbf{n}}\langle \phi^{(\mathbf{n})} | \xi^{(\mathbf{n})} \rangle = -\sum_{p \in \mathcal{P}_n} \operatorname*{Res}_{z_n = p} \Big({}_{\mathbf{n-1}}\langle \phi^{(\mathbf{n})} | h_i^{(\mathbf{n-1})} \rangle \, \psi_i^{(n)} \Big) \ ,$$

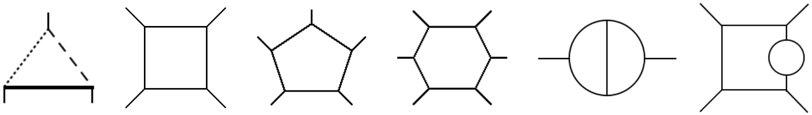
$$\Big(\delta_{ij}\, \partial_{z_n} - \hat{\mathbf{\Omega}}_{ij}^{(n)} \Big) \psi_j^{(n)} = \hat{\xi}_i^{(n)} \ ,$$

$$\hat{\mathbf{\Omega}}_{ij}^{(n)} = -\big(\mathbf{C}_{(\mathbf{n-1})}^{-1} \big)_{ik} \; {}_{\mathbf{n-1}}\langle e_k^{(\mathbf{n-1})} | (\partial_{z_n} - \hat{\omega}_n) h_j^{(\mathbf{n-1})} \rangle \ ,$$

$$\xi_i^{(n)} = \big(\mathbf{C}_{(\mathbf{n-1})}^{-1} \big)_{ij} \; {}_{\mathbf{n-1}}\langle e_j^{(\mathbf{n-1})} | \xi^{(\mathbf{n})} \rangle \ ,$$

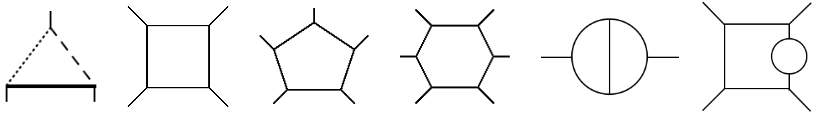
$$\big(\mathbf{C}_{(\mathbf{n-1})} \big)_{ij} \equiv {}_{\mathbf{n-1}}\langle e_i^{(\mathbf{n-1})} | h_j^{(\mathbf{n-1})} \rangle \ .$$

We have done the full reduction of

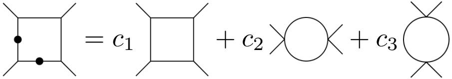


multivariate

We have done the full reduction of



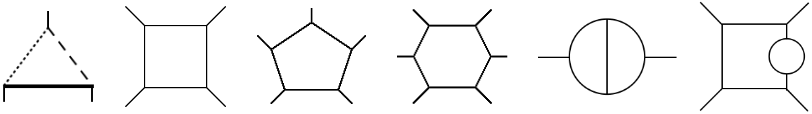
In particular



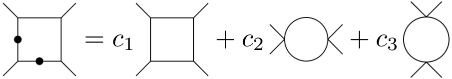
$$u(\mathbf{x}) = \left((st - sx_4 - tx_3)^2 - 2tx_1(s(t+2x_3-x_2-x_4)+tx_3) + s^2x_2^2 + t^2x_1^2 - 2sx_2(t(s-x_3)+x_4(s+2t))\right)^{\frac{d-5}{2}}.$$

Applying *regulators* to uncut propagators $u \to u\prod_i x_i^{\rho_i}$ gives $\nu = 3$

We have done the full reduction of



In particular



$$u(\mathbf{x}) = \big((st - sx_4 - tx_3)^2 - 2tx_1(s(t+2x_3-x_2-x_4)+tx_3)$$
$$+ s^2x_2^2 + t^2x_1^2 - 2sx_2(t(s-x_3)+x_4(s+2t))\big)^{\frac{d-5}{2}}.$$

Applying *regulators* to uncut propagators $u \to u\prod_i x_i^{\rho_i}$ gives $\nu = 3$

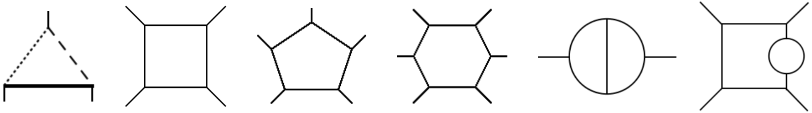
$$\hat{\phi} = (x_1^2 x_2^2 x_3 x_4)^{-1} \quad \hat{\phi}_1 = (x_1x_2x_3x_4)^{-1} \quad \hat{\phi}_2 = (x_1x_3)^{-1} \quad \hat{\phi}_3 = (x_2x_4)^{-1}$$

$$c_i = \langle \phi|\xi_j\rangle(\mathbf{C}^{-1})_{ji} \quad \text{with} \quad \mathbf{C}_{ij} = \langle \phi_i|\xi_j\rangle$$

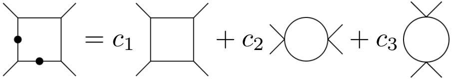
Università degli Studi di Padova

Marie Curie Actions

We have done the full reduction of



In particular



$$u(\mathbf{x}) = \left((st - sx_4 - tx_3)^2 - 2tx_1(s(t+2x_3-x_2-x_4)+tx_3) + s^2x_2^2 + t^2x_1^2 - 2sx_2(t(s-x_3)+x_4(s+2t))\right)^{\frac{d-5}{2}}.$$

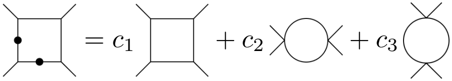
Applying *regulators* to uncut propagators $u \to u \prod_i x_i^{\rho_i}$ gives $\nu = 3$

$$\hat{\phi} = (x_1^2 x_2^2 x_3 x_4)^{-1} \quad \hat{\phi}_1 = (x_1 x_2 x_3 x_4)^{-1} \quad \hat{\phi}_2 = (x_1 x_3)^{-1} \quad \hat{\phi}_3 = (x_2 x_4)^{-1}$$

$$c_i = \langle \phi | \xi_j \rangle (\mathbf{C}^{-1})_{ji} \quad \text{with} \quad \mathbf{C}_{ij} = \langle \phi_i | \xi_j \rangle$$

$$c_1 = \frac{(d-5)(d-6)}{st}, \quad c_2 = \frac{-4(d-5)(d-3)}{s^3 t}, \quad c_3 = \frac{-4(d-5)(d-3)}{st^3},$$

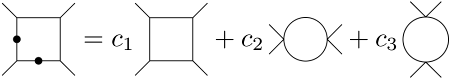
in agreement with FIRE



The cut of the s-channel bubble: cut $\{x_2, x_4\}$.

$$\int \frac{u\,\mathrm{d}^4 x}{x_1^2 x_2^2 x_3 x_4}\bigg|_{\text{cut}2,4} = \int \frac{\partial_{x_2} u}{x_1^2 x_3}\bigg|_{\{x_2,x_4\}\to 0} \mathrm{d}x_1 \mathrm{d}x_3 = \int \left(u_{\{x_2,x_4\}\to 0}\ \frac{\partial_{x_2} u}{u x_1^2 x_3}\bigg|_{\{x_2,x_4\}\to 0} \mathrm{d}x_1 \mathrm{d}x_3 \right)$$

multivariate



The cut of the s-channel bubble: cut $\{x_2, x_4\}$.

$$\int \frac{u\,\mathrm{d}^4x}{x_1^2 x_2^2 x_3 x_4}\bigg|_{\text{cut}2,4} = \int \frac{\partial_{x_2} u}{x_1^2 x_3}\bigg|_{\{x_2,x_4\}\to 0} \mathrm{d}x_1 \mathrm{d}x_3 = \int \left(u_{\{x_2,x_4\}\to 0}\ \frac{\partial_{x_2} u}{u x_1^2 x_3}\bigg|_{\{x_2,x_4\}\to 0} \mathrm{d}x_1 \mathrm{d}x_3 \right)$$

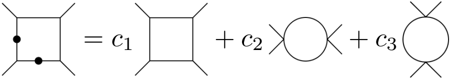
$$u_{24\text{cut}} = \left(s^2 t + t(x_1 - x_3)^2 - 2s(2x_1x_3 + t(x_1 + x_3))\right)^{\frac{d-5}{2}}$$

and now $u_{24\text{cut}} \to u_{24\text{cut}} x_1^{\rho_1} x_3^{\rho_3}$ gives $\nu = 2$.

UNIVERSITÀ DEGLI STUDI DI PADOVA

MARIE CURIE ACTIONS

multivariate



The cut of the s-channel bubble: cut $\{x_2, x_4\}$.

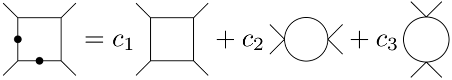
$$\int \frac{u\,\mathrm{d}^4x}{x_1^2 x_2^2 x_3 x_4}\bigg|_{\text{cut}2,4} = \int \frac{\partial_{x_2} u}{x_1^2 x_3}\bigg|_{\{x_2,x_4\}\to 0} \mathrm{d}x_1 \mathrm{d}x_3 = \int \left(u_{\{x_2,x_4\}\to 0} \frac{\partial_{x_2} u}{u x_1^2 x_3}\bigg|_{\{x_2,x_4\}\to 0} \mathrm{d}x_1 \mathrm{d}x_3 \right)$$

$$u_{24\text{cut}} = \left(s^2 t + t(x_1 - x_3)^2 - 2s(2x_1x_3 + t(x_1 + x_3))\right)^{\frac{d-5}{2}}$$

and now $u_{24\text{cut}} \to u_{24\text{cut}} x_1^{\rho_1} x_3^{\rho_3}$ gives $\nu = 2$.

$$\hat{\phi} = \frac{(d-5)s(x_1+x_3-s)}{\left(s^2t+t(x_1-x_3)^2-2s(2x_1x_3+t(x_1+x_3))\right)x_1x_3}\,, \quad \hat{\phi}_1 = (x_1x_3)^{-1}\,, \quad \hat{\phi}_2 = 1\,.$$

$c_i = \langle\phi|\xi_j\rangle(\mathbf{C}^{-1})_{ji}$ with $\mathbf{C}_{ij} = \langle\phi_i|\xi_j\rangle$ for c_1 and c_2



The cut of the s-channel bubble: cut $\{x_2, x_4\}$.

$$\int \frac{u\,\mathrm{d}^4 x}{x_1^2 x_2^2 x_3 x_4}\bigg|_{\mathsf{cut2,4}} = \int \frac{\partial_{x_2} u}{x_1^2 x_3}\bigg|_{\{x_2,x_4\}\to 0} \mathrm{d}x_1 \mathrm{d}x_3 = \int \left(u_{\{x_2,x_4\}\to 0}\; \frac{\partial_{x_2} u}{u x_1^2 x_3}\bigg|_{\{x_2,x_4\}\to 0} \mathrm{d}x_1 \mathrm{d}x_3 \right)$$

$$u_{24\mathsf{cut}} = \left(s^2 t + t(x_1 - x_3)^2 - 2s(2x_1 x_3 + t(x_1 + x_3))\right)^{\frac{d-5}{2}}$$

and now $u_{24\mathsf{cut}} \to u_{24\mathsf{cut}} x_1^{\rho_1} x_3^{\rho_3}$ gives $\nu = 2$.

$$\hat{\phi} = \frac{(d-5)s(x_1+x_3-s)}{\left(s^2 t+t(x_1-x_3)^2-2s(2x_1x_3+t(x_1+x_3))\right)x_1x_3}\,, \quad \hat{\phi}_1 = (x_1 x_3)^{-1}\,, \quad \hat{\phi}_2 = 1\,.$$

$$c_i = \langle \phi | \xi_j \rangle (\mathbf{C}^{-1})_{ji} \quad \text{with} \quad \mathbf{C}_{ij} = \langle \phi_i | \xi_j \rangle \text{ for } c_1 \text{ and } c_2$$

The $\{x_1, x_3\}$-cut would yield c_3. Combine *spanning cuts*: The bottom-up approch.

Perspectives

$$I = \int_{\mathcal{C}} u\phi \quad \to \quad I = \sum_i c_i I_i \qquad \text{with} \qquad c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad C_{ij} = \langle \phi_i | \xi_j \rangle$$

For one-forms:

$$\langle \phi | \xi \rangle = \sum_{p \in \mathcal{P}} \mathsf{Res}_{z=p}(\psi_p \xi) \qquad\qquad (\mathrm{d} + \omega)\psi_p = \phi$$

For multivariate forms it is more involved but similar.

Perspectives

$$I = \int_{\mathcal{C}} u\phi \quad \rightarrow \quad I = \sum_i c_i I_i \qquad \text{with} \qquad c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad C_{ij} = \langle \phi_i | \xi_j \rangle$$

For one-forms:

$$\langle \phi | \xi \rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p}(\psi_p \xi) \qquad\qquad (\mathrm{d} + \omega)\psi_p = \phi$$

For multivariate forms it is more involved but similar.

Not only for integral relations:

Differential equations:

$$\partial_s I = \int_{\mathcal{C}} \partial_s (u\phi) = \int_{\mathcal{C}} u\tilde{\phi}$$

Dimension shift relations:

$$I_{d \to d \pm 2n} = \int_{\mathcal{C}} (u\phi)_{d \to d \pm 2n} = \int_{\mathcal{C}} u\tilde{\tilde{\phi}}$$

Perspectives

$$I = \int_{\mathcal{C}} u\phi \quad \to \quad I = \sum_i c_i I_i \qquad \text{with} \qquad c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad C_{ij} = \langle \phi_i | \xi_j \rangle$$

For one-forms:

$$\langle \phi | \xi \rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p}(\psi_p \xi) \qquad (\mathrm{d} + \omega)\psi_p = \phi$$

For multivariate forms it is more involved but similar.

Not only for integral relations:

Differential equations:

$$\partial_s I = \int_{\mathcal{C}} \partial_s (u\phi) = \int_{\mathcal{C}} u\tilde{\phi}$$

Dimension shift relations:

$$I_{d\to d\pm 2n} = \int_{\mathcal{C}} (u\phi)_{d\to d\pm 2n} = \int_{\mathcal{C}} u\tilde{\tilde{\phi}}$$

Also for contiguity relations for (generalized) hypergeometric functions

$${}_2F_1\left({a,\, b+2 \atop c+2}, x\right) = C_1\; {}_2F_1\left({a,\, b \atop c}, x\right) + C_2\; {}_2F_1\left({a,\, b+1 \atop c+1}, x\right)$$

Perspectives

$$I = \int_{\mathcal{C}} u\phi \quad \rightarrow \quad I = \sum_i c_i I_i \qquad \text{with} \qquad c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad C_{ij} = \langle \phi_i | \xi_j \rangle$$

For one-forms:

$$\langle \phi | \xi \rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p}(\psi_p \xi) \qquad (\mathrm{d} + \omega)\psi_p = \phi$$

For multivariate forms it is more involved but similar.

Not only for integral relations:

Differential equations:

$$\partial_s I = \int_{\mathcal{C}} \partial_s(u\phi) = \int_{\mathcal{C}} u\tilde{\phi}$$

Dimension shift relations:

$$I_{d \to d \pm 2n} = \int_{\mathcal{C}} (u\phi)_{d \to d \pm 2n} = \int_{\mathcal{C}} u\tilde{\tilde{\phi}}$$

Also for contiguity relations for (generalized) hypergeometric functions

$${}_2F_1\left({a,\, b+2 \atop c+2}, x \right) = C_1 \; {}_2F_1\left({a,\, b \atop c}, x \right) + C_2 \; {}_2F_1\left({a,\, b+1 \atop c+1}, x \right)$$

Can find integral relations without the use of IBPs.

$$I = \int_{\mathcal{C}} u\phi \quad \rightarrow \quad I = \sum_i c_i I_i \qquad \text{with} \qquad c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad C_{ij} = \langle \phi_i | \xi_j \rangle$$

For one-forms:

$$\langle \phi | \xi \rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p}(\psi_p \xi) \qquad (\mathrm{d} + \omega)\psi_p = \phi$$

For multivariate forms it is more involved but similar.

Not only for integral relations:

Differential equations:

$$\partial_s I = \int_{\mathcal{C}} \partial_s (u\phi) = \int_{\mathcal{C}} u\tilde{\phi}$$

Dimension shift relations:

$$I_{d \to d \pm 2n} = \int_{\mathcal{C}} (u\phi)_{d \to d \pm 2n} = \int_{\mathcal{C}} u\tilde{\tilde{\phi}}$$

Also for contiguity relations for (generalized) hypergeometric functions

$$_2F_1\left({a,\, b+2 \atop c+2}, x\right) = C_1 \; _2F_1\left({a,\, b \atop c}, x\right) + C_2 \; _2F_1\left({a,\, b+1 \atop c+1}, x\right)$$

Can find integral relations without the use of IBPs.

"IBPs without IBPs"

Università degli Studi di Padova

Marie Curie Actions

Future work:

- Full understanding of the multivariate case

Future work:

- Full understanding of the multivariate case
- Classify hypergeometric functions
- Clarify connection to *co-action*
- Combine with rational reconstruction
- Optimized algorithm for sub-sectors

Future work:

- Full understanding of the multivariate case
- Classify hypergeometric functions
- Clarify connection to *co-action*
- Combine with rational reconstruction
- Optimized algorithm for sub-sectors
- Make an extremely fast code
- ...

Future work:

- Full understanding of the multivariate case
- Classify hypergeometric functions
- Clarify connection to *co-action*
- Combine with rational reconstruction
- Optimized algorithm for sub-sectors
- Make an extremely fast code
- ...

Thank you for the invitation to speak,
and thank you for listening!

Hjalte Frellesvig

UNIVERSITÀ DEGLI STUDI DI PADOVA

MARIE CURIE ACTIONS