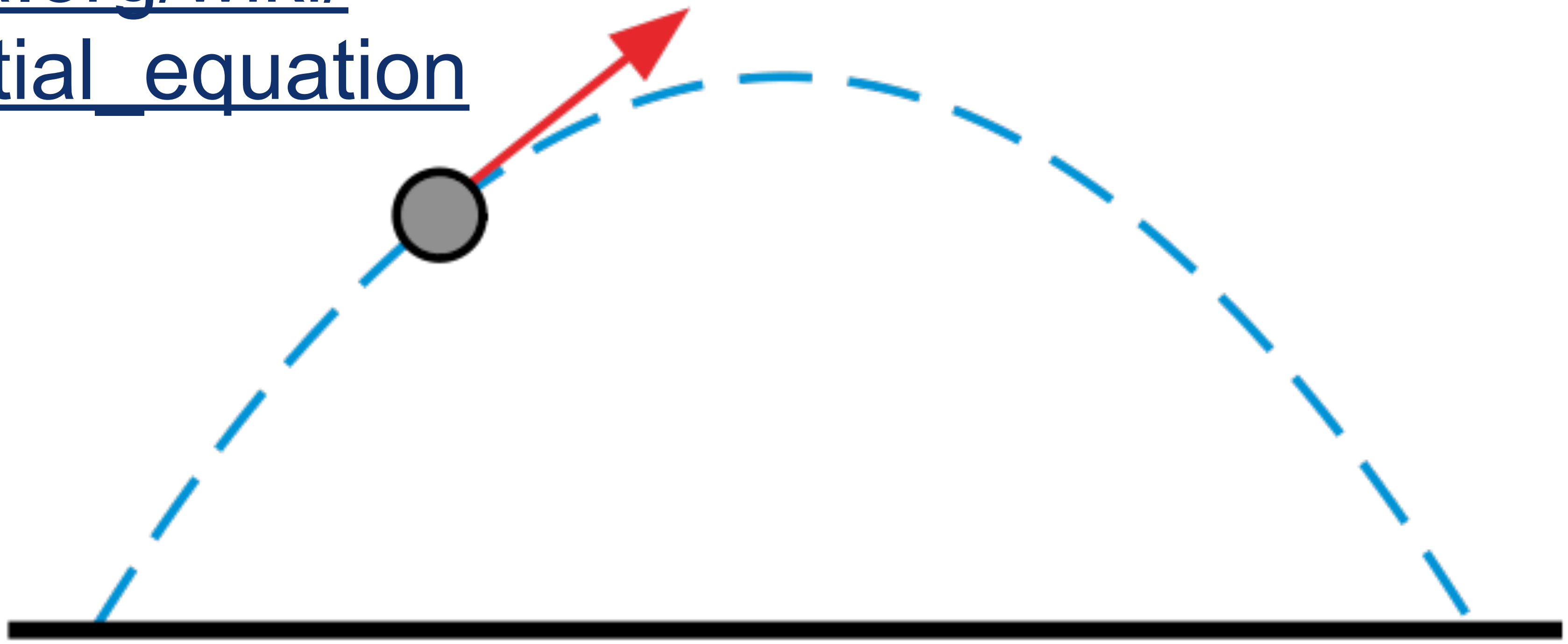
PY410 / 505
Computational Physics 1

Salvatore Rappoccio

Ordinary differential equations

- Finally, we're able to solve the most trivial thing in your physics knowledge : the ODE
  - http://en.wikipedia.org/wiki/Ordinary_differential_equation



$$\begin{aligned}\ddot{x}(t) &= -g \\ \dot{x}(t) &= -gt + v_0 t \\ x(t) &= -\frac{1}{2}gt^2 + v_0 t + x_0\end{aligned}$$

Ordinary differential equations

- Garcia spends a lot of time on ODE’s and PDE’s, and does a pretty good job of explaining it
  - Chapter 2, 3
- It’s also in Numerical Recipes chapter 16

- One of the more developed portions of scientific computing, so clearly a lot to say on the topic

- Also lots of fun applications! We all like trajectories :)

Ordinary differential equations

- Like we did for derivatives and integrals, we'll go back a step from the continuum limit to a finite difference equation

$$\dot{x}(t) = v(t) = \frac{dx(t)}{dt} \approx \frac{x(t-dt) - x(t)}{dt}$$

- We can rearrange this (i.e. do a Taylor expansion) to get

$$x(t+dt) \approx x(t) + \frac{dx}{dt}dt + \mathcal{O}(dt^2)$$

- So, given x(0) and dx/dt, we can determine the entire series!
  – This is, in a nutshell, Euler's method

Ordinary differential equations

- Explicitly, we define time steps for a time-slice τ :

$$f_n = f(t_n), \quad t_n = n\tau$$

- Euler's method updates the velocity and the position :

$$\begin{aligned} \mathbf{v}_{n+1} &= \mathbf{v}_n + \tau \mathbf{a}_n \\ \mathbf{x}_{n+1} &= \mathbf{x}_n + \tau \mathbf{v}_n \end{aligned}$$

Ordinary differential equations

- The pseudocode is therefore :
  - Specify initial conditions r0 and v0
  - Choose a time step tau
  - Calculate acceleration given current r and v
  - Update r and v at the new step
  - Continue until you get to the end

Ordinary differential equations

- As we did for the case of derivatives and integrals, this is a "one-point" method (but we need the velocities too)
- Unsurprisingly, there are multiple-point methods with higher accuracy

- Is this "one-point" method good enough?
  - Usually, not really.
  - Looking at a series, so each step has errors $\mathcal{O}(\tau^2)$
  - After N steps, if N = T / τ, then the total error due to the Euler method is actually $\mathcal{O}(\tau^2 N) \sim \mathcal{O}(\tau)$

- Booooo, this is terrible

Ordinary differential equations

- Something more like a two-point method : Euler-Cromer method

$$
\begin{aligned}
\mathbf{v}_{n+1} &= \mathbf{v}_n + \tau \mathbf{a}_n \\
\mathbf{x}_{n+1} &= \mathbf{x}_n + \tau \mathbf{v}_{n+1}
\end{aligned}
$$

  – Updates the position using the velocity at the next point, unlike the Euler method

  – A. Cromer, "Stable Solutions using the Euler Approximation", Am. J. Phys. 49, 455 (1981)

Ordinary differential equations

- What about a midpoint method?
- Let's say we average the two velocities (at n+1 and n):

$$\begin{aligned} \mathbf{v}_{n+1} &= \mathbf{v}_n + \tau \mathbf{a}_n \\ \mathbf{x}_{n+1} &= \mathbf{x}_n + \tau \frac{\mathbf{v}_{n+1} + \mathbf{v}_n}{2} \end{aligned}$$

- Then our position equation would be

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \tau \mathbf{v}_n + \frac{1}{2}\tau^2 \mathbf{a}_n$$

- This is effectively a midpoint method, and reduces the uncertainty now to $\mathcal{O}(\tau^2)$
  - –Better, but still not fantastic

Ordinary differential equations

- Projectile motion is America's favorite pastime!

- Let's analyze the situation with our current numerical techniques to see how well we do

Jon Lester, Game 5 2013 World Series

Also add air resistance!
https://en.wikipedia.org/wiki/Drag_equation

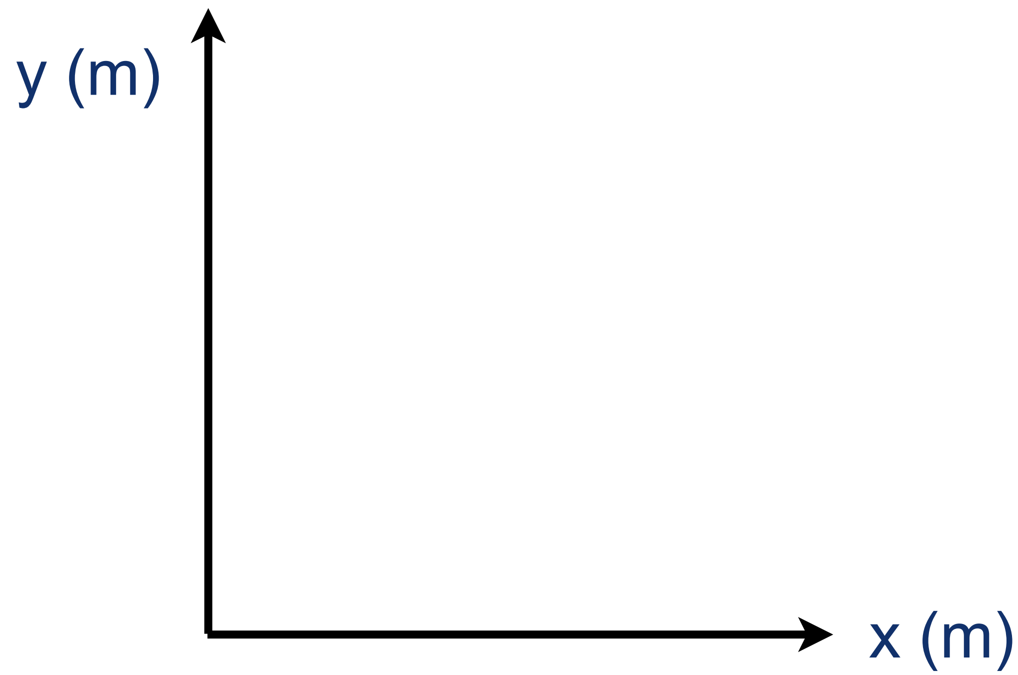
Ordinary differential equations

- Pseudocode :
  - Set initial position and velocity
  - Set physical parameters
  - Loop until ball hits the ground (or max reached):
    - Record position (computed and exact) for plotting
    - Compute acceleration
    - Calculate new position and velocity
    - If ball reaches ground or max reached, quit
  - Print max range, time of flight
  - Plot the trajectory

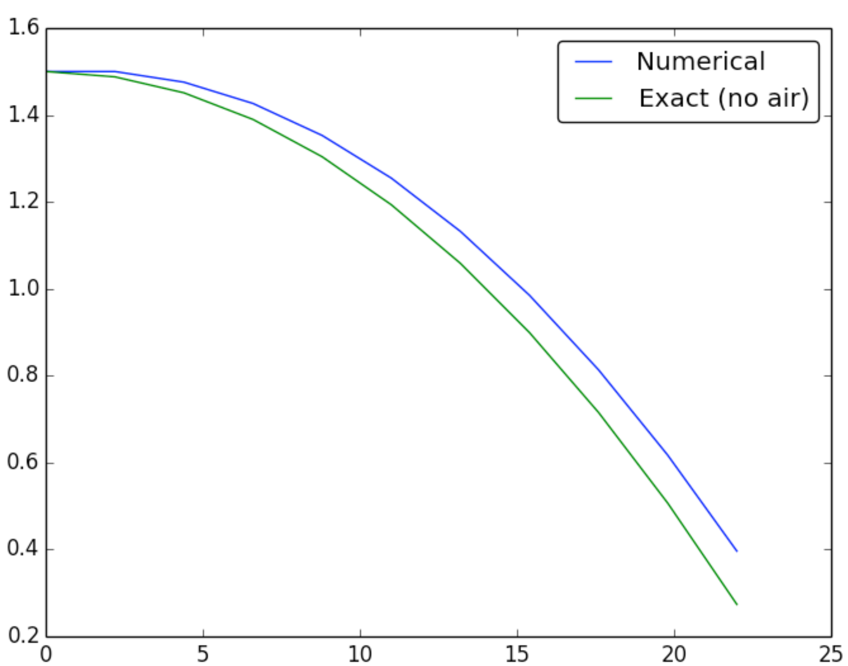
Ordinary differential equations

- Drag force:
  - Opposite the velocity
  - Magnitude is:
    - $F_D = \frac{1}{2}\rho v^2 C_D A$
    - where:
      - ρ: mass density of the fluid
      - v velocity of projectile
      - C_D drag coefficient (dimensionless)
      - A reference area

Ordinary differential equations

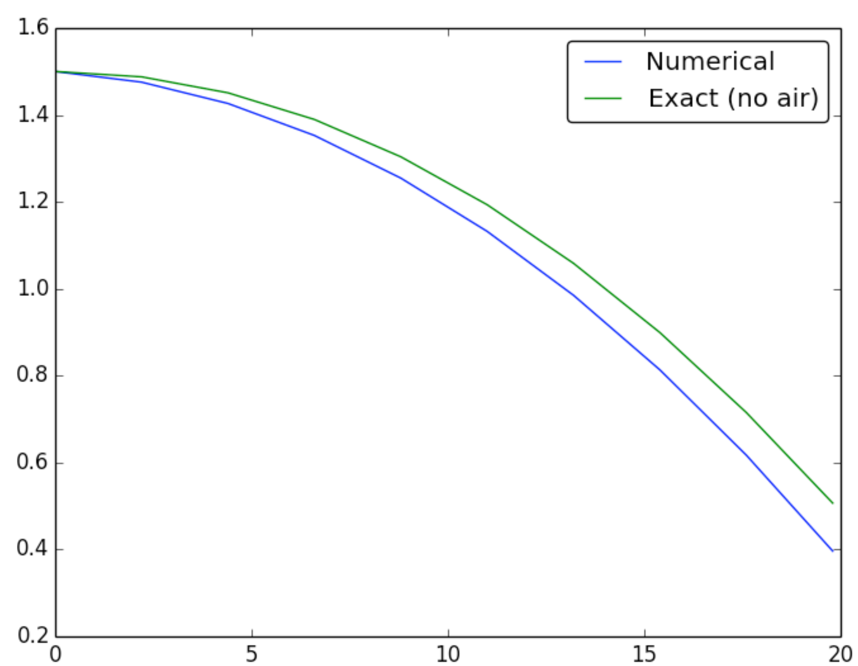


Euler



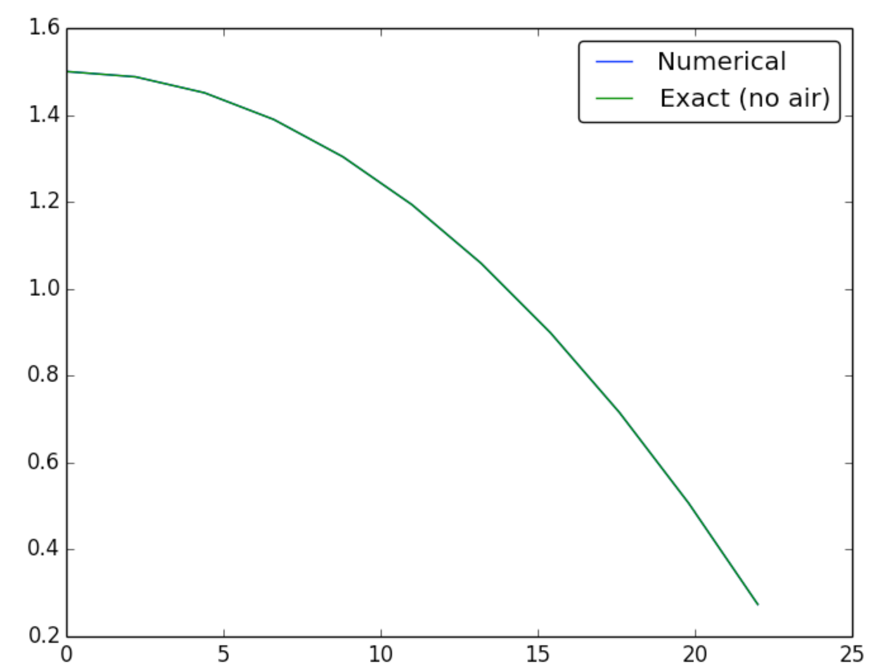
Overshoots

Euler-Cromer



Undershoots

Midpoint



Just right!

Ordinary differential equations

- Projectile motion : use midpoint method, and good enough

- Is that all?
  - Let's try something else

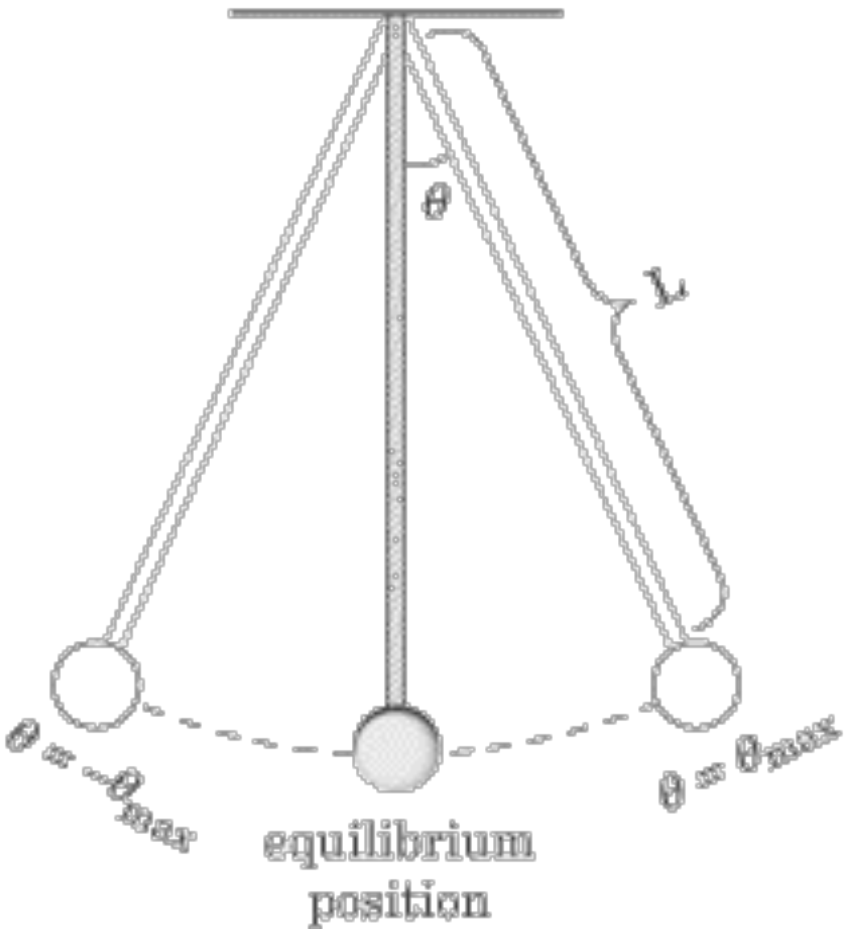
Ordinary differential equations

- Try a simple harmonic oscillator (like a pendulum)!

- Try a damped, driven pendulum
- Equation of motion is:

$$\frac{d^2\theta}{dt^2} = -\omega_0^2\theta - \gamma\frac{d\theta}{dt} + F_D\cos(\omega_D t) \; ,$$

$$\omega_0 = \sqrt{\frac{g}{L}} \; .$$



- Have transient motion related to the damping force
- Analytically, steady-state solution is :

$$\theta(t) = \frac{F_D\cos\left(\omega_D t + \phi\right)}{\sqrt{(\omega_0^2 - \omega_D^2)^2 + (\gamma\omega_D)^2}} \; .$$

Ordinary differential equations

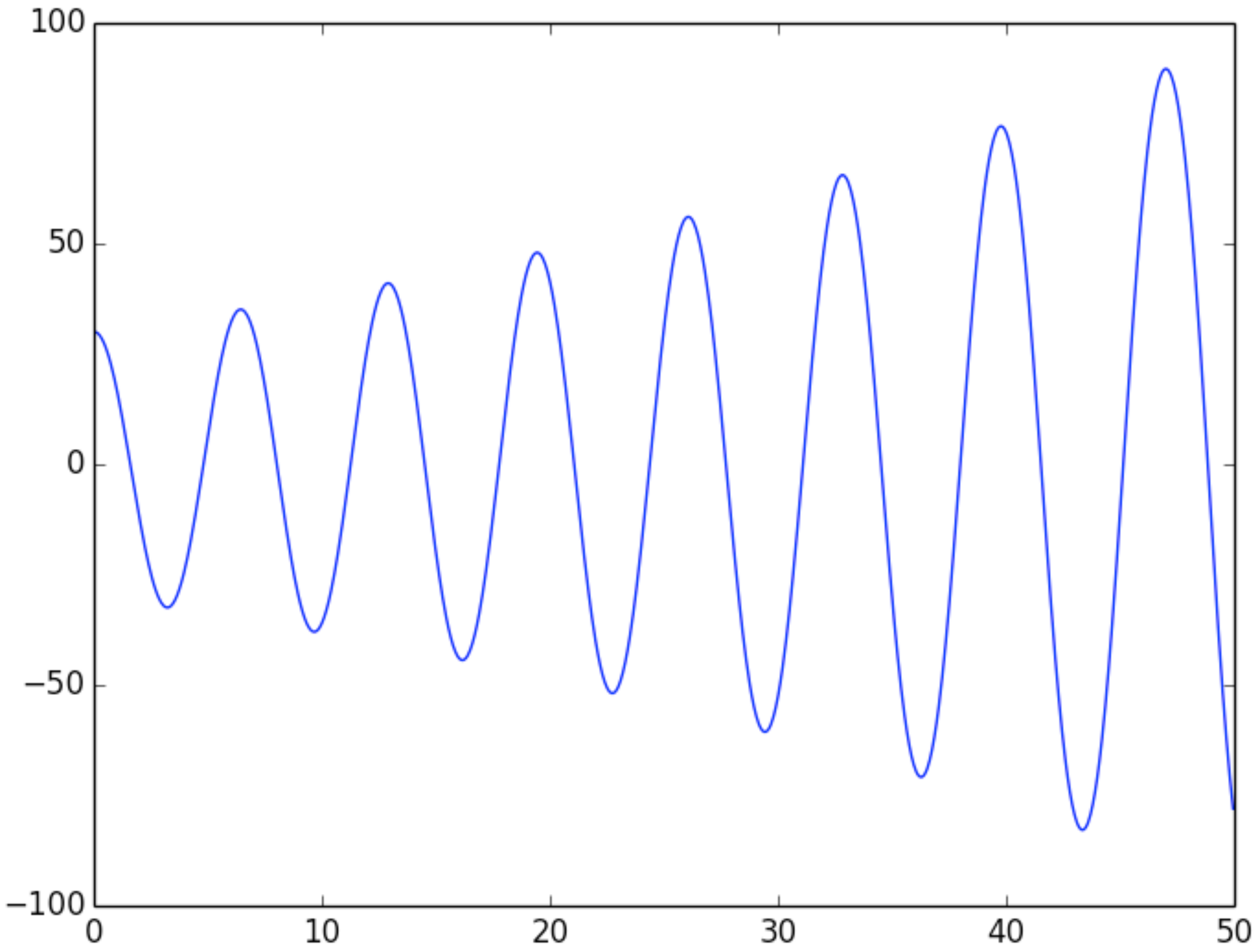
- Already can guess that Euler will fail miserably, and you'd be right.

Ordinary differential equations

- Pseudocode :
  - Set initial position and velocity of the pendulum
  - Set physical constants
  - Loop over desired number of steps with given time step
    - Record angle and time
    - Compute new position and velocity
    - Test if pendulum has passed through theta=0, use t to estimate period
  - Estimate average period, including uncertainty
  - Plot theta vs t

Ordinary differential equations

- Yipes! Totally unstable!
  - Turns out midpoint method also has this property
- Well NOW what do we do???



Ordinary differential equations

- What about a CENTERED three-point method?

$$f'(t) = \lim_{\tau \to 0} \frac{f(t+\tau) - f(t-\tau)}{2\tau}$$

- As before, we get uncertainties better than expected because the second term in the Taylor expansion cancels:

$$\begin{aligned} f(t+\tau) &= f(t) + \tau f'(t) + \frac{1}{2}\tau^2 f''(t) + \frac{1}{6}\tau^3 f'''(\chi) \\ f(t-\tau) &= f(t) - \tau f'(t) + \frac{1}{2}\tau^2 f''(t) - \frac{1}{6}\tau^3 f'''(\chi) \end{aligned}$$

  – Here, chi is some point between t-tau and t+tau

- So:

$$f'(t) = \frac{f(t+\tau) - f(t-\tau)}{2\tau} - \frac{1}{6}\tau^2 f'''(\chi)$$

Ordinary differential equations

- We get the same mileage out of the centered three-point method as we did with general derivatives

- How can we use this to improve even further?

- Start again from equations of motion:

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \mathbf{a}(\mathbf{x}(t)) \\ \frac{d\mathbf{x}}{dt} &= \mathbf{v}(t)\end{aligned}$$

- Then write

$$\frac{\mathbf{v}(t+\tau) - \mathbf{v}(t-\tau)}{2\tau} + \mathcal{O}(\tau^2) = \mathbf{a}(\mathbf{x}(t))$$

Ordinary differential equations

- Strategy here is to FIRST update the velocity, and then use THAT to update the position
- So, position is centered in t, t+2tau:

$$\frac{\mathbf{x}(t+2\tau)-\mathbf{x}(t)}{2\tau}+\mathcal{O}(\tau^2)=\mathbf{v}(t)$$

- Then we get:

$$\begin{aligned}\mathbf{v}_{n+1} &= \mathbf{v}_{n-1}+2\tau\mathbf{a}_n+\mathcal{O}(\tau^3)\\ \mathbf{x}_{n+2} &= \mathbf{x}_n+2\tau\mathbf{v}_{n+1}+\mathcal{O}(\tau^3)\end{aligned}$$

Ordinary differential equations

- Strategy here is to FIRST update the velocity, and then use THAT to update the position
- So, position is centered in t, t+2tau:

$$\frac{\mathbf{x}(t+2\tau)-\mathbf{x}(t)}{2\tau}+\mathcal{O}(\tau^2)=\mathbf{v}(t)$$

- Then we get:

$$\begin{aligned}\mathbf{v}_{n+1} &= \mathbf{v}_{n-1}+2\tau\mathbf{a}_n+\mathcal{O}(\tau^3)\\ \mathbf{x}_{n+2} &= \mathbf{x}_n+2\tau\mathbf{v}_{n+1}+\mathcal{O}(\tau^3)\end{aligned}$$

- The “Leapfrog” scheme!
  - Leaps x(0), x(2), x(4),
    v(1), v(3), v(5)

Ordinary differential equations

- Final example : Verlet scheme
- Can use :

$$\begin{aligned} \frac{d^2\mathbf{x}}{dt^2} &= \mathbf{a}(\mathbf{x}(t)) \\ \frac{d\mathbf{x}}{dt} &= \mathbf{v}(t) \end{aligned}$$

- The discretization becomes :

$$\begin{aligned} \frac{\mathbf{x}_{n+1} - \mathbf{x}_{n-1}}{2\tau} + \mathcal{O}(\tau^2) &= \mathbf{v}_n \\ \frac{\mathbf{x}_{n+1} - \mathbf{x}_{n-1} - 2\mathbf{x}_n}{2\tau} + \mathcal{O}(\tau^2) &= \mathbf{a}_n \end{aligned}$$

Ordinary differential equations

- Rearranging terms, we get

$$\mathbf{v}_n = \frac{\mathbf{x}_{n+1} - \mathbf{x}_{n-1}}{2\tau} + \mathcal{O}\left(\tau^2\right)$$

$$\mathbf{x}_{n+1} = 2\mathbf{x}_n - \mathbf{x}_{n-1} + \tau^2 \mathbf{a}_n$$

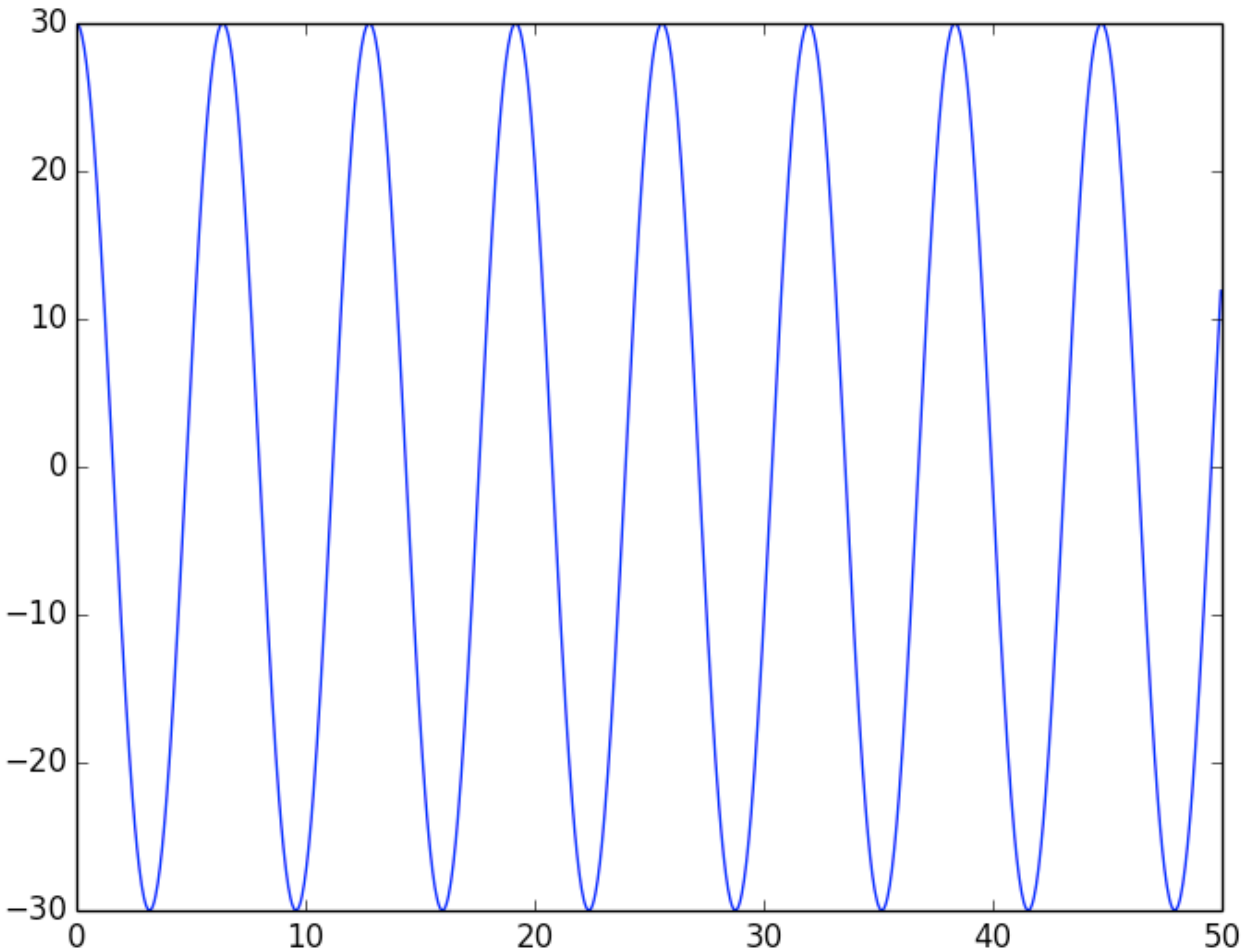
- If we know x0 and x1, we can then get the remaining values

Ordinary differential equations

- Verlet and leap-frog are, unfortunately, not “self starting”
- So, we usually use an Euler step in the backwards direction to start things off
- So, how does it do?

Ordinary differential equations

- Now solves pendulum problem nicely!



Next steps : Runge-Kutta

- To set up the Runge-Kutta algorithm, let's rewrite the Euler algorithm in vector notation :

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), t) \qquad \mathbf{x}(t+\tau) = \mathbf{x}(t) + \tau\mathbf{f}(\mathbf{x}, t)$$

- What are we really doing here? Consider our Taylor expansion again :

$$x(t+\tau) = x(t) + \tau\frac{dx(\psi)}{dt}$$

$$= x(t) + \tau f(x(\psi), \psi)$$

- We don't know which psi to take, but for some value it is exact
  - Euler : $\psi = t$
  - Euler-Cromer : $\psi = t$ (for velocity) $\psi = t + \tau$ (for position)
  - Now, Runge-Kutta : $\psi = t + \frac{1}{2}\tau$

Runge-Kutta

- But! We don't know $x(t + \frac{1}{2}\tau)$
- Estimate that from a simple Euler step

$$\mathbf{x}^*(t + \frac{1}{2}\tau) = \mathbf{x}(t) + \frac{1}{2}\tau\mathbf{f}(\mathbf{x}(t), t)$$

- Then, we get:

$$\mathbf{x}(t + \tau) = \mathbf{x}(t) + \tau\mathbf{f}(\mathbf{x}^*(t + \frac{1}{2}\tau), t + \frac{1}{2}\tau)$$

Runge-Kutta

- Can also try a two-point estimate for f:

$$\mathbf{x}(t+\tau) = \mathbf{x}(t) + \frac{1}{2}\tau\left[\mathbf{f}(\mathbf{x}(t),t) + \mathbf{f}(\mathbf{x}(t+\tau),t+\tau)\right]$$

  - Here we approximated:

$$= x(t) + \tau f(x(\psi),\psi)$$

  - Second order Runge-Kutta!

Runge-Kutta

- There is also a fourth-order Runge-Kutta

$$\mathbf{x}(t+\tau) = \mathbf{x}(t) + \frac{1}{6}\left[\mathbf{F}_1 + 2\mathbf{F}_2 + 2\mathbf{F}_3 + \mathbf{F}_4\right]$$

- where :

$$\begin{aligned}
\mathbf{F}_1 &= \mathbf{f}(\mathbf{x}, t) \\
\mathbf{F}_2 &= \mathbf{f}(\mathbf{x} + \frac{1}{2}\tau\mathbf{F}_1, t + \frac{1}{2}\tau) \\
\mathbf{F}_3 &= \mathbf{f}(\mathbf{x} + \frac{1}{2}\tau\mathbf{F}_2, t + \frac{1}{2}\tau) \\
\mathbf{F}_4 &= \mathbf{f}(\mathbf{x} + \tau\mathbf{F}_3, t + \tau)
\end{aligned}$$

- Better and better estimates for $= x(t) + \tau f(x(\psi), \psi)$

- Local truncation error $\mathcal{O}(\tau^5)$

Runge-Kutta

- Can go even further (8th order, whatever), but typically it is not required

- One thing that is nice is to have is an adaptive stepsize for the RK method :
  - If smooth, can have larger steps
  - If jagged, need smaller steps

- Adaptive RK computes information about its own performance, and can adjust as needed!

Runge-Kutta

- Adaptive heuristic :
  - Check full step
  - Check half steps
  - If accuracy starts to diverge, decrease the step size

- Loose a bit by doing the half-steps, but it pays off over the length of the ODE
  - Get accuracy around $\mathcal{O}(\tau^6)$

Runge-Kutta

- Pseudocode is :
  - Input x(t), t, tau, f(x,t;lambda), and lambda
  - Adapt dt to reach desired accuracy :
    - Evaluate F1,F2,F3,F4 at half-point until accuracy reached
  - Evaluate F1,F2,F3,F4 at full point
  - Compute x(t+tau) with fourth-order Runge-Kutta

Back to the pendulum

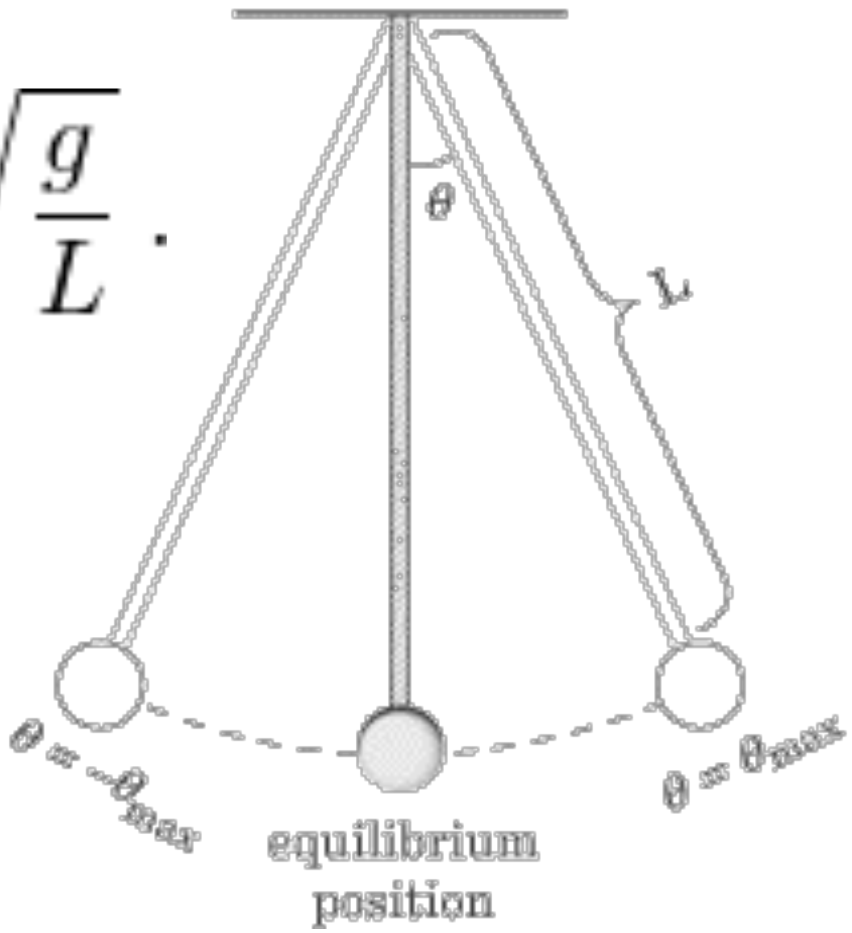
- But this time, with feeling!

- Let's try to expand beyond small perturbations

$$\frac{d^2\theta}{dt^2} = -\omega_0^2\theta - \gamma\frac{d\theta}{dt} + F_D\cos(\omega_D t)\,, \quad \omega_0 = \sqrt{\frac{g}{L}}\,.$$

replace by sin(theta)

$$\theta(t) = \frac{F_D\cos\left(\omega_D t + \phi\right)}{\sqrt{(\omega_0^2 - \omega_D^2)^2 + (\gamma\omega_D)^2}}\,.$$



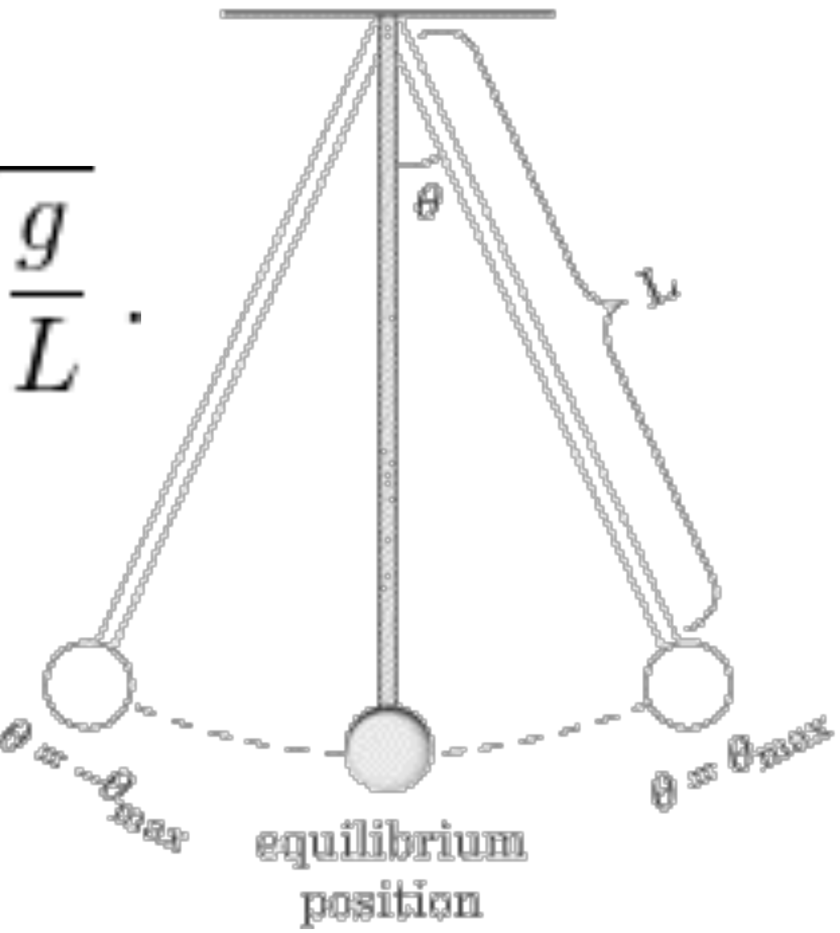
Back to the pendulum

- But this time, with feeling!

- Let's try to expand beyond small perturbations

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin\theta - \gamma\frac{d\theta}{dt} + F_D\cos\omega_D t$$

$$\omega_0 = \sqrt{\frac{g}{L}}\,.$$



$$\xcancel{\theta(t) = \frac{F_D\cos(\omega_D t + \phi)}{\sqrt{(\omega_0^2 - \omega_D^2)^2 + (\gamma\omega_D)^2}}}\,.$$

No analytic solution

Back to the pendulum

- What can we learn from this problem?

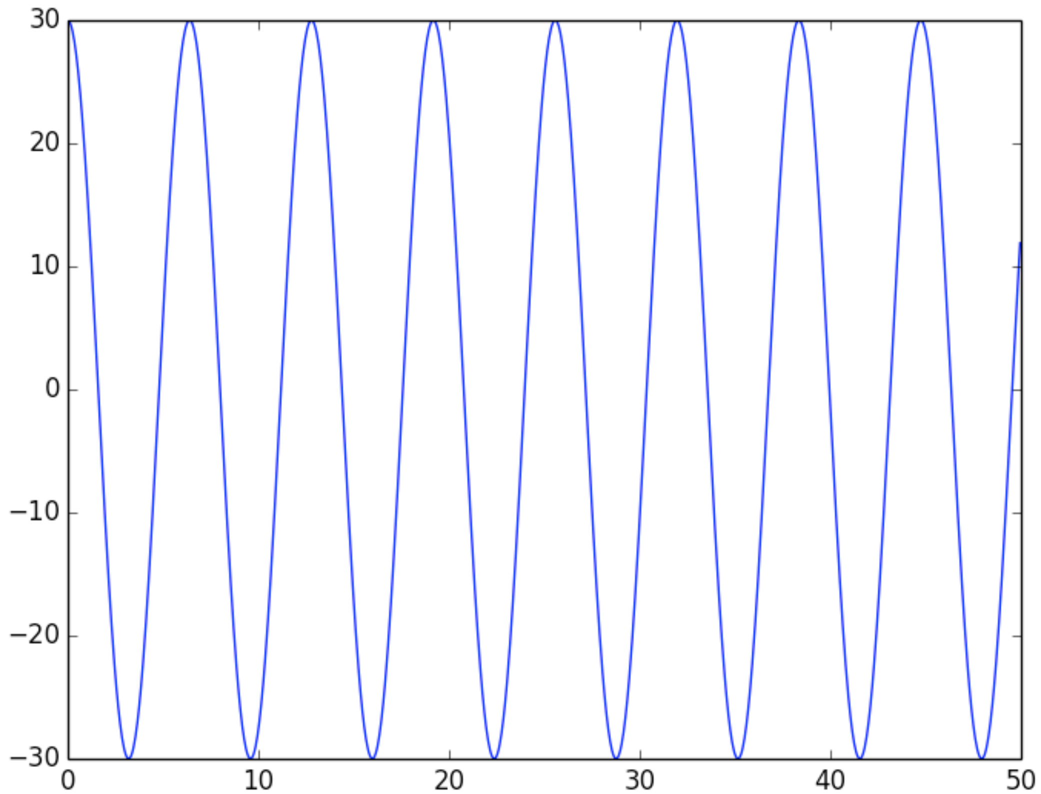
- Simplest "real world" case of nonlinear dynamical system
  - Perfect exhibition of chaotic dynamical system!

- Can see current research on this here :

- http://www.thphys.uni-heidelberg.de/~gasenzer/index.php?n1=teaching&n2=chaos

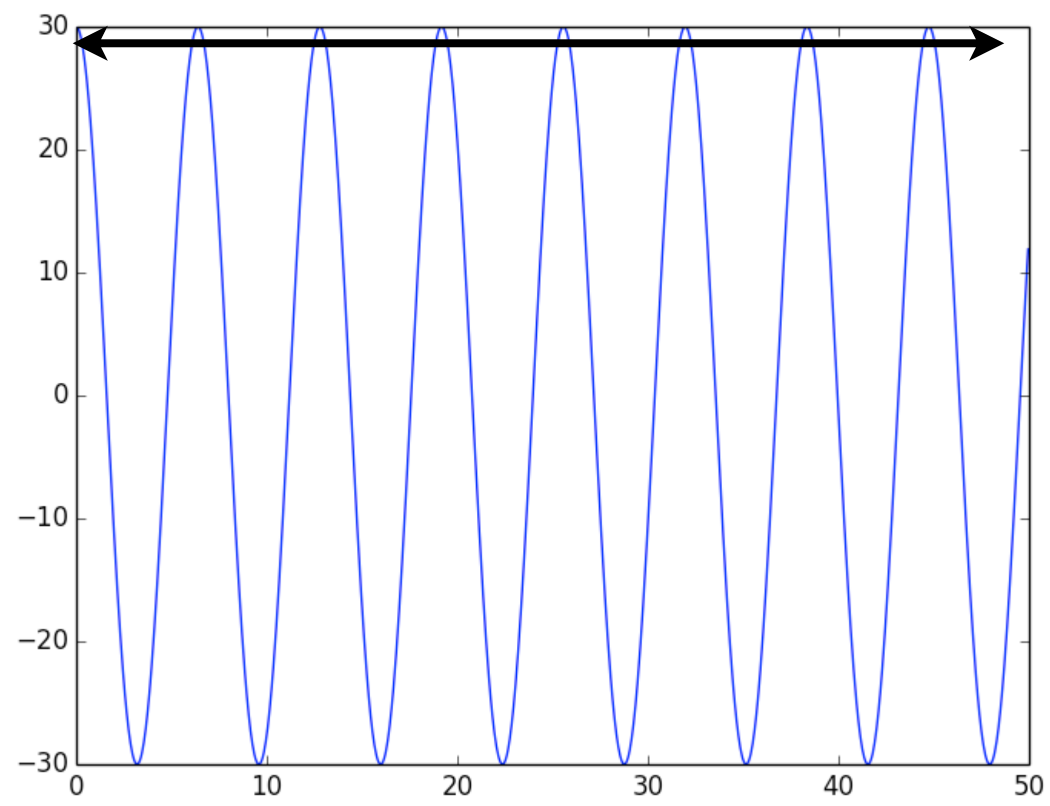
Nonlinear Dynamical Systems

- Before we get into that, need a bit of math
- Need to summarize the dynamical system graphically
- We're looking at full trajectories, but for linear systems, they're "boring"
  - They just go back and forth over the same path



Nonlinear Dynamical Systems

- To summarize this, let's just pick a specific point and see how often it goes through that point



  – Let's pick the maximum value, just for specificity

Nonlinear Dynamical Systems

- Easiest way to think about this is to characterize as a system of equations

- Recall our Verlet method:

$$\begin{aligned}
\mathbf{v}_n &= \frac{\mathbf{r}_{n+1} - \mathbf{r}_{n-1}}{2\tau} + \mathcal{O}(\tau^2) \\
\mathbf{x}_{n+1} &= 2\mathbf{x}_n - \mathbf{x}_{n-1} + \tau^2 \mathbf{a}_n
\end{aligned}$$

- Just generalize :

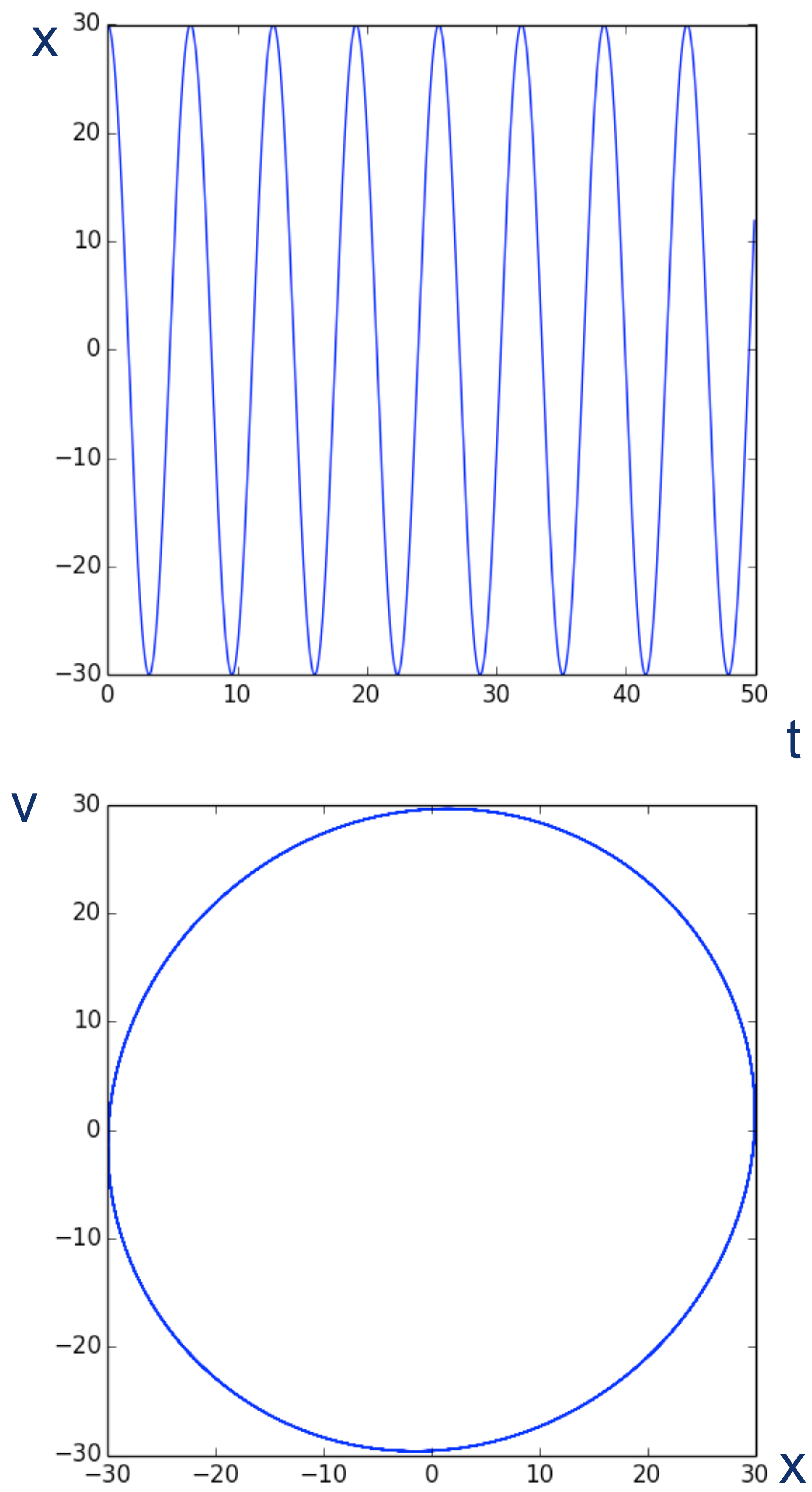
$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{v} \\
\dot{\mathbf{v}} &= \mathbf{f}(\mathbf{x}, \mathbf{v}; \lambda)
\end{aligned}$$

- Can then do lots of graphical analysis on (x,v)

Nonlinear Dynamical Systems

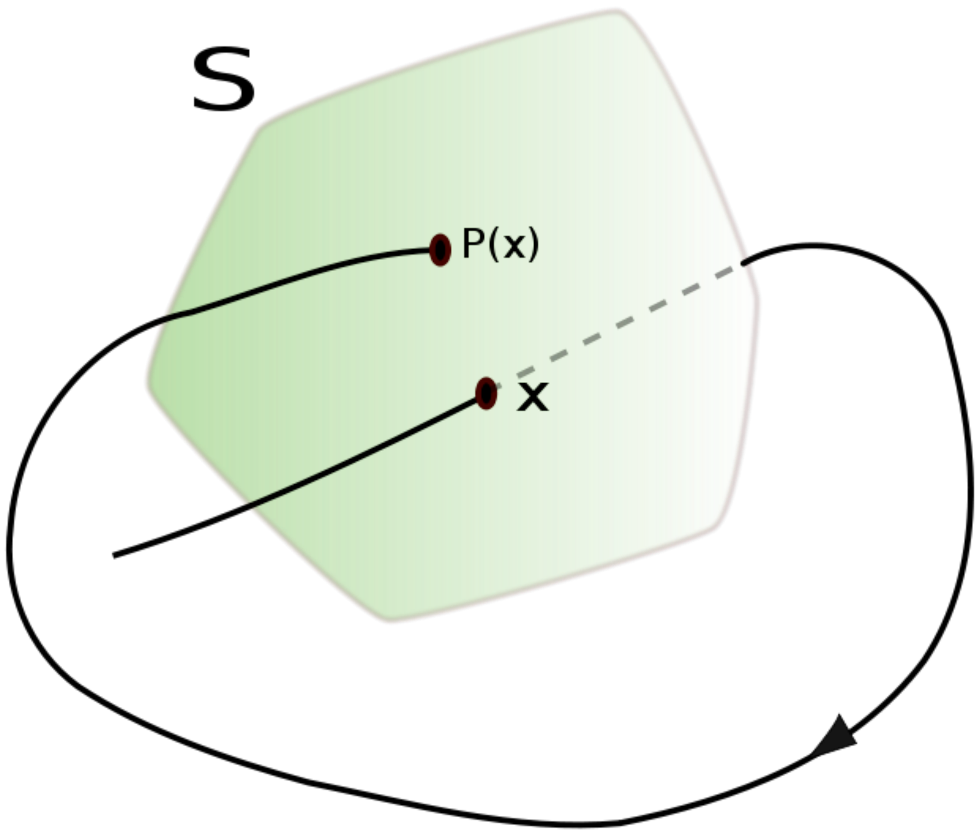
- If we plot (x,v) we get a simple ellipse for linear motion

- If we pick an hyperplane in the (x,v) space, we can easily understand where the dynamics lies
  - This is a Poincare map



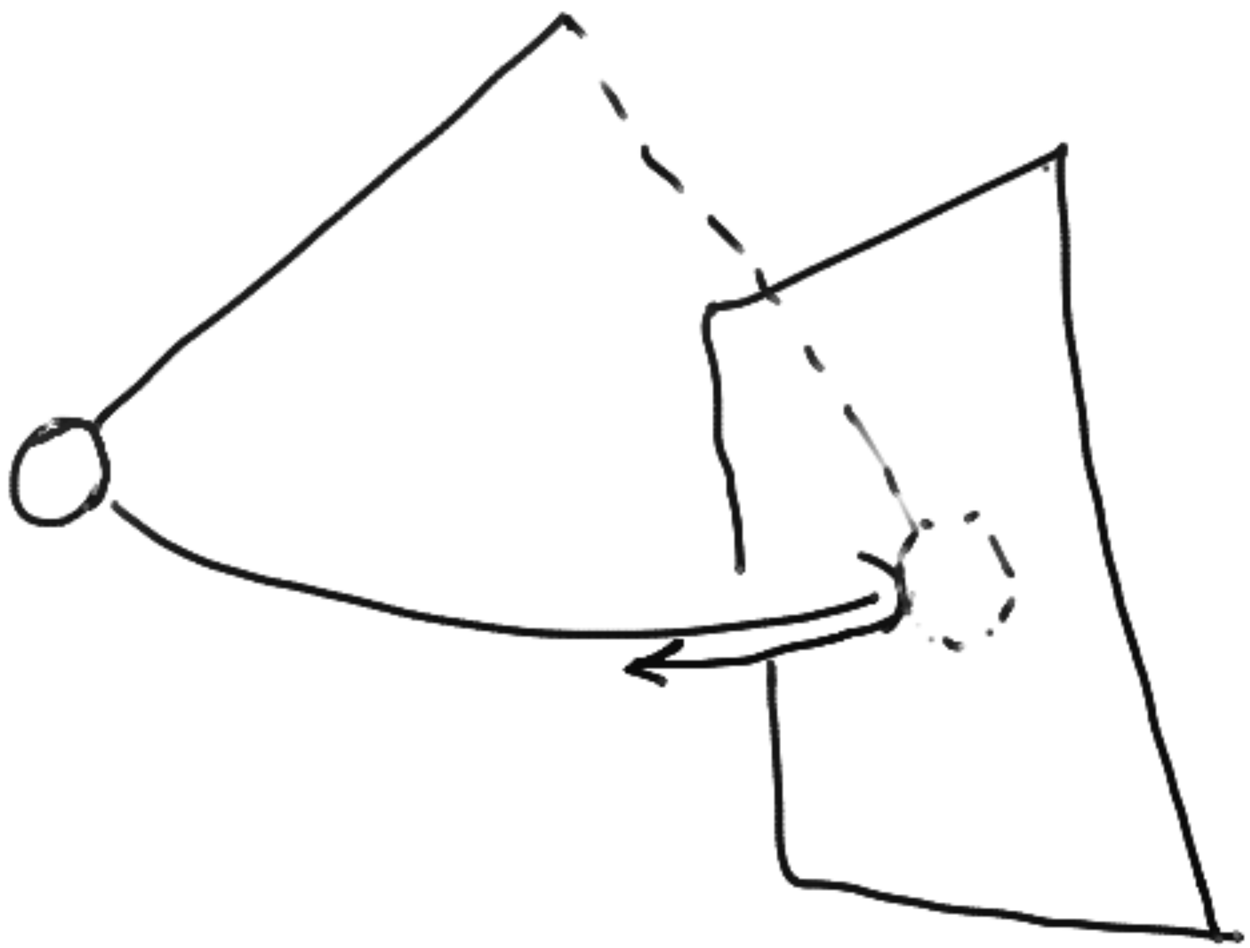
Nonlinear Dynamical Systems

- This is an interpretation as a discrete dynamical system in (n-1)-dim space
- Easier to analyze and visualize, but hard to determine them in practice
  - http://en.wikipedia.org/wiki/Poincaré_map



Pendulum

- Let’s look back at our pendulum
- Pick a Poincare section at a specific period (either natural frequency, or driving frequency)



Pendulum

- Equations are

Let $x = \theta$

$y = \dot{\theta}$

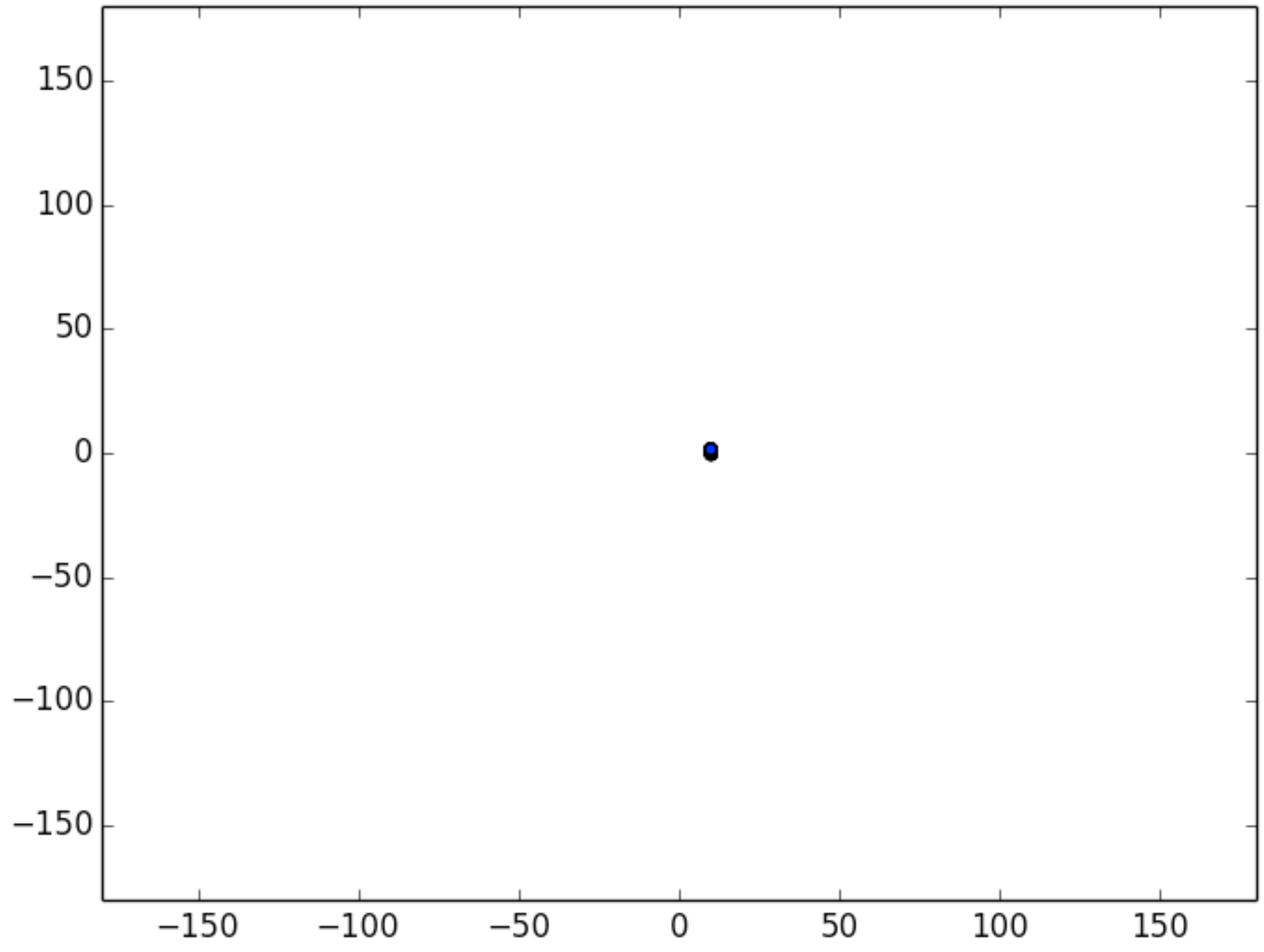
$t =$ time

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{t} \end{pmatrix} = \begin{pmatrix} y \\ -\omega_0^2 \sin x - \gamma y + F_D \cos \omega_D t \\ 1 \end{pmatrix}$$

Nonlinear Dynamical Systems

- Let's look back at our pendulum

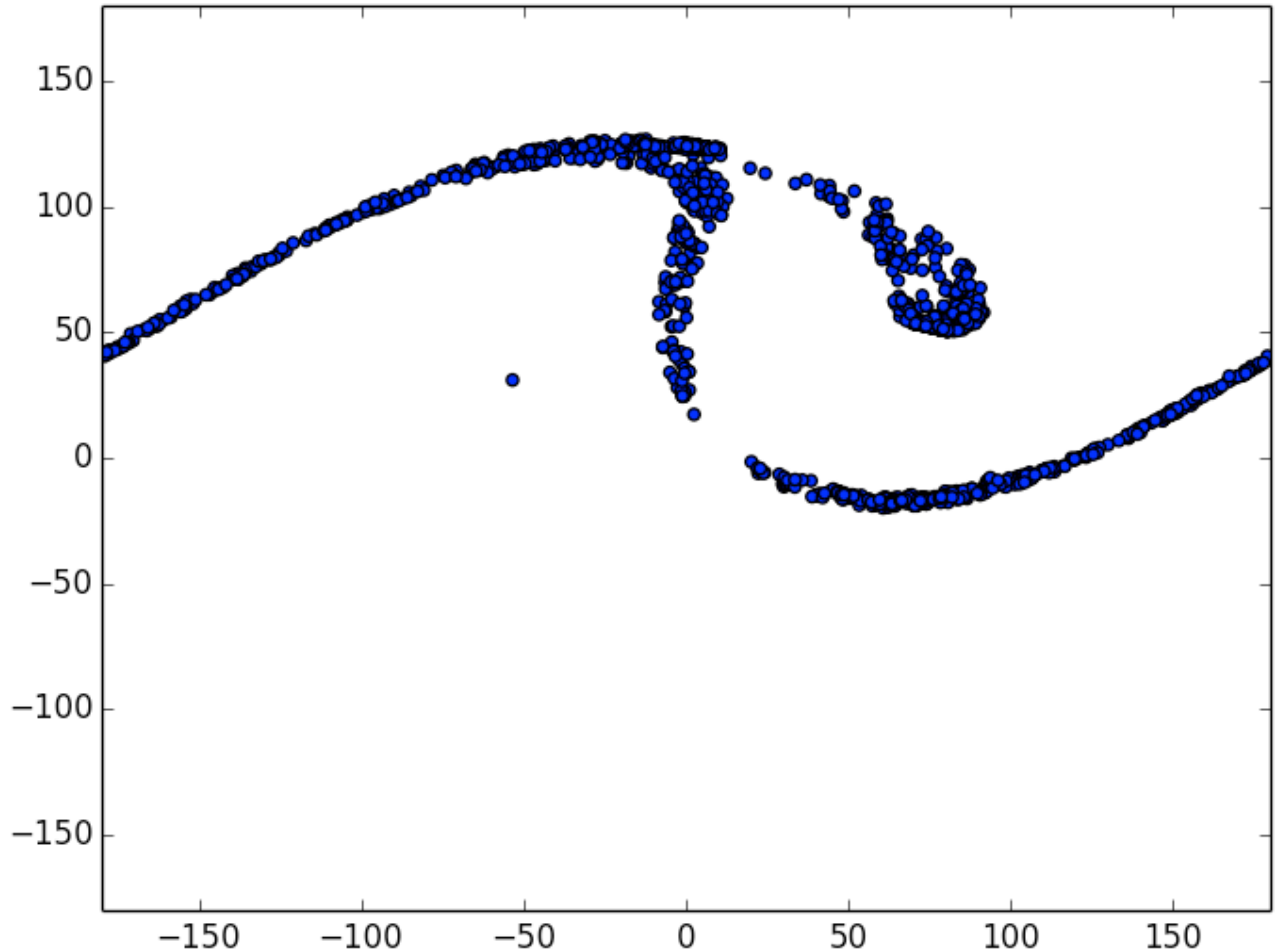
- For linear motion :



Nonlinear Dynamical Systems

- Let's look back at our pendulum

- For nonlinear motion : Chaos!



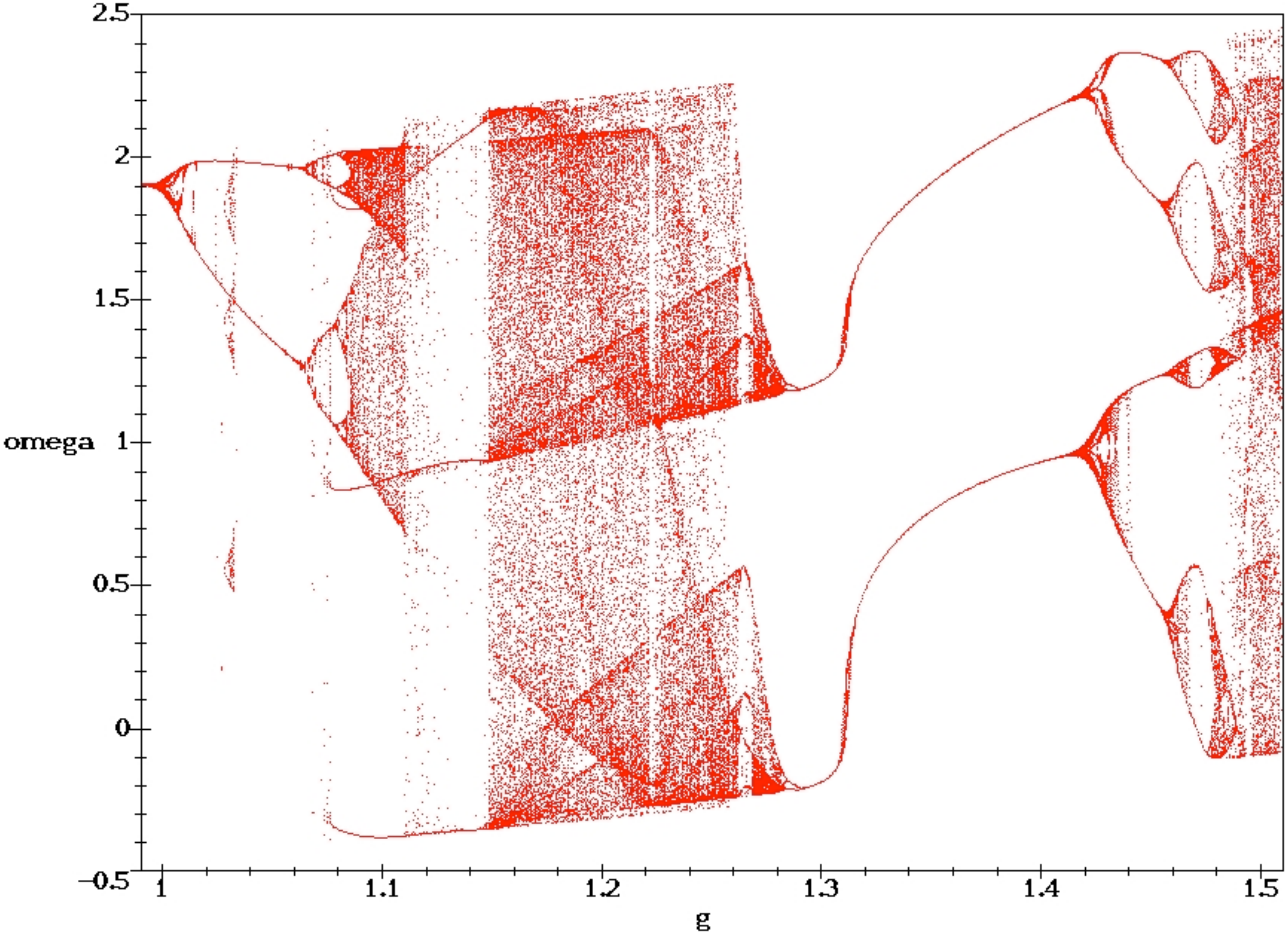
Bifurcation diagrams

- To characterize this, look at
  - http://en.wikipedia.org/wiki/Bifurcation_diagram

- Procedure :
  - Derive a Poincare section for your system
  - Project the section into a single axis (your choice)
  - Plot the projection versus the input strength

Bifurcation diagrams

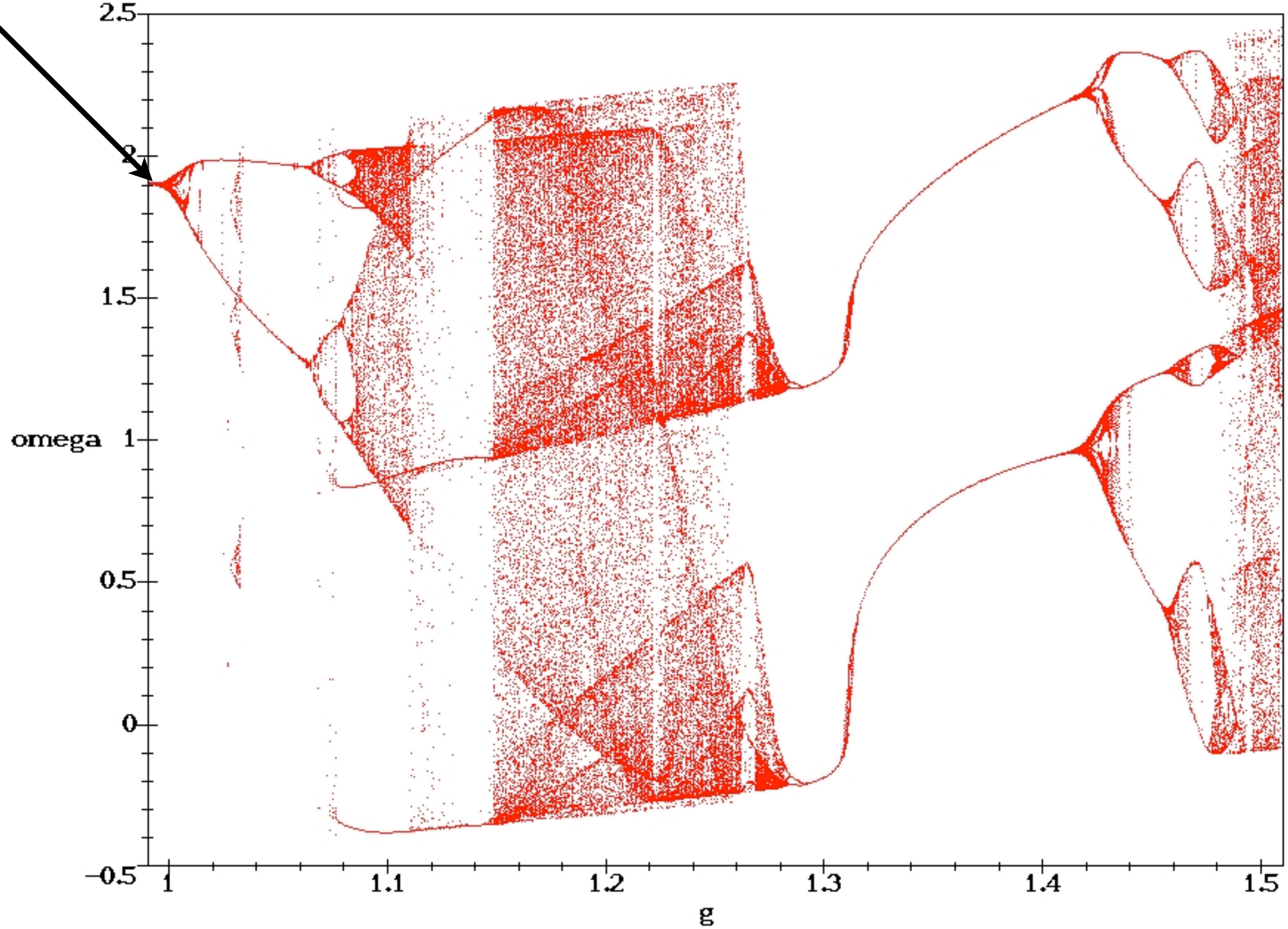
- What does this tell us?



Bifurcation diagrams

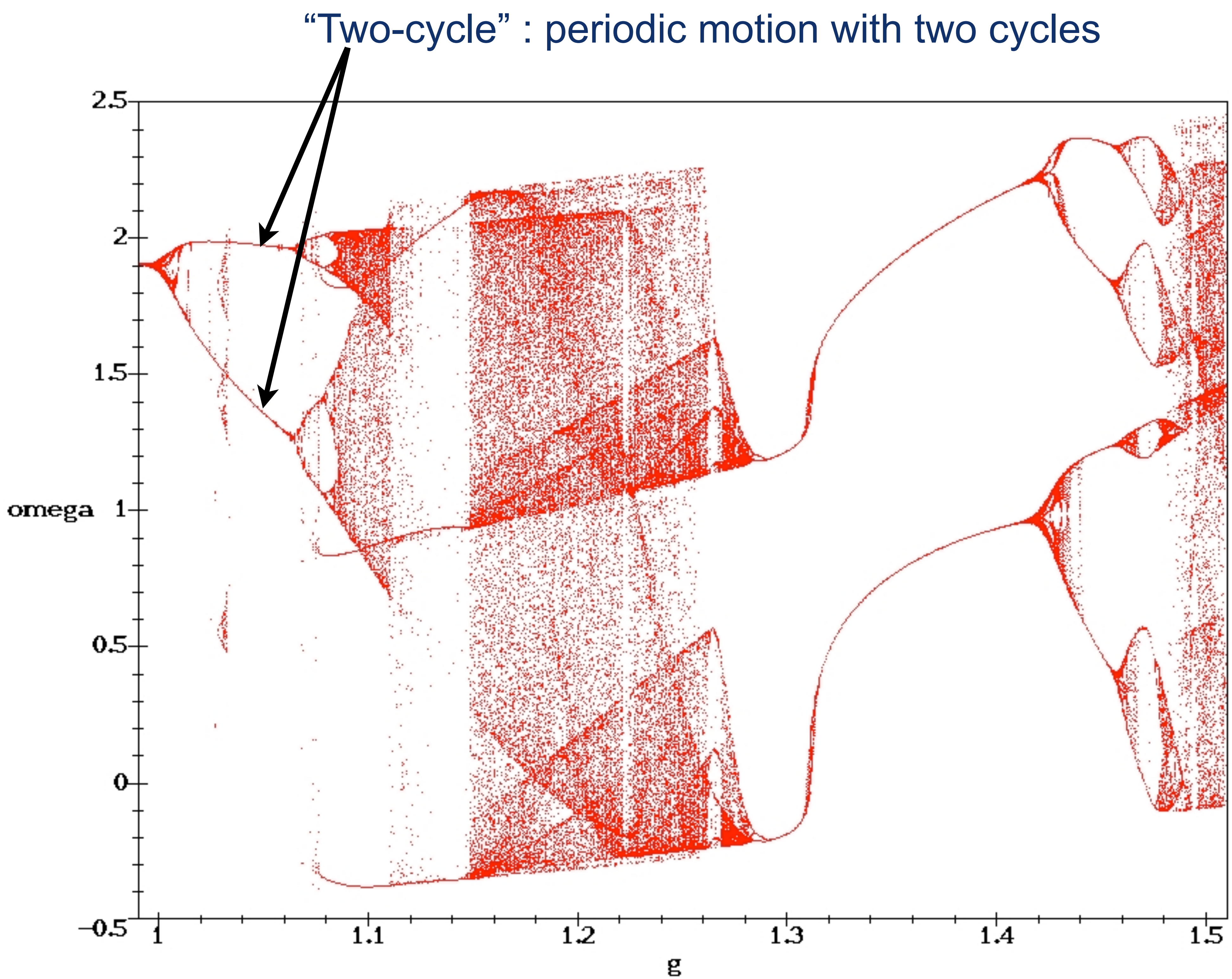
- What does this tell us?

"One-cycle" : periodic motion



Bifurcation diagrams

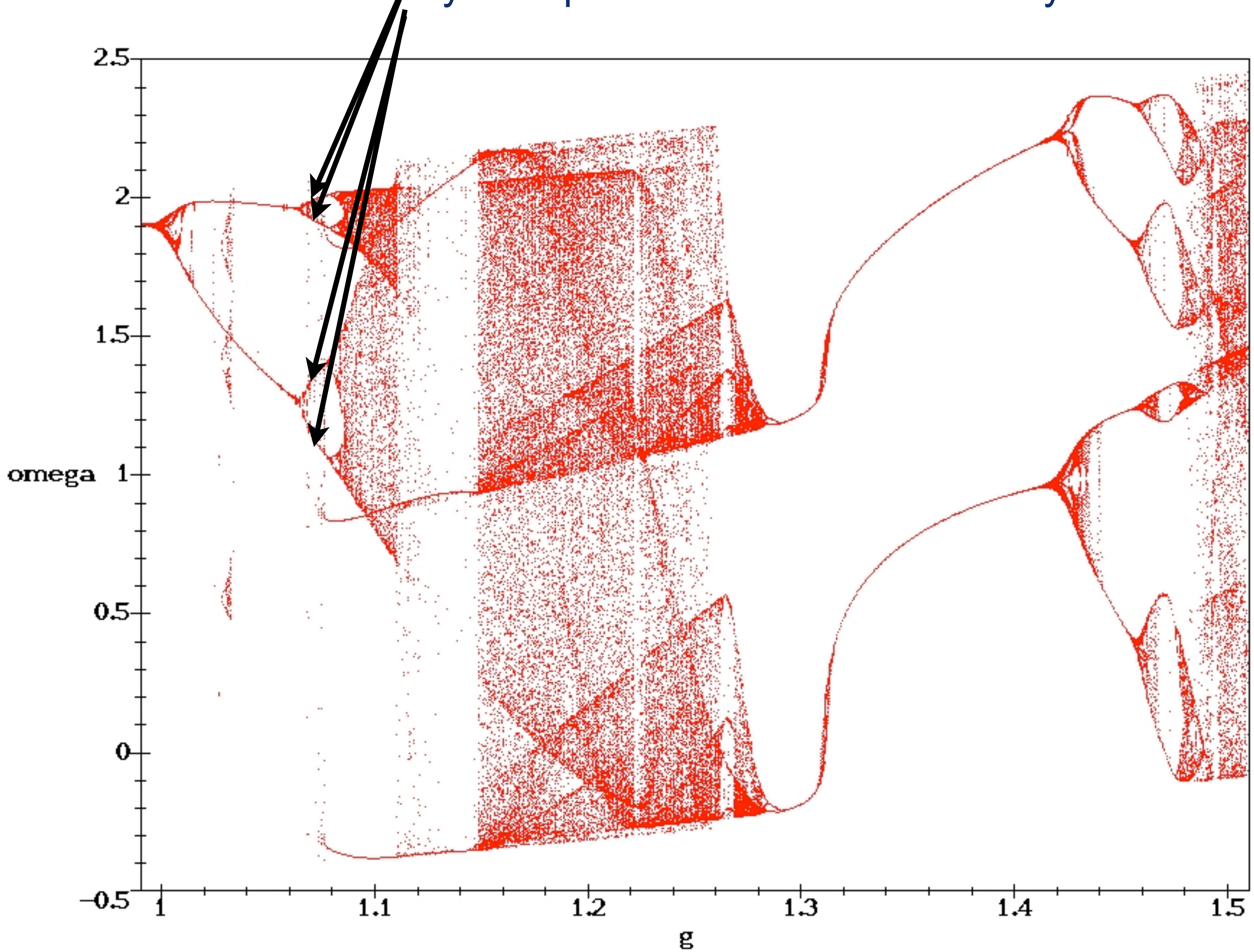
- What does this tell us?



Bifurcation diagrams

- What does this tell us?

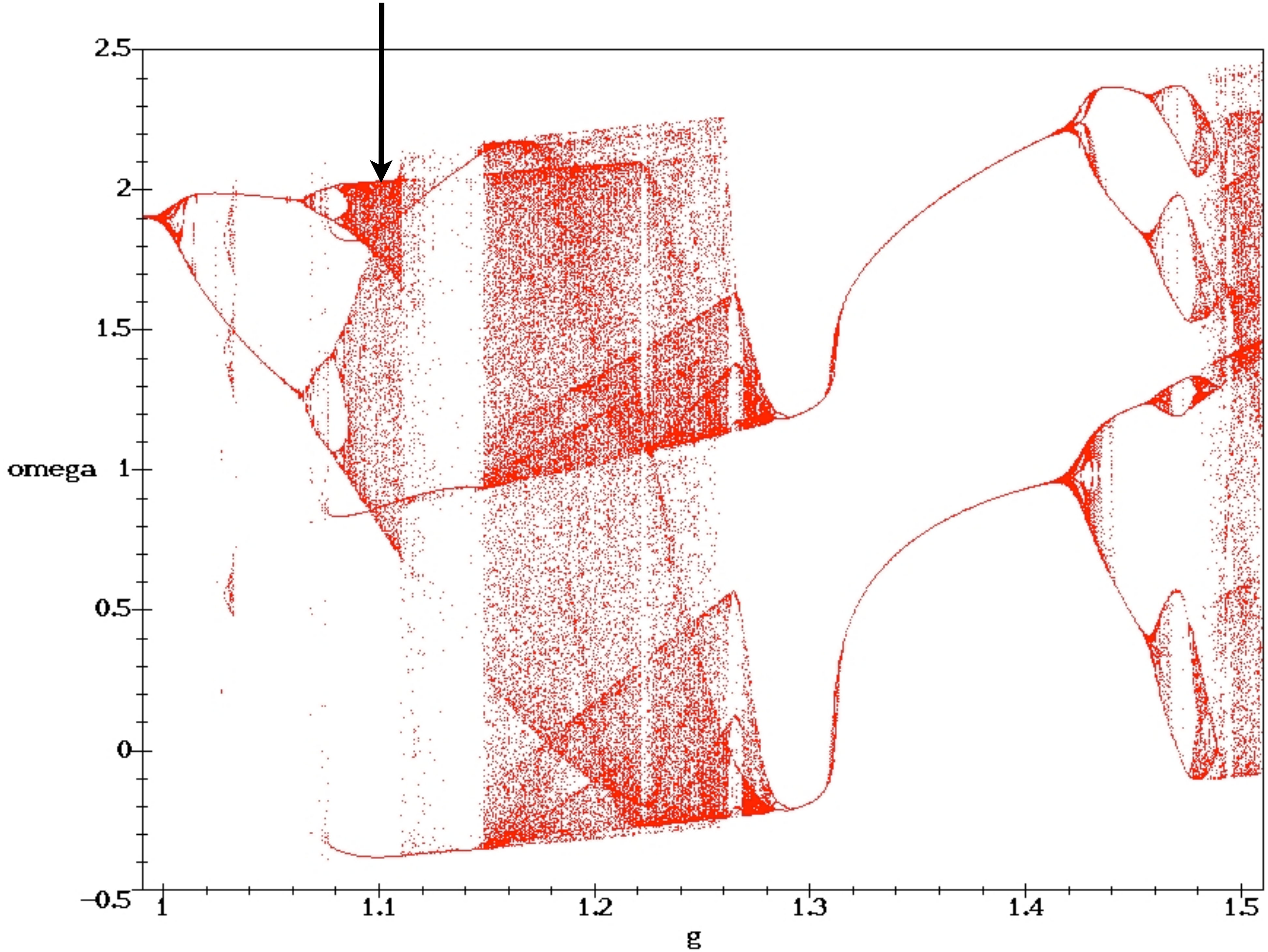
"Four-cycle" : periodic motion with four cycles



Bifurcation diagrams

- What does this tell us?

Chaos!



Bifurcation diagrams

- You’ll play around with these in your homework

Nonlinear Dynamical Systems

- A lot of physical systems exhibit chaos
  - Weather
  - Three-body gravitational problem
  - Cream in your coffee
  - etc

- Let’s take a look at a higher-dimensional one (Lorenz’s weather model)

Nonlinear Dynamical Systems

- Meteorologist Ed Lorenz showed in the 1960's that some simple systems were inherently unpredictable (numerically) due to extreme sensitivity of initial conditions
  - http://en.wikipedia.org/wiki/Lorenz_system
  - The butterfly effect!

- Model is :

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x),$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y,$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z.$$

x = rate of convection overturning

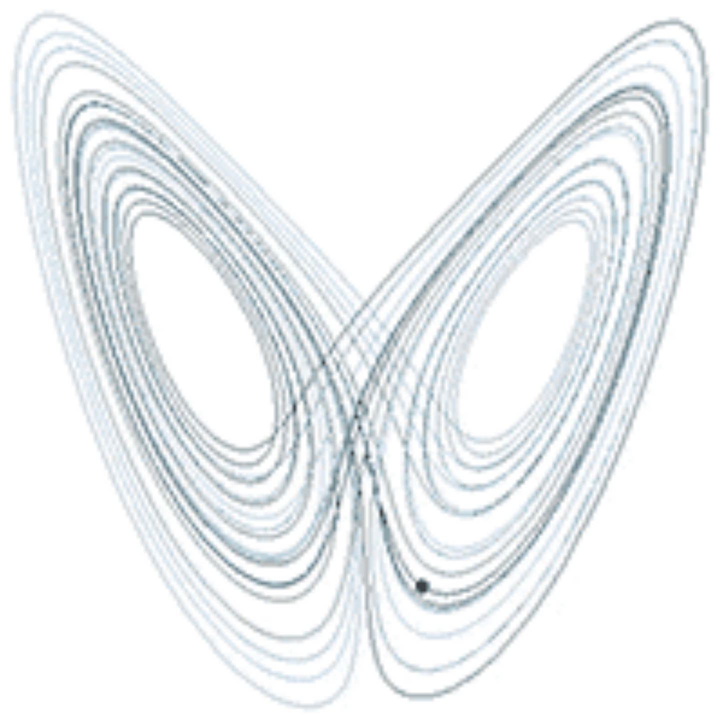
y,z =horizontal and vertical temperature gradients

sigma, rho, beta are positive constants

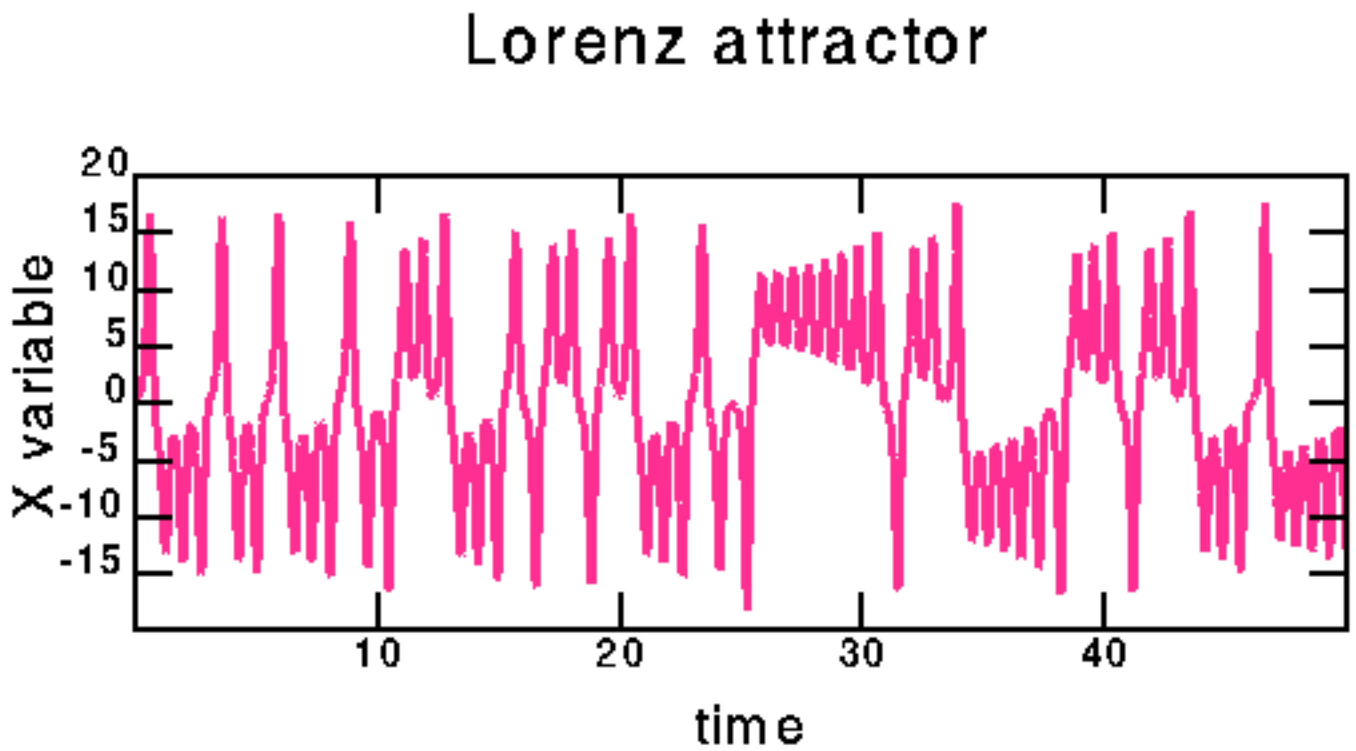
  - Simplified equations describing fluid circulation in a shallow layer of fluid, heated from below and cooled from above

Nonlinear Dynamical Systems

- To cut to the chase, the trajectories look like this :



- The "Lorenz attractor"

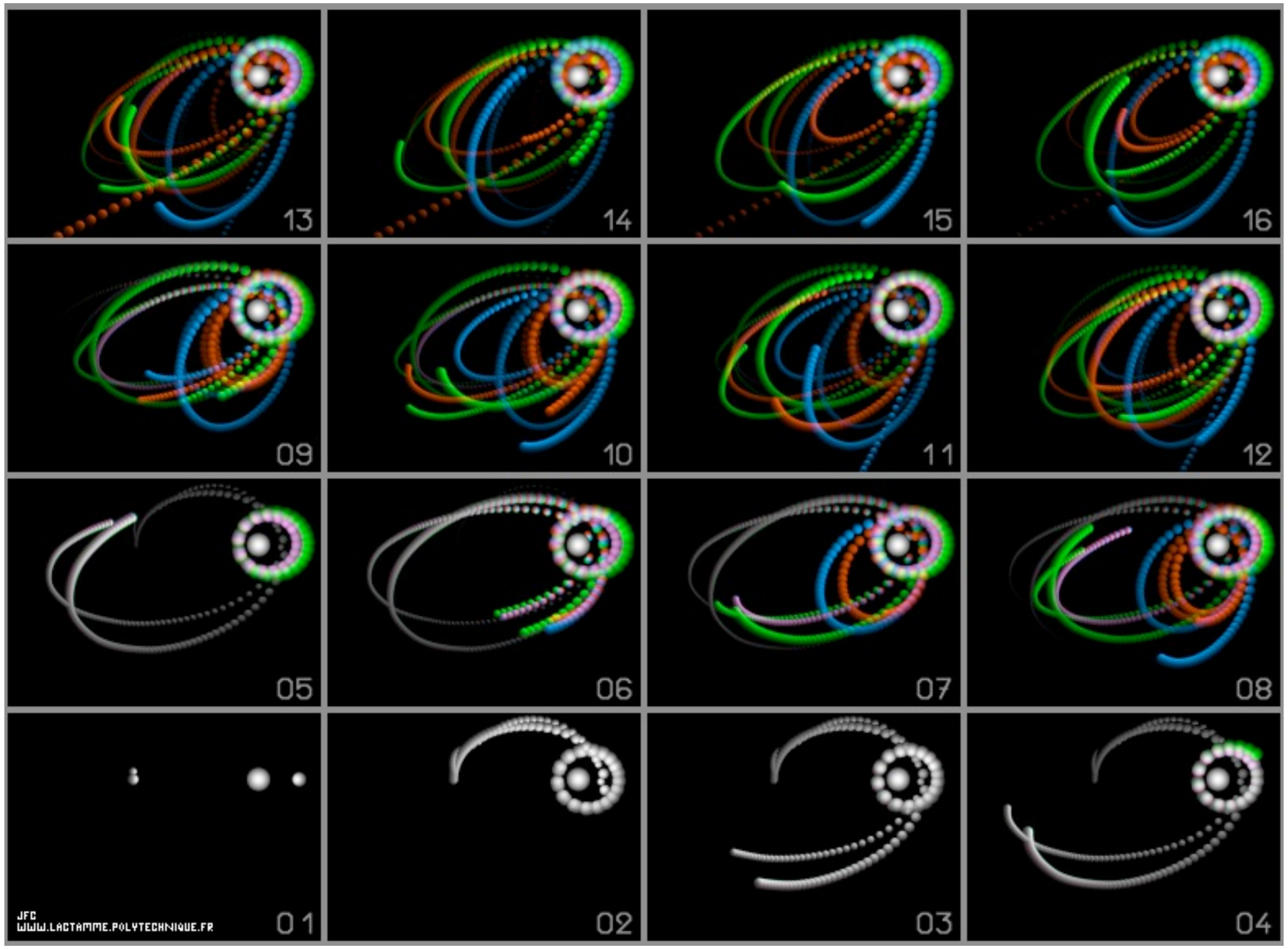


- If you look in a single dimension :

http://www.physics.emory.edu/~weeks/research/tseries1.html

Celestial dynamics

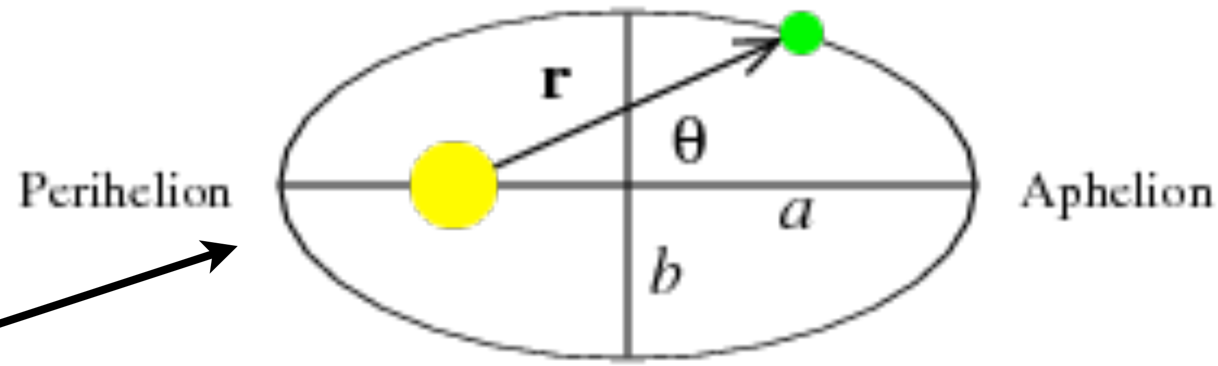
- We can now turn to another popular ODE application : celestial dynamics and the N-body problem



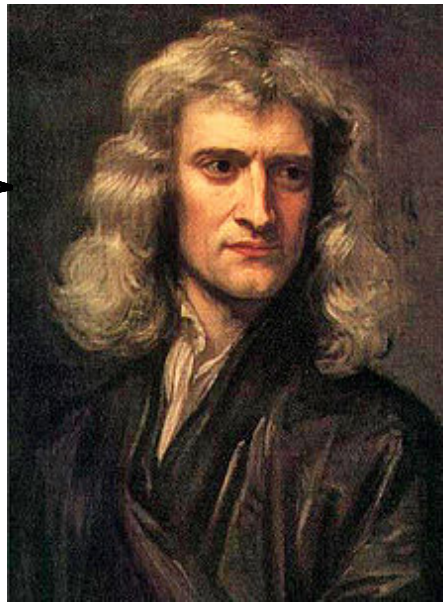
http://www.lactamme.polytechnique.fr/images/NCOR.73.16.D/display.html

Celestial dynamics

- 2-body dynamics : trivial, solved in the 17th century, what we give to 1st year undergrads

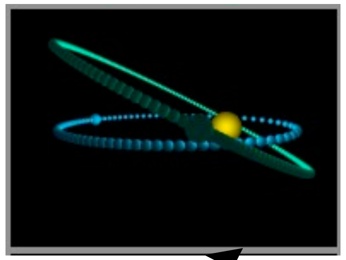


$$\mathbf{F} = \frac{-Gm_1m_2}{r^2}\hat{\mathbf{r}}$$



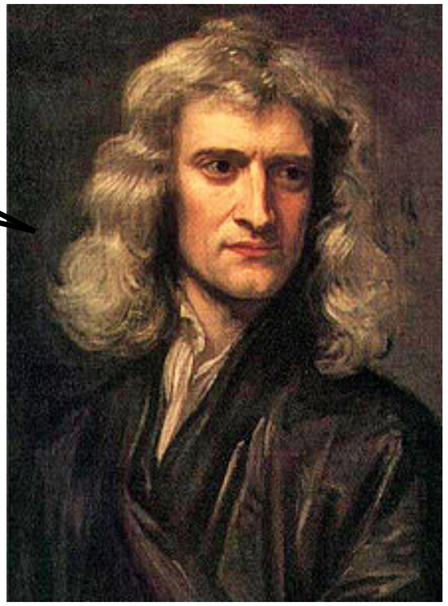
Celestial dynamics

- 3-body dynamics : intractably difficult, chaotic dynamical system, people spend their entire lives on this problem



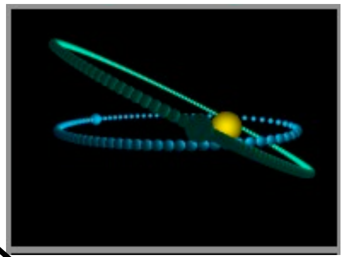
Divine nudges from time to time to maintain stability

(paraphrased)

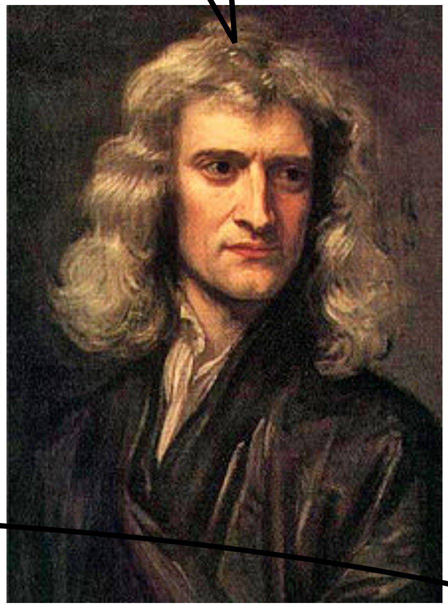


Celestial dynamics

- 3-body dynamics : intractably difficult, chaotic dynamical system, people spend their entire lives on this problem



Laplace

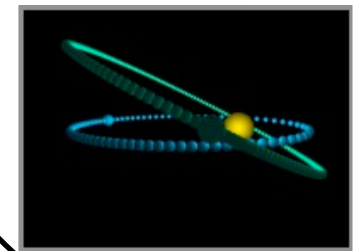


http://arxiv.org/abs/1503.06861v1

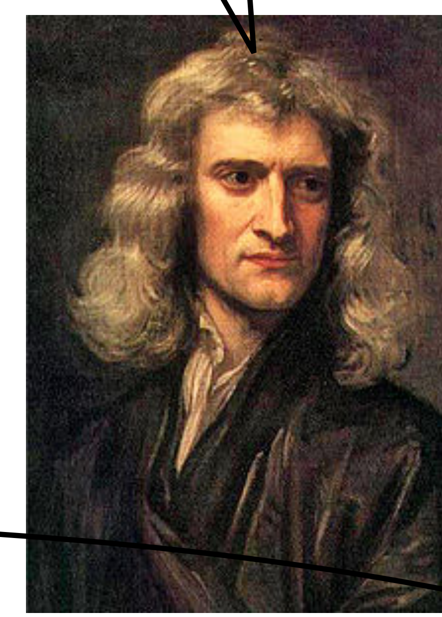
These phenomena and some others similarly explained, lead us to believe that everything depends on these laws by relations more or less hidden, but of which it is wiser to admit ignorance, rather than to substitute imaginary causes solely in order to quiet our uneasiness about the origin of the things that interest us.

Celestial dynamics

- 3-body dynamics : intractably difficult, chaotic dynamical system, people spend their entire lives on this problem



Laplace



Translation : It's okay to just say "I don't know yet"!

Celestial dynamics

- Laplace thought he actually showed the stability of 3-body systems
  - Was eventually shown to be wrong by Poincare

$$\frac{d^2\mathbf{r}_1}{dt^2} = -Gm_2\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3} - Gm_3\frac{\mathbf{r}_1-\mathbf{r}_3}{|\mathbf{r}_1-\mathbf{r}_3|^3}$$

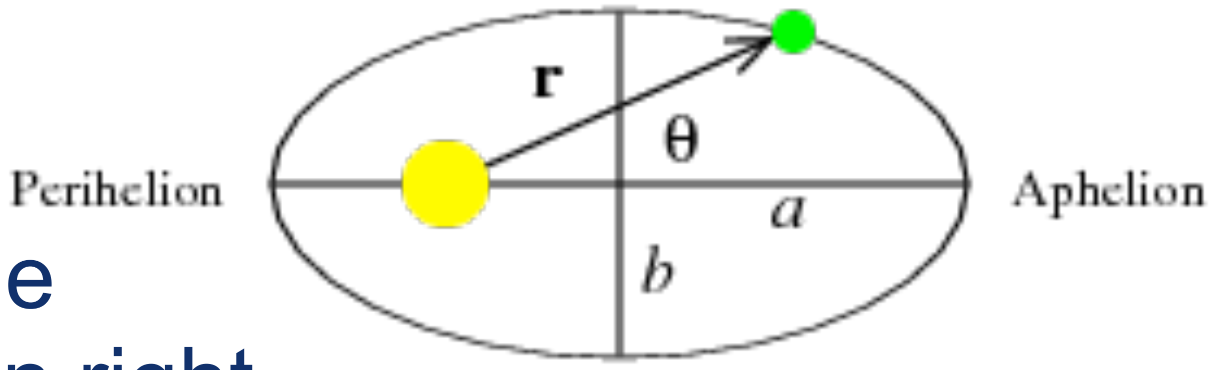
$$\frac{d^2\mathbf{r}_2}{dt^2} = -Gm_3\frac{\mathbf{r}_2-\mathbf{r}_3}{|\mathbf{r}_2-\mathbf{r}_3|^3} - Gm_1\frac{\mathbf{r}_2-\mathbf{r}_1}{|\mathbf{r}_2-\mathbf{r}_1|^3}$$

$$\frac{d^2\mathbf{r}_3}{dt^2} = -Gm_1\frac{\mathbf{r}_3-\mathbf{r}_1}{|\mathbf{r}_3-\mathbf{r}_1|^3} - Gm_2\frac{\mathbf{r}_3-\mathbf{r}_2}{|\mathbf{r}_3-\mathbf{r}_2|^3}$$

  - 9 degrees of freedom (x,y,z of each position)
  - 7 independent conserved quantities :
    - Energy, total momentum, total angular momentum
  - Underconstrained problem, so cannot solve exactly except in certain circumstances

Celestial dynamics

- We turn to numerics!



- First, let's check with a simple 2-d version to get our intuition right

$$\frac{d}{dt}\begin{pmatrix} t \\ x \\ y \\ v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 1 \\ v_x \\ v_y \\ -G(m_1+m_2)\,x/r^3 \\ -G(m_1+m_2)\,y/r^3 \end{pmatrix}$$

Celestial dynamics

- Then, let's work on a co-planar solution

$$\frac{d^2\mathbf{r}_1}{dt^2} = -Gm_2\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3} - Gm_3\frac{\mathbf{r}_1-\mathbf{r}_3}{|\mathbf{r}_1-\mathbf{r}_3|^3}$$
$$\frac{d^2\mathbf{r}_2}{dt^2} = -Gm_3\frac{\mathbf{r}_2-\mathbf{r}_3}{|\mathbf{r}_2-\mathbf{r}_3|^3} - Gm_1\frac{\mathbf{r}_2-\mathbf{r}_1}{|\mathbf{r}_2-\mathbf{r}_1|^3}$$
$$\frac{d^2\mathbf{r}_3}{dt^2} = -Gm_1\frac{\mathbf{r}_3-\mathbf{r}_1}{|\mathbf{r}_3-\mathbf{r}_1|^3} - Gm_2\frac{\mathbf{r}_3-\mathbf{r}_2}{|\mathbf{r}_3-\mathbf{r}_2|^3}$$

- Problem is, this is unstable : you don't expect numerical or actual orbits to stay collinear very long

Celestial dynamics

- Can look at a restricted case where the third body's mass is zero

$$\frac{d^2\mathbf{r}_1}{dt^2} = -Gm_2\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}$$

$$\frac{d^2\mathbf{r}_2}{dt^2} = -Gm_1\frac{\mathbf{r}_2-\mathbf{r}_1}{|\mathbf{r}_2-\mathbf{r}_1|^3}$$

$$\frac{d^2\mathbf{r}_3}{dt^2} = -Gm_1\frac{\mathbf{r}_3-\mathbf{r}_1}{|\mathbf{r}_3-\mathbf{r}_1|^3} - Gm_2\frac{\mathbf{r}_3-\mathbf{r}_2}{|\mathbf{r}_3-\mathbf{r}_2|^3}$$

- The third vector equation is decoupled from the first two, so we have 2-body Kepler motion again!

Celestial dynamics

- Can find particular exact solutions of the 2-body equations with a circular orbit in the CM frame of the two heavier ones
  - Also restrict to 2-d
- In units where the radius of the circular orbit =1 , and time are in in units of the inverse angular speed :

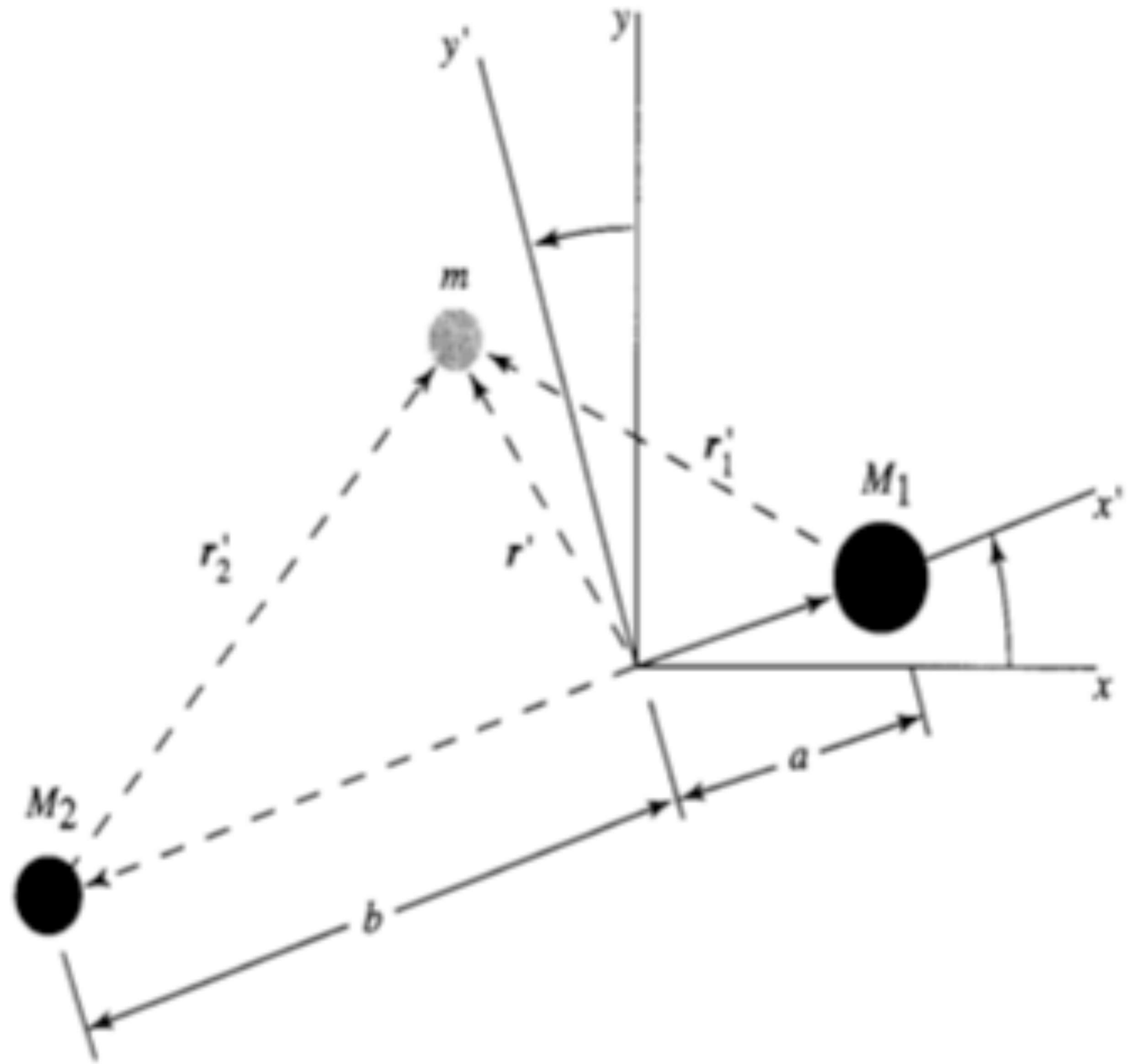
$$|\mathbf{r}_1 - \mathbf{r}_2| = 1 \ , \qquad \omega = \frac{2\pi}{T} = \sqrt{\frac{G(m_1 + m_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}} = 1 \ , \qquad G(m_1 + m_2) = 1$$

- All constants can then be expressed in terms of a single parameter:

$$\alpha \equiv \frac{m_2}{m_1 + m_2} \ , \qquad Gm_2 = \frac{Gm_2}{G(m_1 + m_2)} = \alpha \ , \qquad Gm_1 = 1 - \alpha$$

Celestial dynamics

- Inertial frame (fixed to stars)
  - "Sidereal" frame
  - Rotation like a rigid dumbbell
- Can also choose a rotating coordinate system (Figure 7.4.1 of Fowles/Cassiday)
  - "Synodic" frame
  - Two heavy masses are at rest (and have Coriolis forces), light mass isn't
  - Easy to plot total motion



Celestial dynamics

- If the sidereal and synodic axes coincide at t=0, coordinates of the lowest mass are related by :

$$x(t) = x'(t)\cos t - y'(t)\sin t$$
$$y(t) = x'(t)\sin t + y'(t)\cos t$$

- Take the derivatives :

$$\ddot{x}\cos t + \ddot{y}\sin t = \ddot{x}' - x' - 2\dot{y}'$$
$$-\ddot{x}\sin t + \ddot{y}\cos t = \ddot{y}' - y' + 2\dot{x}'$$

- Solve for synodic acceleration:

$$\ddot{x}\cos t + \ddot{y}\sin t = \ddot{x}' - x' - 2\dot{y}'$$
$$-\ddot{x}\sin t + \ddot{y}\cos t = \ddot{y}' - y' + 2\dot{x}'$$

Celestial dynamics

- Equations of motion in the synodic frame are :

$$\ddot{x}' = -\frac{(1-\alpha)(x'-\alpha)}{((x'-\alpha)^2 + y'^2)^{3/2}} - \frac{\alpha(x'+1-\alpha)}{((x'+1-\alpha)^2 + y'^2)^{3/2}} + x' + 2\dot{y}'$$

$$\ddot{y}' = -\frac{(1-\alpha)y'}{((x'-\alpha)^2 + y'^2)^{3/2}} - \frac{\alpha y'}{((x'+1-\alpha)^2 + y'^2)^{3/2}} + y' - 2\dot{x}'$$

Gravitational force from the heavy masses

Centripetal acceleration

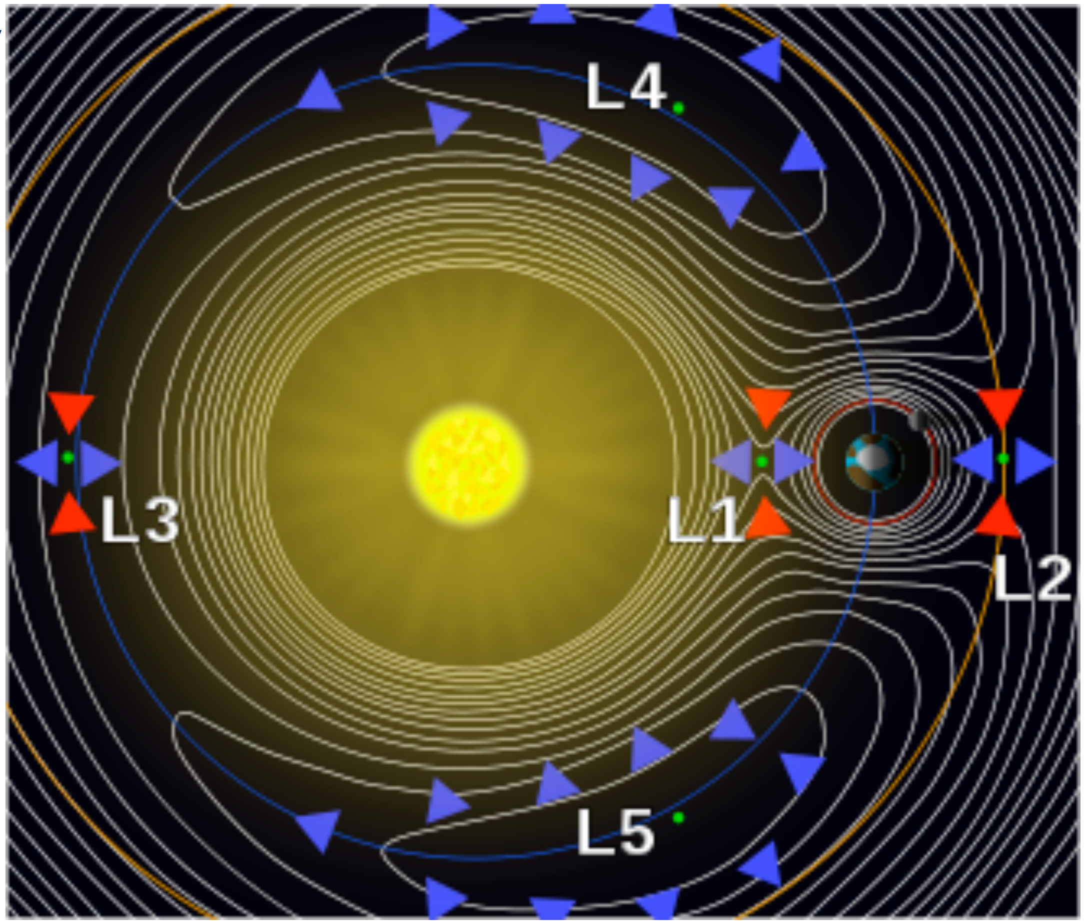
Coriolis acceleration

Celestial dynamics

- The effective potential in the rotating frame is:

$$V(x', y') = -\frac{1-\alpha}{\sqrt{(x'-\alpha)^2 + y'^2}} - \frac{\alpha}{\sqrt{(x'+1-\alpha)^2 + y'^2}} - \frac{x'^2 + y'^2}{2}$$

  - The Coriolis acceleration, however, depends on the velocity so cannot be written in this way



- The effective potential itself has 5 extrema
  - Two maxima, three saddle points
- Here called Lagrangian points
  - http://en.wikipedia.org/wiki/Lagrangian_point

Celestial dynamics

- If the third mass starts at a Lagrangian point, it stays there forever
- It would seem everything else is unstable
  - But! The Coriolis force then “kicks in” when the mass is moving, and you get bands of stability near the LP’s and between the LP’s, some still wander to infinity
  - Can also get chaotic motion!

Celestial dynamics

- Can it be solved exactly?
- Tertiary mass moves in 2d, so have 2 degrees of freedom
  - Similar to Kepler problem
- However, for Kepler problem : energy + angular momentum conserved
- Here : Coriolis force ==> not a conservative force, depends on velocity
- One conserved quantity, however, is the Jacobi Integral :
  - http://en.wikipedia.org/wiki/Jacobi_integral

$$C = x'^2 + y'^2 + \frac{2(1-\alpha)}{\sqrt{(x'-\alpha)^2 + y'^2}} + \frac{2\alpha}{\sqrt{(x'+1-\alpha)^2 + y'^2}} - \dot{x}'^2 - \dot{y}'^2$$

Celestial dynamics

- See here for some fun discussions about this topic :
  - http://pages.physics.cornell.edu/~sethna/teaching/sss/jupiter/jupiter.htm