Modular graph functions and Poincaré series

Axel Kleinschmidt (Albert Einstein Institute, Potsdam)

Elliptics '19

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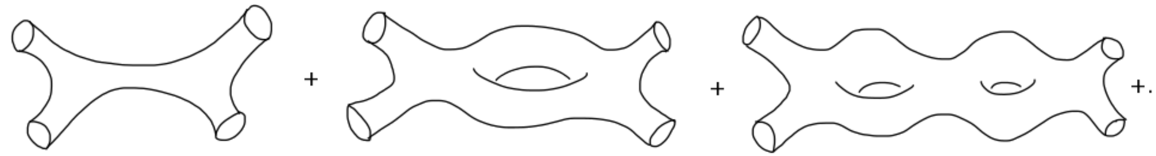
Joint work with Olof Ahlén [1803.10250]
and Daniele Dorigoni [1903.09250]

Two types of modularity in string theory

<u>World-sheet</u>



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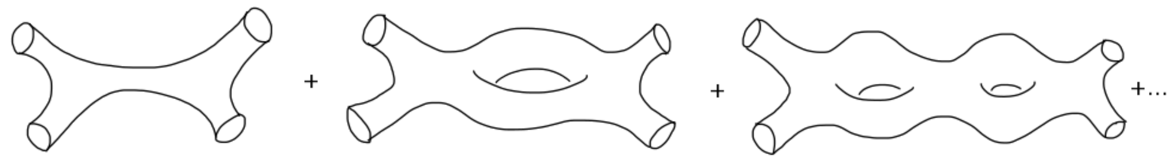

closed n-point amplitude on genus h world-sheet Σ_τ includes integral over cx. structure τ.
Σ_τ related by large diffeos identified $\Rightarrow$ modular inv.

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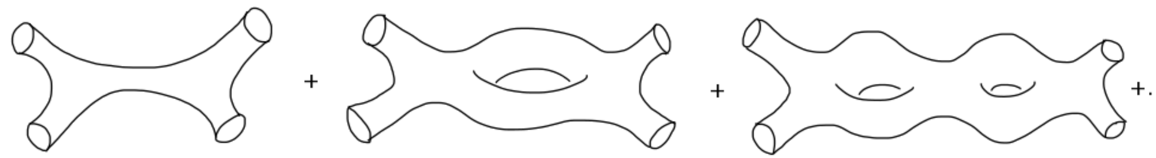

$SL(2,\mathbb{Z})$ $Sp(4,\mathbb{Z})$

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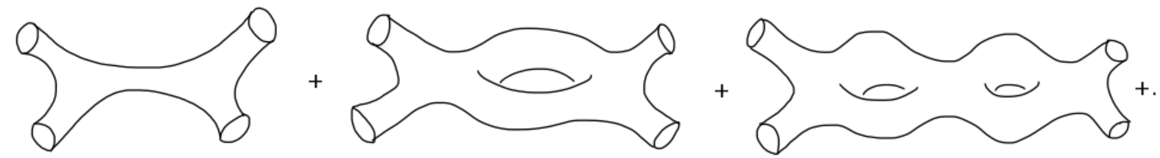

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Compactification of II(B) string on torus T^d
$\Rightarrow$ U-duality $G(\mathbb{Z})$ [Hull, Townsend]

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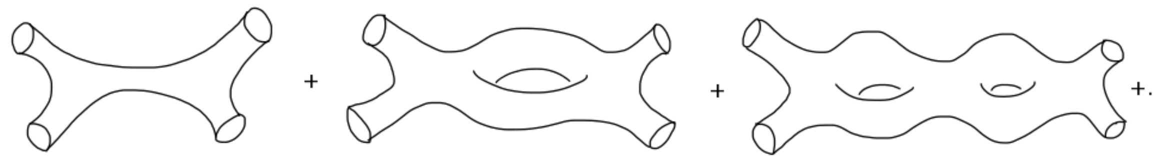

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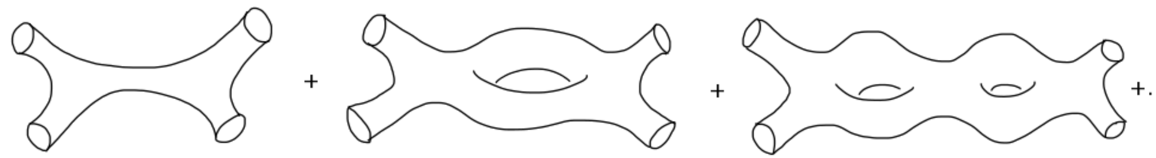

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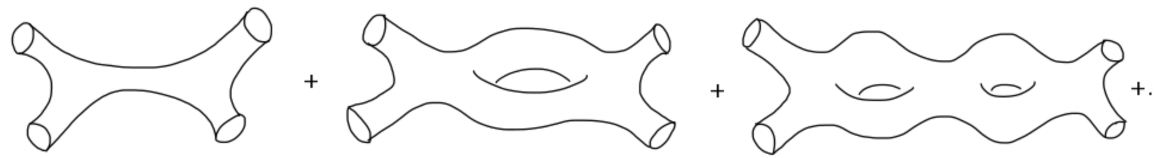

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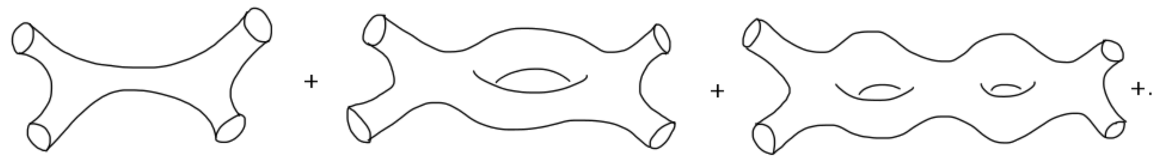

$$\mathcal{A}_h = \int_{\mathcal{M}_h} d\mu_h(\tau) \prod_{i=1}^{n} \int_{\Sigma_\tau} \frac{d^2 z_i}{\text{“vol}(Sp(2h))\text{”}} \, \langle V_1(z_1) \cdots V_n(z_n) \rangle_{\Sigma_\tau}$$

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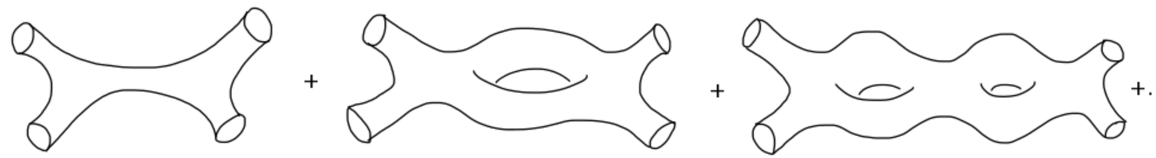

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modular invariant integrand

often computable only in $\alpha' = \ell_s^2$ expansion

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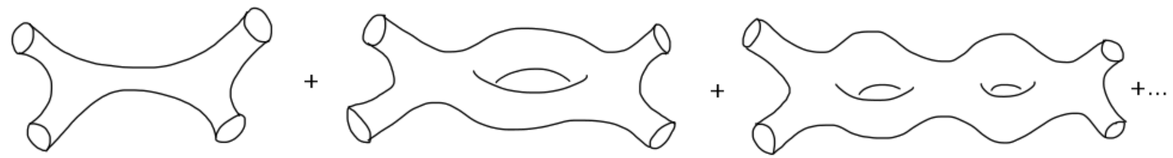

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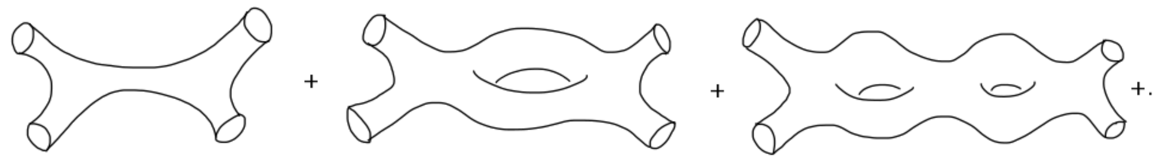

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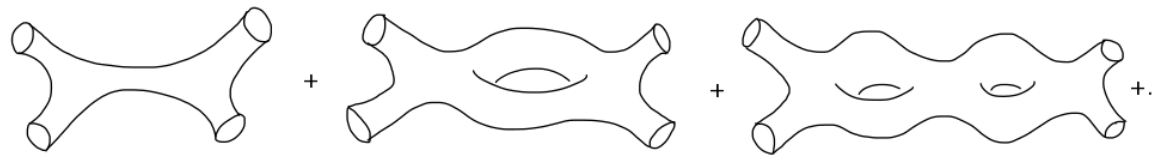

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Low-energy α'-expansion

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$$S = \frac{1}{(\alpha')^4}\int d^{10}x\sqrt{-g}\Big(R + (\alpha')^3 f_{R^4}(\tau)R^4$$

$$+(\alpha')^5 f_{D^4R^4}(\tau)D^4R^4 + (\alpha')^6 f_{D^6R^4}(\tau)D^6R^4 + \dots\Big)$$

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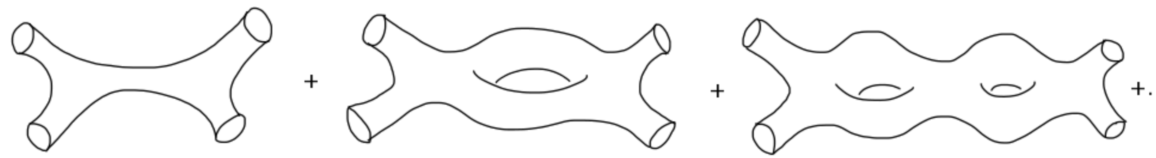

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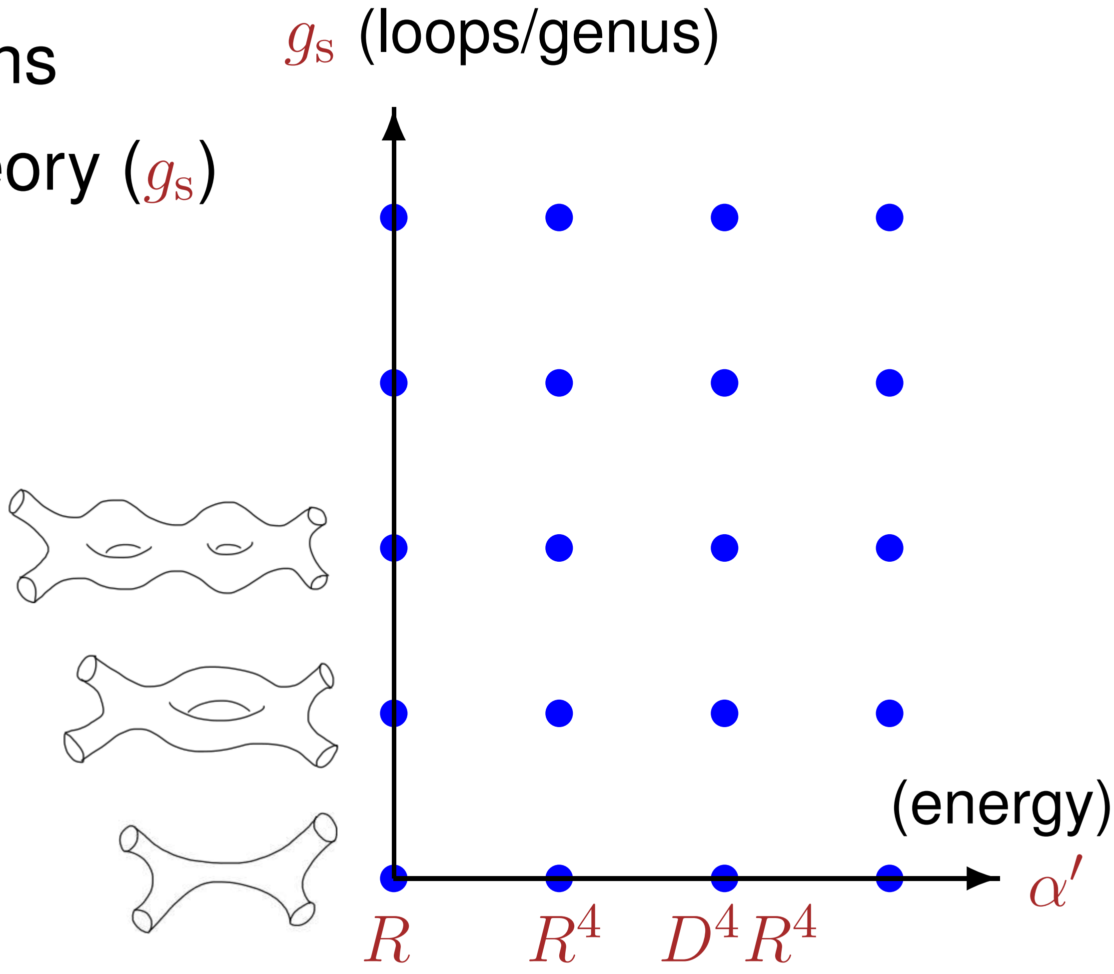
U-duality invariant, sometimes constrained by susy

Double expansion of string amplitudes

Modularity in two expansions

- In loop perturbation theory (g_s)
- In energy (α')



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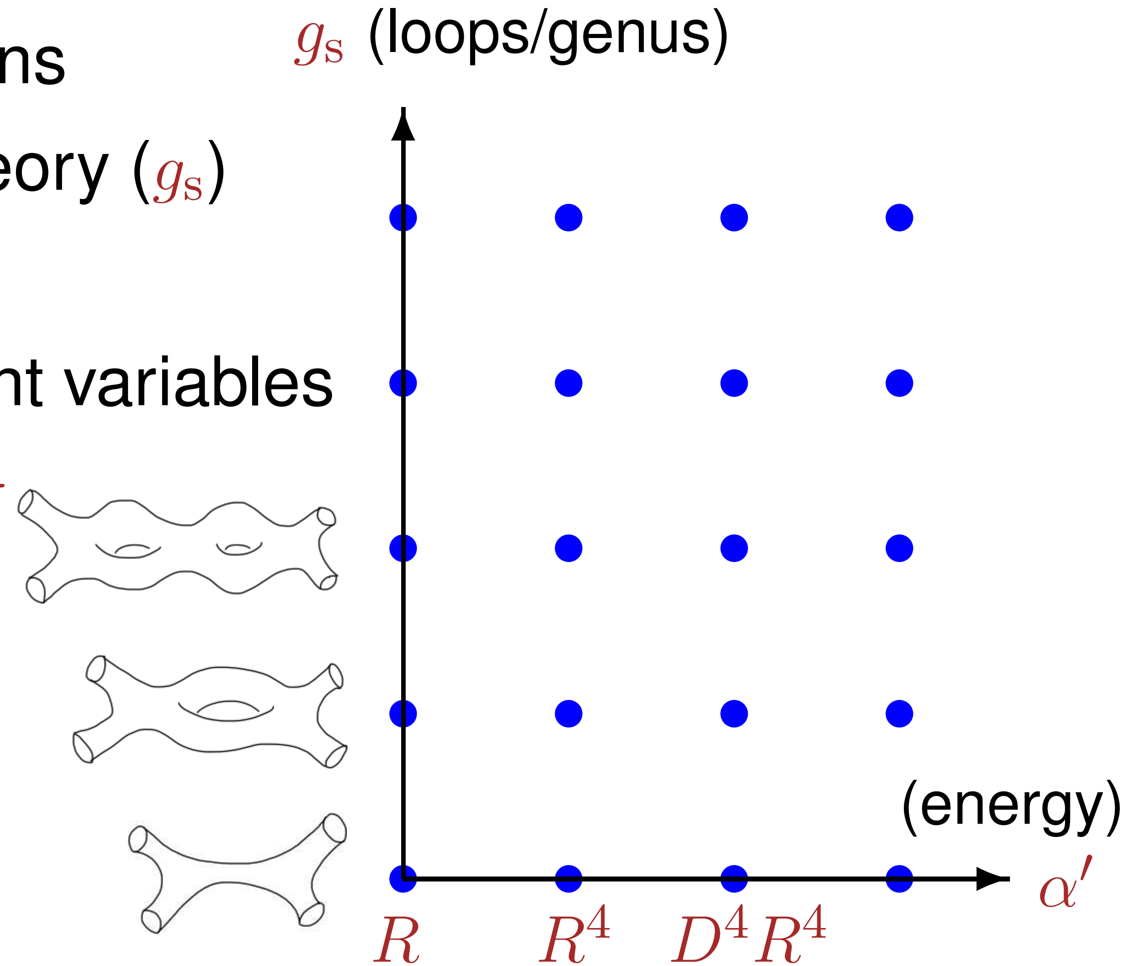
- World-sheet modulus τ



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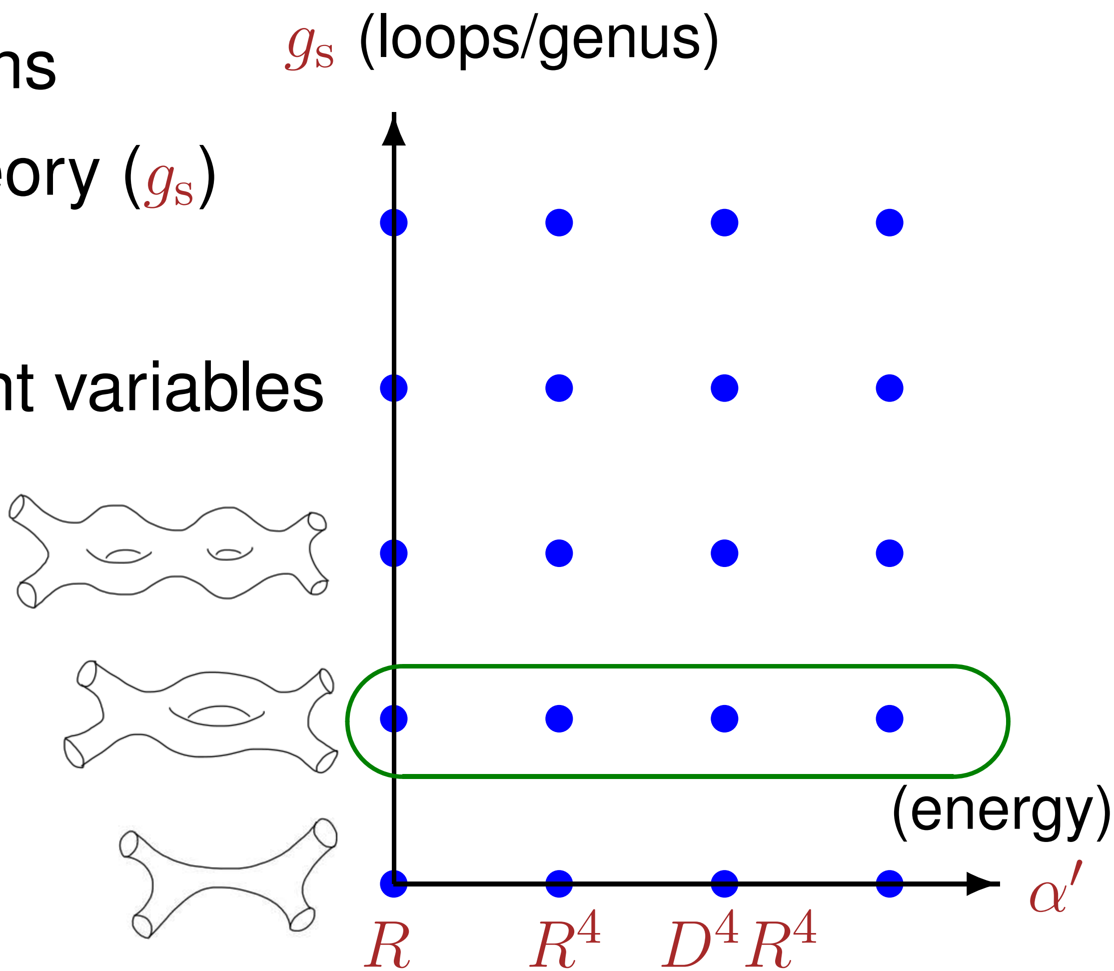

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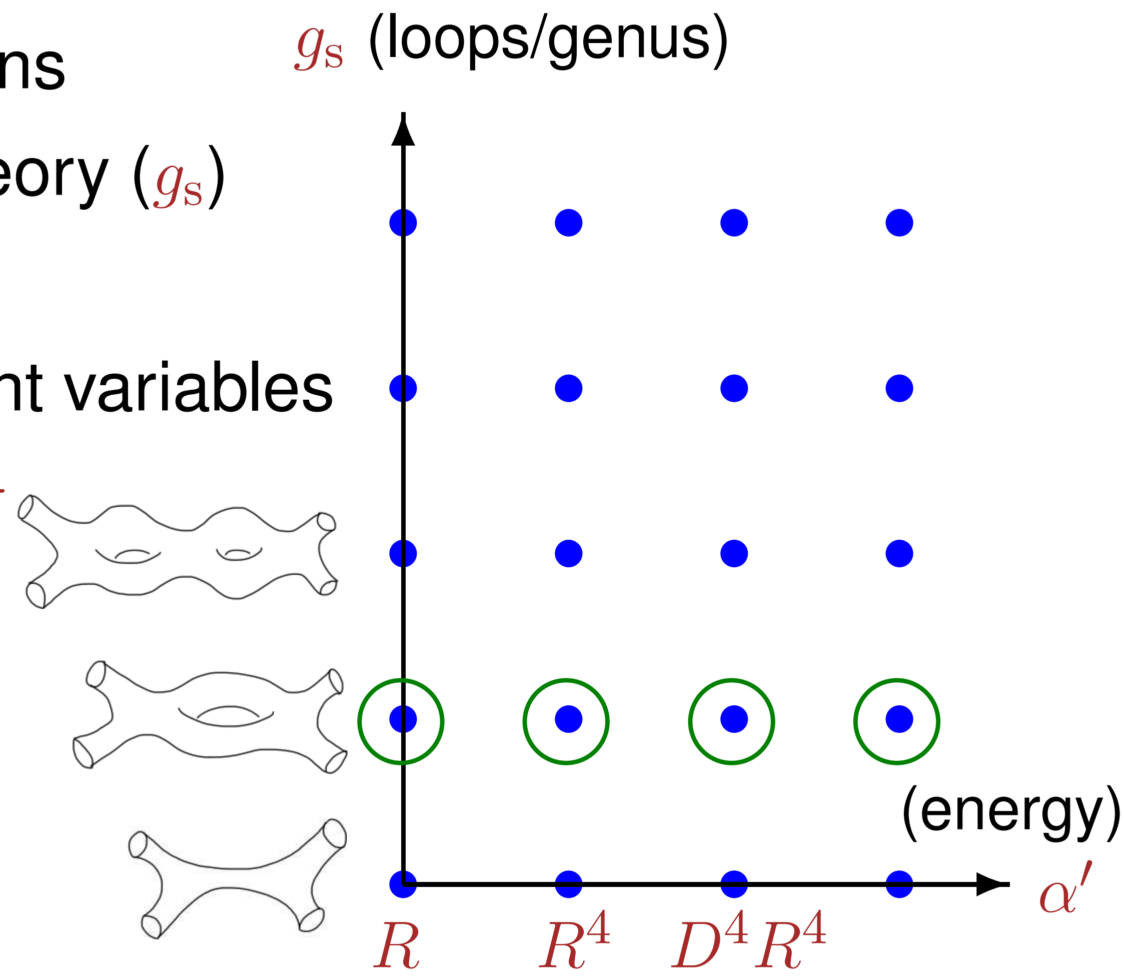

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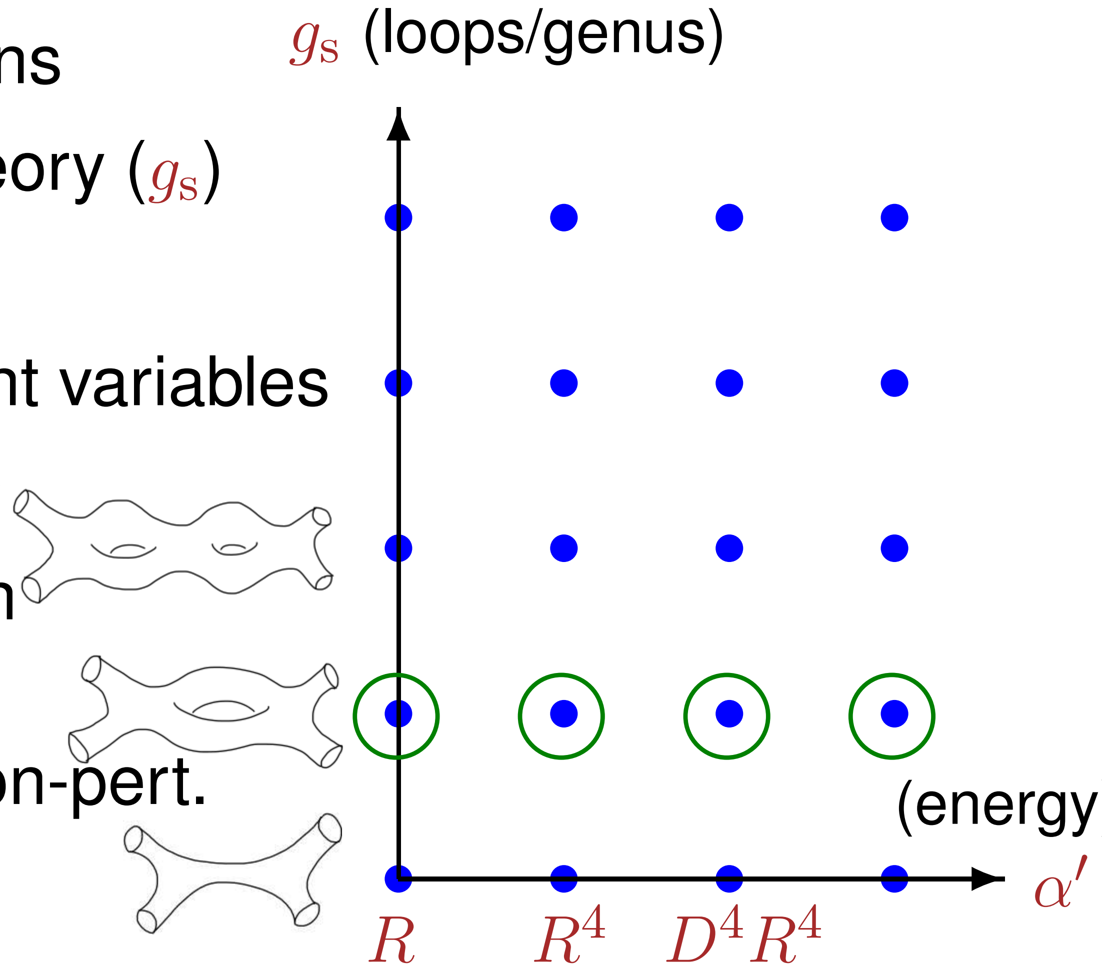
  $z = C_{(0)} + ie^{-\phi}$

  $\langle e^{\phi}\rangle = g_\mathrm{s} \Rightarrow$ includes non-pert.



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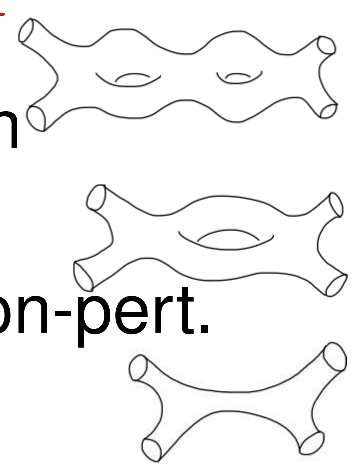

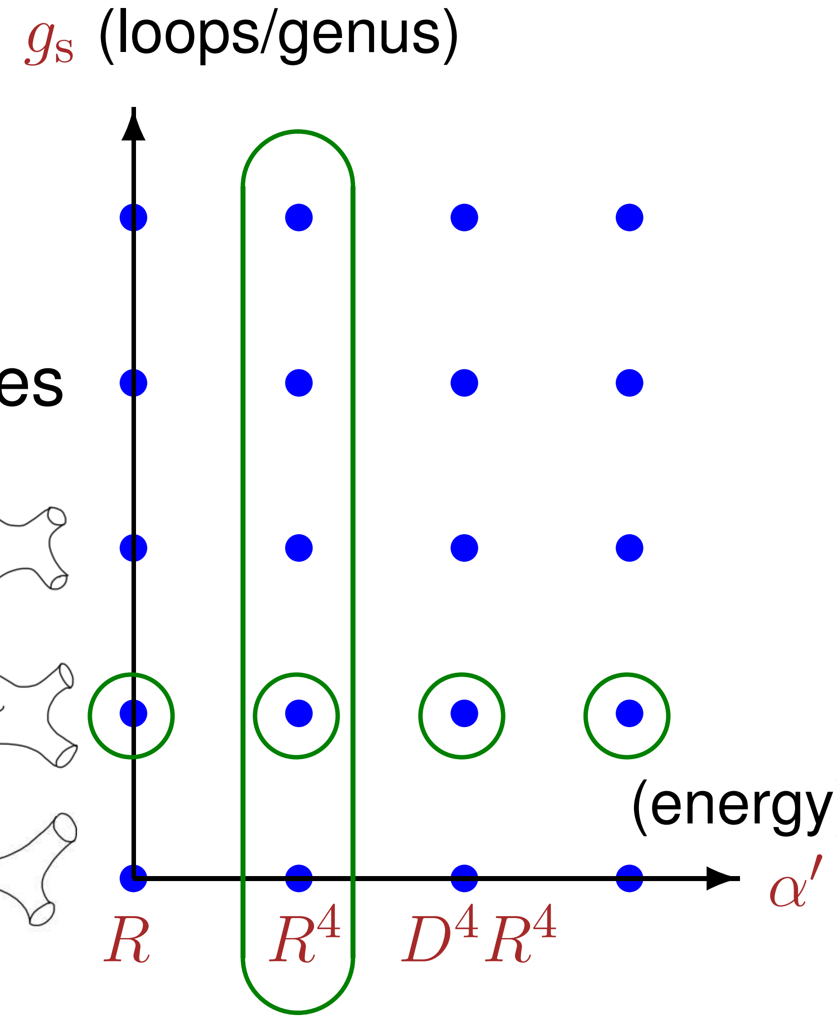

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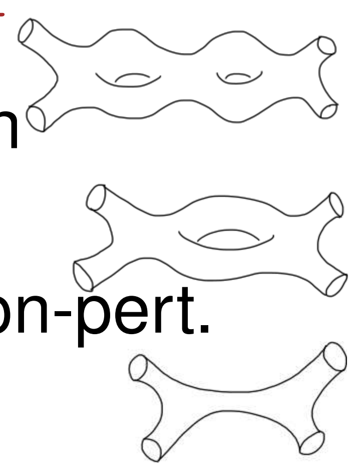
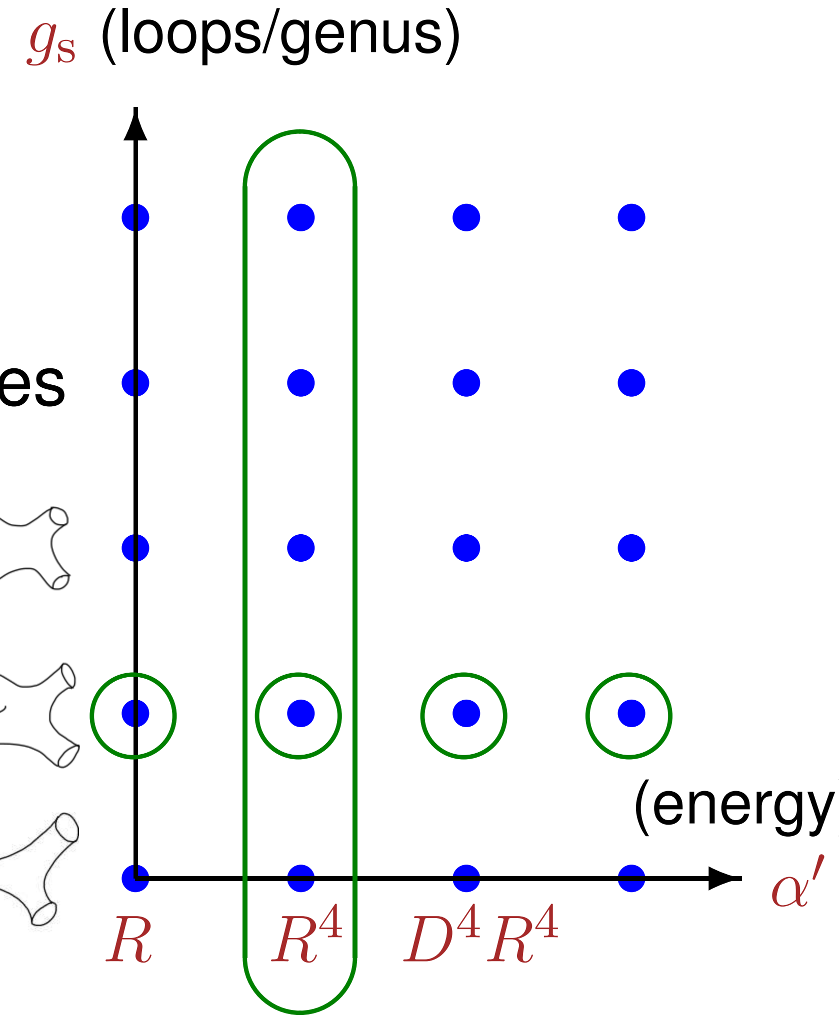
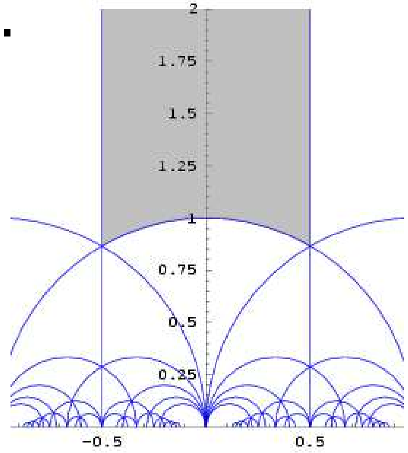




Two types compute aspects of the same quantity and have common features

Modular differential equations

The $SL(2,\mathbb{Z})$-invariant functions $f(\tau)$ that appear in the two cases satisfy (modular) differential equation, e.g.



$$(\Delta - s(s-1))\, f(\tau) = R(\tau)$$

($\Delta = \tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2)$ for $\tau = \tau_1 + i\tau_2$)

modular invariant 'source'

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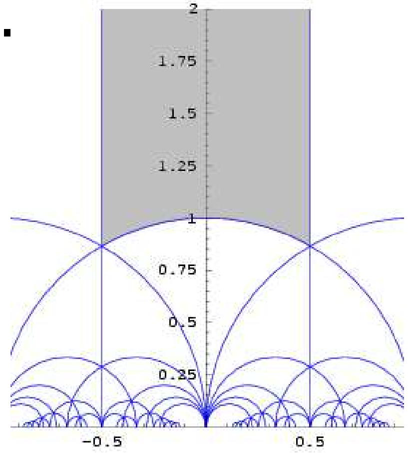

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Origin of differential equations different

- World-sheet Feynman rules give explicit function (as multiple lattice sum) [D'Hoker, Green, Vanhove]
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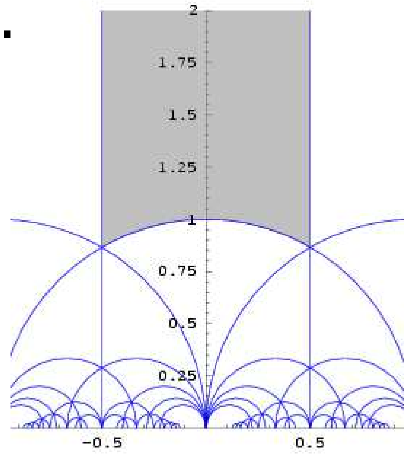

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Use differential equation as common starting point to learn about $f(\tau)$! (Use τ for both cases.)

Interlude: Modular graph functions

At string one-loop encounter things like

$$\prod_{i=1}^{n} \int_{\Sigma_\tau} \frac{d^2 z_i}{\tau_2} \exp\left[\sum_{k<\ell} s_{k\ell} G(z_k - z_\ell|\tau)\right] \qquad (\star)$$

from CFT.

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Mandelstam variables

$s_{k\ell} = -\frac{\alpha'}{4}(p_k + p_\ell)^2$

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from CFT. Scalar Green function on torus ($z = u\tau + v$)

$$G(z|\tau) = -\log\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^2 + \frac{2\pi z_2^2}{\tau_2} - 2\log\left|(2\pi)^{1/2}\eta(q)\right|^2$$

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Integral ($\star$) not known in closed form. Evaluate by expanding in Mandelstam/at low energies, i.e. α'

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not nec.an amplitude

Feynman rules on Σ_τ [D'Hoker, Green, Vanhove]

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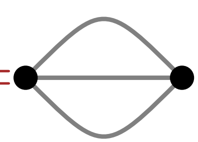
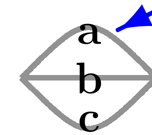
Generates $SL(2,\mathbb{Z})$-invariant functions, e.g.

a propagators in a row

$C_{1,1,1} =$  or $C_{a,b,c} =$ 

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Modular graph functions are nested lattice sums (from loop momenta)

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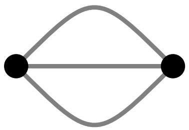

$$= \sum_{\substack{(m_1,n_1)\neq(0,0)\\(m_2,n_2)\neq(0,0)\\(m_1+m_2,n_1+n_2)\neq(0,0)}} \frac{\tau_2^3}{\pi^3|m_1\tau+n_1|^2|m_2\tau+n_2|^2|(m_1+m_2)\tau+n_1+n_2|^2}$$

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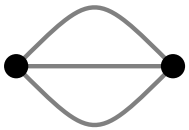
Modular graph functions are nested lattice sums (from loop momenta)



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$$= 2\zeta(6)\pi^{-3}E_3(\tau) + \zeta(3)$$ [Zagier]

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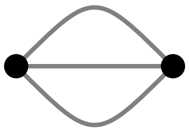

$$= \sum_{\substack{(m_1,n_1)\neq(0,0)\\(m_2,n_2)\neq(0,0)\\(m_1+m_2,n_1+n_2)\neq(0,0)}} \frac{\tau_2^3}{\pi^3|m_1\tau+n_1|^2|m_2\tau+n_2|^2|(m_1+m_2)\tau+n_1+n_2|^2}$$

$$= 2\zeta(6)\pi^{-3}E_3(\tau) + \zeta(3)$$ [Zagier]

non-holomorphic Eisenstein series

$$E_s(\tau) = \frac{1}{2\zeta(2s)} \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m\tau+n|^{2s}}$$

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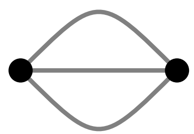

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In general hard to extract information from nested sums; some results see [Basu, Brödel, Brown, D'Hoker, Duke, Gerken, Green, Gürdogan, Kaidi, AK, Matthes, Panzer, Schlotterer, Vanhove, Zerbini,...]

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Solving by Poincaré series [Ahlén, AK]

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Assume that RHS has Poincaré series expansion

$$R(\tau) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \rho(\gamma\tau)$$

$$B(\mathbb{Z}) = \left\{ \begin{pmatrix} \pm 1 & n \\ 0 & \pm 1 \end{pmatrix} \right\}$$

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Idea: Try to solve (1') rather than (1). Reduced complexity!

Example of Poincaré series

Non-holomorphic Eisenstein series

$$E_s(\tau) = \frac{1}{2\zeta(2s)} \sum_{(c,d)\neq(0,0)} \frac{\tau_2^s}{|c\tau + d|^{2s}} = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} [\mathrm{Im}\,(\gamma\tau)]^s$$

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Fourier expansion for $\tau \to \tau+1$

$$E_s(\tau) = \tau_2^s + \frac{\sqrt{\pi}\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\tau_2^{1-s} + \frac{2\pi^s}{\Gamma(s)\zeta(2s)}\tau_2^{1/2}\sum_{n\neq 0}|n|^{s-1/2}\sigma_{1-2s}(n)K_{s-1/2}(2\pi|n|\tau_2)e^{2\pi i n\tau_1}$$

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$\sigma_{1-2s}(n) = \sum_{d|n} d^{1-2s}$

$K_{s-1/2}(2\pi|n|\tau_2) \sim \frac{1}{2\sqrt{|n|\tau_2}}e^{-2\pi|n|\tau_2}\left(1+O(\tau_2^{-1})\right)$

Solving by Poincaré series

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Solve Fourier mode by mode $\sigma(\tau) = \sum_{n\in\mathbb{Z}} c_n(\tau_2)e^{2\pi i n\tau_1}$

$$c_0(\tau_2) = \frac{6}{5}\pi^{-5}\zeta(10)\tau_2^5 + \frac{2\pi^3\zeta(3)}{945}\tau_2^2 + \frac{\zeta(5)}{10(\epsilon(\epsilon-1)-6)}\tau_2^\epsilon$$

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What does that say about $C_{3,1,1}$?

Relating $\sigma(\tau)$ and $f(\tau) = \sum_\gamma \sigma(\gamma\tau)$

Fourier expansions. For

$$\sigma(\tau) = \sum_{n\in\mathbb{Z}} c_n(\tau_2) e^{2\pi i n \tau_1} \quad \text{one has} \quad f(\tau) = \sum_{n\in\mathbb{Z}} a_n(\tau_2) e^{2\pi i n \tau_1}$$

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with

$$a_n(\tau_2) = c_n(\tau_2) + \sum_{c>0} \sum_{m\in\mathbb{Z}} S(m,n;c) \int_{\mathbb{R}} e^{-2\pi i \omega n - 2\pi i m \frac{\omega}{c^2(\omega^2+\tau_2^2)}} c_m\left(\frac{\tau_2}{c^2(\omega^2+\tau_2^2)}\right) d\omega$$

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Kloosterman sum

$$S(m,n;c) = \sum_{q\in(\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i (mq + nq^{-1})/c}$$

(Above formula for Fourier coefficient $a_n(\tau_2)$ assumes absolute convergence.)

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Focus on zero mode (of sum $f(\tau)$)

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Understand things like ($c_n(\tau_2) \sim \sigma_a(n)(4\pi|n|)^b \tau_2^r e^{-2\pi|n|\tau_2}$)

$$\sum_{c>0} \sum_{m>0} \sum_{q\in(\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i m q/c} \sigma_a(m) \left(\frac{4\pi m}{c^2\tau_2}\right)^b \int_{\mathbb{R}} e^{-2\pi m \frac{1+it}{c^2\tau_2(1+t^2)}} \frac{dt}{(1+t^2)^r}$$

Zagier's proposition

$$\sum_{c>0}\sum_{m>0}\sum_{q\in(\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i m q/c}\sigma_a(m)\left(\frac{4\pi m}{c^2\tau_2}\right)^b \int_{\mathbb{R}} e^{-2\pi m \frac{1+it}{c^2\tau_2(1+t^2)}} \frac{dt}{(1+t^2)^r}$$

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'Riemann term'

$I_\varphi = \int_0^\infty \varphi(t)dt$

Zagier's proposition

$$\sum_{c>0}\sum_{m>0}\sum_{q\in(\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i m q/c}\sigma_a(m)\left(\frac{4\pi m}{c^2\tau_2}\right)^b \underbrace{\int_{\mathbb{R}} e^{-2\pi m\frac{1+it}{c^2\tau_2(1+t^2)}}\frac{dt}{(1+t^2)^r}}_{\frac{\pi}{4^{1-r}\Gamma(r)}\sum_{k\geq 0}\left(-\frac{\pi m}{c^2\tau_2}\right)^k\frac{\Gamma(2r+k-1)}{k!\,\Gamma(k+r)}}$$

Note that τ_2 only occurs in the combination $m\tau_2 \Rightarrow$ use a proposition of Zagier's for obtaining the asymptotic $\tau_2 \to \infty$

Suppose $\varphi(t) \sim \sum_{n\geq 0} b_n t^n$ around $t \sim 0$. Then

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$I_\varphi = \int_0^\infty \varphi(t)dt$

'Euler term' (exchange sums)

$\zeta(-n,h) = \sum_{m\geq 0}(m+h)^n$

(Hurwitz zeta, $h > 0$)

Polylog proposition

Applying Zagier's proposition generates expressions like

$$\sum_{c>0} c^{-s} \sum_{q\in(\mathbb{Z}/c\mathbb{Z})^\times} \sum_{h=1}^{c} \zeta\big(1-k,\frac{h}{c}\big)\mathrm{Li}_n(e^{2\pi i hq/c})$$

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This formula can be applied to the solutions for $C_{2,1,1}$, $C_{3,1,1}$ and recovers the known Laurent polynomials

Modular graph functions

Case $C_{2,1,1}$:

$$a_0(\tau_2) = \frac{2\pi^4}{14\,175}\tau_2^4 + \frac{\pi\zeta(3)}{45}\tau_2 + \frac{5\zeta(5)}{12\pi}\tau_2^{-1} - \frac{\zeta(3)^2}{4\pi^2}\tau_2^{-2} + \frac{9\zeta(7)}{16\pi^3}\tau_2^{-3} + O(e^{-4\pi|n|\tau_2})$$

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Agrees with [D'Hoker, Green, Vanhove]

Salient features

- ‘Deformation’ of equation is important for convergence. In particular when there are perfect squares on the RHS, like $E_{3/2}(\tau)^2$ or in $C_{2,1,1}$ case
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- For treating U-duality case $f_{D^6R^4}(\tau)$ need one more ‘epicycle’: Differential operators acting on general seed. Get the correct result ($g_\mathrm{s} = \tau_2^{-1}$)

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- Considering general seed with parameters a, b and r also useful for getting non-perturbative contributions under control: resurgence

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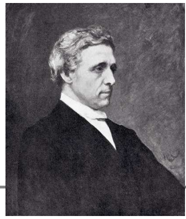

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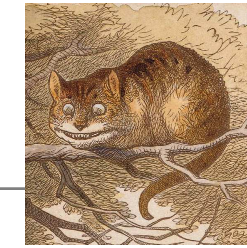

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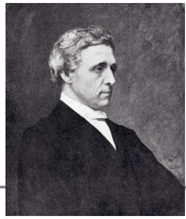

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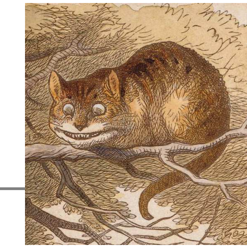

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Performing resurgent analysis *before* limit produces non-perturbative terms that survive in the limit: Cheshire cat resurgence. For $C_{3,1,1}$ OK with [D'Hoker, Kaidi]

$$a_0^{\mathrm{np}}(\tau_2) \sim \sum_{m>0} \sigma_{-3}(m)\sigma_{-5}(m)(\pi\tau_2)^{-2} m e^{-4\pi m \tau_2}\left(1+\frac{1}{\pi m \tau_2}\right)$$

Lambert series and iterated integrals

[w.i.p. with D. Dorigoni]

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Methods help with asymptotics of Lambert series ($q = e^{2\pi i\tau}$)

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For $s = 1-k$ with $k \in \mathbb{N}$ even get $\mathcal{L}_{1-k}(q) = G_k^0(q)$ (hol. Eisenstein without constant term).

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Zagier's method leads quickly to asymptotic expansion

$$\mathcal{L}_s(e^{-2\pi y}) \sim \zeta(1-s)\Gamma(1-s)(2\pi y)^{s-1} + \sum_{k=0}^{\infty} \frac{(-2\pi y)^{k-1}}{\Gamma(k)} \zeta(1-k)\zeta(s+1-k) + O(e^{-2\pi n y})$$

(Cf. [Banerjee, Wilkerson].) Also info on non-pert. terms

Lambert series and iterated integrals

'modularity gap'

Can also obtain the 'S-dual' by similar methods for $s = k - 1 \in \mathbb{Z}$ odd

$$\mathcal{L}_{k-1}(e^{2\pi i\tau}) = \tau^{k-2}\mathcal{L}_{k-1}(e^{-2\pi i/\tau}) + P_k(\tau)$$

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- matches results by [Shimomura]
- example of what Zagier calls a quantum modular form (also close to quasi and mock in this case)

Lambert series and iterated integrals

Note that since $\sigma_s(m) = m^s \sigma_{-s}(m)$:

$$\mathcal{L}_{-s}(q) = (q\partial_q)^s \mathcal{L}_s(q)$$

For negative integer $s = 1 - k$ interpret $q\partial_q$ as iterated integral.

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$$\mathcal{E}_0(k, 0^{k-2}; \tau) = (-1)^{k-1} \int\limits_{0 \le q_1 \le \cdots \le q_{k-1} \le q} d\log q_1 \cdots d\log q_{k-1} \frac{G_k^0(q_1)}{(2\pi i)^k}$$
$$= -\frac{2}{(k-1)!} \mathcal{L}_{k-1}(\tau)$$

Consistent with their modular trms, also for $\mathcal{E}_0(k, 0^{k-2-p})$ etc.

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Generalisation to multiple polylogs and MZV?

Summary and outlook

- General method for solving inhomogeneous modular differential equations using Poincaré series
- Reproduces correctly known results. Arguably more involved where explicit lattice sums available
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Thanks for your attention!

Extra slides

General formula

For seed $c_n(\tau_2) \sim \sigma_a(n)(4\pi|n|)^b \tau_2^r e^{-2\pi|n|\tau_2}$

$$2\tau_2 \sum_{c>0} \sum_{m>0} \sum_{q\in(\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i m q/c} \int_{\mathbb{R}} e^{-2\pi m \frac{1+it}{\tau_2 c^2(1+t^2)}} \sigma_a(m) \frac{(4\pi m)^b}{(\tau_2 c^2(1+t^2))^r} dt$$

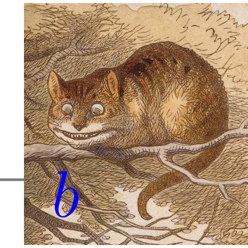
$$\sim \frac{2^{3-2r+2b}\pi\tau_2^{1+b-r}}{\Gamma(r)} \Bigg[\frac{\tau_2}{\pi} \frac{\Gamma(b+1)\Gamma(2r-b-2)}{\Gamma(r-b-1)} \frac{\zeta(2r-a-2b-2)\zeta(1-a)}{\zeta(2r-a-2b-1)}$$

$$+ \left(\frac{\tau_2}{\pi}\right)^{a+1} \frac{\Gamma(a+b+1)\Gamma(2r-a-b-2)}{\Gamma(r-a-b-1)} \frac{\zeta(2r-a-2b-2)\zeta(a+1)}{\zeta(2r-a-2b-1)}$$

$$+ \left(\frac{\pi}{\tau_2}\right)^b \sum_{n\geq 0} \left(\frac{-\pi}{\tau_2}\right)^n \frac{\Gamma(2r+n-1)}{n!\cdot\Gamma(r+n)}$$

$$\times \frac{\zeta(-b-n)\zeta(-a-b-n)\zeta(2r-a-b+n-1)\zeta(2r-b+n-1)}{\zeta(2r+2n)\zeta(2r-a-2b-1)} \Bigg]$$

Resurgence analysis for $C_{3,1,1}$



Asymptotic tail from b-deformation

$$a_0^{\text{asy}}(\tau_2) = \frac{16}{\pi}\sin(\pi b)\sum_{n\geq 0}(4\pi\tau_2)^{-n-5}(6+n)\Gamma(n+2)\frac{\zeta(2+n)\zeta(5+n)\zeta(7+n)\zeta(10+n)}{\zeta(2n+12)}$$

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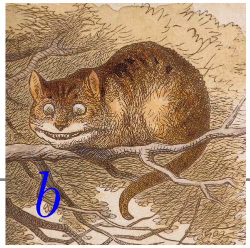

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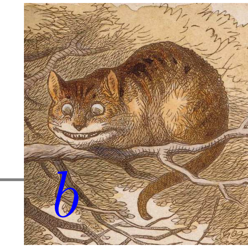

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In limit $b \to 0$ left with

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