The Equal-Mass Banana Integral and Elliptic Curves

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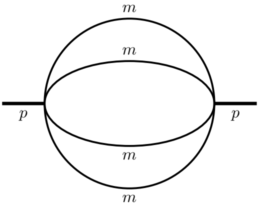
The Banana Family



$$I_{a_1,\ldots,a_9}(p^2, m^2, d) = \int \prod_{i=1}^{3} \mathrm{d}^d k_i$$

$$\frac{(k_3^2)^{a_5}(k_1 \cdot p)^{a_6}(k_2 \cdot p)^{a_7}(k_3 \cdot p)^{a_8}(k_1 \cdot k_2)^{a_9}}{[k_1^2 - m^2]^{a_1}[k_2^2 - m^2]^{a_2}[(k_1 - k_3)^2 - m^2]^{a_3}[(k_2 - k_3 - p)^2 - m^2]^{a_4}}$$

What Lorenzo Said*

Four basis integrals

$$\mathcal{I}_0(\epsilon,x) = I_{2,2,2,0}(p^2,m^2,2-2\epsilon) = 1$$

$$\mathcal{I}_1(\epsilon,x) = (1+2\epsilon)(1+3\epsilon)I_{1,1,1,1}(p^2,m^2,2-2\epsilon)$$
$$\mathcal{I}_2(\epsilon,x) = (1+2\epsilon)I_{2,1,1,1}(p,^2,m^2,2-2\epsilon)$$
$$\mathcal{I}_3(\epsilon,x) = I_{2,2,1,1}(p^2,m^2,2-2\epsilon)$$

$$x = \frac{4m^2}{p^2}$$

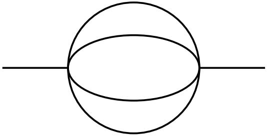
All finite in $d=2$!

The Differential Equation

$$\partial_x \begin{pmatrix} \mathcal{I}_1(x) \\ \mathcal{I}_2(x) \\ \mathcal{I}_3(x) \end{pmatrix} = B(x) \begin{pmatrix} \mathcal{I}_1(x) \\ \mathcal{I}_2(x) \\ \mathcal{I}_3(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2(4x-1)} \end{pmatrix} ,$$

$$B(x) = \begin{pmatrix} \frac{1}{x} & \frac{4}{x} & 0 \\ \frac{1}{4(1-x)} & \frac{1}{x} + \frac{2}{1-x} & \frac{3}{x} + \frac{3}{1-x} \\ -\frac{1}{8(1-x)} + \frac{1}{8(1-4x)} & -\frac{1}{1-x} + \frac{3}{2(1-4x)} & \frac{1}{x} + \frac{6}{1-4x} - \frac{3}{2(1-x)} \end{pmatrix} .$$

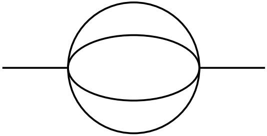
Sun-Ripened Bananas



Underlying geometry is $K3$ surface.

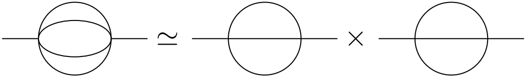
$\rightarrow$ Little known about integrals over $K3$ surfaces

Sun-Ripened Bananas



Underlying geometry is $K3$ surface.

For all equal masses: Fibered by elliptic curve of sunrise!



$$\partial_x \begin{pmatrix} I_1(x) \\ I_2(x) \\ I_3(x) \end{pmatrix} = B(x) \begin{pmatrix} I_1(x) \\ I_2(x) \\ I_3(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2(4x-1)} \end{pmatrix},$$

$$\mathcal{L}_x^{(3)} I_1(x) = 0$$

$$\mathcal{L}_x^{(3)} = \partial_x^3 + \frac{3(8x-5)}{2(x-1)(4x-1)}\partial_x^2 + \frac{4x^2-2x+1}{(x-1)(4x-1)x^2}\partial_x + \frac{1}{x^3(4x-1)}\,.$$

$$I_2(x) = \frac{1}{4}(x\,\partial_x - 1)\,H_1(x)$$
$$I_3(x) = \frac{1}{12}(x^2(1-x)\,\partial_x^2 - x(1+x)\,\partial_x + 1)\,H_1(x)\,,$$

The Differential Equation

$\mathcal{L}_x^{(3)}$ is the symmetric square of some $\mathcal{L}_x^{(2)}$.

Joyce

Changing variables $x(t) = \frac{-4t}{(t-1)(t-9)}$, we find

$$\mathcal{L}_x^{(2)} \to \partial_t^2 + \Big(\frac{1}{t-9} + \frac{1}{t-1}\Big)\partial_t + \Big(\frac{1}{36(t-9)} - \frac{1}{4(t-1)} + \frac{1}{4t^2} + \frac{2}{9t}\Big)$$

$$\mathcal{L}_t^{(2)} = \partial_t^2 + \Big(\frac{1}{t-9} + \frac{1}{t-1} + \frac{1}{t}\Big)\partial_t + \Big(\frac{1}{12(t-9)} + \frac{1}{4(t-1)} - \frac{1}{3t}\Big)$$

$\mathcal{L}_x^{(2)}$ annihilates $\sqrt{t}\Psi_i(t)$

Sun-Ripened Bananas

→ Solutions are products of $\sqrt{t}\Psi_i(t)$.

$$\begin{aligned} H_1(x(t)) &= -\frac{1}{3}\, t\, \Psi_1(t)^2\,, \\ J_1(x(t)) &= \frac{i}{3}\, t\, \Psi_1(t)\,(\Psi_1(t) + \Psi_2(t))\,, \\ I_1(x(t)) &= \frac{1}{3}\, t\,(\Psi_1(t) + \Psi_2(t))\,(\Psi_1(t) + 3\Psi_2(t))\,, \end{aligned}$$

$\Psi_1(t)$, $\Psi_2(t)$ are homogeneous solutions of the banana integral

(c.f. Lorenzo's talk)

The Inhomogeneous Solution

All homogeneous solutions spanned by

$$\mathcal{W}(x) = \begin{pmatrix} H_1(x) & J_1(x) & I_1(x) \\ H_2(x) & J_2(x) & I_2(x) \\ H_3(x) & J_3(x) & I_3(x) \end{pmatrix} .$$

For

$$\vec{\mathcal{I}} = \mathcal{W}\,\vec{M}$$

we get

$$\partial_x \begin{pmatrix} M_1(x) \\ M_2(x) \\ M_3(x) \end{pmatrix} = \mathcal{W}^{-1} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2(4x-1)} \end{pmatrix}$$

Integrals of Modular Forms

Define $\tau = \frac{\Psi_2(t)}{\Psi_1(t)}$, then

- Ψ_1 is a modular form of weight 1 for $\Gamma_1(6)$
- $t(\tau)$ is invariant under $\Gamma_1(6)$

[Adams, Weinzierl]

$\rightarrow$ Convenient basis of modular forms:

[Broedel, Duhr, Dulat, Penante, Tancredi]

$$f_{n,p}(\tau) = t^p \Psi_1(t)^n$$

The Inhomogeneous Solution

$$\partial_\tau \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \frac{f_{4,4} - 10 f_{4,3} + 90 f_{4,1} - 81 f_{4,0}}{36\pi^2} \begin{pmatrix} 3(1 + I(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}; \tau))^2 \\ -2i(2 + 3\, I(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}; \tau)) \\ 1 \end{pmatrix} .$$

$$\int_{i\infty}^{\tau} \mathrm{d}\tau\, f_{n,p}\, I(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}; \tau) = I\left(\begin{smallmatrix} n & 0 \\ p & 0 \end{smallmatrix}; \tau\right)$$

$\Rightarrow$ Straight-forward integration!

Can write result as

$$\tilde{M}_1 = \frac{4\zeta_3}{-\pi^2} - \frac{i}{3\pi^3}\left[81\, I\left(\begin{smallmatrix}0 & 0 & 4\\ 0 & 0 & 0\end{smallmatrix};\tau\right) - 90\, I\left(\begin{smallmatrix}0 & 0 & 4\\ 0 & 0 & 1\end{smallmatrix};\tau\right) + 10\, I\left(\begin{smallmatrix}0 & 0 & 4\\ 0 & 0 & 3\end{smallmatrix};\tau\right) - I\left(\begin{smallmatrix}0 & 0 & 4\\ 0 & 0 & 4\end{smallmatrix};\tau\right)\right]$$

$$\tilde{M}_2 = \frac{1}{3\pi^3}\left(81\, I\left(\begin{smallmatrix}0 & 4\\ 0 & 0\end{smallmatrix};\tau\right) - 90\, I\left(\begin{smallmatrix}0 & 4\\ 0 & 1\end{smallmatrix};\tau\right) + 10\, I\left(\begin{smallmatrix}0 & 4\\ 0 & 3\end{smallmatrix};\tau\right) - I\left(\begin{smallmatrix}0 & 4\\ 0 & 4\end{smallmatrix};\tau\right)\right)$$

$$\tilde{M}_3 = \frac{i}{18\pi^3}\left(81\, I\left(\begin{smallmatrix}4\\ 0\end{smallmatrix};\tau\right) - 90\, I\left(\begin{smallmatrix}4\\ 1\end{smallmatrix};\tau\right) + 10\, I\left(\begin{smallmatrix}4\\ 3\end{smallmatrix};\tau\right) - I\left(\begin{smallmatrix}4\\ 4\end{smallmatrix};\tau\right)\right)$$

$$\begin{pmatrix}\tilde{M}_1\\ \tilde{M}_2\\ \tilde{M}_3\end{pmatrix} = U(\tau)\begin{pmatrix}M_1\\ M_2\\ M_3\end{pmatrix} \quad \text{where} \quad \mathcal{W} = SU$$

The Inhomogeneous Solution and eMPLs

The Inhomogeneous Solution

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z, \tau\right) = \int_0^z \mathrm{d}w\, g^{(n_1)}(w - z_1, \tau)\tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; w, \tau\right)\,,$$

$g^{(n)}(z,\tau)$ - expansion coefficients of Eisenstein-Kronecker series

(c.f. Oli's $f^{(n)}(z,\tau)$)

$$g^{(k)}(z+1,\tau) = g^{(k)}(z,\tau)$$
$$g^{(n)}(z+\tau,\tau) = \sum (-2\pi i)^k g^{(n-k)}(z,\tau)$$

The Inhomogeneous Solution

For $z_i = a + b\,\tau$ with $a, b \in \mathbb{Q}$, we can rewrite $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z_0, \tau\right)$, as iterated integrals of modular forms, using

$$\Delta^{\mathfrak{m}}([M, \gamma, \omega]^{\mathfrak{m}}) = \sum_i [M, \gamma, \omega_i]^{\mathfrak{m}} \otimes [M, \omega_i^{\vee}, \omega]^{\mathfrak{dr}}$$

(c.f. Nils)

$$\Delta\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_m \\ z_1 & \dots & z_m \end{smallmatrix}; z_0, \tau\right) = \sum_k \gamma_k \otimes W_{(m-k)}\,,$$

where

$$\gamma_k \sim \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ w_1 & \dots & w_k \end{smallmatrix}; z_0, \tau\right)$$
$$W_k \sim [g^{(n_1)}(w_1, \tau)\mathrm{d}\tau | \dots | g^{(n_k)}(w_k, \tau)\mathrm{d}\tau]\,.$$

$$g^{(n)}(\tfrac{r}{N} + \tfrac{s}{N}\tau, \tau) = \sum (2\pi i)^k h^{(n-k)}_{N,r,s}(\tau)$$

The Inhomogeneous Solution

Then we get the corresponding expression in terms of iterated integrals of modular forms as

$$(\mathsf{Cusp} \otimes \textstyle\int)\Delta\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_m \\ z_1 & \dots & z_m \end{smallmatrix}; z, \tau\right) = \sum_k \gamma_k \otimes W_{(m-k)}\,,$$

where

$$\int [f_1 \mathrm{d}\tau | \dots | f_n \mathrm{d}\tau] = I(f_1 | \dots | f_n; \tau) = \int_{i\infty}^{\tau} \mathrm{d}\tau' f_1\, I(f_2 | \dots | f_n; \tau')\,,$$

and

$$\mathsf{Cusp}\,\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_m \\ z_1 & \dots & z_m \end{smallmatrix}; z, \tau\right) = \lim_{\tau \to i\infty} \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_m \\ z_1 & \dots & z_m \end{smallmatrix}; z, \tau\right)$$

Elliptic Polylogarithms

- Write a suitable Ansatz in terms of eMPLs
- Rewrite eMPLs as integrals of modular forms
- compare coefficients

$\rightarrow$ We found a corresponding solution!