Using Minkowski Functionals to Analyze CMB Lensing

Yuqi Kang

Jan Hamann

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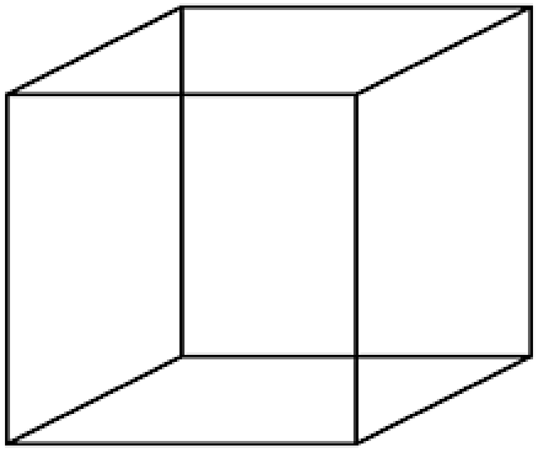
What is Minkowski Functionals?

Euler's formula for convex polyhedra

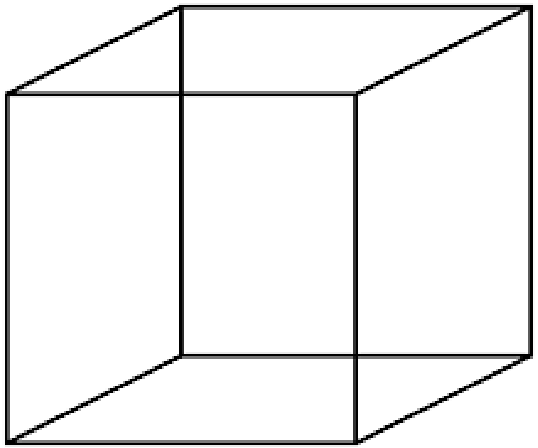


$$V + F - E = 2$$

vertices faces edges

What is Minkowski Functionals?

Euler's formula for convex polyhedra

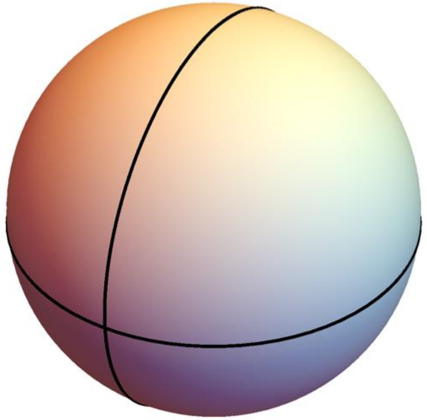


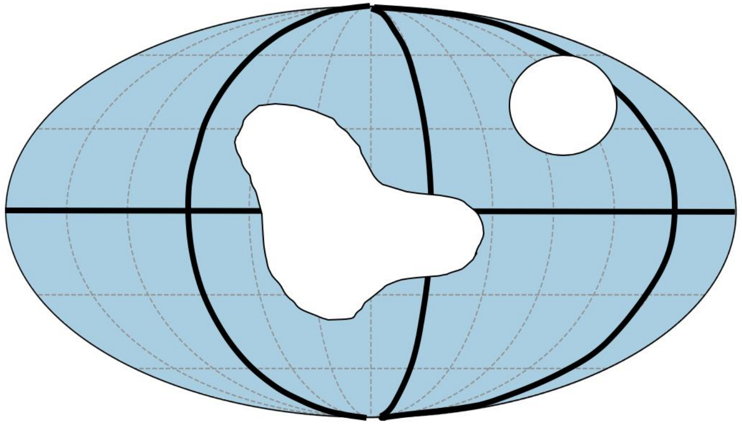
$$V + F - E = 2$$

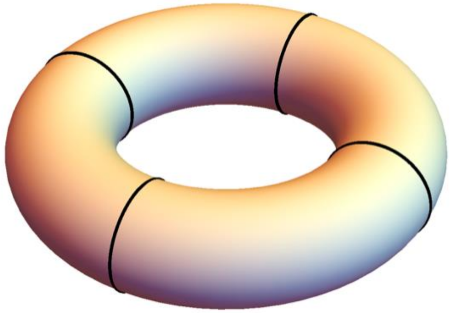
vertices faces edges

$$8 + 6 - 12 = 2$$

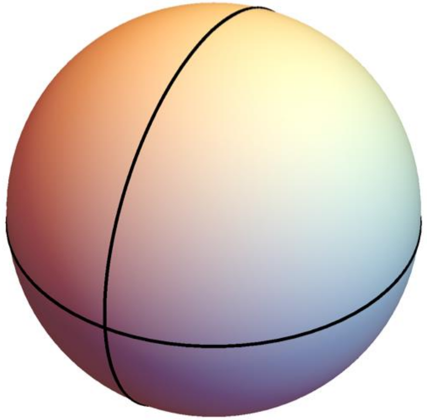
Euler’s formula for general surface





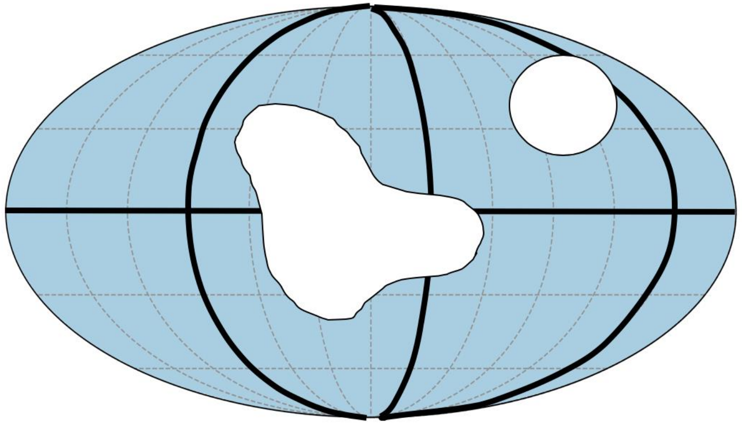


Euler's formula for general surface



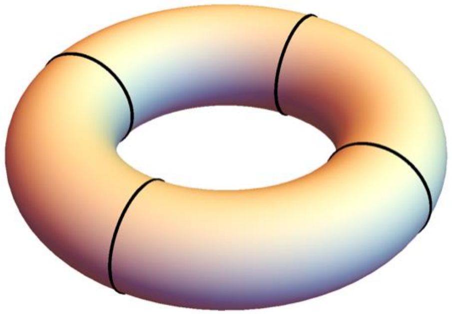
$$2 + 4 - 4 = 2$$

vertices faces edges



$$10 + 6 - 16 = 0$$

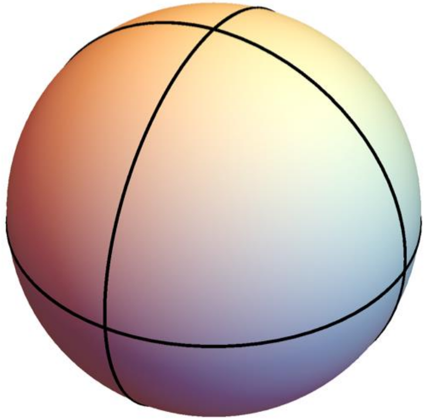
vertices faces edges



$$0 + 4 - 4 = 0$$

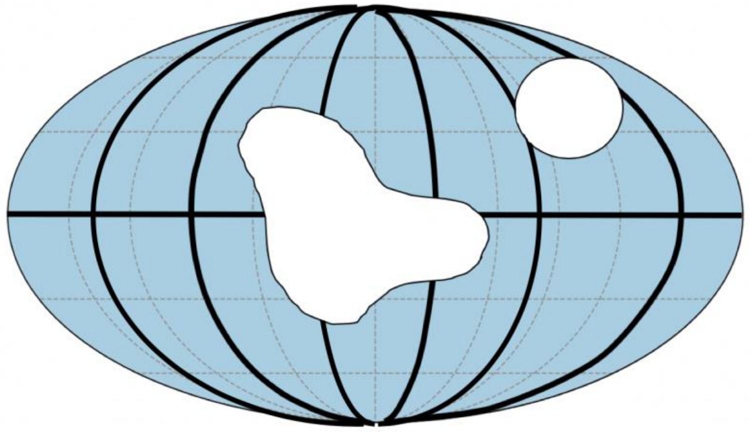
vertices faces edges

Euler's formula for general surface



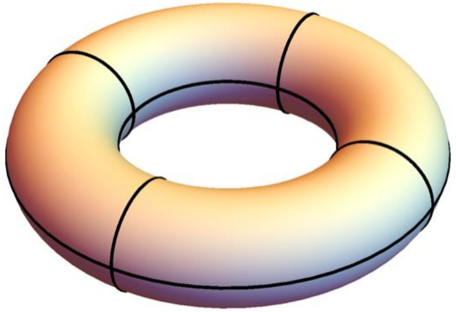
$$6 + 8 - 12 = 2$$

vertices faces edges



$$16 + 13 - 29 = 0$$

vertices faces edges



$$8 + 8 - 16 = 0$$

vertices faces edges

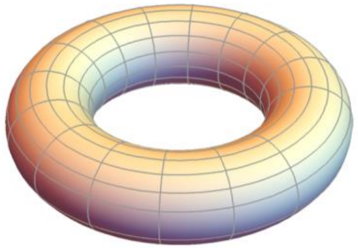
$$V + F - E = \chi$$

vertices faces edges

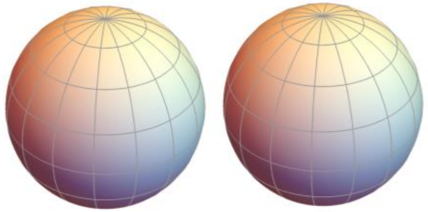
Euler characteristic

$$\chi = 2\ (\#\ \text{of disconnected parts}) - \eta\ (\#\ \text{of } \mathbf{holes})$$

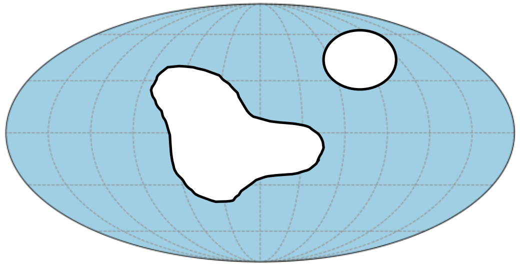
$$\eta = \begin{cases} 1 & \text{non-orientable} \\ 2 & \text{orientable} \end{cases}$$



$\chi = 0$



$\chi = 4$



$\chi = 0$

Minkowski Functionals (MFs)

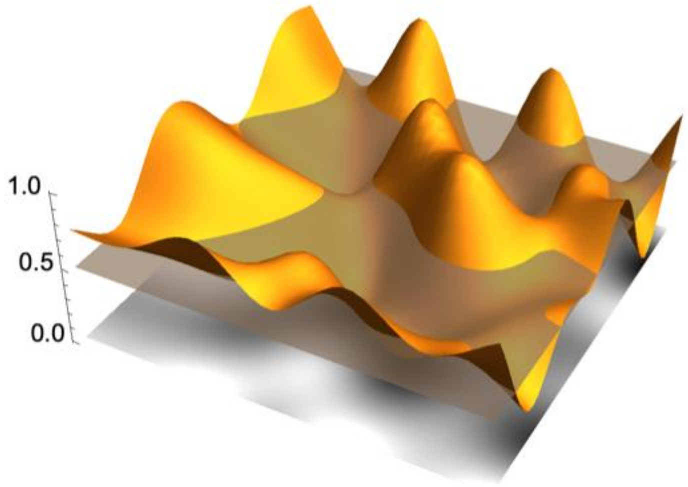
- **morphological** descriptors

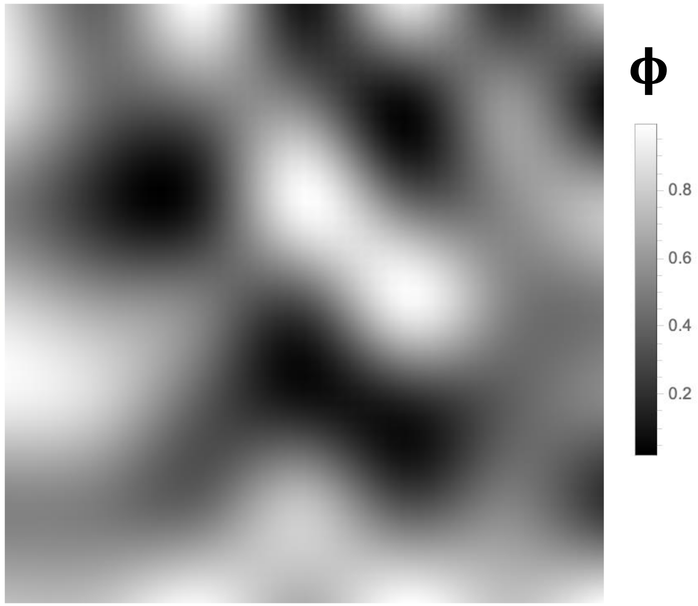
d	1	2	3
V_0	length	area	volume
V_1	χ	circumference	surface area
V_2	–	χ	total mean curvature
V_3	–	–	χ

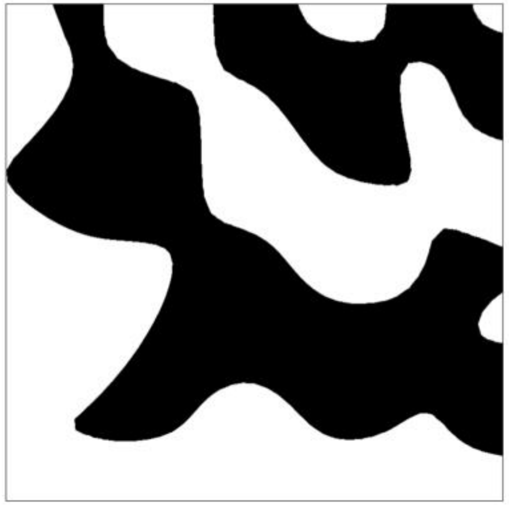
Analysis of 2D scalar field ϕ

- Minkowski Functionals of **excursion sets**
- Define a **threshold value** ν
- $\phi > \nu$ → white, $\phi < \nu$ → Black

excursion sets







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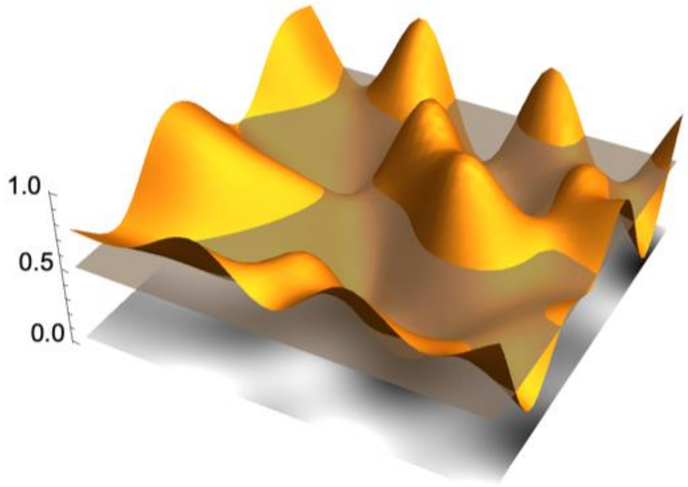

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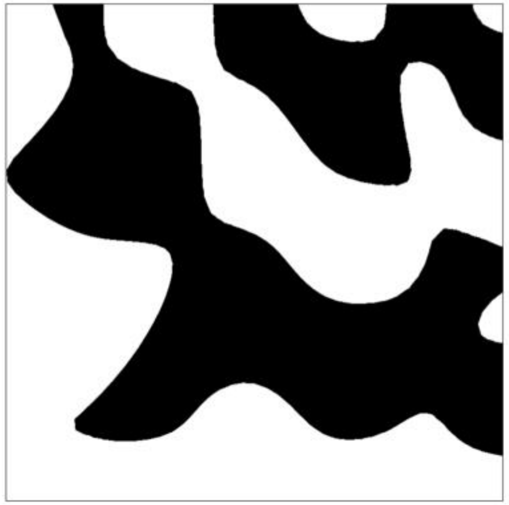

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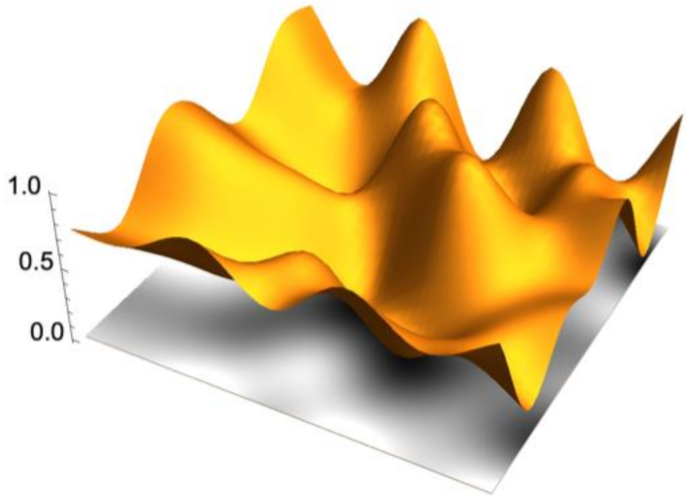
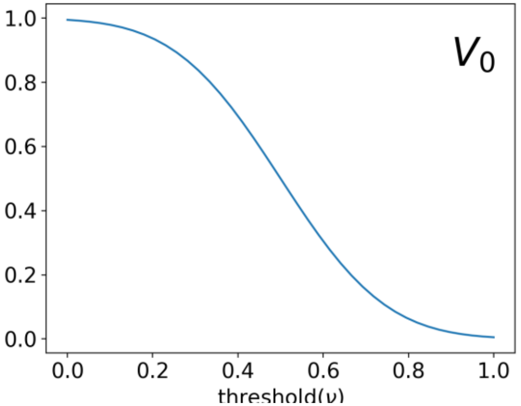
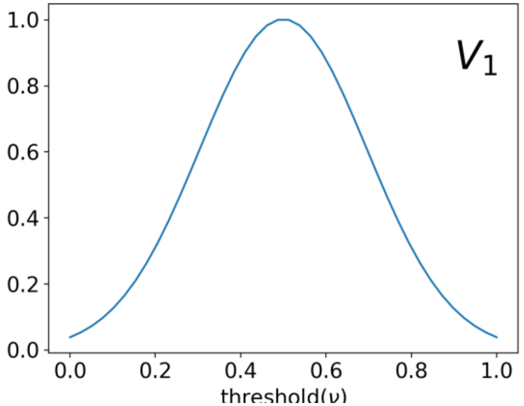
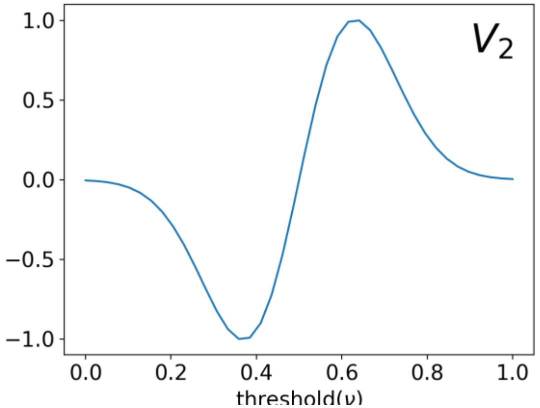




area



circumference



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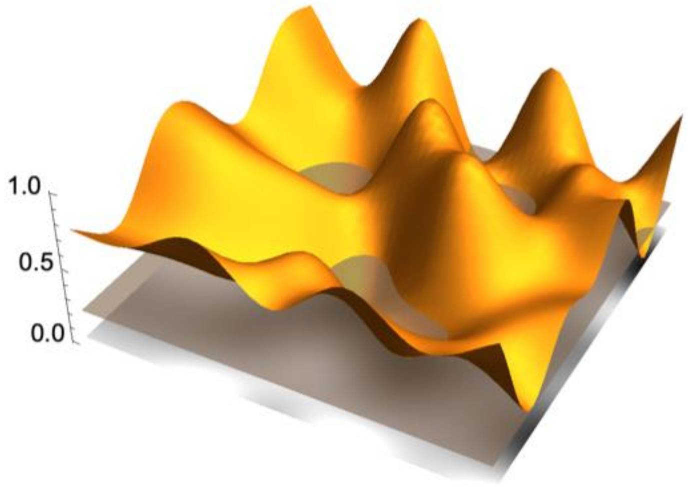

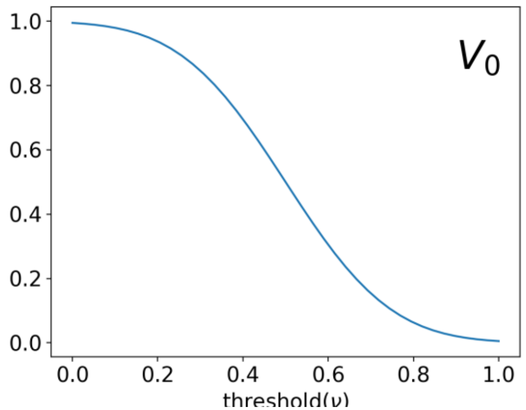

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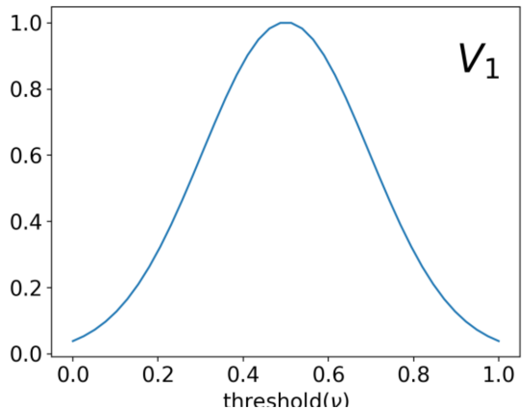

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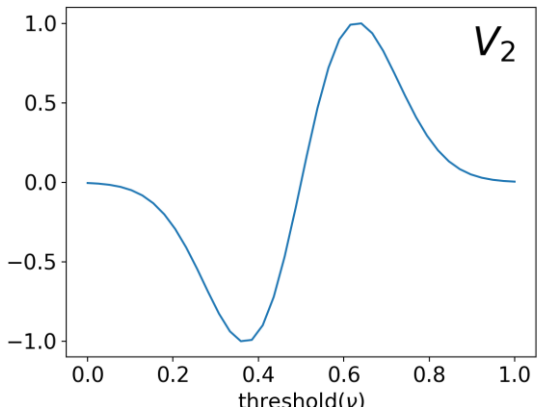

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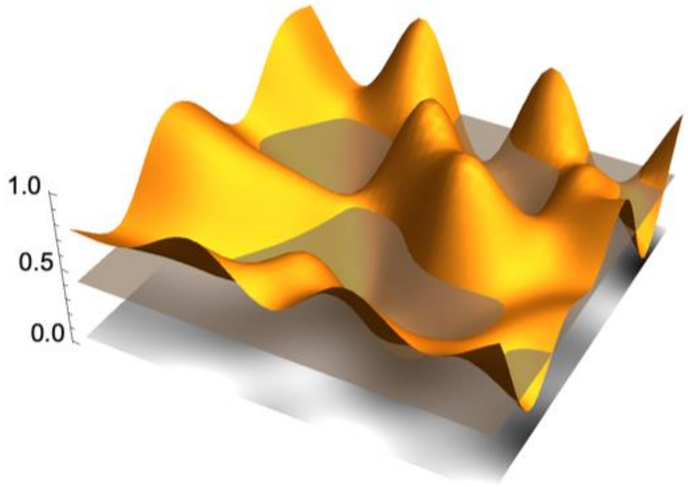

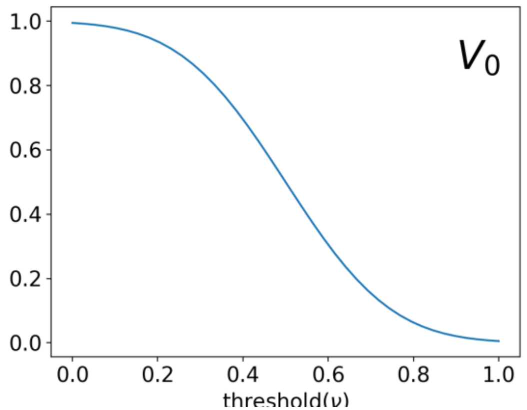

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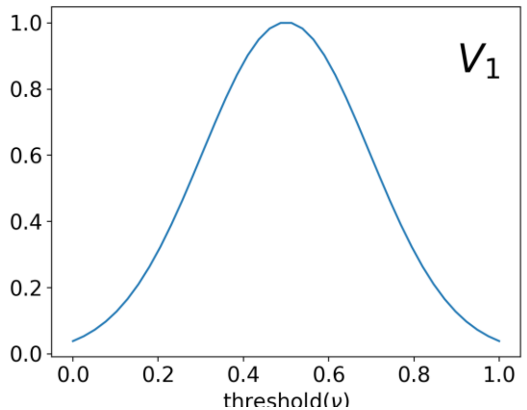

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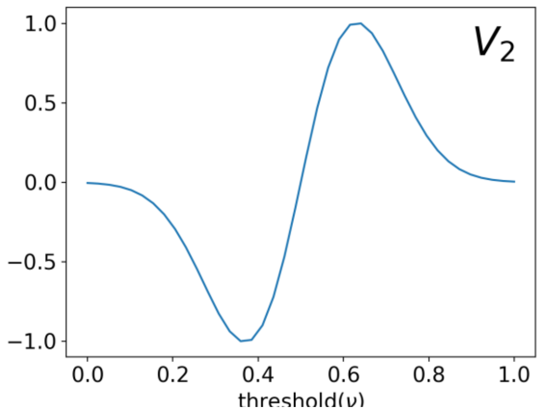

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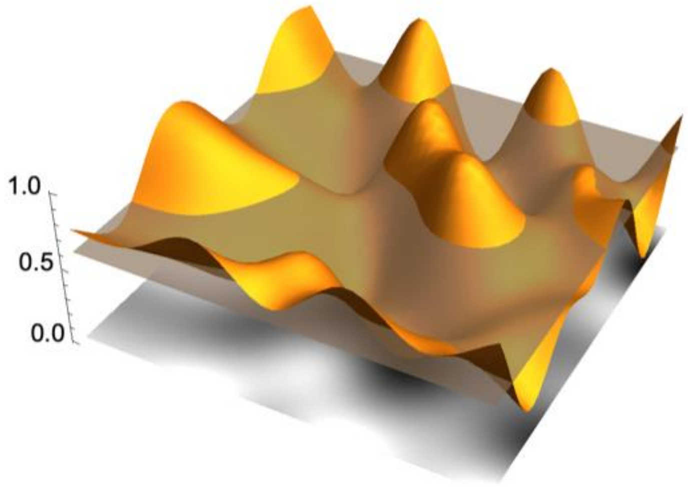

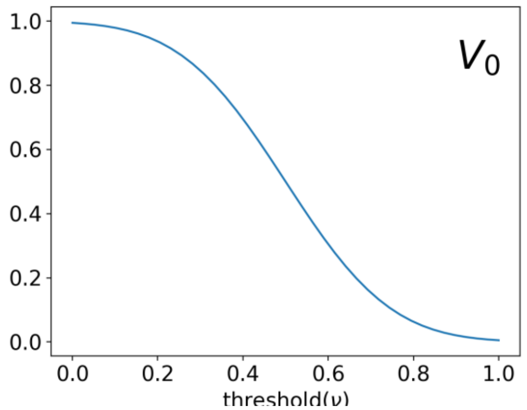

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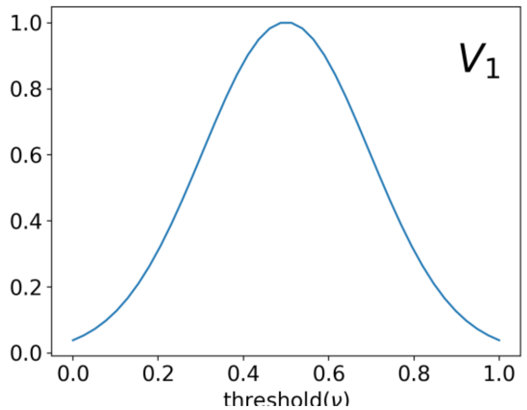

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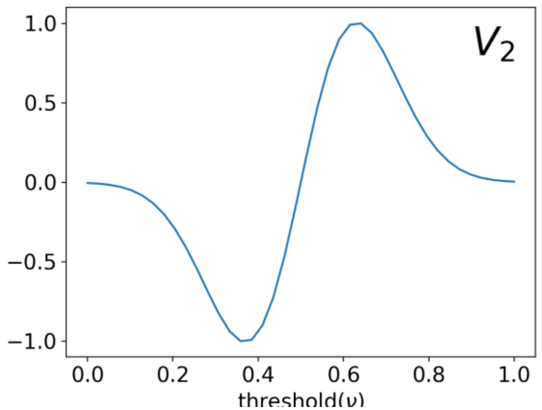

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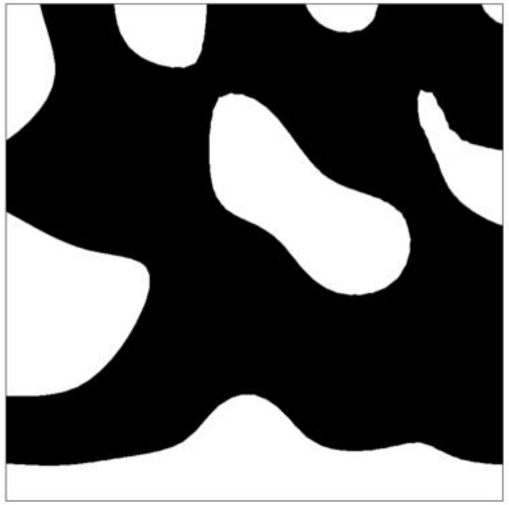

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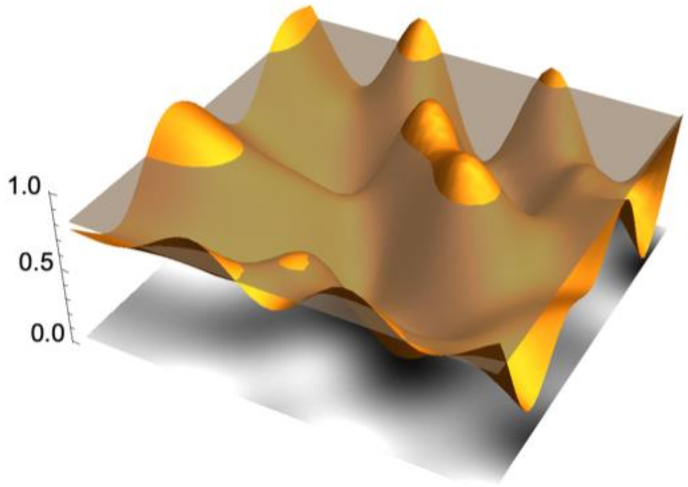

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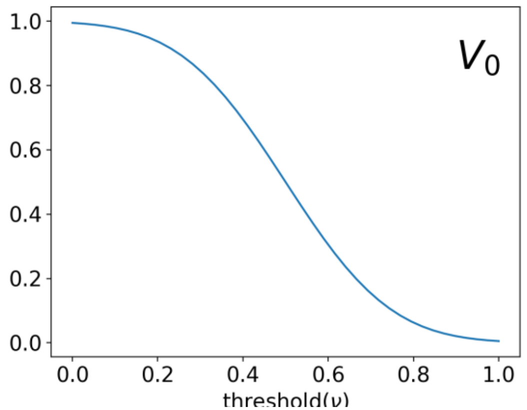

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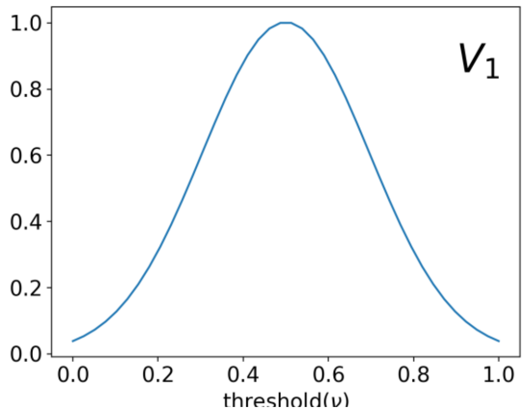

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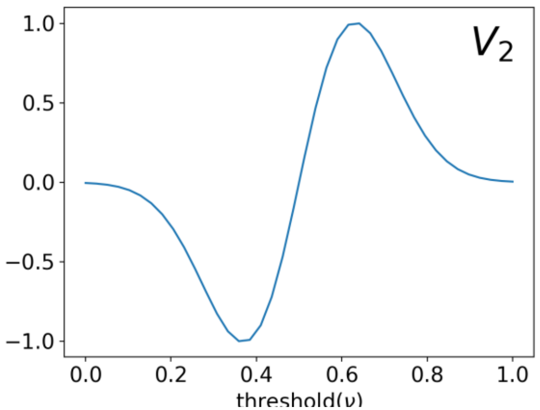

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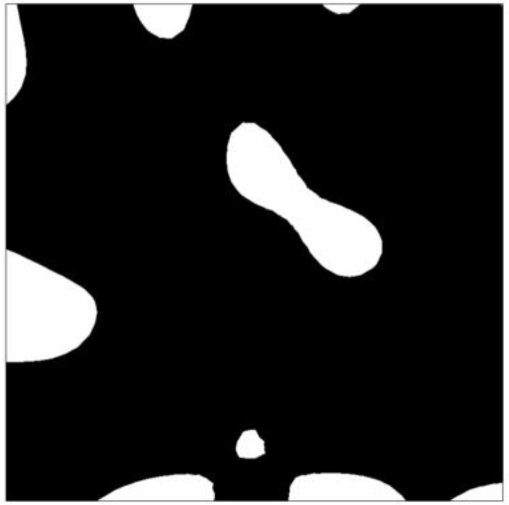

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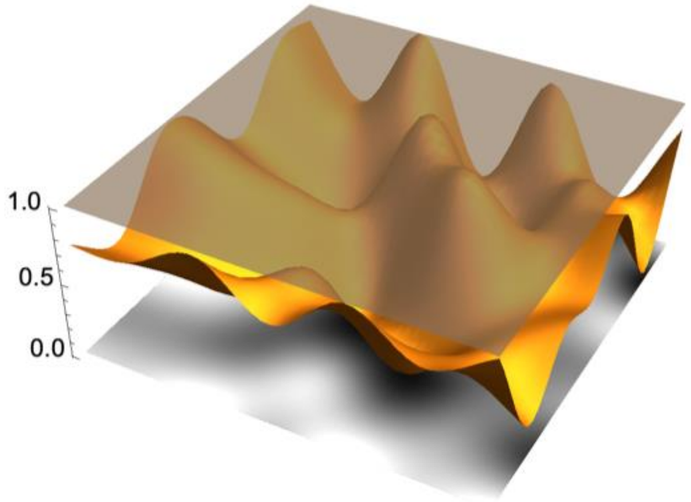

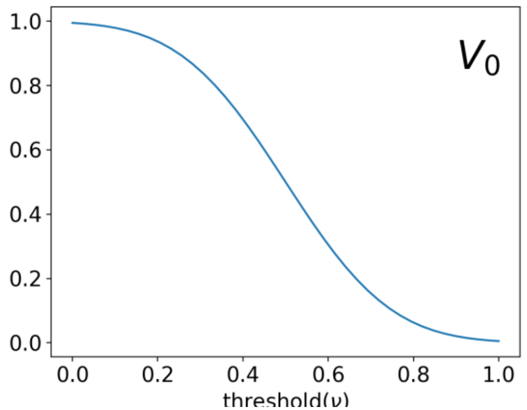

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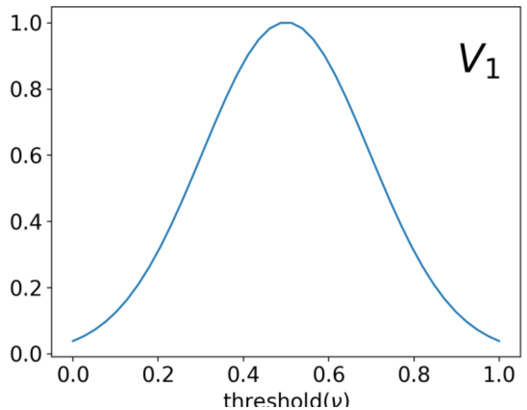

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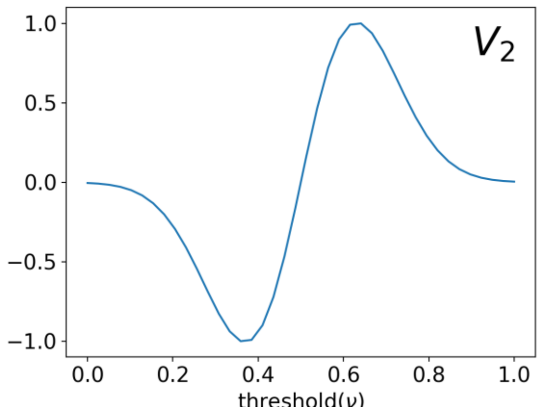

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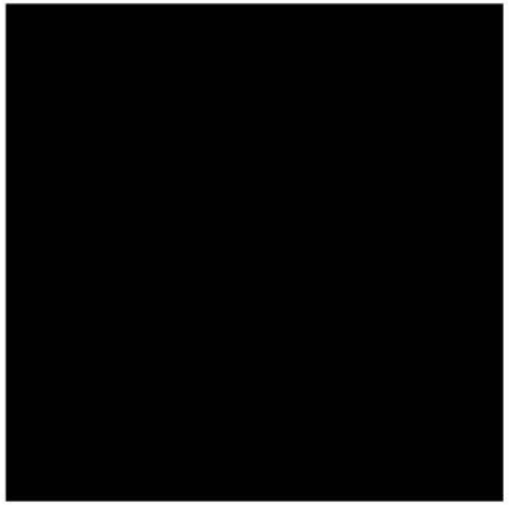

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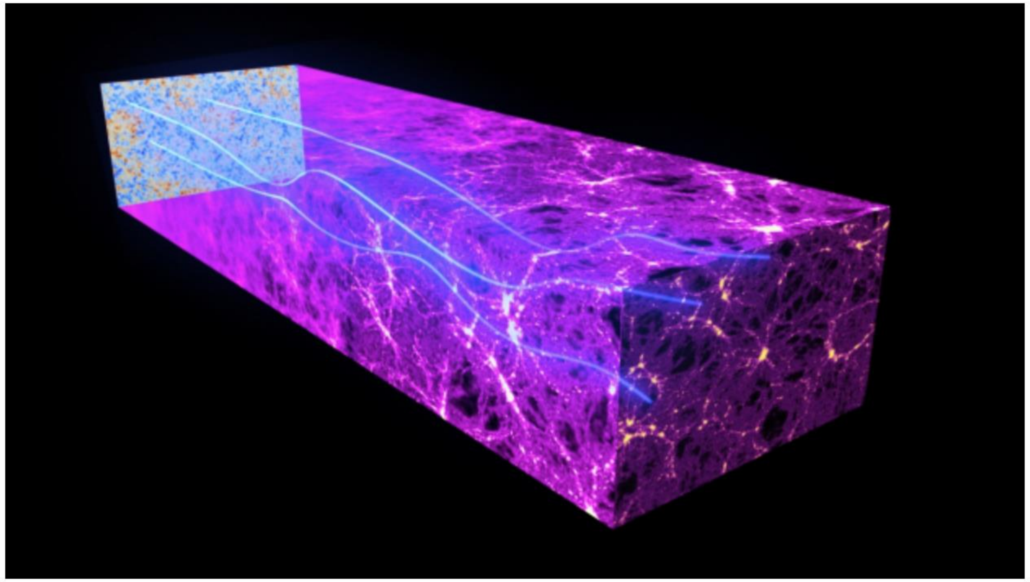
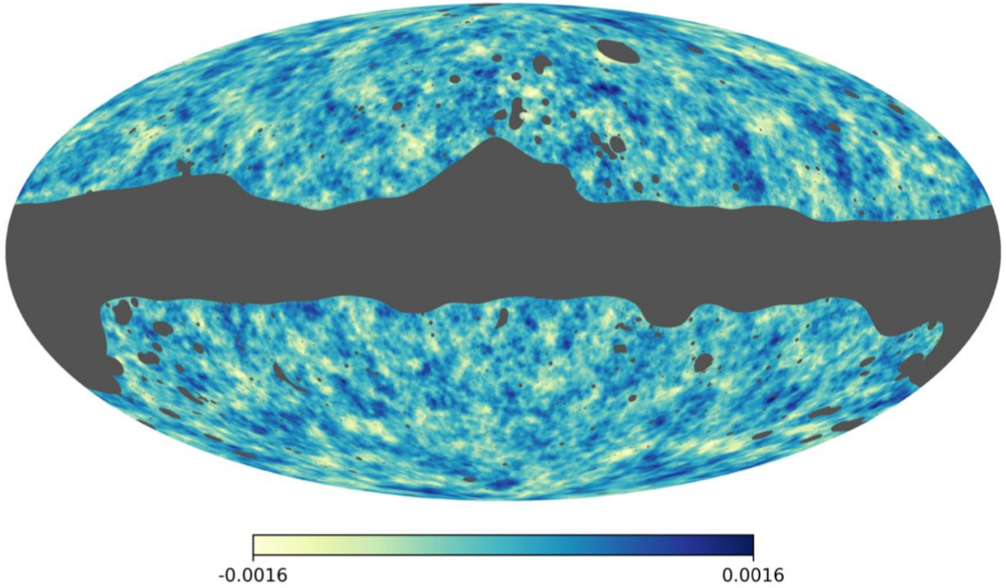
MFs are idea descriptors of CMB Lensing

- Relatively straightforward to evaluate
- Sensitive to all higher order correlations (great for non-Gaussian field)





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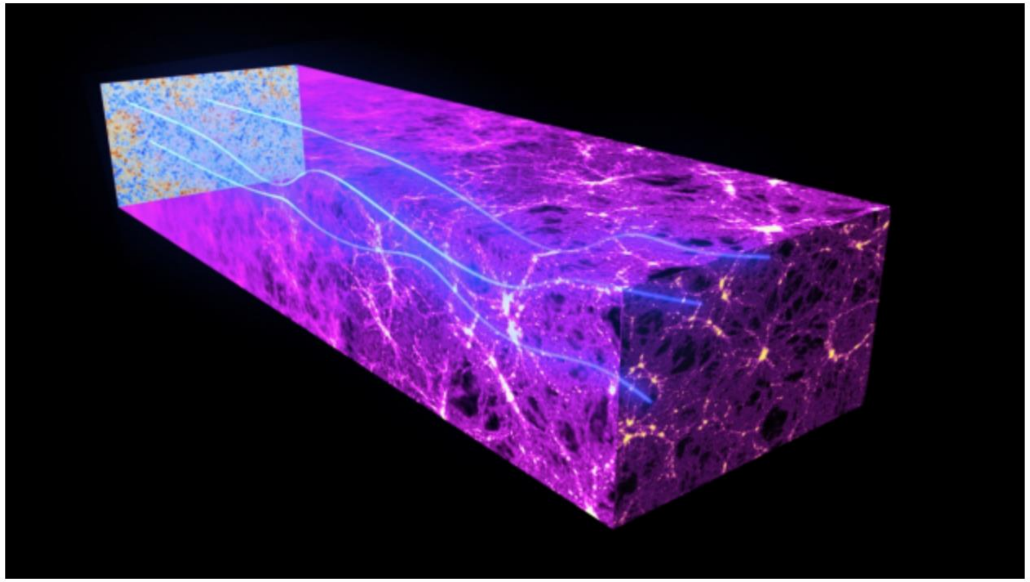

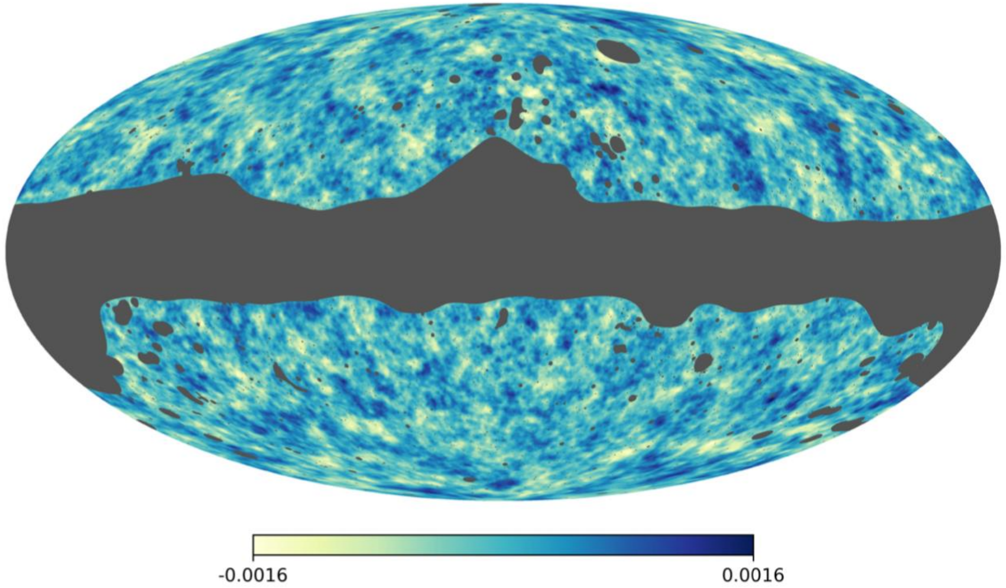

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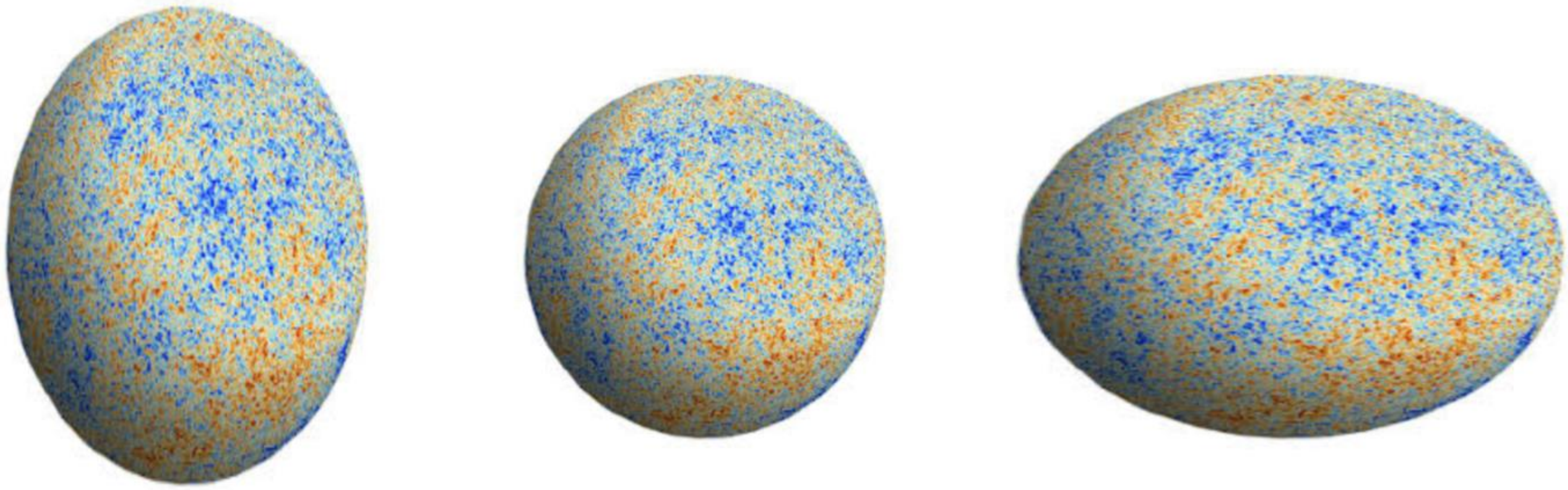

Constrain cosmological parameters

- Likelihood

covariance matrix

$$-2\ln\mathcal{L}(data|\Theta) = \sum_{i,j} \mu_i C_{ij}^{-1} \mu_j$$

$$\mu_i = V_n^{obs}(\nu_i) - V_n^{th}(\nu_i, \Theta)$$

Planck observation

theory prediction

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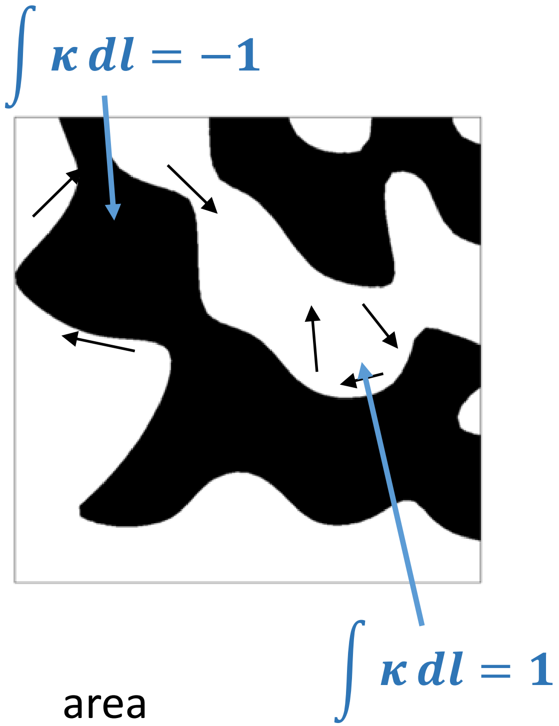
MFs from maps



- Numerically

$$V_0 = \int_\Sigma d\Omega$$ area

$$V_1 = \frac{1}{4}\int_{\partial\Sigma} dl$$ circumference

$$V_2 = \frac{1}{2\pi}\int_{\partial\Sigma} \kappa dl$$ χ

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Prediction of MFs

- N-body simulations + ray-tracing
- Perturbative approach

$$V_i = V_i^G + V_i^{(1)}\,\sigma + V_i^{(2)}\,\sigma^2 + \ldots$$

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Hermite polynomials

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Hermite polynomials

2nd moment (variance)

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Hermite polynomials

2nd moment (variance)

2nd moment of field gradient

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2nd moment

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Planck observation

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2nd moment

Power spectrum

Constrain cosmological parameters

- Likelihood

covariance matrix ↓

$$-2\ln\mathcal{L}(data|\Theta) = \sum_{i,j} \mu_i C_{ij}^{-1} \mu_j$$

$$\mu_i = V_n^{obs}(\nu_i) - V_n^{th}(\nu_i, \Theta)$$

↑ Planck observation

↑ theory prediction

Prediction of MFs

- N-body simulations + ray-tracing
- Perturbative approach

$$V_i = V_i^G + V_i^{(1)}\,\sigma + V_i^{(2)}\,\sigma^2 + \ldots$$

↑ 2nd moment ↑ 3rd moment ↑ 4th moment

↑ Power spectrum

Constrain cosmological parameters

- Likelihood

covariance matrix

$$-2\ln\mathcal{L}(data|\Theta) = \sum_{i,j} \mu_i C_{ij}^{-1} \mu_j$$

$$\mu_i = V_n^{obs}(\nu_i) - V_n^{th}(\nu_i, \Theta)$$

Planck observation ↑ V_n^{obs}; theory prediction ↑ V_n^{th}

Prediction of MFs

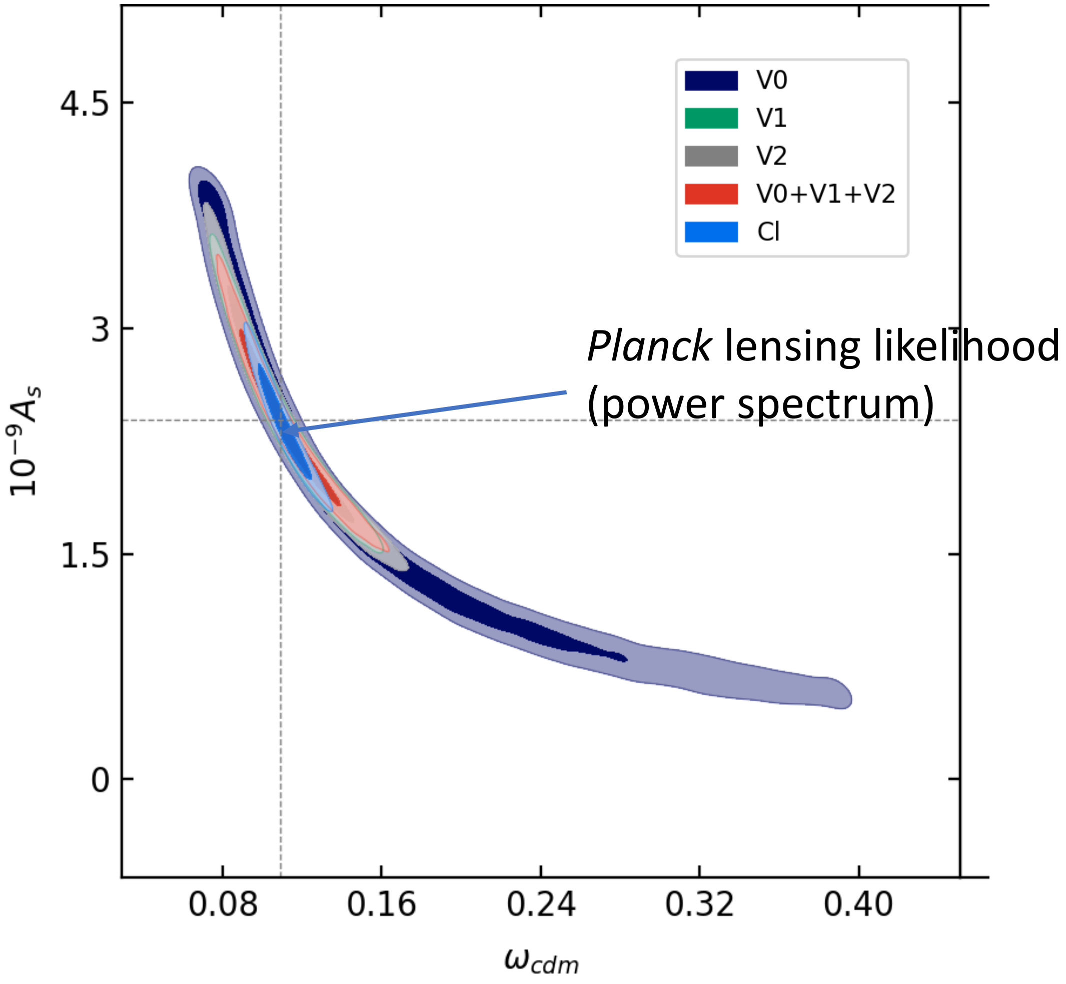
- N-body simulations + ray-tracing
- Perturbative approach

$$V_i = V_i^G + V_i^{(1)}\,\sigma + V_i^{(2)}\,\sigma^2 + \ldots$$

V_i^G	$V_i^{(1)}$	$V_i^{(2)}$
2nd moment	3rd moment	4th moment
Power spectrum	Bispectrum	Trispectrum

Current Result

- Gaussian approximation $V_i \approx V_i^G$
- Forecast constrain with Planck-like fiducial data (best-fit, noise, mask)



Current Result

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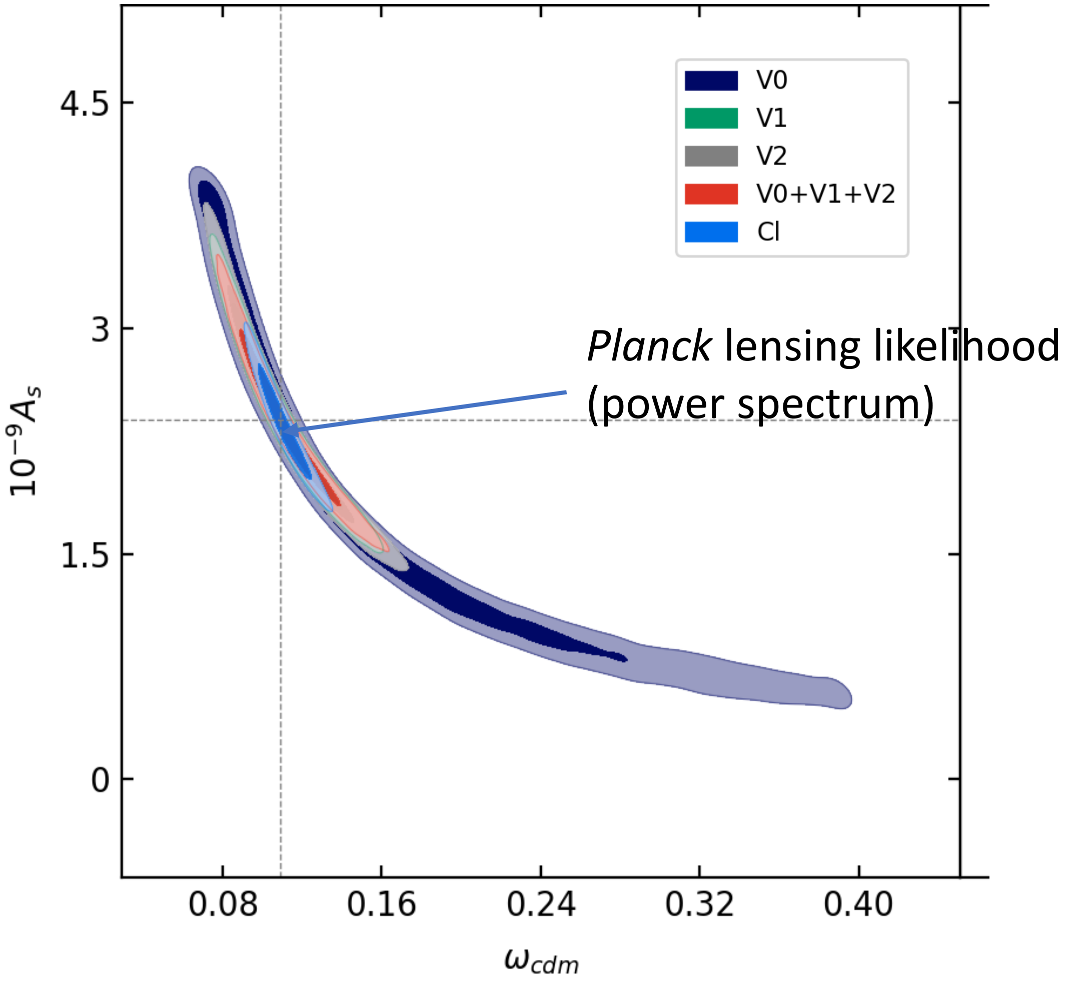
- Treat low multipoles and high multipoles separately



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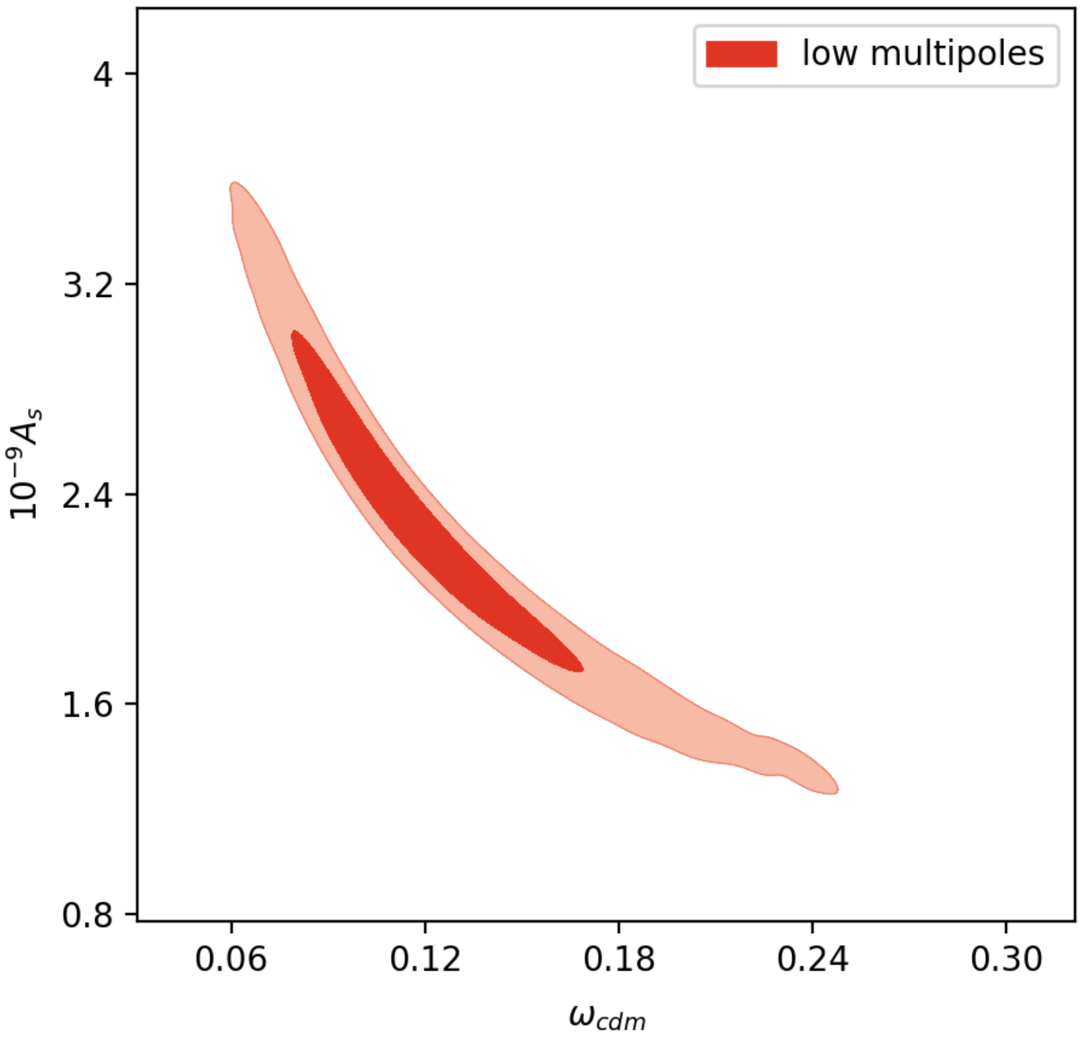

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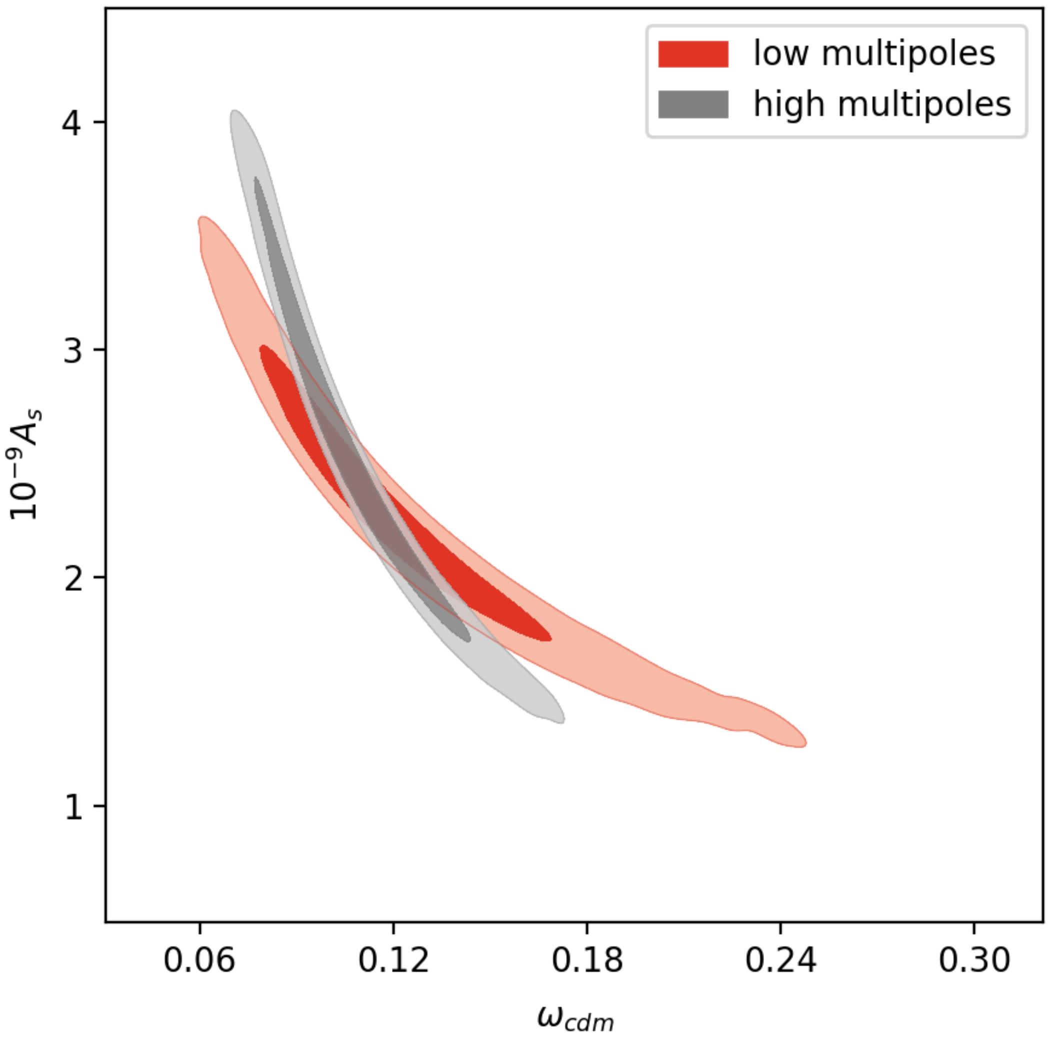

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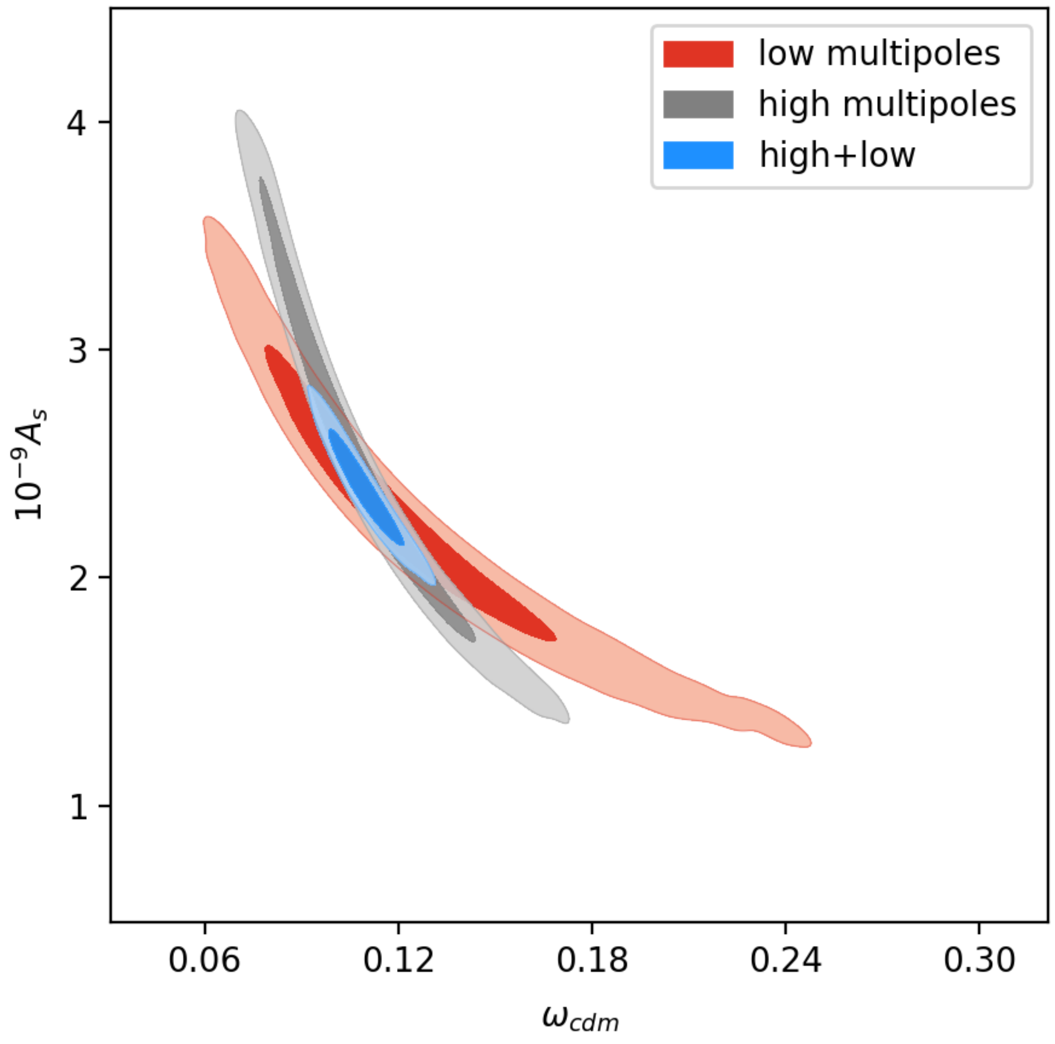

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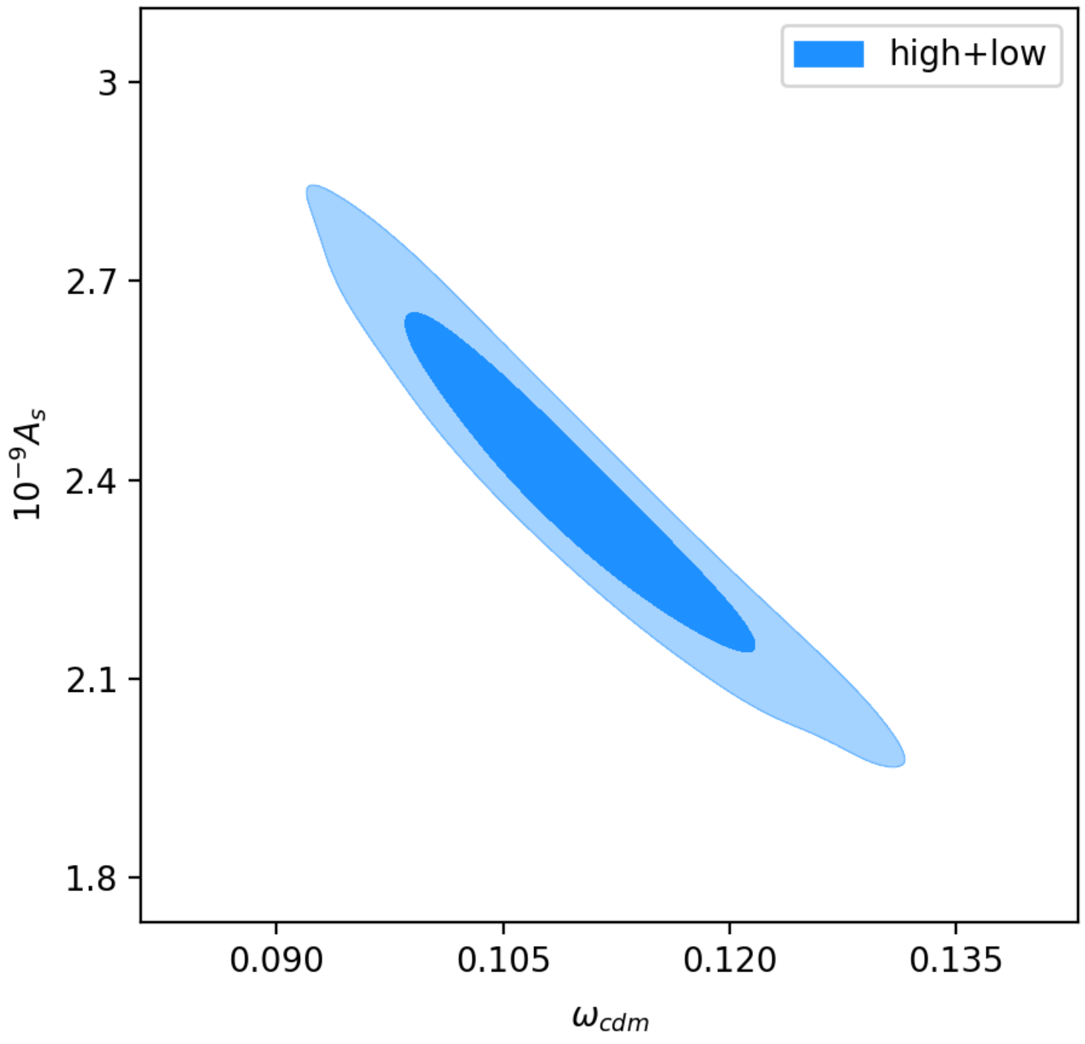

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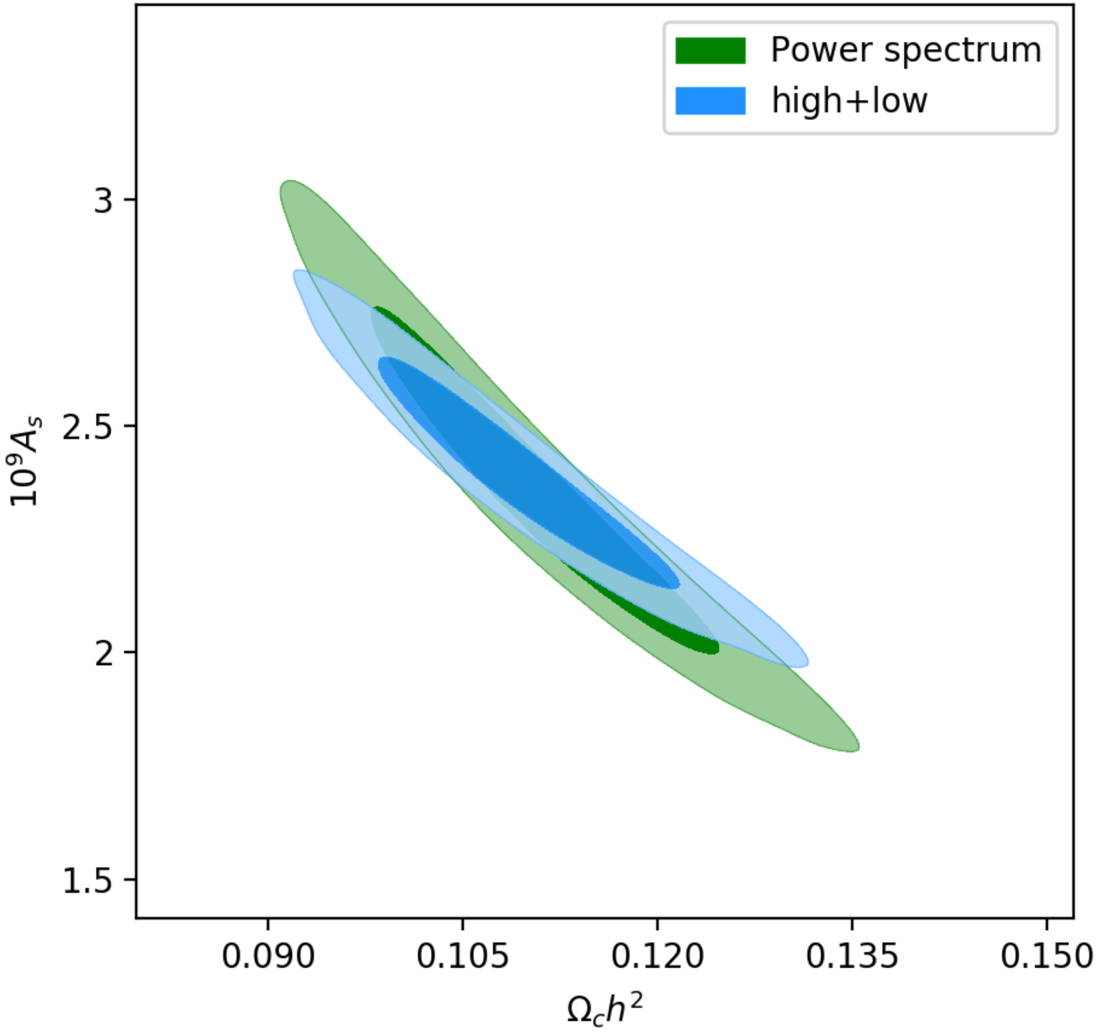

Summary

- Minkowski functionals describe topological and morphological properties of maps
- For non-Gaussian maps, MFs can add information that is missed by the power spectrum
- MFs are promising tools for current and future CMB lensing observations

Back up Slides

Current Result

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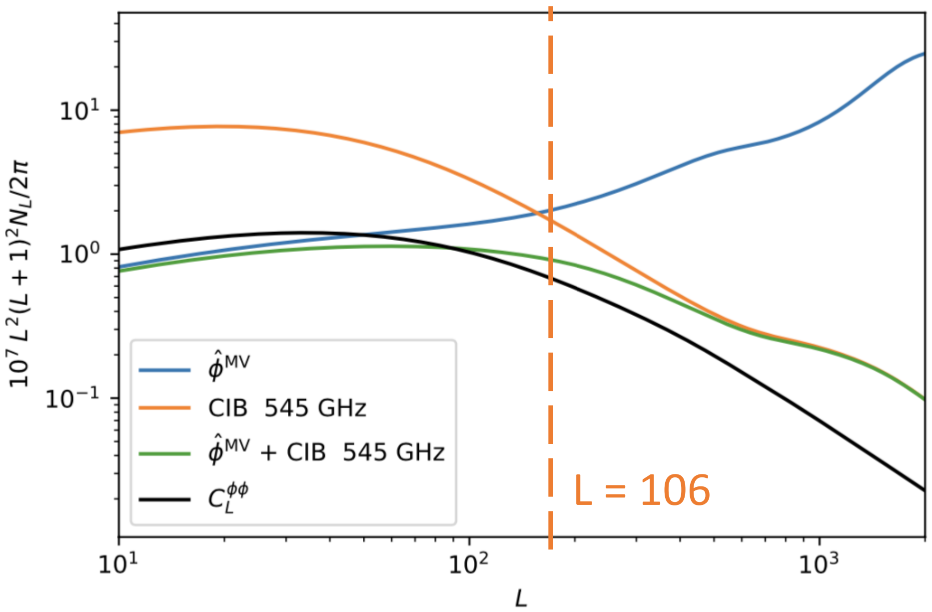
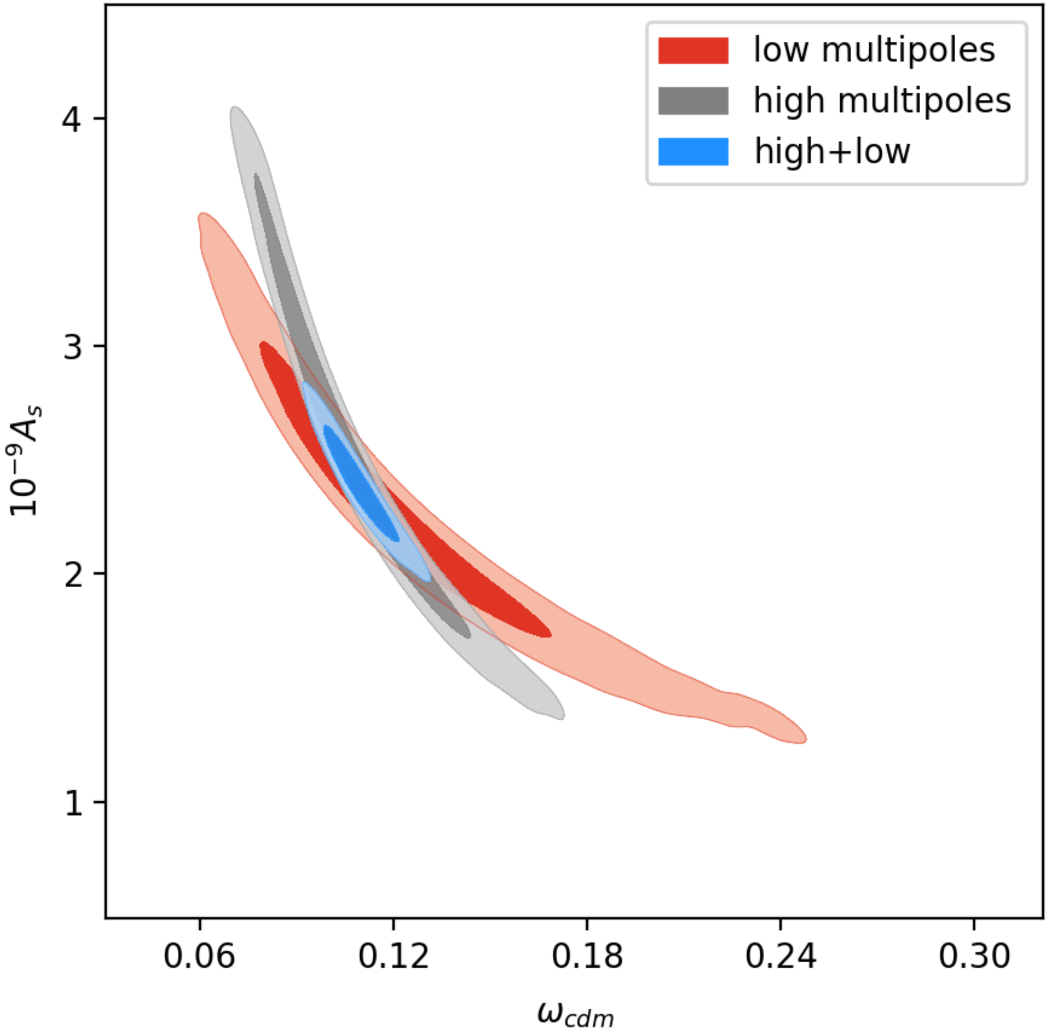


[Planck,2018]



Covariance Matrix