Experience with the symplectic 3D SC Kick

Frank Schmidt in collaboration with Yuri Alexahin (Fermilab)
• Theory and MAD-X Implementation have been published and Yuri gave a nice presentation during last year’s “2nd SC collaboration meeting at CERN”.

• The main emphasis is here to discuss how the SigmaMatrix approach allows for beam-based optics calculations and how well the tool describes the PS experiment with a moderate SC tune-shift of $\Delta Q_x = 0.05$.

• One of the main issues are unphysical code properties. In particular, one finds that the SigmaMatrix approach is very sensitive to special features of the particle distribution. In fact, one may find singularities.

• Due to the fact that the emittances are renormalized every turn in the adaptive mode, noise is being introduced into the simulations. Measures to deal with this problem will be discussed.
• Emittance blow-up and beam profiles are being compared with the 2012 PS experiments.

• In the coming years the plan is to deal with certain limitations of the code and to expand the features of the code to fully treat any ring accelerator up to the ultimate tune-shift of ~1.

• Disclaimer: Malte Titze will discuss on Tuesday a very systematic SC study for the PS, albeit in a quite different state, E.G. twice as large tune-shift etc, and also the SPS. He uses different tools and in particular also PIC codes. One could argue that these two studies are complementary to each other.
### TABLE II. Beam and machine parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intensity $N_p \times 10^{10}$ p</td>
<td>55</td>
</tr>
<tr>
<td>Normalized horizontal rms emittance $\sigma_x$ [mm mrad]</td>
<td>3.6</td>
</tr>
<tr>
<td>Normalized vertical rms emittance $\sigma_y$ [mm mrad]</td>
<td>2.2</td>
</tr>
<tr>
<td>Bunch length $\sigma_L$ [ns]</td>
<td>33</td>
</tr>
<tr>
<td>Momentum spread $\Delta p / p$ $\times 10^{-3}$</td>
<td>0.95</td>
</tr>
<tr>
<td>Horizontal maximum tune spread $\Delta Q_x \max$</td>
<td>-0.05</td>
</tr>
<tr>
<td>Vertical maximum tune spread $\Delta Q_y \max$</td>
<td>0.07</td>
</tr>
<tr>
<td>Sextupole current $I_{3X}$ [A]</td>
<td>2</td>
</tr>
<tr>
<td>Harmonic number $h$</td>
<td>8</td>
</tr>
<tr>
<td>RF voltage $V_{RF}$ [kV]</td>
<td>20.5</td>
</tr>
<tr>
<td>Natural horizontal chromaticity $Q_x$</td>
<td>-5.30</td>
</tr>
<tr>
<td>Natural vertical chromaticity $Q_y$</td>
<td>-7.02</td>
</tr>
<tr>
<td>Kinetic energy of the stored beam [GeV]</td>
<td>2</td>
</tr>
<tr>
<td>Number of stored turns</td>
<td>495646</td>
</tr>
<tr>
<td>Storage time [s]</td>
<td>1.1</td>
</tr>
<tr>
<td>Relativistic $\alpha$</td>
<td>0.948</td>
</tr>
<tr>
<td>Relativistic $\gamma$</td>
<td>3.14</td>
</tr>
<tr>
<td>Synchrotron period [turns]</td>
<td>1164</td>
</tr>
<tr>
<td>$\beta_x$ at the horizontal wire scanner in SS68 [m]</td>
<td>12.40</td>
</tr>
<tr>
<td>$\beta_y$ at the vertical wire scanner in SS64 [m]</td>
<td>21.75</td>
</tr>
</tbody>
</table>

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**3D SC Kick**

**F. Schmidt**

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[Diagram showing integer resonance, coupled resonance, and safest spot on a plot with tune axis and quality factor axis.]
The flow of derivation goes as follows:

1. Starting point is the 3D bunch charge density
2. In the long bunch approximation the 3D SC potential can be factorized into two factors, the longitudinal charge density $\lambda(z - v_0 t)$ and the transverse potential $\Phi(x, y)$.
3. In this approximation symplecticity is guaranteed: the transverse field (and the associated kick) is proportional to the charge density at the particle location $\lambda(z)$, but also it is complemented by a longitudinal kick dependent on the transverse coordinates.
4. The transverse potential can then be brought into its final form.
5. Traditionally, the transverse potential will be treated in three regimes: small, large and intermediate transverse displacements.
6. The compromise between speed and precision has been decided to allow for a precision of 6 digits in all 3 regimes.
7. Essential is that the symplecticity can be kept in all operations (other than through the unavoidable rounding error in double precision data operations).
Most relevant equations are shown here:

1. 3D bunch charge density

\[ \rho(x, y, z, t) = \frac{\lambda(z - v_0 t)}{2\pi \sigma_x \sigma_y} \exp \left( -\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} \right) \]

2. 3D SC potential

\[ \phi(x, y, z, t) \simeq \lambda(z - \nu_0 t) \cdot \Phi(x, y) \]

4. Final transverse potential

\[ \Phi(x, y) = \int_0^1 \left\{ \exp \left( -\frac{x^2 t}{2\sigma_x^2} - \frac{y^2 r^2 t}{2\sigma_y^2 [1 + (r^2 - 1)t]} \right) - 1 \right\} \frac{dt}{t \sqrt{1 + (r^2 - 1)t}} \]
## Evolution of the Implementation

<table>
<thead>
<tr>
<th>Version I: True frozen and adaptive - 2012</th>
<th>Version II: 3D symplectic kick - 2018</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ad-hoc only the transverse kick due the longitudinal density is considered ➔ loss of symplecticity</td>
<td>Due to 3D Potential, kicks in both planes ➔ Symplecticity guaranteed</td>
</tr>
<tr>
<td>Code functioning requires stable Twiss ➔ Close to low order resonance even for small SC tune-shift not guaranteed!</td>
<td>Introduction of the SigmaMatrix from beam data to derive optics, dispersion. Works great except when it fails! ➔ see later!</td>
</tr>
<tr>
<td>Emittances derived by formula.</td>
<td>Iterative process to derive effective Beam Sigmas from the SigmaMatrix.</td>
</tr>
<tr>
<td>Using Twiss &amp; Dispersion from MAD-X ➔ effective Beam Sigmas at every SC Kick.</td>
<td>Transfer of the SigmaMatrix to all SC kick locations. (See next slide for details)</td>
</tr>
<tr>
<td>Too aggressive adaptive technique in the longitudinal plane had to be suppressed. ➔ artificial emittance blow-up</td>
<td>These iteration procedures may not converge or not fast enough. Therefore damping or weighting terms had to be introduced.</td>
</tr>
</tbody>
</table>
Transfer of the sigma-matrix from observation point to the SC elements around the ring can be performed in two ways:

1) "free sigma" (open line) mode - \( \sigma_2 = T \sigma_1 T_{\text{transposed}} \) - corresponds to free oscillations of the beam envelope;

2) “periodic sigma“ mode: the periodicity of the beam envelope is imposed every turn (requires stable optics to exist)

The "free sigma" mode allows to study Gluckstern’s particle-envelope resonance but is sensitive to statistical noise.

The “periodic sigma“ mode is more robust and can be used for long-term tracking.
# SigmaMatrix: Singularity & Correlation

<table>
<thead>
<tr>
<th>Case</th>
<th>Twiss Method</th>
<th>Issues</th>
<th>Npart</th>
<th>$\beta_{x11}$</th>
<th>$\beta_{x12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FNAL Booster</td>
<td>MAD-X</td>
<td>-</td>
<td>-</td>
<td>33.66</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>SigmaMatrix</td>
<td>Equal transverse Emittances</td>
<td>1’000’000</td>
<td>22.31</td>
<td>12.40</td>
</tr>
<tr>
<td>CERN PS</td>
<td>MAD-X</td>
<td>-</td>
<td>-</td>
<td>19.483</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>SigmaMatrix</td>
<td>No issue</td>
<td>2’000</td>
<td>19.225</td>
<td>9.570 $10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>SigmaMatrix</td>
<td>No issue</td>
<td>455’000</td>
<td>19.488</td>
<td>2.953 $10^{-8}$</td>
</tr>
<tr>
<td></td>
<td>SigmaMatrix</td>
<td>Data Correlation</td>
<td>2’000</td>
<td>Error: Iteration Failure</td>
<td></td>
</tr>
<tr>
<td></td>
<td>SigmaMatrix</td>
<td>Data Correlation</td>
<td>1’000’000</td>
<td>1.501</td>
<td>10.68</td>
</tr>
</tbody>
</table>
PS Example:
$Q_x = 6.039$
500’000 Turns

Emittance Spikes

$\xi_x$, $\xi_y$

Spikes
Emittance Blow-Up in the Free Mode

![Graph showing the emittance blow-up in the free mode. The x-axis represents the number of turns, and the y-axis represents the transverse emittance in meters. The graph compares the emittance for 8,000 and 16,000 particles, showing a steady increase with increasing turns.]
Adaptive Noise Reduction

![Graph showing horizontal emittance over turns with 1,000 and 4,000 particles.

- **Horizontal Emittance**
  - Y-axis: 3.0e-06 to 5.5e-06
  - Values are plotted for different numbers of particles.

- **Turns**
  - X-axis: 1*10^5 to 5*10^5

- **Legend**
  - 1,000 Particles
  - 4,000 Particles

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I. The SigmaMatrix is very sensitive due certain properties of the distribution:
   
a) Equal transverse Tunes
b) Data correlation in the distribution
c) General artificial coupling is always found.

II. Long-term studies (500’000 turns) show occasional emittance spikes. Most likely it could be a weakness in the SigmaMatrix iteration procedure. ➔ Certainly on our Todo list!

III. The noise in the FREE propagating Mode decreases significantly doubling the macro particle numbers from 8’000 to 16’000, yet the artificial emittance blow-up remains noticeable after only 2500 turns.

IV. In the last simulation campaign of 2017 with the former adaptive mode it could be shown that both an overall and locale blow-up could be noticed over 500’000 turns with 1’000 macro particles. Both effects have been reduced substantially for a four-fold increase of the macro particles. We are expecting similar results for “Periodic” mode but tests need to be repeated. As a consequence we typically use 2’000 macro particles for most studies.
Dispersion: Map Property
Horizontal Dispersion

![Graph showing horizontal dispersion]

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Horizontal Dispersion (Blow UP)

![Graph of Horizontal Dispersion](image)

- **Shifted MAD-X no SC**
- **MAD-X SC**
- **SigmaMatrix**

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Dispersion: Map Property

- Dispersion is a 5D concept
  - i.e. the delta-p/p dependent 4D closed can be expressed as a Taylor series in $D^n$.
  - the lowest order is what we call dispersion
- Unfortunately, in MAD-X like all delta-p/p dependent parameters the Dispersion must be multiplied by the relativistic beta → In MAD-X the 5th coordinate is $p_t$ and not delta-p/p.
- It can be calculated from the transfer map $M$ via:
  $$d_x = -\frac{(d_1 - M_{22}d_1 + M_{12}d_2)}{(-1 + M_{11} + M_{12}M_{21} + M_{22} - M_{11}M_{22})}$$
  $$d'_x = -\frac{(M_{21}d_1 + d_2 - M_{11}d_2)}{(-1 + M_{11} + M_{12}M_{21} + M_{22} - M_{11}M_{22})}$$
  $$d_1 = M_{15}$$
  $$d_2 = M_{25}$$
- With linear coupling the 4D dispersion version is needed: ~50 terms for each 4 components.
- Yuri Alexahin and more recently Malte (→ see tomorrow!) have made an attempt to get dispersion directly from components of the SigmaMatrix.
- I haven’t been very successful with Yuri’s approach but by reverse engineering of the transfer map $M$ by using the eigenvector matrix $A$ from the SigmaMatrix, its inverse $A^{-1}$ and rotation $R$ by calculating the SC tune-shift with the simulation of the zero amplitude particle together with the macro particle distribution. The transfer map is then determine via:
  $$M = A^{-1}.R.A$$
- I tried to determine dispersion in the 6D case, not the 5D case.
• Here the horizontal Betafunction is shown.

• Up to this point Dispersion and Betafunctions have been evaluated at the starting point of the machine ➔ Next what happens around the ring?

• Again ➔ Malte will add much more on this subject tomorrow!
Horizontal $\beta$ Function

![Graph of Horizontal $\beta$ Function](image)

- Shifted MAD-X no SC
- MAD-X SC
- SigmaMatrix
H β & Dispersion at $Q_x = 6.244$
H β & Dispersion at $Q_x = 6.060$
H β & Dispersion at $Q_x = 6.039$
SC Tune Spreads
Tune Spread $Q_x = 6.224$
Tune Spread $Q_x = 6.060$

A few particles locked to the integer resonance
Tune Spread $Q_x = 6.039$

![Graph showing the spread of tune values with $Q_x = 6.039$.](image)
• A lot can be learned from the SC tune spread (aka “Necktie”)

• Typically one should find very well defined shapes when calculating tunes. However, it turns out that one needs multiple synchrotron periods for well defined tunes. In fact, a minimum of two such periods is just sufficient!

• On the other hand using the phase advance over one turn the images are very fuzzy and tend to be all over the place.

• The bare tunes are the matched tunes without SC.

• Most relevant parameter is of course the SC tune-shift and it is essential that it agrees with the analytical estimate! In the above figures this has been shown implicitly: The MAD-X calculation agrees very well with the tunes calculated from the zero amplitude particles.

• This SC tune-shift test can be seen as an essential sanity check of any SC code and should be applied for all relevant SC simulation studies.

• Yes Again! ➔ Malte will also add more on this subject tomorrow!
Back to PS close to Integer
Tune Spread $Q_x = 6.039$
Tune Spread $Q_x = 6.039$ Close-Up

Tune Spread $Q_x=6.039$
SC Tune-Shift

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H Phase Space $Q_x = 6.039$ (0 Ampl.)

$\sim 1/8$ of $2\sigma$
H Phase Space $Q_x = 6.039$

(0 Ampl. and Island)
The population at or close to the integer could be addressed to particles close small amplitudes but more detailed studies are needed.

The appearance of two islands seems to imply that this is in fact a SC resonance. Since we are close to the integer a particle can only appear in one of the islands, of course.

This is just a conjecture which needs to be confirmed by a more in-depth investigation.

In the meantime, traditional perturbation theory has been extended to SC Foteini & Hannes have just published a report on the subject.

Last, one of the essential MAD-X extension will be to include the 3D symplectic SC kick into PTC so that resonances will be determined automatically with NormalForm techniques.
• In the experiment the beam is injected at safe tunes and only then shifted to the tune of interest.

• Therefore, in the simulation a particle distribution is created at the safe tunes and used at all studied tunes.

• This is in particular important at \( Q_x = 6.039 \) close to the integer. In fact, a particle distribution created based on the optics at the integer is not matched and blows up.

• To create the following beam profiles the distributions are simulated for a synchrotron period so that slight mismatches have been “washed” out.
Profile $Q_x = 6.244$

H

Start

End

V

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Profile $Q_x = 6.104$

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3D SC Kick - F. Schmidt
Profile $Q_x = 6.039$

Hor. $Q_x=6.039$, $Q_y=6.479$

Start

End

Hor. $Q_x=6.039$, $Q_y=6.479$

Vert. $Q_x=6.039$, $Q_y=6.479$

V

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Conclusion

• The 3D symplectic SC kick and the SigmaMatrix have been successfully installed into a special MAD-X version. The legacy version has been kept except for some bug fixing. In particular, the old version no longer leads to artificial emittance blow-up at the integer.
• It was found that a fully selfconsistent adaption of the effective Beam Sigmas, the so-called “Free” mode, leads to unacceptable high transverse emittance blow-up. This is clearly noise that is introduced and indeed can be reduced by increasing the number of Macro-particles. However, this render this “Free” mode almost useless for long-term studies.
• We are therefore using the so-called “Periodic” mode. This allows to perform long-term studies over 500’000 turns for the PS, because the noise could be sufficiently reduced with some 2’000 - 4’000 Macro-particles. This holds for both versions.
• Tests at the PS with a relative small SC tune-shift of ~0.05 have been successfully completed.
• Supportive studies concerning the optics and dispersion functions under the influence of SC have also been presented.
The present implementation is in the process of being ported to the standard Mad-X version.

This will also include the set-up and matching routines such that the code can be used without the external preparation procedure presently be required.

Harry Renshall is continuously helping us with the OPENMP instrumentation (almost done) and various issues like proper treatment of “lost particles” and MAD-X pitfalls.

We need to overcome the emittance spike issue.

We need progress on minimizing artificial transverse coupling and introduce real coupling. Both may be combined to achieve the actual coupling present in the machine.

We still need to fully decouple the code from MAD-X Twiss and rely solely on the SigmaMatrix.

One would need a fresh start to attack the noise problem as one of the serious limiting issue in all modes. It also needs to be better categorized with respect a wide range of applications.

It is urgent to allow for distributions different from Gaussian. This is particularly important in the longitudinal plane since it may be far from Gaussian. Just the fact that it needs to be typically cut at 2.5 $\sigma$ to fit into the bucket is a deviation from a true Gaussian profile.

The end goal is an implementation that covers any application up to the ultimate space charge tune shift of $\sim 1$.

Equally relevant is to include the 3D symplectic SC kick into PTC and NormalForm.
Reserve
Horizontal $D_{px}$
H β & Dispersion at $Q_x = 6.104$
Tune Spread $Q_x = 6.104$
V Phase Space $Q_x = 6.039$ (0 Ampl.)

3D SC Kick - F. Schmidt
L Phase Space $Q_x = 6.039$ (0 Ampl.)