Black hole metric from scattering amplitudes in various dimensions

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Progress on old and new themes in Cosmology

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based on 2010.08882 with Stavros Mougiakakos

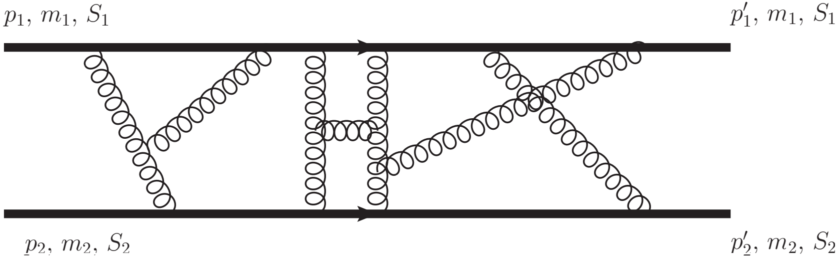
Gravity effective field theories

[...] as an open theory, quantum gravity is arguably our best quantum field theory, not the worst. [J. D. Bjorken, ``The Future of particle physics,'' hep-ph/0006180]

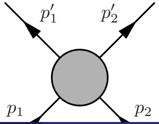
$$\mathcal{S}_{eff} = \frac{1}{16\pi G_N} \int d^4x \, \sqrt{g} \mathcal{R} + \mathcal{S}_{eff}^{\text{matter}} + \cdots$$

Einstein's theory gravity is the first term of an effective field theory coupling gravity to matter [Donoghue]

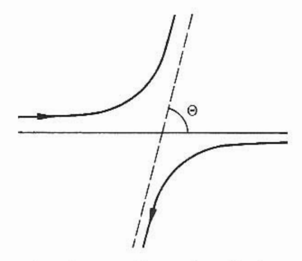
- Standard QFT (local, unitary, lorentz invariant, ...)
- The low-energy DOF: graviton, usual matter fields
- Standard symmetries: General relativity as we know it

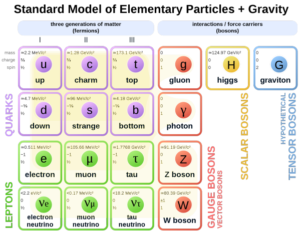


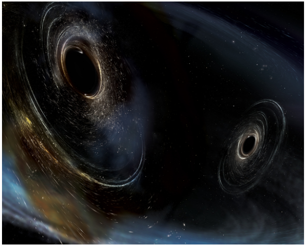
One important insight is that the **classical** gravitational two-body interactions needed for the GW signals can be extracted from quantum scattering amplitudes from gravity effective actions : **exact post-Minkowskian expansion**



$$= \sum_{L=0}^{+\infty} G_N^{L+1} \mathcal{M}^{\mathrm{L-loop}}$$







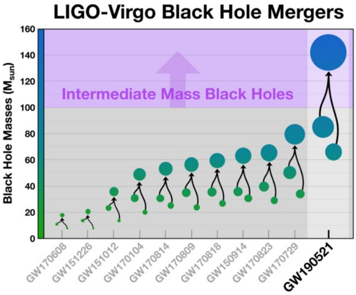
- **Classical scattering:** scattering angle χ : a lot of physical information for bound orbits
- **Quantum scattering:** probability amplitude $\mathcal{M}$: for generic EFT of gravity in various dimensions
- Extending our understanding of black holes

[Damour; Veneziano et al.; Porto et al.; Bern et al.; Goldberger, Rothstein; Damgaard, Bjerrum-Bohr, Vanhove, $\cdots$]

Black holes

Black holes play an important role in the gravitational wave physics and it is important to make sure we understand them well.

The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concept of space and time (Subrahmanyan Chandrasekhar)



Black hole formation is a robust prediction of the general theory of relativity (Citation for the 2020 Nobel prize award to Roger Penrose)

Classical solution from quantum field

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Quantum Tree Graphs and the Schwarzschild Solution

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(Received 7 July 1972)

I. INTRODUCTION

In an attempt to find quantum corrections to solutions of Einstein's equations, the question naturally arises as to whether the $\hbar \to 0$ limit of the quantum theory correctly reproduces the classical results. Formally, at least, the correspondence between the tree-graph approximation to quantum field theory and the classical solution of the field equations is well known,[1] i.e., the classical field produced by an external source serves as the generating functional for the connected Green's functions in the tree approximation, the closed-loop contributions vanishing in the limit $\hbar \to 0$. The purpose of this paper is to present an explicit calculation of the vacuum expectation value (VEV) of the gravitational field in the presence of a spherically symmetric source and verify, to second order in perturbation theory, that the result is in agreement with the classical Schwarzschild solution of the Einstein equations. This would appear to be the first step towards tackling the much more ambitious program of including the radiative quantum corrections.

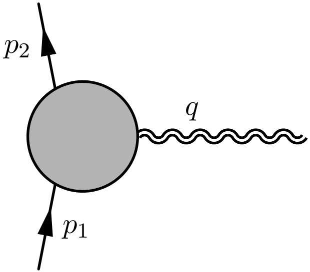
In 1973 Duff asked the question about the classical limit of quantum gravity. He showed how to reproduce the Schwarzschild back hole metric from quantum tree graphs to G_N^3 order

Since then the relation between quantum and classical gravity in amplitude have been rethought with new insights [Donoghue, Holstein], [Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove], [Kosower, Maybee, O'Connell]

Black hole metric from amplitudes

Black hole metric are extracted from the three-point vertex function

- Schwarzschild black hole: Scalar field $S = 0$, mass M
- Reissner-Nordström black hole: Scalar field $S = 0$, charge Q, mass M
- Kerr-Newman black hole: Fermionic field $S = \frac{1}{2}$, charge Q, mass M



$$= -\frac{i\sqrt{32\pi G_N}}{2} \sum_{l\geqslant 0} \langle T^{(l)\,\mu\nu}(q^2)\rangle \epsilon_{\mu\nu}$$

Black hole metric from amplitudes

The scattering amplitudes are done in the *de Donder gauge* coordinate system

$$\eta^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}(g) = \eta^{\mu\nu} g^{\lambda\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right) = 0$$

The Schwarzschild-Tangherlini metric in the de donder coordinate system

$$ds^2 = h_0(r,d)dt^2 - h_1(r,d)d\vec{x}^2 - h_2(r,d)\frac{(\vec{x}\cdot d\vec{x})^2}{\vec{x}^2}$$

$$\begin{aligned}
h_0(r) &:= 1 - 4\frac{d-2}{d-1}\,\frac{\rho(r,d)}{f(r)^{d-2}},\\
h_1(r) &:= f(r)^2,\\
h_2(r) &:= -f(r)^2 - f(r)^{d-2}\frac{\left(f(r) + r\frac{df(r)}{dr}\right)^2}{f(r)^{d-2} - 4\frac{d-2}{d-1}\rho(r,d)}\,.
\end{aligned}$$

The dimensionless parameter is the post-Minkowskian expansion parameter

$$\rho(r,d) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}\,\frac{G_N m}{r^{d-2}}$$

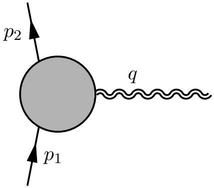
The de Donder gauge metric in four dimensions

$$h_0^{\mathrm{dD}}(r) = 1 - \frac{2G_N m}{r} + 2\left(\frac{G_N m}{r}\right)^2 + 2\left(\frac{G_N m}{r}\right)^3 + \left(\frac{4}{3}\log\left(\frac{rC_3}{G_N m}\right) - 6\right)\left(\frac{G_N m}{r}\right)^4 + \cdots$$

$$\begin{aligned} h_1^{\mathrm{dD}}(r) &= 1 + 2\frac{G_N m}{r} + 5\left(\frac{G_N m}{r}\right)^2 + \left(\frac{4}{3}\log\left(\frac{rC_3}{G_N m}\right) + 4\right)\left(\frac{G_N m}{r}\right)^3 \\ &+ \left(-\frac{4}{3}\log\left(\frac{rC_3}{G_N m}\right) + \frac{16}{3}\right)\left(\frac{G_N m}{r}\right)^4 + \left(\frac{64}{15}\log\left(\frac{rC_3}{G_N m}\right) - \frac{26}{75}\right)\left(\frac{G_N m}{r}\right)^5 \\ &+ \left(\frac{4}{9}\log\left(\frac{rC_3}{G_N m}\right)^2 - \frac{24}{5}\log\left(\frac{rC_3}{G_N m}\right) + \frac{298}{75}\right)\left(\frac{G_N m}{r}\right)^6 + \cdots \end{aligned}$$

- The metric is finite by as powers of $\log(r)$
- The solution has a single constant of integration C_3.

Black hole metric from amplitudes



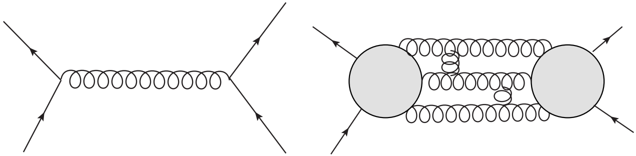
$$= -\frac{i\sqrt{32\pi G_N}}{2}\sum_{l\geqslant 0}\langle T^{(l)\,\mu\nu}(q^2)\rangle\epsilon_{\mu\nu}$$

In the de Donder gauge the metric perturbations are obtained as

$$h_{\mu\nu}^{(l+1)}(\vec{x}) = -16\pi G_N \int \frac{d^d\vec{q}}{(2\pi)^d}\frac{e^{i\vec{q}\cdot\vec{x}}}{\vec{q}^2}\left(\langle T_{\mu\nu}^{(l)}\rangle(q^2) - \frac{1}{d-1}\eta_{\mu\nu}\langle T^{(l)}\rangle(q^2)\right)$$

- The classical metric is obtained by using the classical limit of the quantum amplitude $\langle T_{\mu\nu}^{(l)}\rangle^{\text{class.}}(q^2)$ [Bjerrum-Bohr et al.]
- But one can as well include quantum correction to the metric [Donoghue et al.], [Bjerrum-Bohr et al.]

Classical physics from loops



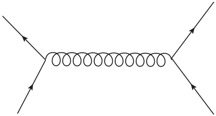
We will be considering the pure gravitational interaction between massive and massless matter of various spin

$$\mathcal{L}_{\rm EH} \sim \int d^4x \left(-\frac{1}{16\pi G_N}\,\mathcal{R} + \sqrt{32\pi G_N} h_{\mu\nu} T^{\mu\nu}_{\rm matter}\right),$$

$$\mathfrak{M} = \frac{1}{\hbar}\,\mathcal{M}^{\text{Post-Min}} + \hbar^0\,\mathfrak{M}^{1-\text{loop}} + \cdots.$$

The classical contribution are the singular pieces when $\hbar \to 0$

One graviton exchange : tree-level amplitude



$$\mathfrak{M}^{(1)} = -\frac{4\pi G_N}{E_1 E_2} \frac{[2(p_1 \cdot p_2)^2 - m_1^2 m_2^2 - |\vec{q}\,|^2 (p_1 \cdot p_2)]}{|\vec{q}\,|^2}$$

The potential is obtained

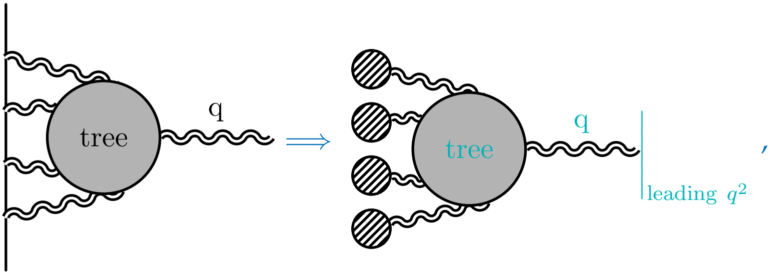
$$\mathfrak{M}^{(1)}(q) = \mathcal{M}^{(1)}(q) + \frac{4\pi G_N p_1 \cdot p_2}{E_1 E_2}$$

The classical potential is obtained by taking the 3d Fourier transform
$E_i^2 = \vec{p}_i^{\,2} + m_i^2$

$$V_{1PM}(r) = \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{4E_1 E_2} \mathcal{M}^{(1)}(\vec{q})\, e^{i\vec{q}\cdot\vec{r}} = \frac{G_N}{E_1 E_2} \frac{m_1^2 m_2^2 - 2(p_1 \cdot p_2)^2}{r}$$

The higher order in q^2 are quantum with powers of $\hbar$

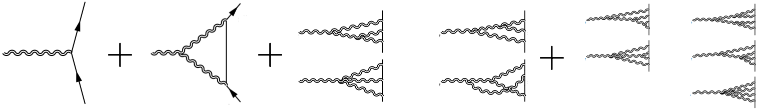
Classical contributions from quantum loops



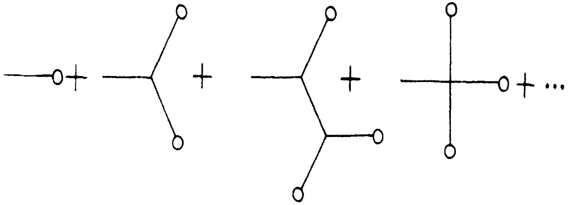
The classical limit results in cutting the massive lines, projecting on the contribution from localised sources at different positions in space and keeping only the leading q^2 contribution from the multi-graviton tree-level amplitudes.

[Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove]

Classical contributions from quantum loops



$$\frac{GM}{r} + \left(\frac{GM}{r}\right)^2 + \left(\frac{GM}{r}\right)^3 + \left(\frac{GM}{r}\right)^4$$



- The tree skeleton graphs are the one computed by Duff

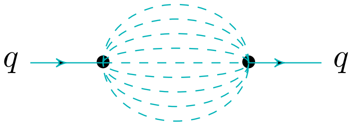
Classical metric from loops

$$h_{\mu\nu}^{(l+1)}(\vec{q}) = -8\left(c_1^{(l)}(d)(2\delta_\mu^0\delta_\nu^0 - \eta_{\mu\nu}) + c_2^{(l)}(d)\left(2\frac{q_\mu q_\nu}{q^2} + (d-2)\eta_{\mu\nu}\right)\right)$$

$$\times\ (\pi G_N m)^{l+1}\frac{J_{(l)}(\vec{q}^2)}{\vec{q}^2}\,.$$

The metric components in the static limit are given by a single master integral

$$J_{(n)}(\vec{q}^2) =$$



$$= \frac{(\vec{q}^2)^{\frac{n(d-2)}{2}}}{(4\pi)^{\frac{nd}{2}}}\frac{\Gamma\left(n+1-\frac{nd}{2}\right)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{(n+1)(d-2)}{2}\right)}$$

Fourier transforming to direct space

$$h_i^{(l+1)}(r,d) = C(d,l)\left(\frac{\rho(r,d)}{4}\right)^{l+1}$$

Divergences

Divergences in stress-tensor and metric components are removed by introducing the non-minimal couplings

$$\delta^{(n)} S^{\text{ct.}} = (G_N m)^{\frac{2n}{d-2}} \int d^{d+1}x \sqrt{-g}\Big(\alpha^{(n)}(d)(\nabla^2)^{n-1} R\partial_\mu\phi\partial^\mu\phi$$
$$+ \left(\beta_0^{(n)}(d)\nabla_\mu\nabla_\nu(\nabla^2)^{n-2}R + \beta_1^{(n)}(d)(\nabla^2)^{n-1}R_{\mu\nu}\right)\partial^\mu\phi\partial^\nu\phi\Big).$$

- $\delta^{(1)} S^{\text{ct.}}$ renormalised the stress-tensor and the metric components [Goldberger, Rothstein; Jakobsen]
- The finite piece after cancelling the divergences matches the log contributions in the metric solution. The constant of integration is the renormalisation scale
- $\delta^{(n)} S^{\text{ct.}}$ with $n \geqslant 2$ renormalise the divergences in the stress-tensor but do not contribute to the metric components

Outlook

it is satisfying to be able to embed such classical solutions in the new understanding of the relation between general relativity and the quantum theory of gravity

1. Our analysis shows that our understanding of the classical limit of quantum scattering amplitudes is complete
2. We have nicely displayed the importance of the coordinate choice and the gauge fixing conditions.
3. The same analysis can be extended easily for recovering the Kerr-Newman (charged spin $\frac{1}{2}$) and Reissner-Nordström metric (charged scalar).
4. **gravity is richer in higher dimensions!** Since the amplitude approach has been validated in higher-dimensions we can explore various interesting classical gravity physics in higher dimensions
5. We can include the quantum corrections to the metric
6. We can do this in any EFT which have amplitude description, and derive new space-time metrics