Elliptic multiple polylogarithms

an algorithmic approach

The calculation of two and three loop scalar diagrams

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Outline

Elliptic multiple polylogarithms [Brown and Levin(2013)], [Broedel(2015)], [Weinzierl(2016)],[Duhr et al.(2018)]

- Definition eMPLs

$$E_4\left(\begin{smallmatrix} n_1,\ldots,n_k \\ c_1,\ldots,c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \varphi_{n_1}(c_1, t, \vec{a})\, E_4\left(\begin{smallmatrix} n_2,\ldots,n_k \\ c_2,\ldots,c_k \end{smallmatrix}; t, \vec{a}\right) ,$$

Integration kernels eMPLs [Duhr et al.(2018)]

$$\varphi_0(0,x) = \frac{c_4}{y}$$

$$\varphi_1(c,x) = \frac{1}{x-c} \qquad \varphi_{-1}(c,x) = \frac{y_c}{y(x-c)}$$

$$\varphi_{-1}(\infty,x) = \frac{x}{y} \qquad \varphi_1(\infty,x) = \frac{c_4}{y}\, Z_4(x)$$

$$\varphi_n(\infty,x) = \frac{c_4}{y}\, Z_4^{(n)}(x)$$

$$\varphi_{-n}(\infty,x) = \frac{x}{y}\, Z_4^{(n-1)}(x) - \frac{\delta_{n2}}{c_4}$$

$$\varphi_n(c,x) = \frac{1}{x-c}\, Z_4^{(n-1)}(x) - \delta_{n2}\,\Phi_4(x)$$

$$\varphi_{-n}(c,x) = \frac{y_c}{y(x-c)}\, Z_4^{(n-1)}(x)$$

Integrals

- General integral

$$\int dx R(x,y)\chi(x) \qquad \text{with} \qquad \chi(x) = Z_4^{(m)} E_4\left(\begin{smallmatrix}\vec{n}\\ \vec{c}\end{smallmatrix}; x\right)$$

- After Partial fraction decomposition

$$A_k[\chi] = \int dx\, x^k\, \chi(x) \qquad B_{c,k}[\chi] = \int \frac{dx}{(x-c)^k}\,\chi(x)$$

$$C_k[\chi] = \int \frac{dx}{y}\, x^k\, \chi(x) \qquad D_{c,k}[\chi] = \int \frac{dx}{y\,(x-c)^k}\,\chi(x)$$

IBP reduction algorithms [Duhr et al.(2018)]

- Algorithm A

$$A_k[\chi(x)] = \frac{x^{k+1}}{k+1}\chi(x) - \frac{1}{k+1}\, A_{k+1}[\partial_x \chi(x)]$$

- Algorithm B

$$B_{c,k}[\chi(x)] = \frac{\chi(x)}{(k-1)\,(x-c)^{k-1}} - \frac{1}{k-1}\, B_{c,k-1}[\partial_x \chi(x)]$$

- Algorithm C

$$y\, x^{k-2}\,\chi(x) = \frac{1}{2}\sum_{l=0}^{3}(-1)^l\, s_l(a_1, a_2, a_3, a_4) \\ [(2k-1-l)\; C_{k-l}[\chi(x)] + 2\, C_{k+1-l}[\partial_x \chi(x)]]$$

- Algorithm D

$$\frac{y}{(x-c)^{k-1}}\,\chi(x) = -\frac{1}{2}\sum_{l=0}^{3}\left[(2k-2-l)\,D_{c,k-l}[\chi] - 2\,D_{c,k-1-l}[\partial_x\chi]\right]$$

$$\frac{1}{l!}\partial_c^l y_c^2$$

- Algorithm D where $c = a_i$, a root of y

$$\frac{y}{(x-a_1)^{k}}\,\chi(x) = -\frac{1}{2}\sum_{l=0}^{3}\left[(2k-1-l)\,D_{a_1,k-l}[\chi] - 2\,D_{a_1,k-1-l}[\partial_x\chi]\right]$$

$$\frac{1}{l!}\partial_{a_1}^l(a_{12}a_{13}a_{14})$$

Base integrals

- Performing partial fractioning and using the IBP relation iteratively will lead to 7 base integrals

$$A_{-1}[\chi] = \int \frac{dx}{x}\,\chi(x) \qquad B_{c,1}[\chi] = \int \frac{dx}{x-c}\,\chi(x)$$

$$C_2[\chi] = \int \frac{x^2\,dx}{y}\,\chi(x) \qquad C_1[\chi] = \int \frac{x\,dx}{y}\,\chi(x)$$

$$C_0[\chi] = \int \frac{dx}{y}\,\chi(x) \qquad C_{-1}[\chi] = \int \frac{dx}{y\,x}\,\chi(x)$$

$$D_{c,1}[\chi] = \int \frac{dx}{y\,(x-c)}\,\chi(x)$$

Feynman integrals

- Feynman integrals with Feynman parametrization

$$I_{\nu_1,\ldots,\nu_n}(\{s\}) = N\left(i\pi^{\frac{d}{2}}\right)^l \frac{\Gamma(\nu - \frac{ld}{2})}{\prod_{i=1}^n \Gamma(\nu_i)} \int_\Delta d^n\vec{\alpha} \left(\prod_{i=1}^n \alpha_i^{\nu_i - 1}\right) \mathcal{U}^{\nu - \frac{d}{2}(l+1)} \mathcal{F}^{-\nu + \frac{ld}{2}},$$

- Integration domain

$$\Delta = \left\{ \vec{\alpha} \,\middle|\, \alpha_i \geq 0, \sum_{i \in \tilde{S}} \alpha_i = 1 \right\}$$

Linearly reducible elliptic Feynman integrals

- Linear reducible Feynman integrals: at every integration step one can do a linear factorization $\rightarrow$ final result in terms of MPLs. [Brown(2009), Panzer(2015)]

- Linear reducible elliptic Feynman integrals: linearly reducible if one excludes the last integration. [Hidding and Moriello(2019)]

$$\mathcal{I} = \int dx\, MPL(x,y) = \int dx\, R(x,y) G(f_1(x,y), \ldots, f_n(x,y); x)$$

Statement

Linearly reducible elliptic Feynman integrals can always be expressed as a linear combination of eMPLs.

Solution strategy

I Choose a Feynman diagram (n propagators)

II Use Feynman parameter representation for the corresponding integral (n Feynman parameters)

III Perform n-1 integrations in terms of MPLs

- Use the Cheng-Wu theorem and a convenient integration order [Cheng and Wu(1987)]
- Use e.g. HyperInt to perform the n-1 integrations [Panzer(2015)]

IV Obtain a linearly reducible elliptic Feynman integral

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IV Obtain a linearly reducible elliptic Feynman integral

V Rewrite MPLs to eMPL

VI Perform the last integration and obtain the primitive

VII Regularize the primitive

VIII Perform a numerical check

Banana like diagrams

Tree-loop Banana A and Banana B

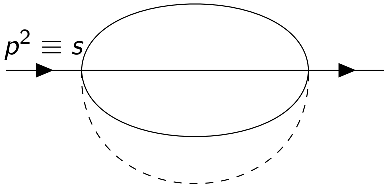


Figure: Banana A

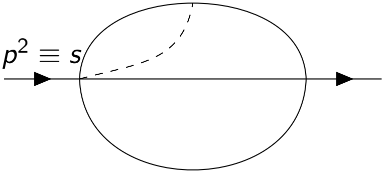
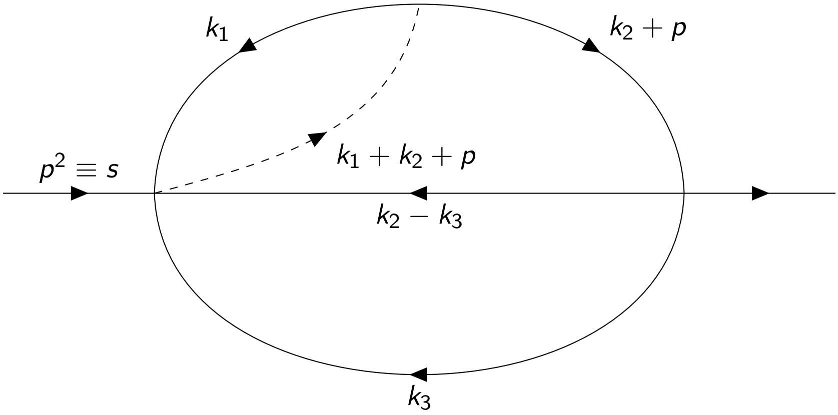


Figure: Banana B

Calculation banana B

- Choose Feynman diagram, 5 propagators



Calculation banana B

II Integral in Feynman parameter representation, 5 parameters. Finite integral with $d = 4 - 2\epsilon$ and S_{21221}

$$S_{21221}(S, m^2)\Big|_{\epsilon^0} = \int_\Delta d^5\vec{\alpha}\, \alpha_1\alpha_3\alpha_4\mathcal{U}^0\mathcal{F}^{-2}$$

- Symanzik polynomials

$$\mathcal{U} = \alpha_2\left(\alpha_4\alpha_5 + \alpha_3\left(\alpha_4 + \alpha_5\right)\right) + \alpha_1\left(\alpha_2\left(\alpha_3 + \alpha_4\right) + \alpha_4\alpha_5 + \alpha_3\left(\alpha_4 + \alpha_5\right)\right)$$

$$\begin{aligned}\mathcal{F} = &\left(\alpha_2\left(\alpha_3 + \alpha_4\right) + \alpha_4\alpha_5 + \alpha_3\left(\alpha_4 + \alpha_5\right)\right)\alpha_1^2 m^2 + \alpha_1\Big(\alpha_3^2\left(\alpha_4 + \alpha_5\right) m^2 \\ &+ \alpha_4\alpha_5\left(\alpha_4 + \alpha_5\right) m^2 + \alpha_3\left(\alpha_4^2 m^2 + \alpha_5^2 m^2 + \alpha_5\alpha_4\left(3m^2 + S\right)\right) \\ &+ \alpha_2\Big(\alpha_3^2 m^2 + \alpha_4\left(\alpha_4 + 2\alpha_5\right) m^2 + \alpha_3\left(2\alpha_5 m^2 + \alpha_4\left(3m^2 + S\right)\right)\Big)\Big) \\ &+ \alpha_2\Big(\alpha_3^2\left(\alpha_4 + \alpha_5\right) m^2 + \alpha_4\alpha_5\left(\alpha_4 + \alpha_5\right) m^2 \\ &+ \alpha_3\left(\alpha_4^2 m^2 + \alpha_5^2 m^2 + \alpha_5\alpha_4\left(3m^2 + S\right)\right)\Big)\end{aligned}$$

Calculation banana B

III Perform n-1 integrations in terms of MPLs by using Cheng-Wu

Calculation banana B

IV Obtain a linearly reducible elliptic Feynman integral

$$
\begin{aligned}
S_{21221}(S, m^2)\Big|_{\epsilon^0} = \int_0^1 dx \Big[& P_1(x) G\left(0, \gamma(x), 1\right) \\
& - \frac{1}{2} P_1(x) \log^2\left(\frac{m^2 x}{m^2 - S(x-1)x}\right) + P_2(x,y) G\left(0, \phi(x,y), 1\right) \\
& + P_2(x,-y) G\left(0, \phi(x,-y), 1\right) - P_2(x,y) G\left(\phi(x,-y), \phi(x,y), 1\right) \\
& - P_2(x,-y) G\left(\phi(x,y), \phi(x,-y), 1\right) - P_1(x) G\left(\phi(x,y), \gamma(x), 1\right) \\
& - P_1(x) G\left(\phi(x,-y), \gamma(x), 1\right) - P_2(x,y) \log^2\left(\rho(x,y)\right) \\
& - P_2(x,-y) \log^2\left(\rho(x,-y)\right) \Big]
\end{aligned}
$$

V Rewrite MPLs to eMPLs by taking the derivative and integrate back

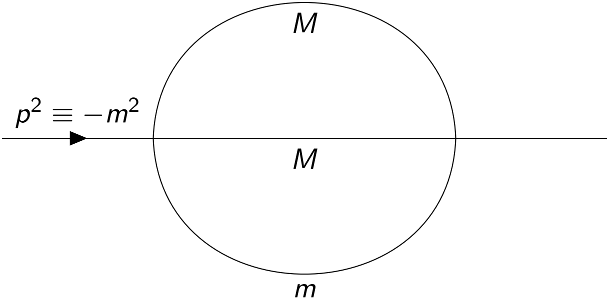
VI Perform the last integration and obtain primitive

Calculation banana B

$$
\begin{aligned}
S_{21221}(S,m^2)\Big|_{\epsilon^0} = -\frac{1}{2c_4m^2S}\Bigg[& 2\bigg(E_4\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & a_3 & 0 \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & a_4 & 0 \end{smallmatrix};1\right)\bigg) + 2E_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & \infty \end{smallmatrix};1\right) \\
& + 2E_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & 0 \end{smallmatrix};1\right) + 2E_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_4 & \infty \end{smallmatrix};1\right) + 2E_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_4 & 0 \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & c_1 & \infty \end{smallmatrix};1\right) \\
& + E_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & c_1 & 0 \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & c_1 & 0 \end{smallmatrix};1\right) - 3E_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 0 & \infty \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 1 & \infty \end{smallmatrix};1\right) \\
& - 3E_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{smallmatrix};1\right) - 3E_4\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{smallmatrix};1\right) \\
& - c_4\bigg(2E_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ \infty & a_3 & 0 \end{smallmatrix};1\right) + 2E_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ \infty & a_3 & \infty \end{smallmatrix};1\right) + 2E_4\left(\begin{smallmatrix} -1 & 1 & 1 \\ \infty & a_3 & 0 \end{smallmatrix};1\right) \\
& + 2E_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ \infty & a_4 & 0 \end{smallmatrix};1\right) + 2E_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ \infty & a_4 & \infty \end{smallmatrix};1\right) + 2E_4\left(\begin{smallmatrix} -1 & 1 & 1 \\ \infty & a_4 & 0 \end{smallmatrix};1\right) \\
& + E_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ \infty & c_1 & 0 \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ \infty & c_1 & \infty \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} -1 & 1 & 1 \\ \infty & c_1 & 0 \end{smallmatrix};1\right) - 3\bigg(E_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ c_1 & 0 & \infty \end{smallmatrix};1\right) \\
& + E_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ c_1 & \infty & 0 \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ c_1 & \infty & \infty \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} 1 & -1 & 1 \\ c_1 & \infty & 0 \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} 1 & 1 & -1 \\ c_1 & 0 & \infty \end{smallmatrix};1\right) \\
& + E_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ c_1 & 0 & 0 \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} 1 & -1 & 1 \\ c_1 & 0 & 0 \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} 1 & 1 & -1 \\ c_1 & 0 & 0 \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} 1 & 1 & 1 \\ c_1 & 0 & 0 \end{smallmatrix};1\right)\bigg) \\
& - 3E_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ \infty & 0 & 0 \end{smallmatrix};1\right) - 3E_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ \infty & 0 & \infty \end{smallmatrix};1\right) - 3E_4\left(\begin{smallmatrix} -1 & 1 & 1 \\ \infty & 0 & 0 \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ \infty & 1 & 0 \end{smallmatrix};1\right) \\
& + E_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ \infty & 1 & \infty \end{smallmatrix};1\right) + E_4\left(\begin{smallmatrix} -1 & 1 & 1 \\ \infty & 1 & 0 \end{smallmatrix};1\right)\bigg)\Bigg].
\end{aligned}
$$

All order sunrise diagram

All order sunrise diagram



- Computed in d=4-2ϵ

An all order sunrise solution [Kalmykov and Kniehl(2009)]

$$
\begin{aligned}
J_{122}(m^2, M^2) = & -(M^2)^{-2-\epsilon}(m^2)^{1-\epsilon}\gamma^2(1+\epsilon)\frac{(1+\epsilon)}{\epsilon(1-\epsilon)} \\
& \Bigg[\frac{1}{6}\,{}_4F_3\left(\begin{matrix}1,\frac{3}{2},1+\frac{\epsilon}{2},\frac{3}{2}+\frac{\epsilon}{2}\\ 2-\epsilon,\frac{5}{4},\frac{7}{4}\end{matrix}\,\middle|\,-t^2\right) \\
& -\left(\frac{M^2}{m^2}\right)^{1-\epsilon}\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}\,{}_4F_3\left(\begin{matrix}1,\frac{1}{2}+\epsilon,1+\frac{\epsilon}{2},1+\epsilon\\ \frac{3}{2}-\frac{\epsilon}{2},\frac{3}{4}+\frac{\epsilon}{2},\frac{5}{4}+\frac{\epsilon}{2}\end{matrix}\,\middle|\,-t^2\right) \\
& -\left(\frac{M^2}{m^2}\right)^{-\epsilon}\frac{(1-\epsilon)}{(2-\epsilon)(3+2\epsilon)}\,{}_4F_3\left(\begin{matrix}1,\frac{3}{2}+\frac{\epsilon}{2},\frac{3}{2}+\epsilon,1+\epsilon\\ 2-\frac{\epsilon}{2},\frac{5}{4}+\frac{\epsilon}{2},\frac{7}{4}+\frac{\epsilon}{2}\end{matrix}\,\middle|\,-t^2\right)\Bigg]
\end{aligned}
$$

$$
t = \frac{m^2}{2M^2}
$$

- For an all order sunrise with equal masses see [Adams and Weinzierl(2018)]

An all order sunrise solution [Kalmykov and Kniehl(2009)]

$$J_{122}(m^2, M^2) = -(M^2)^{-2-\epsilon}(m^2)^{1-\epsilon}\gamma^2(1+\epsilon)\frac{(1+\epsilon)}{\epsilon(1-\epsilon)}$$
$$\Bigg[\frac{1}{6}\,{}_4F_3\left(\begin{matrix}1,\frac{3}{2},1+\frac{\epsilon}{2},\frac{3}{2}+\frac{\epsilon}{2}\\ 2-\epsilon,\frac{5}{4},\frac{7}{4}\end{matrix}\,\middle|\,-t^2\right)$$
$$-(\frac{M^2}{m^2})^{1-\epsilon}\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}\,{}_4F_3\left(\begin{matrix}1,\frac{1}{2}+\epsilon,1+\frac{\epsilon}{2},1+\epsilon\\ \frac{3}{2}-\frac{\epsilon}{2},\frac{3}{4}+\frac{\epsilon}{2},\frac{5}{4}+\frac{\epsilon}{2}\end{matrix}\,\middle|\,-t^2\right)$$
$$-(\frac{M^2}{m^2})^{-\epsilon}\frac{(1-\epsilon)}{(2-\epsilon)(3+2\epsilon)}\,{}_4F_3\left(\begin{matrix}1,\frac{3}{2}+\frac{\epsilon}{2},\frac{3}{2}+\epsilon,1+\epsilon\\ 2-\frac{\epsilon}{2},\frac{5}{4}+\frac{\epsilon}{2},\frac{7}{4}+\frac{\epsilon}{2}\end{matrix}\,\middle|\,-t^2\right)\Bigg]$$

$$t = \frac{m^2}{2M^2} \qquad\qquad {}_4F_3^{(1)}(t)$$

- For an all order sunrise with equal masses see [Adams and Weinzierl(2018)]

Hypergeometric function [Kotikov, Moriello, Campert]

- Hypergeometric function expressed as a one-fold integral

$$\frac{{}_4F_3^{(1)}(t)}{K(\epsilon,t)} = \int_0^1 dx \Bigg[-\frac{2}{(1-x^2)}\left(\frac{1-x^2}{x^2}\right)^{\epsilon} + \frac{2}{(1-x^2)ty}\left(\frac{(1-x^2)(1+ty)^2}{4x^2(ty)^2}\right)^{\epsilon} - \frac{2}{(1-x^2)ty}\left(\frac{(1-x^2)(1+ty)^2}{4x^2(ty)^2}\right)^{\epsilon}\mathcal{S}_\epsilon \Bigg]$$

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- where we define

$$S_\epsilon = \sum_{l=0}^{\infty}(-2)^{l+1}\sum_{n=0}^{\infty}\epsilon^{n+l+2}\frac{(-1)^{n+l+1}}{n!(l+1)!}S_{n+1,l+1}$$

$$S_{n+1,l+1} = \int_0^{\chi}\frac{dz}{z}\left(\log(z)-\log(\chi)\right)^n\left(\log(1+z)\right)^{l+1}$$

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$$S_\epsilon = \sum_{l=0}^{\infty}(-2)^{l+1}\sum_{n=0}^{\infty}\epsilon^{n+l+2}\frac{(-1)^{n+l+1}}{n!(l+1)!}S_{n+1,l+1}$$

$$S_{n+1,l+1} = \int_0^{\chi}\frac{dz}{z}\,(\log(z)-\log(\chi))^n(\log(1+z))^{l+1}$$

An all order result

$$\frac{{}_4F_3^{(1)}(t)}{K(\epsilon,t)} = -\frac{\Gamma\left(\frac{1}{2}-\epsilon\right)\Gamma(\epsilon)}{\sqrt{\pi}} + P_1 \Big| \sum_{i=0}^{\infty} \epsilon^i \left(\frac{L_A^i}{i!}\right)$$
$$- P_1 \Big| \sum_{i=2}^{\infty} \epsilon^i \sum_{k=0}^{i-2} \frac{L_A^k}{k!} \sum_{n=0}^{i-k-2} (-2)^{i-k-n-1} S_{n+1,i-k-n-1} \,.$$

- Where the Nielson polylogs in terms of eMPLs

$$S_{n+1,l+1} = \sum_{k=0}^{n} \binom{n}{k} (-L_{S_A})^{n-k} \left[P_0 \big| (L_{S_A}^k L_{S_B}^{l+1}) + I_{k,l}^{(S)}(0)\right]$$

- The concatenation $|$ is defined as

$$E_4\left(\begin{smallmatrix}\vec{n}\\ \vec{c}\end{smallmatrix};x\right) \big| E_4\left(\begin{smallmatrix}\vec{m}\\ \vec{d}\end{smallmatrix};x\right) = E_4\left(\begin{smallmatrix}\vec{n} & \vec{m}\\ \vec{c} & \vec{d}\end{smallmatrix};x\right)$$

Results in terms of eMPLs

$$P_1 \Big| \sum_{i=0}^{\infty} \epsilon^i \left(\frac{L_A^i}{i!} \right)$$

$$\begin{aligned} P_1 =& E_4\left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}; x\right) - E_4\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}; x\right), \\ L_A =& - E_4\left(\begin{smallmatrix} 1 \\ a_1 \end{smallmatrix}; x\right) - E_4\left(\begin{smallmatrix} 1 \\ a_2 \end{smallmatrix}; x\right) - E_4\left(\begin{smallmatrix} 1 \\ a_3 \end{smallmatrix}; x\right) - E_4\left(\begin{smallmatrix} 1 \\ a_4 \end{smallmatrix}; x\right) - 2E_4\left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}; x\right) \\ & - 2E_4\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}; x\right) + 3E_4\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}; x\right) - 2E_4\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; x\right) + 3E_4\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}; x\right) \\ & - \log\left(t^2+1\right) + 2\log\left(\sqrt{t^2+1}+1\right) - \log(4), \end{aligned}$$

Results in terms of eMPLs

$$S_{n+1,l+1} = \sum_{k=0}^{n} \binom{n}{k} (-L_{S_A})^{n-k} \left[P_0 | (L_{S_A}^k L_{S_B}^{l+1}) + I_{k,l}^{(S)}(0) \right]$$

$$\begin{aligned} P_0 =& 2E_4\left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}; x\right) + 2E_4\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}; x\right) \\ L_{S_A} =& 2E_4\left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}; x\right) + 2E_4\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}; x\right) + \log\left(\sqrt{t^2+1} - 1\right) \\ & - \log\left(\sqrt{t^2+1} + 1\right) \\ L_{S_B} =& E_4\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}; x\right) + E_4\left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}; x\right) - E_4\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}; x\right) + \frac{1}{2}E_4\left(\begin{smallmatrix} 1 \\ a_1 \end{smallmatrix}; x\right) \\ & + \frac{1}{2}E_4\left(\begin{smallmatrix} 1 \\ a_2 \end{smallmatrix}; x\right) + \frac{1}{2}E_4\left(\begin{smallmatrix} 1 \\ a_3 \end{smallmatrix}; x\right) + \frac{1}{2}E_4\left(\begin{smallmatrix} 1 \\ a_4 \end{smallmatrix}; x\right) - E_4\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}; x\right) \\ & + \frac{1}{2}\log\left(t^2+1\right) - \log\left(\sqrt{t^2+1} + 1\right) + \log(2) \end{aligned}$$

Summary

- General solution strategy for linearly reducible elliptic Feynman integrals.
- Solutions of Banana like diagrams.
- An all order epsilon result for a sunrise diagram.
- Future perspective: This strategy can be used to calculate diagrams of phenomenological interest.

Thank you for your attention :)

Bibliography I

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